Multiple linear regression

Prof. Maria Tackett

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Review

Vocabulary

- Response variable: Variable whose behavior or variation you are trying to understand.
- Explanatory variables: Other variables that you want to use to explain the variation in the response.
- Predicted value: Output of the **model function**
  - The model function gives the typical value of the response variable *conditioning* on the explanatory variables.
- Residuals: Shows how far each case is from its predicted value
  - **Residual = Observed value - Predicted value**

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The linear model with a single predictor

- We're interested in the β_0 (population parameter for the intercept) and the β_1 (population parameter for the slope) in the following model:

$$\hat{y} = \beta_0 + \beta_1 \; x$$

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The linear model with a single predictor

- We're interested in the β_0 (population parameter for the intercept) and the β_1 (population parameter for the slope) in the following model:

$$\hat{y} = \beta_0 + \beta_1 \ x$$

- Unfortunately, we can't get these values
- So we use sample statistics to estimate them:

$$\hat{y} = b_0 + b_1 \ x$$

Least squares regression

The regression line minimizes the sum of squared residuals.

- Residuals: $e_i = y_i - \hat{y}_i$,
- The regression line minimizes $\sum_{i=1}^{n} e_i^2$.
- Equivalently, minimizing $\sum_{i=1}^{n} [y_i - (b_0 + b_1\ x_i)]^2$

Data and Packages

```
library(tidyverse)
library(broom)
```

```
paris_paintings <- read_csv("data/paris_paintings.csv",
                            na = c("n/a", "", "NA"))
```

- Paris Paintings Codebook
- Source: Printed catalogues from 28 auction sales held in Paris 1764 - 1780
- 3,393 paintings, prices, descriptive details, characteristics of the auction and buyer (over 60 variables)

Single numerical predictor

```
m_ht_wd <- lm(Height_in ~ Width_in, data = paris_paintings)
tidy(m_ht_wd)
```

```
## # A tibble: 2 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)    3.62    0.254        14.3 8.82e-45
## 2 Width_in       0.781   0.00950      82.1 0.
```

$$\widehat{Height_{in}} = 3.62 + 0.78\ Width_{in}$$

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Single categorical predictor (2 levels)

```
m_ht_lands <- lm(Height_in ~ factor(landsALL), data = paris_paintings)
tidy(m_ht_lands)
```

```
## # A tibble: 2 x 5
##   term              estimate std.error statistic  p.value
##   <chr>                <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)          22.7      0.328      69.1 0.
## 2 factor(landsALL)1    -5.65     0.532     -10.6 7.97e-26
```

$$\widehat{Height_{in}} = 22.68 - 5.65\ landsALL$$

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Single categorical predictor (> 2 levels)

```
m_ht_sch <- lm(Height_in ~ school_pntg, data = paris_paintings)
tidy(m_ht_sch)
```

```
## # A tibble: 7 x 5
##   term              estimate std.error statistic p.value
##   <chr>                <dbl>     <dbl>     <dbl>   <dbl>
## 1 (Intercept)          14.        10.0     1.40  0.162
## 2 school_pntgD/FL       2.33      10.0     0.232 0.816
## 3 school_pntgF         10.2       10.0     1.02  0.309
## 4 school_pntgG          1.65      11.9     0.139 0.889
## 5 school_pntgI         10.3       10.0     1.02  0.306
## 6 school_pntgS         30.4       11.4     2.68  0.00744
## 7 school_pntgX          2.87      10.3     0.279 0.780
```

$$\widehat{Height_{in}} = 14 + 2.33\ sch_{D/FL} + 10.2\ sch_F + 1.65\ sch_G + 10.3\ sch_I + 30.4\ sch_S + 2.87\ sch_X$$

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The linear model with multiple predictors

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The linear model with multiple predictors

- Population model:

$$\hat{y} = \beta_0 + \beta_1\ x_1 + \beta_2\ x_2 + \cdots + \beta_k\ x_k$$

The linear model with multiple predictors

- Population model:

$$\hat{y} = \beta_0 + \beta_1 \, x_1 + \beta_2 \, x_2 + \cdots + \beta_k \, x_k$$

- Sample model that we use to estimate the population model:

$$\hat{y} = b_0 + b_1 \, x_1 + b_2 \, x_2 + \cdots + b_k \, x_k$$

Data

The data set contains prices for Porsche and Jaguar cars for sale on cars.com.

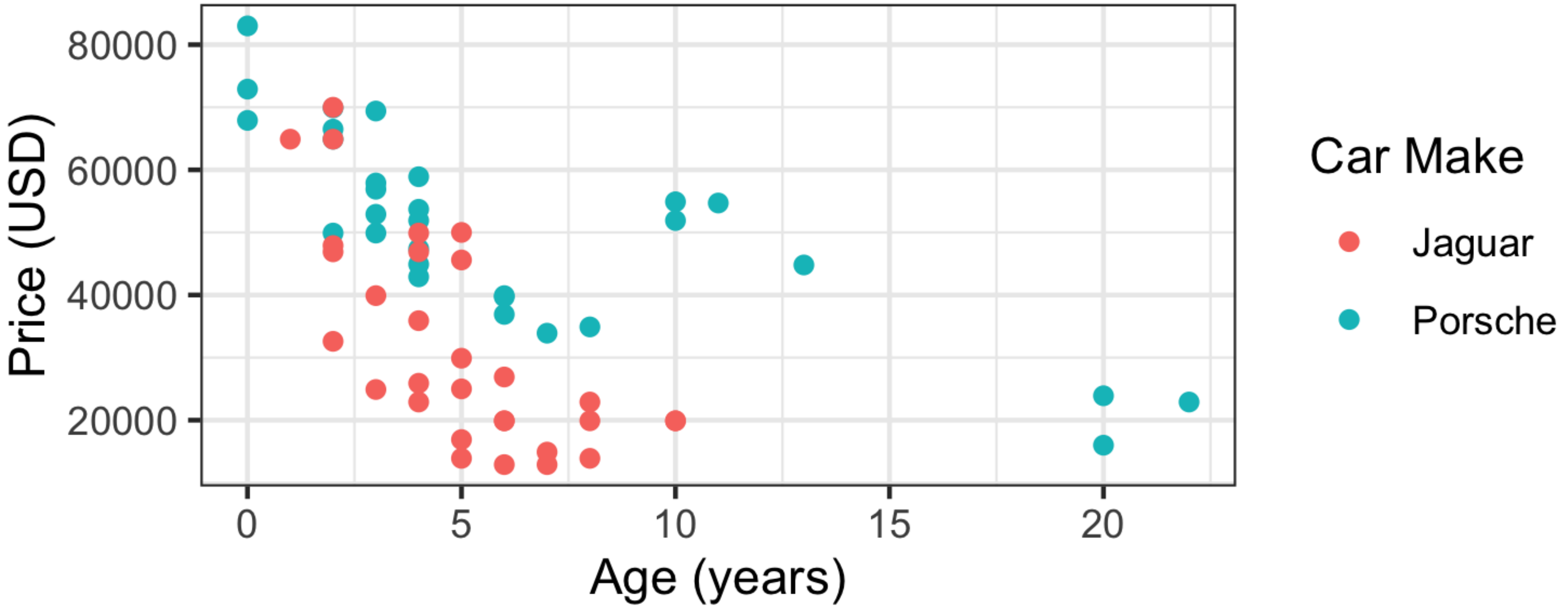
`car`: car make (Jaguar or Porsche)

`price`: price in USD

`age`: age of the car in years

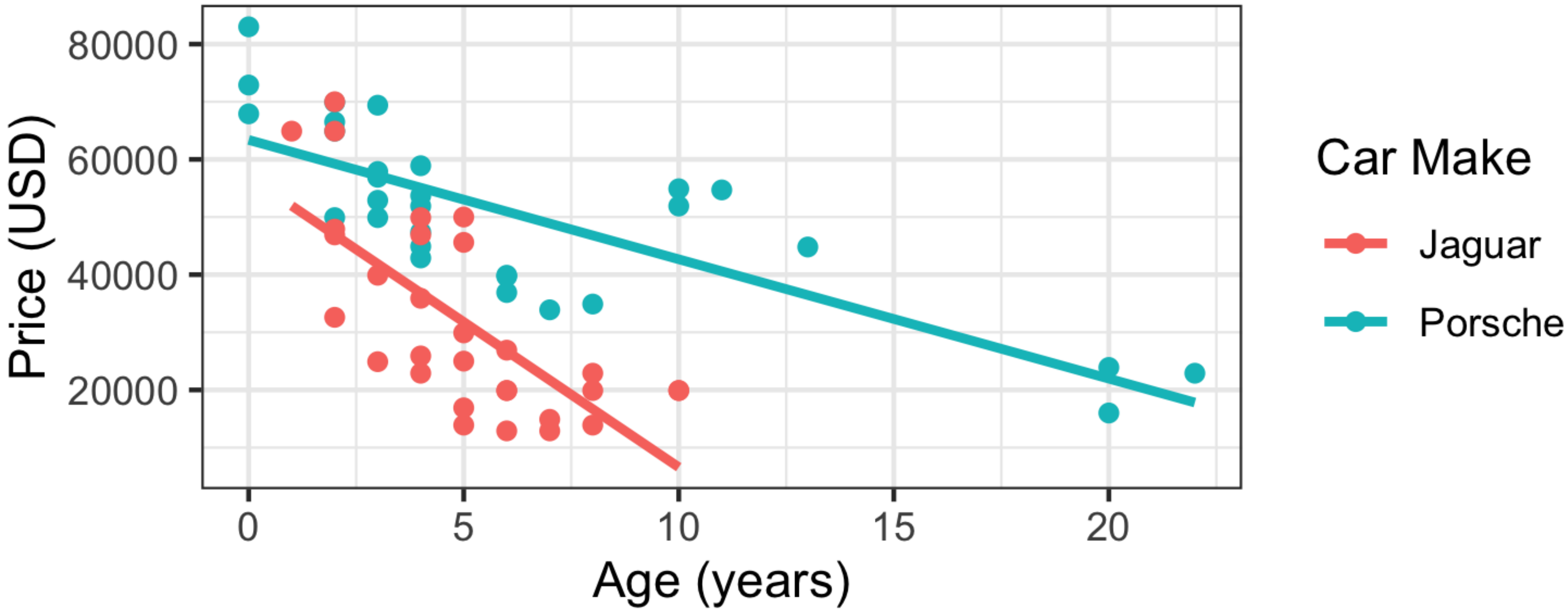
`mileage`: previous miles driven

Price, age, and make



Price vs. age and make

Does the relationship between age and price depend on the make of the car?



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Modeling with main effects

```r
m_main <- lm(price ~ age + car, data = sports_car_prices)

m_main %>%
  tidy() %>%
  select(term, estimate)
```

```
## # A tibble: 3 x 2
##   term        estimate
##   <chr>          <dbl>
## 1 (Intercept)   44310.
## 2 age           -2487.
## 3 carPorsche    21648.
```

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Modeling with main effects

```
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$$\widehat{price} = 44310 - 2487\ age + 21648\ carPorsche$$

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- Plug in 0 for **`carPorsche`** to get the linear model for Jaguars.

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$$\begin{aligned}\widehat{price} &= 44310 - 2487\ age + 21648 \times 0 \\ &= 44310 - 2487\ age\end{aligned}$$

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- Plug in 0 for **`carPorsche`** to get the linear model for Jaguars.

$$\begin{aligned}\widehat{price} &= 44310 - 2487\ age + 21648 \times 0 \\ &= 44310 - 2487\ age\end{aligned}$$

- Plug in 1 for **`carPorsche`** to get the linear model for Porsches.

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- Plug in 0 for **`carPorsche`** to get the linear model for Jaguars.

$$\begin{aligned}\widehat{price} &= 44310 - 2487\ age + 21648 \times 0 \\ &= 44310 - 2487\ age\end{aligned}$$

- Plug in 1 for **`carPorsche`** to get the linear model for Porsches.

$$\begin{aligned}\widehat{price} &= 44310 - 2487\ age + 21648 \times 1 \\ &= 65958 - 2487\ age\end{aligned}$$

Jaguar

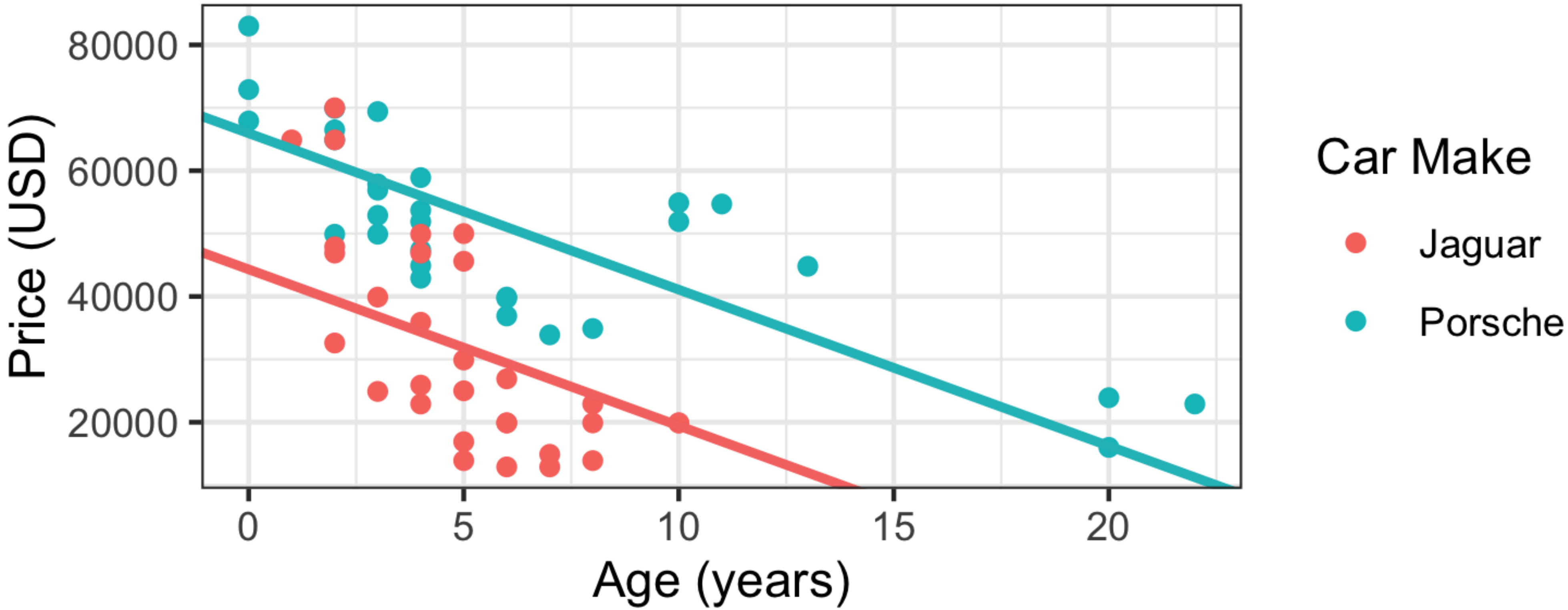
$$
\begin{aligned}
\widehat{price} &= 44310 - 2487\ age + 21648 \times 0 \\
&= 44310 - 2487\ age
\end{aligned}
$$

Porsche

$$
\begin{aligned}
\widehat{price} &= 44310 - 2487\ age + 21648 \times 1 \\
&= 65958 - 2487\ age
\end{aligned}
$$

- Rate of change in price as the age of the car increases does not depend on make of car (same slopes)
- Porsches are consistently more expensive than Jaguars (different intercepts)

Interpretation of main effects



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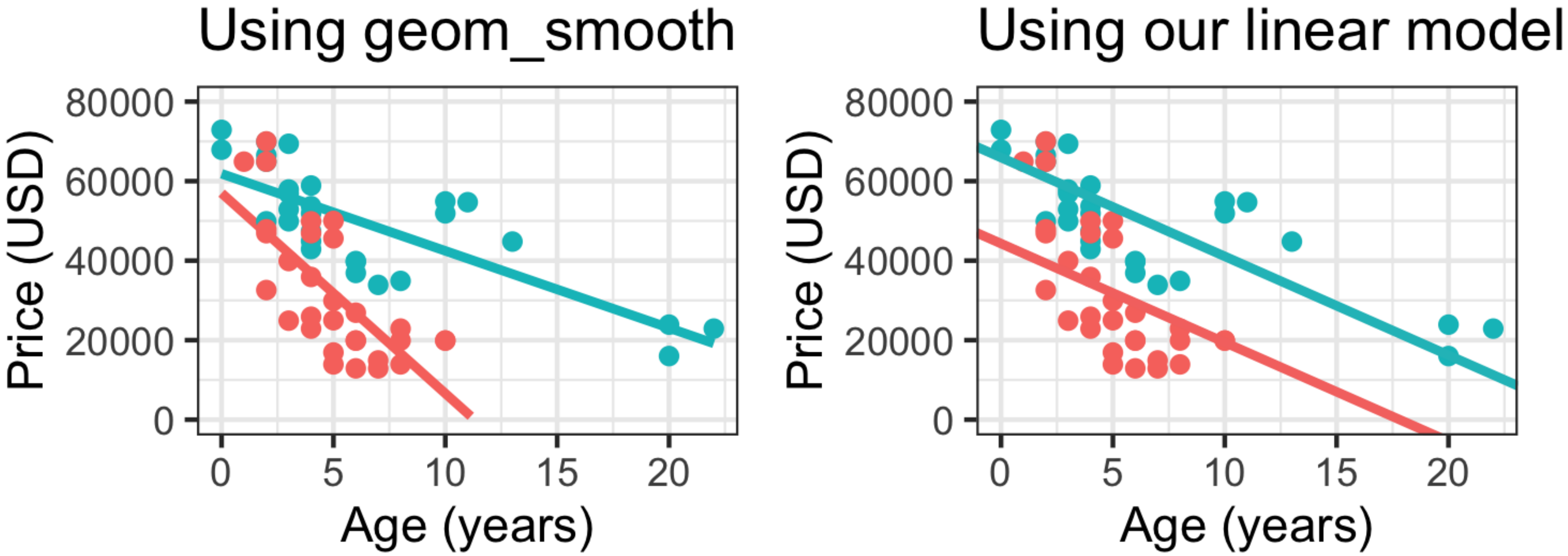
Main effects

```
## # A tibble: 3 x 2
##   term          estimate
##   <chr>            <dbl>
## 1 (Intercept)     44310.
## 2 age             -2487.
## 3 carPorsche      21648.
```

- **All else held constant**, for each additional year of a car's age, the price of the car is predicted to decrease, on average, by $2,487.
- **All else held constant**, Porsches are predicted, on average, to have a price that is $21,647 greater than Jaguars.
- Jaguars that are new (age = 0) are predicted, on average, to have a price of $44,309.

Why is our linear regression model different from what we got from **`geom_smooth(method = "lm")`**?



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What went wrong?

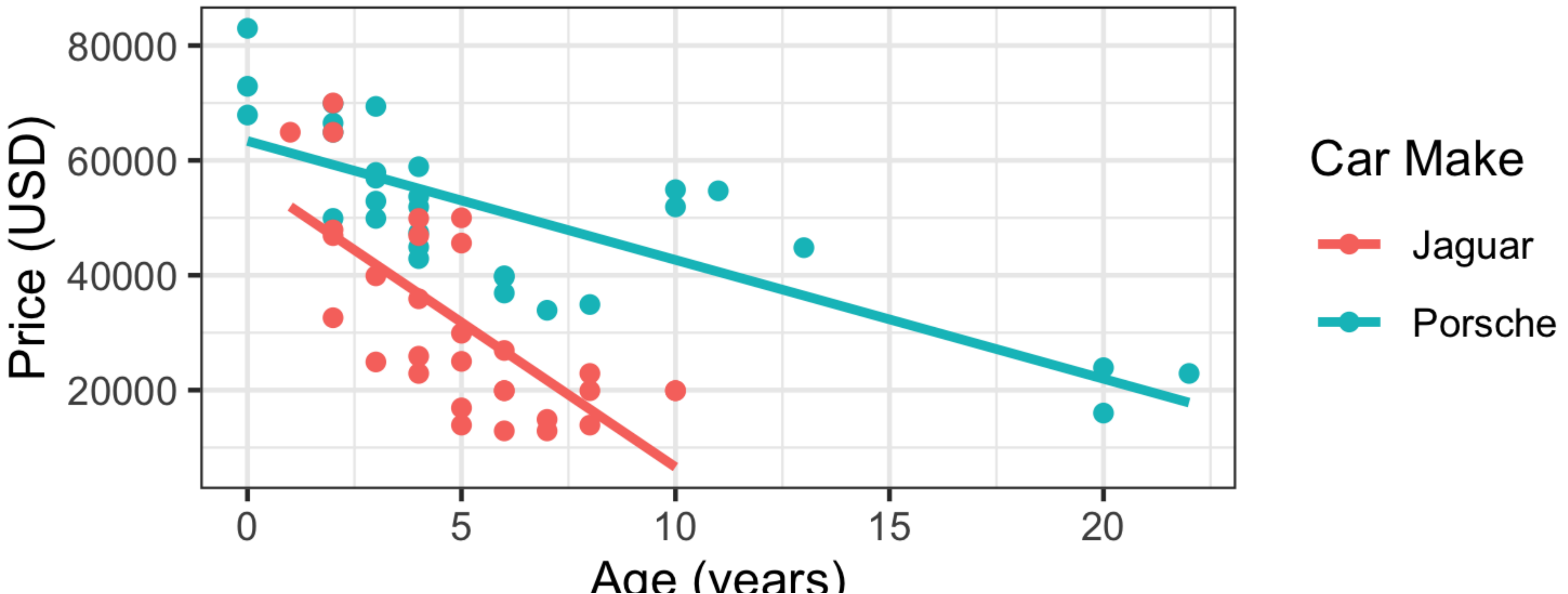
- `car` is the only variable in our model that affects the intercept.
- The model we specified assumes Jaguars and Porsches have the **`same slope`** and **`different intercepts`**.
- What is the most appropriate model for these data?
  - same slope and intercept for Jaguars and Porsches?
  - same slope and different intercept for Jaguars and Porsches?
  - different slope and different intercept for Jaguars and Porsches?

Interacting explanatory variables

- Including an interaction effect in the model allows for different slopes, i.e. nonparallel lines.
- This means that the relationship between an explanatory variable and the response depends on another explanatory variable.
- We can accomplish this by adding an **interaction variable**. This is the product of two explanatory variables.

Price vs. age and car interacting

```r
ggplot(data = sports_car_prices,
       mapping = aes(y = price, x = age, color = car)) +
  geom_point() +
  geom_smooth(method = "lm", se = FALSE) +
  labs(x = "Age (years)", y = "Price (USD)", color = "Car Make")
```



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Modeling with interaction effects

```
m_int <- lm(price ~ age + car + age * car, data = sports_car_prices)
m_int %>%
  tidy() %>%
  select(term, estimate)
```

```
## # A tibble: 4 x 2
##   term            estimate
##   <chr>              <dbl>
## 1 (Intercept)       56988.
## 2 age               -5040.
## 3 carPorsche         6387.
## 4 age:carPorsche     2969.
```

$$\widehat{price} = 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche$$

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Interpretation of interaction effects

$$\widehat{price} = 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche$$

Interpretation of interaction effects

$$\widehat{price} = 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche$$

- Plug in 0 for **`carPorsche`** to get the linear model for Jaguars.

$$\begin{aligned}\widehat{price} &= 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche \\ &= 56988 - 5040\ age + 6387 \times 0 + 2969\ age \times 0 \\ &= 56988 - 5040\ age\end{aligned}$$

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Interpretation of interaction effects

$$\widehat{price} = 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche$$

- Plug in 0 for **`carPorsche`** to get the linear model for Jaguars.

$$\begin{aligned}
\widehat{price} &= 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche \\
&= 56988 - 5040\ age + 6387 \times 0 + 2969\ age \times 0 \\
&= 56988 - 5040\ age
\end{aligned}$$

- Plug in 1 for **`carPorsche`** to get the linear model for Porsches.

$$\begin{aligned}
\widehat{price} &= 56988 - 5040\ age + 6387\ carPorsche + 2969\ age \times carPorsche \\
&= 56988 - 5040\ age + 6387 \times 1 + 2969\ age \times 1 \\
&= 63375 - 2071\ age
\end{aligned}$$

Interpretation of interaction effects

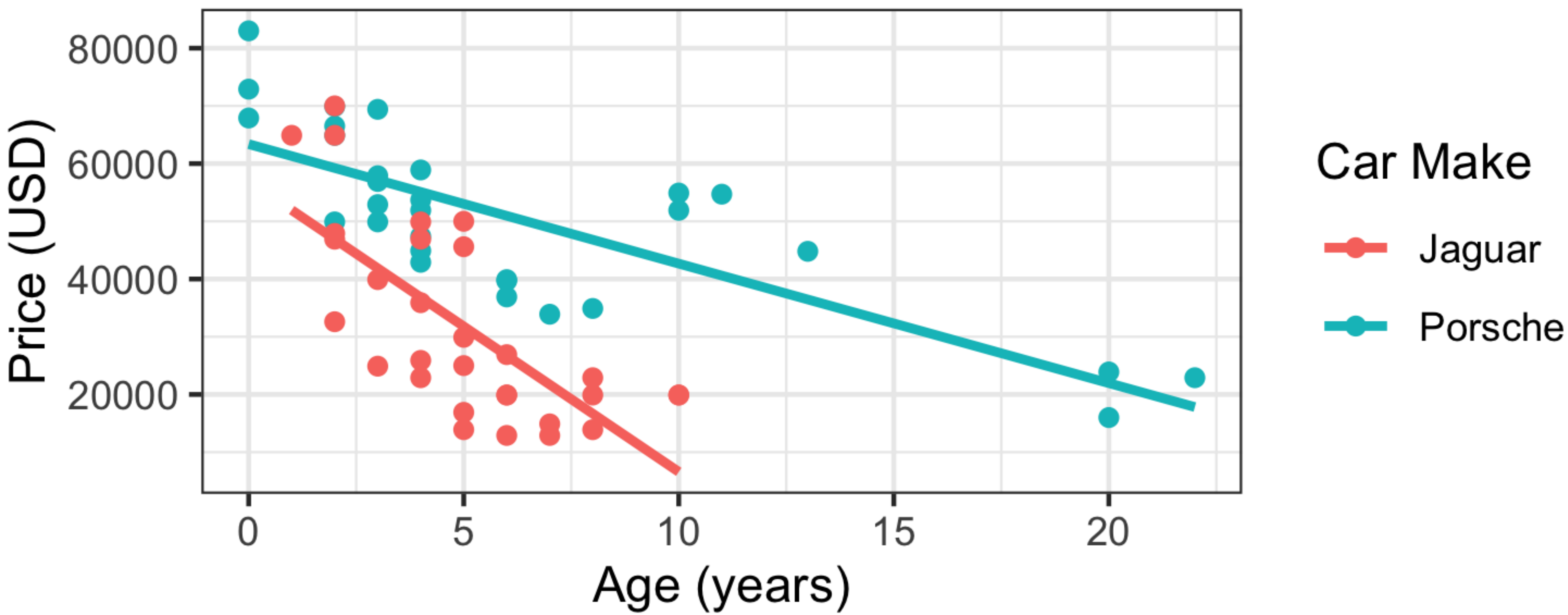
Jaguar

$$\widehat{price} = 56988 - 5040\ age$$

Porsche

$$\widehat{price} = 63375 - 2071\ age$$

- Rate of change in price as the age of the car increases depends on the make of the car (different slopes).
- Porsches are consistently more expensive than Jaguars (different intercepts).


Price (USD)
80000
60000
40000
20000
0
5
10
15
20
Age (years)
Car Make
Jaguar
Porsche

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Continuous by continuous interactions

- Interpretation becomes trickier
- Slopes conditional on values of explanatory variables

Continuous by continuous interactions

- Interpretation becomes trickier
- Slopes conditional on values of explanatory variables

Third order interactions

- Can you? Yes
- Should you? Probably not if you want to interpret these interactions in context of the data.

Assessing quality of model fit

Assessing the quality of the fit

- The strength of the fit of a linear model is commonly evaluated using R^2.
- It tells us what percentage of the variability in the response variable is explained by the model. The remainder of the variability is unexplained.
- R^2 is sometimes called the coefficient of determination.

What does "explained variability in the response variable" mean?

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Obtaining R^2 in R

`price` vs. **`age`** and **`make`**

```
glance(m_main)
```

```
## # A tibble: 1 x 12
##   r.squared adj.r.squared  sigma statistic  p.value    df logLik   AIC   BI
##       <dbl>         <dbl>  <dbl>     <dbl>    <dbl> <dbl>  <dbl> <dbl> <db
## 1     0.607         0.593 11848.      44.0 2.73e-12     2  -646. 1301. 130
## # … with 3 more variables: deviance <dbl>, df.residual <int>, nobs <int>
```

```
glance(m_main)$r.squared
```

```
## [1] 0.6071375
```

Obtaining R^2 in R

`price` vs. **`age`** and **`make`**

```
glance(m_main)
```

```
## # A tibble: 1 x 12
##   r.squared adj.r.squared  sigma statistic  p.value    df logLik   AIC   BI
##       <dbl>         <dbl>  <dbl>     <dbl>    <dbl> <dbl>  <dbl> <dbl> <db
## 1     0.607         0.593 11848.      44.0 2.73e-12     2  -646. 1301. 130
## # … with 3 more variables: deviance <dbl>, df.residual <int>, nobs <int>
```

```
glance(m_main)$r.squared
```

```
## [1] 0.6071375
```

About 60.7% of the variability in price of used cars can be explained by age and make.

R^2

```r
glance(m_main) %>% #model with main effects
  pull(r.squared)
```

```
## [1] 0.6071375
```

```r
glance(m_int) %>% #model with main effects + interactions
  pull(r.squared)
```

```
## [1] 0.6677881
```

R^2

```r
glance(m_main) %>% #model with main effects
  pull(r.squared)
```

```
## [1] 0.6071375
```

```r
glance(m_int) %>% #model with main effects + interactions
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```

```
## [1] 0.6677881
```

- The model with interactions has a higher R^2.

R^2

```
glance(m_main) %>% #model with main effects
  pull(r.squared)
```

```
## [1] 0.6071375
```

```
glance(m_int) %>% #model with main effects + interactions
  pull(r.squared)
```

```
## [1] 0.6677881
```

- The model with interactions has a higher R^2.
- Using R^2 for model selection in models with multiple explanatory variables is <u>not</u> a good idea as R^2 increases when **any** variable is added to the model.

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R^2 - first principles

- We can write explained variation using the following ratio of sums of squares:

$$R^2 = 1 - \left(\frac{\text{variability in residuals}}{\text{variability in response}} \right)$$

Why does this expression make sense?

- But remember, adding **any** explanatory variable will always increase R^2

Adjusted R^2

$$R^2_{adj} = 1 - \left(\frac{\text{variability in residuals}}{\text{variability in response}} \times \frac{n-1}{n-k-1} \right)$$

where n is the number of observations and k is the number of predictors in the model.

Adjusted R^2

$$R^2_{adj} = 1 - \left(\frac{\text{variability in residuals}}{\text{variability in response}} \times \frac{n-1}{n-k-1} \right)$$

where n is the number of observations and k is the number of predictors in the model.

- Adjusted R^2 doesn't increase if the new variable does not provide any new information or is completely unrelated.

Adjusted R^2

$$R^2_{adj} = 1 - \left(\frac{\text{variability in residuals}}{\text{variability in response}} \times \frac{n-1}{n-k-1} \right)$$

where n is the number of observations and k is the number of predictors in the model.

- Adjusted R^2 doesn't increase if the new variable does not provide any new information or is completely unrelated.
- This makes adjusted R^2 a preferable metric for model selection in multiple regression models.

Comparing models

```r
glance(m_main)$r.squared
```

```
## [1] 0.6071375
```

```r
glance(m_int)$r.squared
```

```
## [1] 0.6677881
```

```r
glance(m_main)$adj.r.squared
```

```
## [1] 0.5933529
```

```r
glance(m_int)$adj.r.squared
```

```
## [1] 0.649991
```

In pursuit of Occam's Razor

- Occam's Razor states that among competing hypotheses that predict equally well, the one with the fewest assumptions should be selected.
- Model selection follows this principle.
- We only want to add another variable to the model if the addition of that variable brings something valuable in terms of predictive power to the model.
- In other words, we prefer the simplest best model, i.e. parsimonious model.