Firstly, $P(A|B,C) = \frac{P(A,B,C)}{P(B,C)}$ is the correct definition of conditional probabilities with multiple random variables.
In turn, this means that $P(A|B,C) = \frac{P(A,B,C)}{P(B)P(C)}$ iff B and C are independent. For example, in general the following expansion holds:

$$P(A,B) = \sum_C P(A,B,C) = \sum_C P(A|B,C)P(B,C) = \sum_C P(A|B,C) * P(B|C) * P(C)$$

If we now B and C to be independent we can write:

$$P(A,B) = \sum_C P(A,B,C) = \sum_C P(A|B,C)P(A,B) = \sum_C P(A|B,C)P(B)P(C)$$

From the graph in the letter of recommendation model: i and d are independent only if g is not observed. However, when we calculate the joint probabilities in the following way

$$P(i=0,g=3) = \sum_{d^* \in \{0,1\}} P(g=3|i=0,d=d^*)P(i=0)P(d=d^*)$$

we do not observe the grade (which would mean to condition on the grade) but calculate a joint probability for a particular outcome. This is why we can use the information that the two nodes are independent here and factorize.
The same holds true for calculating $P(i=1,g=3)$, $P(g=3)$ and $P(g=3,d=1)$. For the calculations that involve observing g, we do not assume independence of the two nodes. In particular, the **explaining away** effect on i after observing d is only possible because observing g makes nodes i and d dependent. For the calculations when we condition on g we have

$$P(i=1|g=3) = \frac{P(i=1,g=3)}{P(g=3)}$$

and

$$P(i=1|g=3,d=1) = \frac{P(i=1,g=3,d=1)}{P(g=3,d=1)}$$

For the joint probability $P(i=1,g=3,d=1)$ we do not condition on g and we can use the information in the graph that i and d are independent.

$$\frac{P(i=1,g=3,d=1)}{P(g=3,d=1)} = \frac{P(g=3|i=1,d=1)p(i=1)p(d=1)}{P(g=3,d=1)}$$

In general, when you see a joint probability distribution corresponding to a graph you can use the dependency information of the graph. You can also always use the correct rules of conditional, join, marginal, ... probabilities to manipulate the formula. A joint probability $P(A,B,C,D,E)$ corresponds to no observed nodes in the graph, where as $P(A,B,C|D,E)$ corresponds to observing nodes D and E and expands to $\frac{P(A,B,C)}{P(D,E)}$.