Conditional Probability

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STA 199

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Conditional probability

The probability an event will occur *given* that another event has already occurred is a conditional probability. The conditional probability of event A given event B is:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Conditional probabilities

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Examples come up all the time in the real world:

- *Given* that it rained yesterday, what is the probability that it will rain today?
- *Given* that a mammogram comes back positive, what is the probability that a woman has breast cancer?
- *Given* that I've already watched six episodes of How I Met Your Mother tonight, what is the probability that I'll get any work done this evening?

Coffee and mortality

ORIGINAL RESEARCH

Annals of Internal Medicine

Coffee Drinking and Mortality in 10 European Countries

A Multinational Cohort Study

	Did not die	Died
Does not drink coffee	5438	1039
Drinks coffee occasionally	29712	4440
Drinks coffee regularly	24934	3601

Three probabilities

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Does not drink coffee	5438	1039
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Define events A = died and B = non-coffee drinker. Calculate the following for a randomly selected person in the cohort:

- Marginal probability: $P(A)$, $P(B)$
- Joint probability: $P(A \text{ and } B)$
- Conditional probability: $P(A|B)$, $P(B|A)$

Independence

The multiplicative rule

We can write the definition of condition probability

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Using the equation above, we get...

$$P(B) \times P(A|B) = P(A|B)$$

What does the multiplicative rule mean in plain English?

Defining independence

Events A and B are said to be **independent** when

$$P(A|B) = P(A) \quad \textbf{OR} \quad P(B|A) = P(B)$$

In other words, knowing that one event has occurred doesn't cause us to "adjust" the probability we assign to another event.

Checking independence

We can use the multiplicative rule to see if two events are independent.

If events A and B are independent, then

$$P(A \text{ and } B) = P(A) \times P(B)$$

Independent vs. disjoint events

Since for two independent events $P(A|B) = P(A)$ and $P(B|A) = P(B)$, knowing that one event has occurred tells us nothing more about the probability of the other occurring.

For two disjoint events A and B, knowing that one has occurred tells us that the other definitely has not occurred: $P(A \text{ and } B) = 0$.

Disjoint events are <u>not</u> independent!

Checking independence

	Did not die	Died
Does not drink coffee	5438	1039
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Are dying and abstaining from coffee independent events? How might we check?

Bayes' Rule

An example

In an introductory statistics course, 50% of students were first years, 30% were sophomores, and 20% were upperclassmen.

80% of the first years didn't get enough sleep, 40% of the sophomores didn't get enough sleep, and 10% of the upperclassmen didn't get enough sleep.

What is the probability that a randomly selected student in this class didn't get enough sleep?

Bayes' Rule

As we saw before, the two conditional probabilities $P(A|B)$ and $P(B|A)$ are not the same. But are they related in some way?

Bayes' Rule

As we saw before, the two conditional probabilities $P(A|B)$ and $P(B|A)$ are not the same. But are they related in some way?

Yes they are (!) using **Bayes' rule**:

Bayes' rule:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B)}$$

Bayes' Rule (continued)

By using the rules of probability we've learned so far, we have

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Let's took at an example to see how this works.

Diagnostic Testing

Definitions

Suppose we're interested in the performance of a diagnostic test. Let D be the event that a patient has the disease, and let T be the event that the test is positive for that disease.

- Prevalence: $P(D)$
- Sensitivity: $P(T|D)$
- Specificity: $P(T^c|D^c)$
- Positive predictive value: $P(D|T)$
- Negative predictive value: $P(D^c|T^c)$

What do these probabilities mean in plain English?

Rapid self-administered HIV tests

From the FDA package insert for the Oraquick ADVANCE Rapid HIV-1/2 Antibody Test,

- Sensitivity, $P(T|D)$, is 99.3%
- Specificity, $P(T^c|D^c)$, is 99.8%
  From CDC statistics in 2016, 14.3/100,000 Americans aged 13 or older are HIV+.

Suppose a randomly selected American aged 13+ has a positive test result. What do you think is the probability they have HIV?

Using Bayes' Rule

$$
\begin{aligned}
P(D|T) &= \frac{P(D \text{ and } T)}{P(T)} \\
&= \frac{P(T|D)P(D)}{P(T)} \\
&= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} \\
&= \frac{P(T|D)P(D)}{P(T|D)P(D) + (1 - P(T^c|D^c))(1 - P(D))}
\end{aligned}
$$

What does all of this mean? Let's take a look!

Work through example

A discussion

Think about the following questions:

- Is this calculation surprising?
- What is the explanation?
- Was this calculation actually reasonable to perform?
- What if we tested in a different population, such as high-risk individuals?
- What if we were to test a random individual in a country where the prevalence of HIV is approximately 25%?