Model diagnostics

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STA 210

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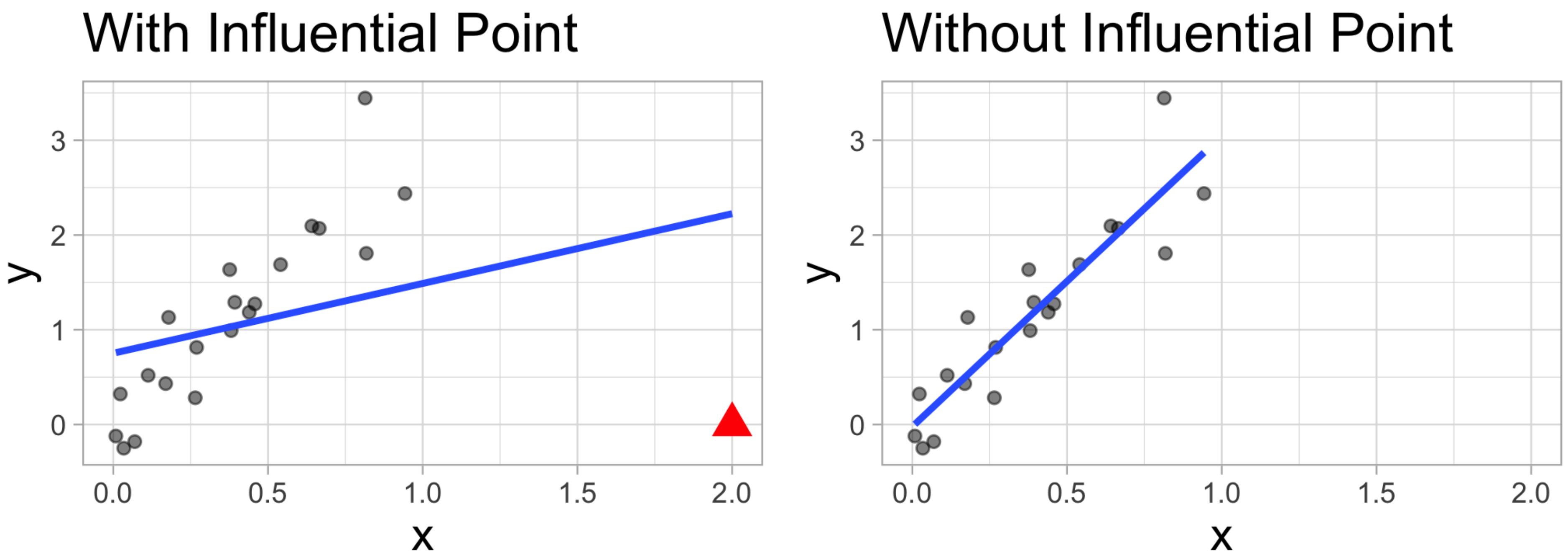
Topics

- Identifying influential points
  - Leverage
  - Standardized residuals
  - Cook's Distance

Influential points

Influential Point

An observation is **influential** if removing it substantially changes the coefficients of the regression model



Influential points

- Influential points have a large impact on the coefficients and standard errors used for inference
- These points can sometimes be identified in a scatterplot if there is only one predictor variable
  - This is often not the case when there are multiple predictors
- We will use measures to quantify an individual observation's influence on the regression model
  - **leverage**, **standardized residuals**, and **Cook's distance**

Model diagnostics in R

Use the **`augment`** function in the broom package to output the model diagnostics (along with the predicted values and residuals)

- response and predictor variables in the model
- **`.fitted`**: predicted values
- **`.se.fit`**: standard errors of predicted values
- **`.resid`**: residuals
- **`.hat`**: leverage
- **`.sigma`**: estimate of residual standard deviation when the corresponding observation is dropped from model
- **`.cooksd`**: Cook's distance
- **`.std.resid`**: standardized residuals

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Example: Average SAT scores by state

- This data set contains the average SAT score (out of 1600) and other variables that may be associated with SAT performance for each of the 50 U.S. states. The data is based on test takers for the 1982 exam.

- **Response** - **`SAT`**: average total SAT score

- **Predictor** - **`Public`**: percentage of test-takers who attended public high schools

Data comes from **`case1201`** data set in the **`Sleuth3`** package

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Model

```
sat_scores <- Sleuth3::case1201
```

```
sat_model <- lm(SAT ~ Public, data = sat_scores)
tidy(sat_model) %>%
  kable(digits = 3)
```

term	estimate	std.error	statistic	p.value
(Intercept)	994.971	84.807	11.732	0.000
Public	-0.579	1.037	-0.559	0.579

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SAT: Augmented Data

```
sat_aug <- augment(sat_model) %>%
  #add observation number for plots
  mutate(obs_num = row_number())
```

```
glimpse(sat_aug)
```

```
## Rows: 50
## Columns: 9
## $ SAT        <int> 1088, 1075, 1068, 1045, 1045, 1033, 1028, 1022, 1017, 1011,…
## $ Public     <dbl> 87.8, 86.2, 88.3, 83.9, 83.6, 93.7, 78.3, 75.2, 97.0, 77.3,…
## $ .fitted    <dbl> 944.1198, 945.0465, 943.8302, 946.3786, 946.5523, 940.7027,…
## $ .resid     <dbl> 143.880224, 129.953547, 124.169810, 98.621450, 98.447698, 9…
## $ .hat       <dbl> 0.02918707, 0.02527061, 0.03063269, 0.02153481, 0.02121224,…
## $ .sigma     <dbl> 68.89683, 69.51144, 69.72849, 70.63271, 70.63847, 70.77489,…
## $ .cooksd    <dbl> 0.0629494764, 0.0441056591, 0.0493526954, 0.0214814500, 0.0…
## $ .std.resid <dbl> 2.0463672, 1.8445751, 1.7673480, 1.3971689, 1.3944776, 1.32…
## $ obs_num    <int> 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, …
```

Leverage

Leverage

- **Leverage:** measure of the distance between an observation's values of the predictor variables and the average values of the predictor variables for the entire data set

- An observation has **high leverage** if its combination of values for the predictor variables is very far from the typical combination of values in the data

- Observations with high leverage should be considered as *potential* influential points

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Calculating leverage

Simple Regression: leverage of the i^{th} observation

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{j=1}^{n}(x_j - \bar{x})^2}$$

- *Note*: Leverage only depends on values of the <u>**predictor**</u> variable(s)

High Leverage

The sum of the leverages for all points is $p + 1$

- p is the number of predictors
- In the case of SLR $\sum_{i=1}^{n} h_i = 2$
- The "typical" leverage is $\frac{(p+1)}{n}$

An observation has **high leverage** if

$$h_i > \frac{2(p+1)}{n}$$

High Leverage

If there is point with high leverage, ask

❓ Is there a data entry error?

❓ Is this observation within the scope of individuals for which you want to make predictions and draw conclusions?

❓ Is this observation impacting the estimates of the model coefficients, especially for interactions?

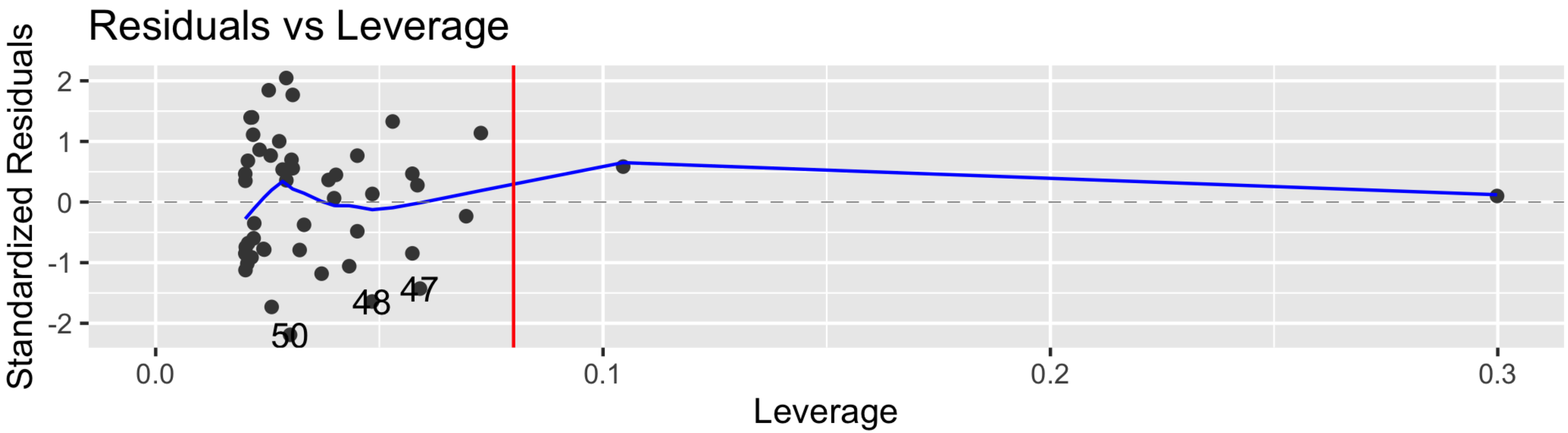
Just because a point has high leverage does not necessarily mean it will have a substantial impact on the regression. Therefore we need to check other measures.

SAT: Leverage

```
(leverage_threshold <- 2*(1+1)/nrow(sat_aug))
```

```
## [1] 0.08
```

```
autoplot(sat_model,which = 5, ncol = 1) +
  geom_vline(xintercept = leverage_threshold, color = "red")
```



Observations with high leverage

```
## # A tibble: 2 × 2
##   obs_num Public
##     <int>  <dbl>
## 1      13   61.2
## 2      22   44.8
```

Why do you think these observations have high leverage?

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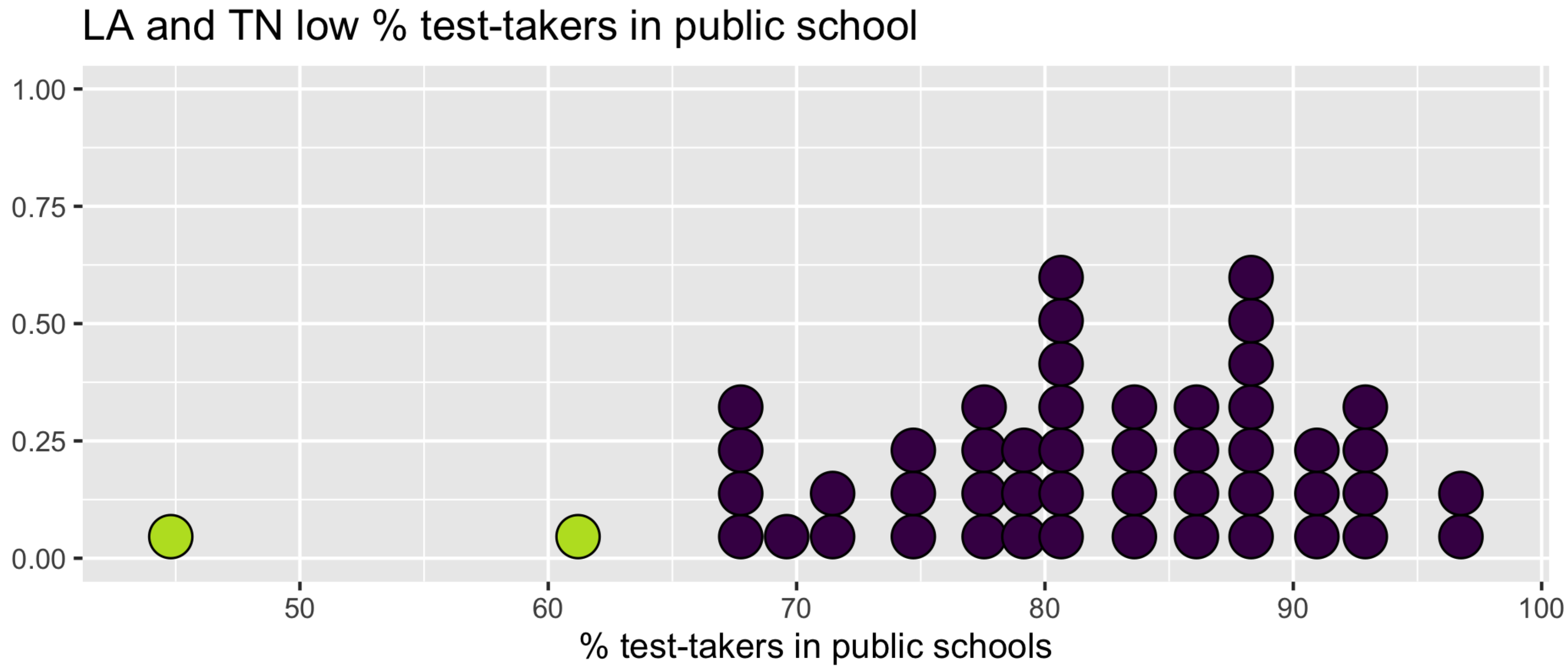
Let's dig into the data



Standardized residuals

Standardized residuals

- What is the best way to identify outliers (points that don't fit the pattern from the regression line)?

- Look for points that have large residuals

- We want a common scale, so we can more easily identify "large" residuals

- We will look at each residual divided by its standard error

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Standardized residuals

$$std.res_i = \frac{y_i - \hat{y}_i}{\hat{\sigma}_\epsilon \sqrt{1 - h_i}}$$

where $\hat{\sigma}_\epsilon$ is the regression standard error

- Standardized residuals are produced by **`augment`** in the column **`.std.resid`**

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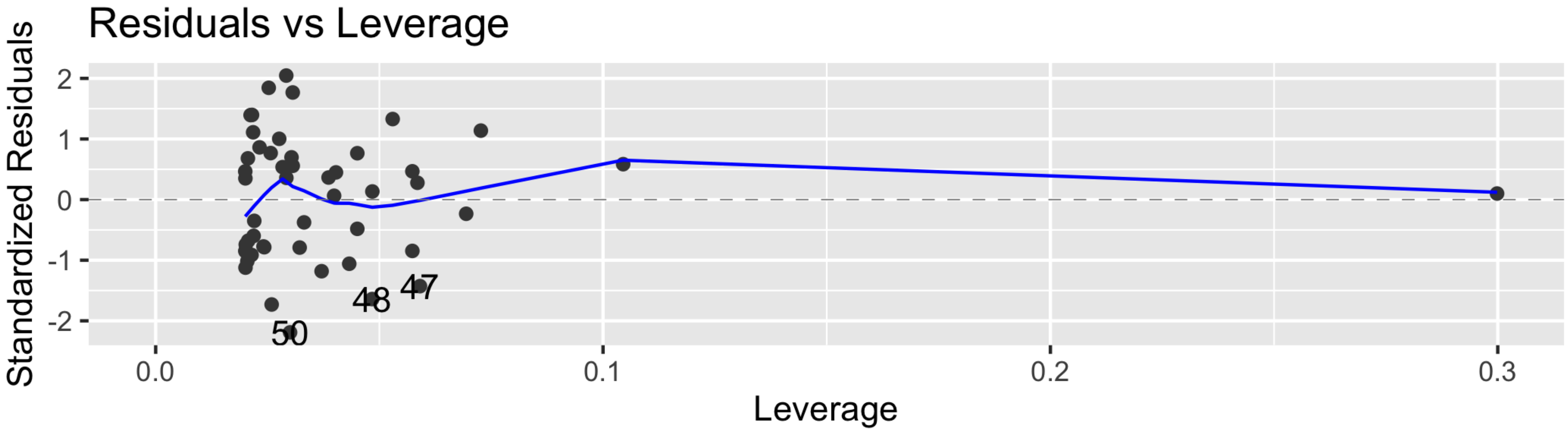
Standardized residuals

Observations with high leverage tend to have low values of standardized residuals because they pull the regression line towards them

```r
autoplot(sat_model, which = 5, ncol = 1)
```



Using standardized residuals

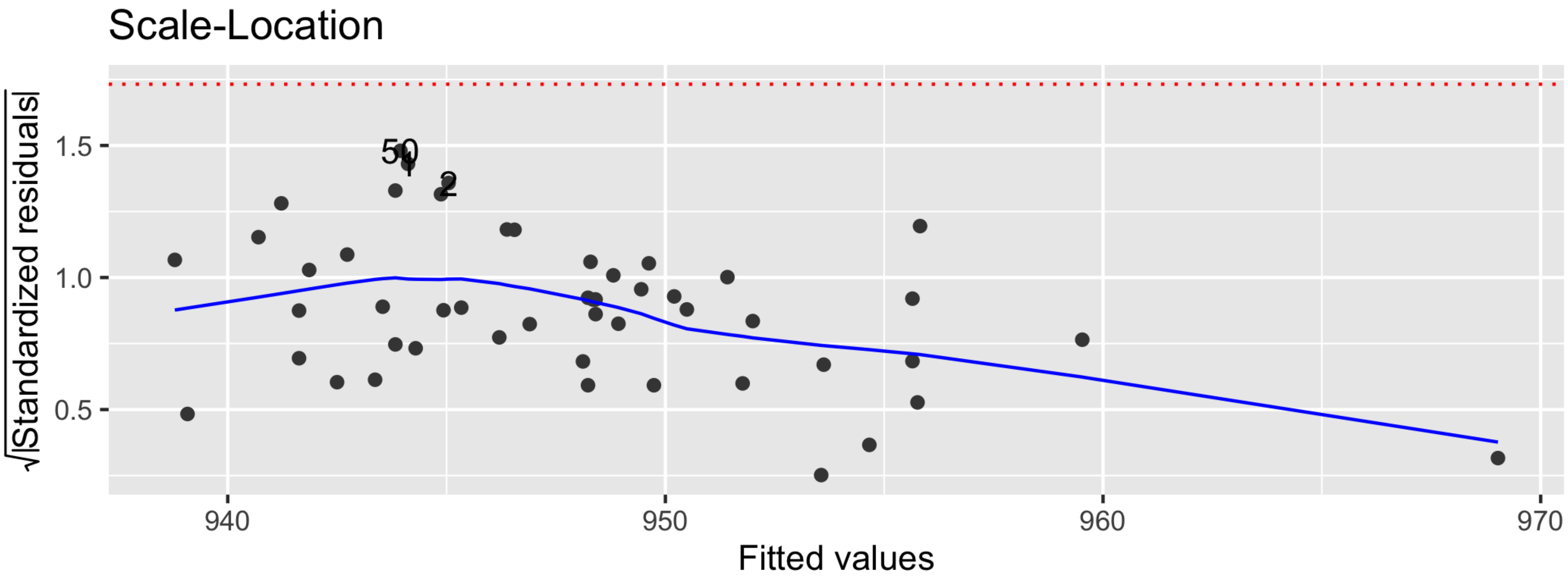
Observations that have standardized residuals of large magnitude are outliers, since they don't fit the pattern determined by the regression model

> An observation is a *potential outlier* if its standardized residual is beyond ± 3.

Make residual plots with standardized residuals to make it easier to identify outliers

SAT: sqrt(Standardized residuals) vs. fitted

```
autoplot(sat_model, which = 3, ncol = 1) +
  geom_hline(yintercept = sqrt(3),color = "red",linetype = "dotted")
```



Cook's Distance

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Motivating Cook's Distance

An observation's influence on the regression line depends on

- How close it lies to the general trend of the data - (Standardized residual)
- Its leverage - h_i

Cook's Distance is a statistic that includes both of these components to measure an observation's overall impact on the model

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Cook's Distance

Cook's distance for the i^{th} observation

$$D_i = \frac{(std.res_i)^2}{p+1}\left(\frac{h_i}{1-h_i}\right)$$

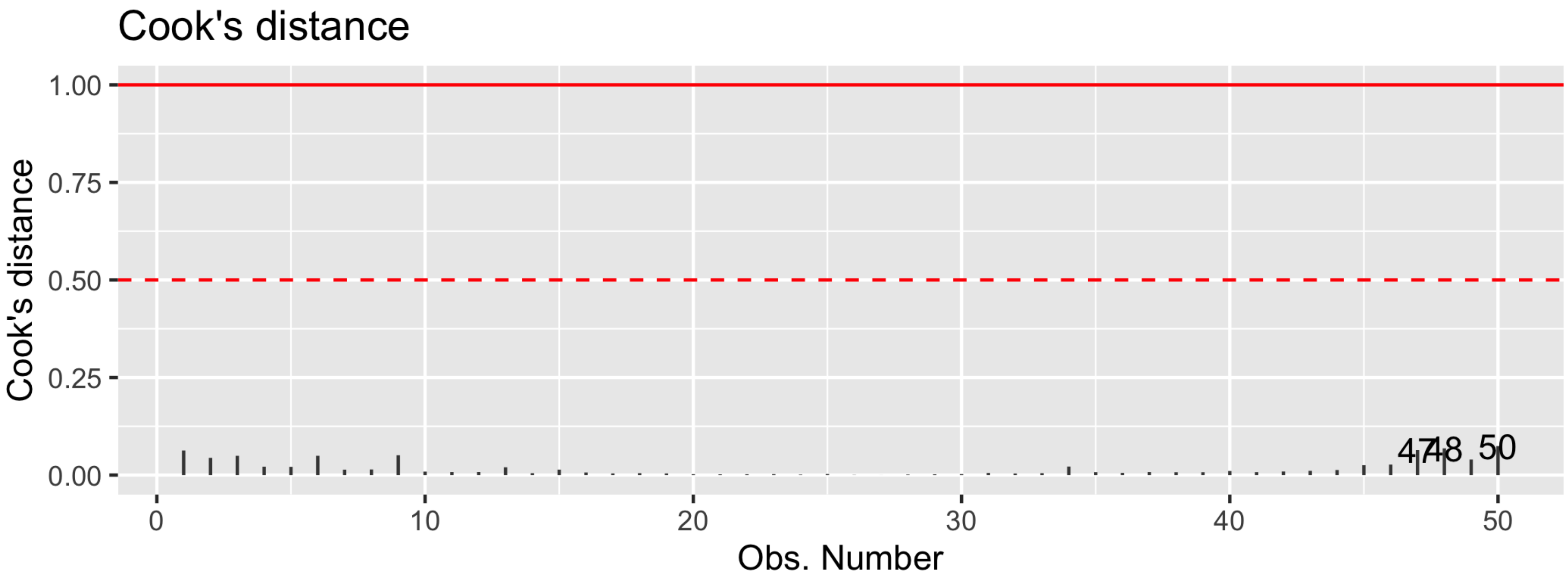
An observation with large D_i is said to have a strong influence on the predicted values

An observation with

- $D_i > 0.5$ is **moderately influential**
- $D_i > 1$ is **very influential**

Cook's Distance

```
autoplot(sat_model, which = 4, ncol = 1) +
  geom_hline(yintercept = 0.5, color = "red", lty = 2) +
  geom_hline(yintercept = 1,color = "red")
```



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Using these measures

- Standardized residuals, leverage, and Cook's Distance should all be examined together
- Examine plots of the measures to identify observations that are outliers, high leverage, and may potentially impact the model.

What to do with outliers/influential points?

It is OK to drop an observation based on the predictor variables if…

- It is meaningful to drop the observation given the context of the problem
- You intended to build a model on a smaller range of the predictor variables. Mention this in the write up of the results and be careful to avoid extrapolation when making predictions

What to do with outliers/influential points?

It is **not OK** to drop an observation based on the response variable

- These are legitimate observations and should be in the model
- You can try transformations or increasing the sample size by collecting more data

In either instance, you can try building the model with and without the outliers/influential observations

See the supplemental notes [Details on Model Diagnostics] for more mathematical details about standardized residuals, leverage points, and Cook's distance.

Recap

- Identifying influential points
  - Leverage
  - Standardized residuals
  - Cook's Distance