Real Analysis: Chapter 1 - Set Theory

- Sets are a collection of elements having a common property.

$A \subset B$  proper subset, $A \subseteq B$ is just subset.

- for $A = B$, need to prove $A \subseteq B$ and $B \subseteq A$.

$\mathbb{N} \to$ countable
$\mathbb{Z} \to$ countable } discrete → between any two ~~conse~~ numbers, there are finite number of numbers.
$\mathbb{Q} \to$ countable.
$\mathbb{R} \to$ uncountable infinity.

- $S := \{x \in S : f(x)\}$, e.g. $S := \{x \in S : x^2 - 3x + 2 = 0\}$.

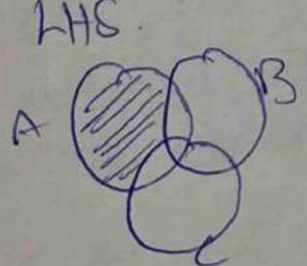
→ Operations on Sets:

1) $A \cup B$
2) $A \cap B$
3) $A \setminus B$ or $(A - B) = \{x \in A : x \notin B\}$.

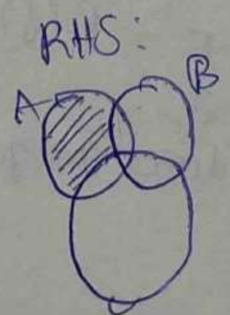
- Theorem(a): $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

i) venn diagram proof.

LHS.



RHS:



ii) $x \in A \setminus (B \cup C)$

$\Rightarrow x \in A,\ x \notin (B \cup C)$

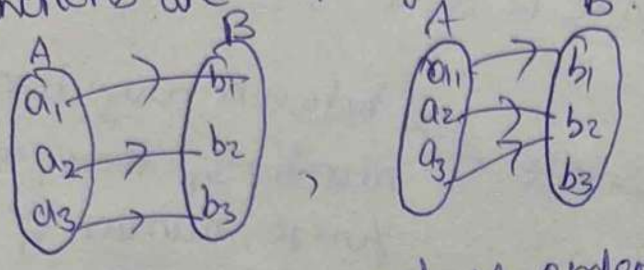
$\Rightarrow x \notin B,\ x \notin C$

→ Cartesian product $A \times B$:

- If A and B are non-empty sets, then the cartesian product is the set of all ordered pairs (a, b) where $a \in A$, $b \in B$.

→ Functions:

- functions are mappings of each and every element in set A to a singular element in set B.



∴ "A function is a set of ordered pairs (a, b) where $a \in A$, $b \in B$ with the condition for each a_i, we have a singular b_i."

→ Direct and Inverse Images:

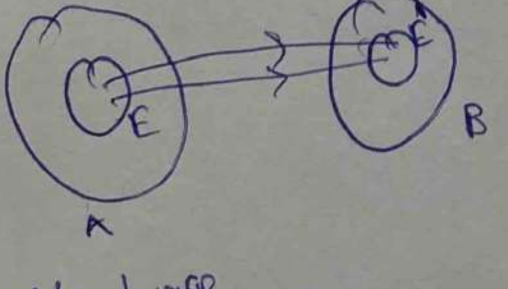
- Direct image - if E is a subset of A, then the direct image of E under f is the subset $f(E)$ of B

$$f(E) := \{f(x) ; x \in E\}.$$

- Inverse image - if $H \subseteq$ (subset) B, then inverse image of H

$$f^{-1}(H) := \{x \in A : f(x) \in H\}.$$

- Direct Image.

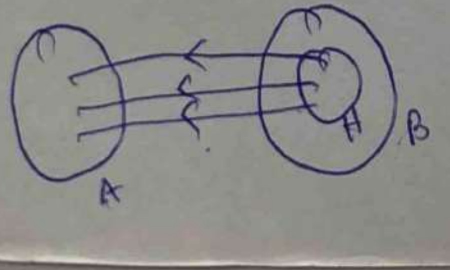


$E' = R(f) \subseteq B$.

↑

direct image of E.

- Inverse Image.



Types of Functions

1) Injective (one-one)
- if for inputs $(x_1, x_2) \in D(f)$, with $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
- outputs are distinct.

2) Surjective (onto)
- if $f(A) = B$
- range of function exhausts B.
- range = codomain

3) Bijective Mapping: one-one and onto;
   ↳ inverse exists.

Composition of a Function.

- $f: A \to B$ and $g: B \to C$
  and if range of f $\subseteq D(g)$
  then ~~fog~~ $g \circ f : A \to C$.
- $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
- $f \circ g \neq g \circ f$.
- Restriction of a function: let $A_1 \subset A$, then $f_1 = f$.

Method of Induction.

- $\mathbb{N} = \{1, 2, 3, \dots\}$
- Well ordering property - Every non-empty subset of $\mathbb{N}$ has at least one element.
  ↳ $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, then ~~if $1 \in S$, and~~ $\exists\ m \in S$, s.t $m \leq k\ \forall k \in S$.

Principle of Mathematical Induction.

- $S \subseteq \mathbb{N}$ and holds the following properties
  1) $1 \in S$
  2) for every $k \in \mathbb{N}$, if $k \in S$
     then $k+1 \in S$.

  Then $S = \mathbb{N}$.

- $1+2+\dots+n = \frac{n(n+1)}{2}$

  $1+2+3+\dots+10{,}000$
  $10{,}000+9999+\dots+\dots+1$ $\Big\}$ $\frac{10{,}000 \quad 10{,}001\text{'s}}{2}$

↳ PMI: Prove $n=1$
assume $n=k$
prove $n=k+1$.

Q) $1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$.

i) $n=1$ $\quad \frac{1\times 4}{4} = 1$ true ✓.

ii) assume $1^3+2^3+3^3+\dots+k^3 = \frac{k^2(k+1)^2}{4}$.

iii) $\frac{k^2(k+1)^2}{4} + \frac{(k+1)^3}{4} = \frac{k^2(k+1)^2+4(k+1)}{4}$

$= \frac{(k+1)[k^2(k+1)+4]}{4}$. $= \frac{(k+1)(k^3+k^2+4)}{4}$

$= \frac{k^4+k^3+4k+k^3+k^2+4}{4}$ $= \frac{k^4+2k^3+k^2+4k+4}{4}$

$= \frac{(k+2)^2+k^3(k+2)}{4} = (k+2)(k^3+(k+2))$.

Rational Numbers

- $\frac{p}{q}$ with $q \neq 0$, Q

1. Algebraic properties
2. Order properties
3. Density properties.

① Algebraic Properties:

A1) if $(a,b) \in Q$, then $a+b \in Q$
A2) $(a+b)+c = a+(b+c)$ associative.
A3) $\exists\ 0 \in Q$, s.t. $a+0=a \rightarrow$ identity element (0)
A4) $\exists\ -a \in Q$, s.t. $\forall a \in Q$, $a+(-a)=0 \rightarrow$ inverse.
A5) $a+b=b+a$ commutative.

rules of addition

M1) $\forall (a,b) \in Q$, $a \cdot b \in Q$.
M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ associative
M3) $\exists\ 1 \in Q$, s.t. $a \cdot 1 = a$ identity element
M4) $\exists\ \frac{1}{a} \in Q$, $\forall a \in Q - \{0\}$, s.t. $a \cdot \frac{1}{a} = 1$ (inverse).
M5) $a \cdot b = b \cdot c$.

rules of multiplication

→ Distributive Property:
$(a+b) \cdot c = ac + bc$.

② Order Properties: To do w inequalities.

O1. If $(a,b) \in \mathbb{Q}$, then only one of the following is true
i) $a < b$, ii) $a > b$, iii) $a = b$. } Law of Trichotomy.

O2. Transitivity:
$a < b < c$, then $a < c$ and $b < c$.

O3. $a < b$
$\Rightarrow a + c < b + c \quad \forall (a,b,c) \in \mathbb{Q}$.

O4. $a < b$ and $a < c$
$ac < bc$.

③ Density Properties:

$\forall (x,y) \in \mathbb{Q}$ with $x < y$
$\exists r \in \mathbb{Q}$ s.t. $x < r < y$.

proof:

$x < y$

$x + y < y + y$.

$\frac{1}{2}(x+y) < \frac{1}{2}(2y)$.

$\frac{x+y}{2} < y$.

$x < y$

$x + x < y + x$

$\frac{1}{2}(2x) < \frac{x+y}{2}$

x is rational. $\therefore y$ is rational

$\frac{x}{2}$ is rational $\quad \frac{y}{2}$ is rational (from O4)

$\frac{x}{2} + \frac{y}{2}$ is rational // (from A1) e.g. Q) There ~~exists~~ cannot exist $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof:

let $r \in \mathbb{Q}$ s.t. $r^2 = 2$

$r = \frac{p}{q}$ $\frac{p^2}{q^2} = 2$.

$p^2 = 2q^2 \Rightarrow p^2$ is even.

$\Rightarrow p$ is even. $p = 2m$.

$2q^2 = 4m^2$

$q^2 = 2m^2$

$\Rightarrow q$ is even.

$\Rightarrow p, q$ have 2 as a common factor. $\therefore$ contradiction

e.g. Q) $\nexists\, r \in \mathbb{Q}$ s.t. $r^2 = m$; $m \in \mathbb{Z}^+$ non-square.

say there exists $m = r^2$; $m \in \mathbb{Z}^+$ non square.

$r = \frac{p}{q}$. $\frac{p^2}{q^2} = m \rightarrow$ simplest.

$p^2 = mq^2$.

prof: $\lambda^2 < m < (\lambda+1)^2$.

$\lambda^2 < \frac{p^2}{q^2} < (\lambda+1)^2$.

$\cancel{\lambda^2 q < \frac{p^2}{q^2} \cdot q < (\lambda+1)^2 q}$.

$\lambda < \frac{p}{q} < \lambda+1$

$\lambda q < p < \lambda q + q$.

$\boxed{0 < p - \lambda q < q}$ — ①.

$m(p-\lambda q)^2$

$= mp^2 - 2\lambda mpq + \lambda^2 q^2$

$= m^2q^2 - 2\lambda mpq + \lambda^2 p^2$

$= (mq - \lambda p)^2 = m(p - \lambda q)^2$

$m = \left(\frac{mq - \lambda p}{p - \lambda q}\right)^2$

$\Rightarrow \boxed{p - \lambda q > q}$ — ②

contradiction

any other rep. $D_R > D_Q$.

$\Rightarrow$ rationals are not enough.

Chapter 2: Real Numbers

Properties of $\mathbb{R}$

1. Algebraic Properties $\rightarrow$ same as rational numbers.
2. Order properties $\Rightarrow$ same as rational numbers
3. Completeness Properties
4. Archimedean Properties
5. Density property

Q) Prove $a \cdot 0 = 0$.

~~according to O1. (Law of trichotamy) there are only 3 possibilities: $a > b$ or $b > a$ or $a = b$.~~
~~$\therefore a \cdot 0 = 0$.~~

Prof:

$0 + 0 = 0$ (identity element of add.)

$a(0+0) = a \cdot 0$

$a \cdot 0 + a \cdot 0 = a \cdot 0$ distributive law.

$a \cdot 0 + (a \cdot 0 + (-a \cdot 0)) = \cancel{a \cdot 0} - a \cdot 0$.

$a \cdot 0 = 0$.

Q) Prove $-(-a) = a$.

$a + (-a) = 0$ additive inverse.

$(-a) + -(-a) = 0$

from ①, $-(-a) = a$

(for $-a + x = 0$
$x = a$).

③ Completeness ~~Archimedean~~ Properties ;

↳ if $S \subseteq \mathbb{R}$,

i) u is said to be upper bound of S if $x \leq u \ \forall x \in S$.

ii) b is lower bound of S if $x \geq b \ \forall x \in S$.

* if set is countable, it is discontinuous.

④ Archimedean Properties ;

↳ " if $(x, y) \in \mathbb{R}$ and $\underline{x > 0, y > 0}$ then $\exists\, n \in \mathbb{N}$ s.t.
$ny > x$. "

Proof: Suppose $\nexists\, n \in \mathbb{N}$ s.t. $ny > x$.

$\Rightarrow \ ny \leq x \ \forall \ n \in \mathbb{N}$

↳ upper bound (x) of all elements (ny) of some set S

$\Rightarrow S = \{ ky ; \ k \in \mathbb{N}, \ y \in \mathbb{R} \}$.

↓

* this set cannot have an upper bound.

↳ $ny \leq b$.

~~ny~~ . $b - y < b$. ∴ $b - y$ cannot be a suprimum.

∴ some element in S will belong to below range.

$\underline{b - y < py \leq b}$; $p \in \mathbb{N}$

↓

$b - y + y < py + y$

$b < (p+1)y$.

$ky > b$ $\underline{\text{contradiction}}$.

Consequences of Archimedean Principle

1) $x \in \mathbb{R}$, $\exists$ $n \in \mathbb{N}$ s.t. $n > x$.
$x > 0$.

case 1: $x > 0$
let $y > 0$, $ny > x$
$y = 1 \Rightarrow n > x$.

case 2: $x \leq 0$.
$n = 1$.

2) $x \in \mathbb{R}, x > 0, \exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < x$

$nx > y$, $y = 1 \Rightarrow nx > 1$
$x > \frac{1}{n}$.

$\because n \in \mathbb{N}, \frac{1}{n} > 0 \Rightarrow 0 < \frac{1}{n} < x$.

3) $x \in \mathbb{R}, x > 0, \exists n \in \mathbb{N}$ s.t. $n-1 \leq x < m$.

i) $ny > x$.
$y = 1 \Rightarrow n > x$.

ii) ? $S = \{k \in \mathbb{N} : k > x\}$
let $m > x$; m is the least no. in S.
$m-1 \notin S \Rightarrow m-1 \leq x$.

4) $x \in \mathbb{R}, \exists m \in \mathbb{Z}$ s.t. $m-1 \leq x < m$.
$m = 1$.
case 1: $x > 0$

⑤ Density Property:

1) for $(x, y) \in \mathbb{R}$. with $x < y$, there exists $r \in \mathbb{Q}$ s.t. $x < r < y$ and there exists $s \notin \mathbb{Q}$ (irrational) s.t. $x < s < y$.

$\cancel{x>y}$ $x < y$

i) $y - x > 0$

$0 < \frac{1}{n} < y - x$. (archimedean).

$ny - nx > 1$.

$ny > nx + 1$. $nx \in \mathbb{R}$.

$m - 1 \leq nx < m$, $m \in \mathbb{N}$

$\Rightarrow m \geq nx$ $\quad$ $m \leq nx + 1 < ny$

$ny - 1 > nx$.

$nx < m < ny$

$x < \frac{m}{n} < ny$ rational.

ii) irrational.

$\sqrt{m}\,x < \sqrt{m}\,y$. $\quad$ $x' < y'$. same.

$\Rightarrow \sqrt{m}\,x < \Omega < \sqrt{m}\,y$ //.

Intervals

- $\{x \in \mathbb{R} : a < x < b\}$ (a, b). open → no greatest / smallest element in this interval.
- $\{x \in \mathbb{R} : a \le x \le b\}$ $[a, b]$. closed → has a strict greatest and least element.
- $\{x \in \mathbb{R} : a < x \le b\}$ $(a, b]$. open-closed.

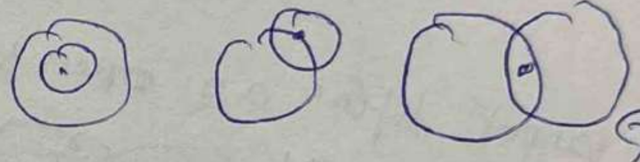
- when a, b are finite, interval is bounded.

Neighbourhood.

→ * a closed bounded interval containing c may not be a nbhd. e.g. $1 \in [1,3]$ but $N(1) \neq [1,3]$.

$c \in \mathbb{R}$, $S \subset \mathbb{R}$ is a neighbourhood of c if $\exists$ an open interval (a, b) s.t.

$c \in (a, b) \subset S$. $N(c)$.

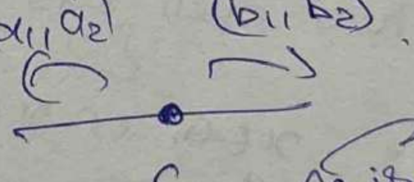


$(c - \epsilon, c + \epsilon)$.
↳ δ nbhd of c. $N(c, \delta)$

- Theorem: If S_1 and S_2 are neighbourhoods of c, then $S_1 \cup S_2$ is also a neighbourhood of c.

$c \in (a_1, b_1) \subset S_1$

$c \in (a_2, b_2) \subset S_2$.



{c is a nbhd of c}

i) $a_3 = \min(a_1, a_2)$ $\quad c \in (a_3, b_3)$

$b_3 = \max(b_1, b_2)$ $\quad \in S_1 \cup S_2$

ii) $a_4 = \max(a_1, a_2)$ $\quad c \in (a_4, b_4)$

$b_4 = \min(b_1, b_2)$ $\quad \in S_1 \cap S_2$.

* no restriction on size of ~~size~~ neighbourhood.

- Interior Point;

$S \subset \mathbb{R}$, x is an interior point of S if ~~at~~ at least one neighbourhood $N(x)$ of x can be defined s.t. this neighbourhood is a proper subset of S.

- In an open interval, all points are interior points.

- An open set is a set for which ~~$S = \text{int } S$~~

if $x \in S$, then $x \in \text{int } S$ (interior point set)

$\Downarrow$ $S = \text{int } S$ ($\hookrightarrow$ all points are interior points in open intervals)

- cannot define an open set on discrete sets ~~of~~ $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$

(* $S = \phi$ has no interior point)

Theorem:

Union of two open sets is an open set.

$G_1 \in \mathbb{R}$. $\quad x \in G_1$ or $x \in G_2$

$G_2 \subset \mathbb{R}$. $\quad \Downarrow$

$x \in G_1$

$N(x) \in G_1 \subset G_1 \cup G_2$

$\therefore$ any nbhd on G_1 can be shown on $G_1 \cup G_2$.

$\text{int } G_1 \cup \text{int } G_2$

$N(x, \delta_1) \subset G_1$ $\quad N(x,\delta) \subseteq N(x,\delta_1) \subset G_1 \cup G_2$

$N(x, \delta_2) \subset G_2$. $\quad N(x,\delta) \subseteq N(x,\delta_2) \subset G_1 \cup G_2$.

$\delta = \min\{\delta_1, \delta_2\}$.

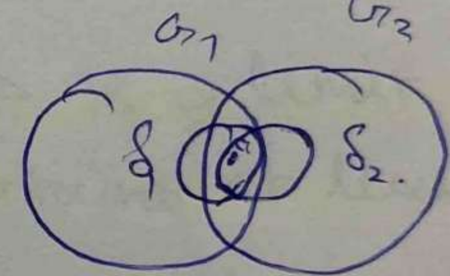
δs are radii.

~~to~~ Theorem:

The intersection of two open sets is an open set.

i) $G_1 \cap G_2 = \phi$; ϕ is an openset.

ii) $x \in G_1$ and $x \in G_2$.



$\downarrow \qquad \downarrow$

$N(x, \delta_1) \subset G_1$, $N(x, \delta_2) \subset G_2$.

$x \in G_1 \cap G_2$,

$\cancel{N(x,\delta_1) \cap N(x,\delta_2)}$ $\quad \delta \min\{\delta_1, \delta_2\}$.

$N(x,\delta) \subset G_1 \cap G_2$.

Quiz1: class 1 to wed class.

Theorem :

- Let $S \subseteq \mathbb{R}$, then int. S is an open set.

  1) int $S = \phi$; ϕ is an open set.

  2) int $S \neq \phi$; $x \in$ int S, x is an interior point of S. $\exists\, N(x) \subset S$

let $y \in N(x)$, $\Rightarrow y \neq x$, then $N(x)$ is also a nbhd of y.

then y is also $\in$ int S

Therefore $N(x) \subset$ int S.

x is also an interior point of int S.

$\Rightarrow$ int S is an open set.

- Let $S \subset \mathbb{R}$: int S is the largest open set contained in S.

- Limit Point : $S \subseteq \mathbb{R}$, A point p is a limit point of S if every deleted neighbourhood of p contains at least an element of S.

  ↳ all interior points are limit points.

- Isolated Point : not a limit point and p is in S

  ↳ e.g. every element of a discrete set is isolated

- $S \subset \mathbb{R}$ and p be a limit point of S

  Then every ~~element~~ neighbourhood of p contains infinitely many elements of S.

Bolzano-Weierstrass Theorem

- Every bounded infinite subset of $\mathbb{R}$ has at least one limit point.

Derived Set $S \subset \mathbb{R}$

↳ set of all limit point. (S').

- Closed Set: if all limit points are contained in S.

$S \subseteq \mathbb{R}$, S is said to be a closed set if $S' \subset S$

- $S = \{ \frac{1}{m} + \frac{1}{n}, m \in \mathbb{N}, n \in \mathbb{N} \}$

0 is a limit point of S.

~~Proof~~ ?

Let $\epsilon > 0$ $\exists$ p, q s.t. $0 < \frac{1}{p} < \frac{\epsilon}{2}$ $\quad$ ~~0<~~ $\frac{1}{p} + \frac{1}{q} < \epsilon$ //

$0 < \frac{1}{q} < \frac{\epsilon}{2}$

Derived Sets

- $S \subset \mathbb{R}$. Then the set of all limit points of S is called a derived set of S (S').

- $S = \{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots, n, \frac{1}{n}\}$ $S' = \phi$

* if S is any finite set, $S' = \phi$, it has no interiors, no limit.
↳ there are many nbhds which don't contain an element from S.

$N(P, \epsilon) - \{P\} = N'(P, \epsilon)$ ↳ this N' must contain at least 1 element that's also in S for P to be a limit point.

Q) set of all ~~rationals~~ naturals.

$S' = \phi$

* $\mathbb{Q}$ (set of Rationals) derived set $S' = \mathbb{R}$.

a (ϵ) ⤳ there is at least a rational b/w two reals (archimedean).

- Open set: all points are interior points

closed set: $S' \subseteq S$ (derived set is a subset of S).

e.g.) $\mathbb{R}$.

1   2

$1 \leq x \leq 2$ ↳ all the points of this set are limit points.

$S = S'$

$1 < x < 2$ ↳ ~~except~~ all points except 1, 2 are interior + limit.
1, 2 are limit points only not interior points.

↳ $S \subset S'$

↳ S' includes 1 and 2
S does not

Theorem

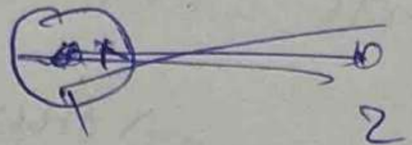
• Union of finite number of closed sets is a closed set.

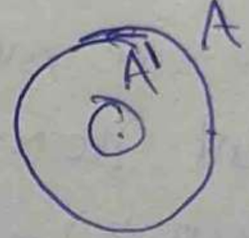
Proof;

Take two closed sets, P.T. $S' \subseteq S$ holds.

$A' \subseteq A$ $\quad$ $A' \cup B' \subseteq A \cup B$ needs to be proved

$B' \subseteq B$.





if $x \in A'$, $x \in A$

if $x \in B'$, $x \in B$.

if $x \in A' \cup B'$, $x \in A \cup B$.

Theorem

• Intersection of finite no. of closed sets is a closed set.

Theorem

- $F \subseteq \mathbb{R}$ is a closed set. Then its complement F^c is an open set.

1) $F = \mathbb{R}$

$F^c = \emptyset \rightarrow$ open set.

2) $F \subset \mathbb{R}$, $F^c \neq \emptyset$, let $x \in F^c$

$\because x \notin F$, x is not a limit point of F

$\exists N(x)$ for such

$N'(x) \cap F = \emptyset$. (not a limit point).

$N(x)$ (including x) is $\subset F^c$.

$x \in \text{int } F^c$.

$\because$ F is a closed set. $F' \subset F$, all limit points $\subset F$. if $x \notin F$, $x \notin F'$ $\therefore x$ is not a limit point of F.

Nested Intervals

- If $\{I_n ; n \in \mathbb{N}\}$ be a family of intervals $I_1, I_2 \ldots I_n$, such that $I_{n+1} \subset I_n$, then this is a family of intervals.

$I_n = \{x : 0 < x < \frac{1}{n}$ with $n \in \mathbb{N}\}$.

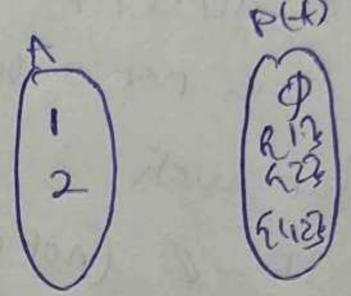
Enumerable Sets

- Let $S \subset \mathbb{R}$. S is enumerable (or denumerable) if there exists a bijective mapping $f : \mathbb{N} \longrightarrow S$
- enumerable sets is the definition of countable ∞.

~~Cantor's~~ Theorem * * *

• If A be a ~~non-empty~~ subset of $\mathbb{R}$, ~~then~~ $\exists$ no surjection $\phi : A \to P(A)$; $P(A)$ is the power set of A.

↳ it is sufficient to prove $\exists$ at least one set in $P(A)$ to which no element in A is mapped. ↳ $|A| < |P(A)|$



Proof:

Let $a \in A$

$f : A \to P(A)$

$\therefore f(a) \in P(A)$, $f(a) \subseteq A$

$a \in f(a)$ or $a \notin f(a)$.

define $S = \{a \in A \mid a \notin f(a)\}$. $S \subseteq A$.

Let $a_0 \in S \to a_0 \notin f(a_0)$ by definition of S

* i.e. $a_0 \in S \Rightarrow a_0 \notin f(a_0)$, i.e. $a_0 \notin S$.

*??? $a_0 \notin S \Rightarrow a_0 \in f(a_0)$, i.e. $a_0 \in S$.

Algebra of Functions

i) $(f+g)(x) = f(x) + g(x)$

ii) $fg(x) = f(x) \cdot g(x)$

iii) $kf(x) = k \cdot f(x)$

iv) $g \neq 0$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

Monotone Function

- if $I \subset \mathbb{R}$. $f: I \to \mathbb{R}$, then f is a monotonically increasing function on I, if $\forall \{x_1, x_2\} \in I$ s.t. $x_1 < x_2$, $f(x_1) \leq f(x_2)$

decreasing $\rightarrow$ $f(x_1) \geq f(x_2)$.

Strictly monotonic $\rightarrow$ equality is not included.

- $I = (-a, a)$.

f is even $\Rightarrow$ $f(-a) = f(a)$ in I

f is odd $\Rightarrow$ $f(-a) = -f(a)$.

Chapter 3: Sequences, Series,

- all problems boil down to pattern recognition.
- Sequence;
- A sequence is a fun^c whose domain is $\mathbb{N}$, codomain is $\mathbb{R}$.

If the fun^c is such a sequence, let $f(x) = x_n$, $n \in \mathbb{N}$.

$(x)_1^\infty$ / (x_n) / $\{x_n\}$ $\rightarrow$ elements of the sets are arranged in a sequence.

$x_n = ar^n$, $x_n = \sin(\frac{n\pi}{2})$ are e.g.s of sequences.

Bounded Sequence

- A sequence (x_n) is said to be
  - i) bounded above; if $\exists K \in \mathbb{R}$, s.t. $x_n \leq K$ $\forall n \in \mathbb{N}$.
  - ii) bounded below; if $\exists K \in \mathbb{R}$, s.t. $x_n \geq K$ $\forall n \in \mathbb{N}$.
  - iii) bounded; if it is bounded both above & below

    e.g. $(\frac{1}{n})$

Convergent Sequence

- A sequence is said to be convergent at $l \in \mathbb{R}$, if for a given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|x_n - l| < \epsilon \quad \forall n \geq N.$$

* has to be an ∞ sequence for convergence, and bounded too.

Divergent Sequence

- $x_n \rightarrow \infty$ as $n \rightarrow \infty$

  e.g. ar^{n-1}; $r > 1$ is diverging.

• Some sequences may not be conv. or diverg.

Q) prove $\lim_{n\to\infty} \frac{1}{n} = 0.$

$\equiv$ Prove $\left|\frac{1}{n} - 0\right| < \epsilon \quad \forall\ n \geq N;$

$0 < \frac{1}{N} < \epsilon$, when $n \geq N$

therefore $\left|\frac{1}{n} - 0\right| < \epsilon.\ \forall n \geq$

let $\epsilon > 0 \in \mathbb{R}$.
By archimidean property
$\exists N \in \mathbb{N}$ s.t. $0 < \frac{1}{N} < \epsilon$
$\forall n \geq N$, $0 < \frac{1}{n} < \epsilon$ holds
$\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon \ \forall n \geq N$
$\Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$ $\square$//
Q.E.D.

$< \epsilon.$

convergent $\Rightarrow$ bounded, but bounded $\not\Rightarrow$ convergent

Q) $\lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1.$

$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \epsilon \quad \forall\ n \geq N.$

$\frac{1}{2^n} < \epsilon. \quad \forall\ n \geq N.$

$0 < \frac{1}{2^N} < \epsilon.$

<u>proof;</u>

$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \epsilon$

$\rightarrow \left|\frac{1}{2^n}\right| < \epsilon \quad = \frac{1}{2^n} < \epsilon.$

$\frac{1}{(1+1)^n} < \epsilon.$

$\frac{1}{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}} < \epsilon.$

$(1+1)^n \geq 1 + n$

$\frac{1}{(1+1)^n} \leq \frac{1}{1+n} < \frac{1}{n} < \epsilon.$

* ## Theorem

- Let (S_n) and (t_n) be two sequences of real numbers. Take a real no.* $s \in \mathbb{R}$, if for $k > 0, k \in \mathbb{R}$ and $N_1 \in \mathbb{N}$ we have

$$|S_n - s| \leq k|t_n| \quad \forall n \geq N_1$$ where t_n converges to 0 $\left(\lim_{n\to\infty} t_n = 0.\right)$

then $\lim_{n\to\infty} S_n = s$.

Proof:

let $\epsilon > 0$.

$$|t_n - 0| < \frac{\epsilon}{k} \quad \forall n \geq N_2 \quad \text{for some } n, N_2 \in \mathbb{N}.$$

$\therefore \quad |t_n| < \frac{\epsilon}{k} \quad \forall n \geq N_2$. take $N = \max\{N_1, N_2\}$

for N both theorem and $|t_n - 0|$ ineq. hold.

$$|S_n - s| \leq k|t_n| < \epsilon \quad \forall n \geq N.$$

$$\therefore |S_n - s| < \epsilon \quad \forall n \geq N.$$

$$\Rightarrow \lim_{n\to\infty} S_n = s. \;//$$

**

Q) Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \quad \forall n \in \mathbb{N}$.

fact: $\sqrt[n]{n} \geq 1$, let $\sqrt[n]{n} = 1 + a_n$,

$a_n \in \mathbb{R}, a_n \geq 0$.

$$n = (1 + a_n)^n$$

$$n = 1 + na_n + \frac{n(n-1)}{2}a_n^2 + \dots + a_n^n$$

$$n \geq 1 + \frac{n(n-1)}{2}a_n^2$$

$n-1 \geq \frac{n(n-1)}{2} a_n^2$.

$a_n \leq \sqrt{\frac{2}{n}}$

$k = \sqrt{2}$

$|\sqrt[n]{n} - 1| = a_n \leq \sqrt{2} \frac{1}{\sqrt{n}}$ $\quad t_n = \frac{1}{\sqrt{n}}$ (another sequence)

$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$. by previous theorem

~~$|\sqrt[n]{n} - 1| < \epsilon \ \forall \ n \geq N \in \mathbb{N}$~~

$\lim_{n \to \infty} \sqrt[n]{n} = 1$.

Theorem: Uniqueness of Limit.

- $\lim_{n \to \infty} S_n = l_1$ and $\lim_{n \to \infty} S_n = l_2$, then $l_1 = l_2$.

i.e. a sequence has only at most one limit.

- Proof:

Proof by contradiction, let $l_1 \neq l_2$.

$|S_n - l_1| < \frac{\epsilon}{2} \ \forall \ n \geq N_1$

$|S_n - l_2| < \frac{\epsilon}{2} \ \forall \ n \geq N_2$.

take $N = \max\{N_1, N_2\}$.

then $\forall \ n \geq N$ both inequalities are satisfied.

consider $|l_1 - l_2|$

$|l_1 - l_2| = |l_1 - S_n + S_n - l_2| \leq |l_1 - S_n| + |S_n - l_2|$

by Δ inequality.

$|l_1 - l_2| \leq |S_n - l_1| + |S_n - l_2|$

$< \epsilon \quad \forall \; n \geq N.$

$0 \leq |l_1 - l_2| < \epsilon$, since ϵ is arbitrary, $|l_1 - l_2| = 0.$

$\Rightarrow l_1 = l_2$ //.

- Proposition;

A sequence (S_n) converges to a real l, iff for each $\epsilon > 0$.

the set $\{ n : x_n \notin (l-\epsilon, l+\epsilon)\}$ is finite.

↳ ~~for~~ the elements $n \leq N$ which don't lie in the nbhd of l are finite.

∞ elements in nbhd of l for convergent sequences.

↳ $|S_n - l_1| < \epsilon \quad \forall \; n \geq N$ → ∞ elements.

$l - \epsilon < S_n < l + \epsilon$. $\quad \therefore \; \forall \; n < N$ finite elements.

*Theorem:

- Every convergent sequence of real numbers is bounded.

Proof:

$\Rightarrow$ Let (S_n) be a seq. that converges to S.

i.e. $\lim_{n\to\infty} S_n = s$

$\Rightarrow |S_n - s| < \epsilon \quad \forall n \geq N$; for some $N \in \mathbb{N}$.

why ~~is inequality staying like this~~?

$\to$ choose $\epsilon = 1$

$|S_n - s| < 1 \quad \forall n \geq N$

$|S_n| = |S_n - S + S| \leq |S_n - s| + |S|$

$|S_n| < 1 + |S| \quad \because |S_n - S| < 1.$

~~bounded $[S_n]$~~ $\forall n \geq N$

~~What about previous terms~~?

consider;

$M = \max\{|S_1|, |S_2|, \ldots |S_N|, 1+|S|\}.$

$|S_m| \leq M.$

(S_n) is bounded $\forall$ ~~its elements~~ $n \in \mathbb{N}$ //.

* **Theorem**: Squeeze theorem of limit

Suppose (S_n), (t_n), and (u_n) are three sequences s.t.

$$S_n \le t_n \le u_n, \ \forall n \in \mathbb{N}$$

(i.e $S_1 \le t_1 \le u_1$, $S_2 \le t_2 \le u_2$ etc.)

then if $\lim_{m\to\infty} S_m = \lim_{n\to\infty} U_n = l$. ($S_n, U_n$ converge to l)

&

then $\lim_{m\to\infty} t_m = l$ (then t_n also converges to l).

Proof;

$|S_n - l| < \frac{\epsilon}{2} \ \forall n \ge N_1$

$|u_n - l| < \frac{\epsilon}{2} \ \forall n \ge N_2$

Let $N = \max\{N_1, N_2\}$.

$l - \epsilon < S_n < l + \epsilon$

$l - \epsilon < u_n < l + \epsilon$.

$l - \epsilon < S_n \le t_n \le u_n < l + \epsilon. \ \forall n \ge N$.

$|t_n - l| < \epsilon \ \forall n \ge N$.

$\lim_{n\to\infty} t_n = l$.

Theorem:

- Let $S \subset \mathbb{R}$ be bounded from above.

Then $\exists (S_n)$ ~~for~~ which $\lim_{n\to\infty} S_n = \sup S$.

Proof:

$c = \sup S$. For each $n \in \mathbb{N}$, $\exists$ S_n s.t. $c - \frac{1}{n} < S_n \leq c$.

$S \subset \mathbb{R}$

$$\lim_{n\to\infty} \left(c - \frac{1}{n}\right) = c.$$

$$\lim_{n\to\infty} S_n = c$$ from squeeze Theorem.

Algebra of Lts of Sequences

$\lim_{n\to\infty} S_n = s$ and $\lim_{n\to\infty} t_n = t$

1) $\lim_{n\to\infty} (S_n + t_n) = s + t$

2) $\lim_{n\to\infty} S_n \cdot t_n = st$.

3) $\lim_{n\to\infty} \frac{S_n}{t_n} = \frac{s}{t}$; [if $t_n \neq 0$ $\forall n \in \mathbb{N}$].

Proof

1) given $\epsilon > 0$

$|S_n - s| < \frac{\epsilon}{2}$ $\forall n \geq N_1$.

$|t_n - t| < \frac{\epsilon}{2}$ $\forall n \geq N_2$.

$\rightarrow$ choose $N = \max\{N_1, N_2\}$.

$|S_n + t_n - s - t| \leq |S_n - s| + |t_n - t|$

$\hookrightarrow$ Δ inequality

$< \epsilon$

$\forall n \geq N$.

ii) given $\epsilon > 0$

~~$|s_n - s| < \sqrt{\epsilon} \quad \forall n \geq N_1$~~

~~$|t_n - t| < \sqrt{\epsilon} \quad \forall n \geq N_2$~~

~~choose $N = \max\{N_1, N_2\}$.~~

$|s_n t_n - s.t|$ consider.

$= |s_n t_n - s.t. + st_n - st_n|$

$= |(s_n - s)t_n + (t_n - t)s| \leq |t_n(s_n - s)| + |s(t_n - t)|$

$\leq |t_n||s_n - s| + |s||t_n - t|$

- (t_n) converges to t, hence (t_n) is bounded.

then $|t_n| \leq K \in \mathbb{R}^+$,

$|s_n - s| < \frac{\epsilon}{2M} \quad \forall n \geq N_1$

$\frac{\epsilon}{2M} \quad \forall n \geq N_2$

$\leq K|s_n - s| + |s|\,|t_n - t|$.

$M = \max\{K, |s|\}$.

$\leq M(|s_n - s| + |t_n - t|)$.

$< \cancel{M}\left(\frac{\epsilon}{\cancel{M}}\right)$

iii) $\left|\frac{s_n}{t_n} - \frac{s}{t}\right|$.

$$\left|\frac{s_n t - t_n s}{t t_n}\right| = \left|\frac{s_n t + s_n t_n - s_n t_n - t_n s}{t t_n}\right|$$

$$= \cancel{\frac{1}{t_n}}$$

$$\left|\frac{s_n t - t_n s - st + st}{t t_n}\right| = \left|\frac{t(s_n - s) + s(t - t_n)}{t t_n}\right|$$

$$= \left|\frac{t(s_n - s) - s(t_n - t)}{t t_n}\right|.$$ Let $\frac{1}{t_n} = t'_n$.

Monotone Sequences

a) A sequence is monotonically increasing if $S_n \leq S_{n+1}$ $\forall n \in \mathbb{N}$.

b) " is monotonically decreasing if $S_n \geq S_{n+1}$ $\forall n \in \mathbb{N}$.

Theorem:

Take a bounded sequence (S_m). (MCT expanded).

i) if (S_m) is monotonically increasing then $\lim_{n\to\infty} S_n = \sup S_n$.

ii) if (S_m) is monotonically dec. then $\lim_{n\to\infty} S_n = \inf S_n$.

Proof:

Let $S_1 = \sup S_n$, $S_2 = \inf S_n$.

i) $\because S_n$ is increasing

Then $\exists S_{n_0}$ s.t. $S_1 - \epsilon < S_{n_0}$ for some $\epsilon > 0$

$$S_1 - \epsilon < \underbrace{S_{n_0} \leq S_n}_{\text{monotonic}} \underbrace{\leq S_1}_{\text{bounded}} < S_1 + \epsilon, \quad \forall n > n_0$$

$$S_1 - \epsilon < S_n < S_1 + \epsilon \Rightarrow |S_n - S_1| < \epsilon \quad \forall n > n_0.$$

$$\therefore \lim_{n\to\infty} S_n = S_1 (\sup\{S_n\}).$$

ii) $S_{n_0} > S_2$

$S_{n_0} < S_2 + \epsilon$.

$$S_2 - \epsilon < S_n \leq S_{n_0} < S_2 + \epsilon. \quad \forall n > n_0.$$

$$\Rightarrow S_2 - \epsilon < S_n < S_2 + \epsilon.$$

Theorem : MCT

Every monotonic sequence converges iff it is bounded.

- previous proof was suff. part.

- necessary
- sufficient.
  ↳ convergence suff. bounded
  convergens ¬ nec. bounded.

Sub-Sequence ;

- e.g. $\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\}$. → (5)

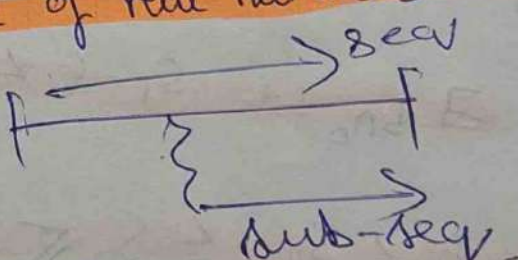
  $\{1, 2, 3, \ldots\}$ sub-sequence

  $\{\frac{1}{n}\}$ sub-sequence

***

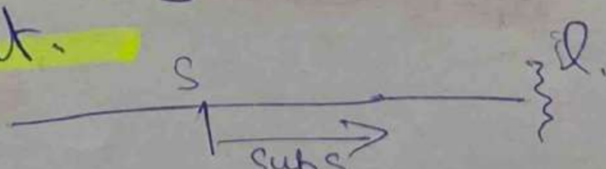
Theorem (BW) on Sub-Sequence

- Every bounded infinite sequence of real numbers has a convergent subsequence.



Theorem

- Any sub-sequence (∞) of a converging sequence is converges to the same limit.



Proof ;

Let (S_{n_k}) be a sub-sequence of (S_n) $\lim_{n \to \infty} S_n = l$.

$|S_n - l| < \epsilon \ \forall n \geq N$. let $K \in \mathbb{N}$ s.t. $K \geq N$,

$|S_{n_k} - l| < \epsilon \quad \forall n_k \geq K \geq N$ □ //

Cauchy Sequence

- A sequence (S_n) is cauchy if for a given $\epsilon > 0$, $\exists N \in \mathbb{N}$

s.t. $|S_n - S_m| < \epsilon \quad \forall n, m \geq N$.

Theorem;

Every convergent sequence is cauchy sequence.

Proof;

Let (S_n) converge at s

Then for a given $\epsilon > 0$

$|S_n - s| < \frac{\epsilon}{2} \quad \forall n \geq N_1$

$|S_m - s| < \frac{\epsilon}{2} \quad \forall m \geq N_2$

why $N_2 \neq N_1$? S_m is just an element $\{S_n\}$??

$N = \max\{N_1, N_2\}$.

Take $|S_n - S_m + s - s| = |(S_n - s) - (S_m - s)|$

$\leq |S_n - s| + |S_m - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$|S_n - S_m| < \epsilon \quad \forall n \geq N$ ||.

Theorem;

- Every cauchy sequence is bounded.

Proof;

Let (S_n) be a cauchy sequence $\lim_{n \to \infty} (S_n) = l$.

~~$|S_n - l| < \epsilon \quad \forall n \geq N_1$~~

~~$|S_n - l| < \epsilon \quad \forall n \geq N_2$.~~ $\quad l - \epsilon < S_m < l + \epsilon$.

~~& $|S_m - l| < \epsilon$~~

~~let $\epsilon = 1$ $\quad |S_n - l| < \epsilon$.~~

~~$l - \epsilon < S_n < l + \epsilon$.~~

$\cancel{|S_m - S_n| = |S_m - S - S_n + S| \le |S_m - S| + |S_n - S|}$

$\cancel{|S_m - S_n| \le}$

$|S_n - S_m| < 1 \quad \forall n, m \ge N$

take $|S_n| = |S_n - S_k + S_k| \le |S_n - S_k| + |S_k| < 1 + |S_k|$

$|S_n| < 1 + |S_k| \quad \forall n \ge N.$ $K > N$

for elements below N;

$M = \max\{S_1, S_2, \dots S_3, |S_N|, 1 + |S_k|\}$ N

Then $|S_n| \le M \quad \forall n \le N,$

$K \in \mathbb{R}$ s.t $S_n \le K \; \forall n$. D//.

$\therefore S_n$ is bounded $\forall$

Theorem

Every cauchy sequence is convergent;

Proof:

Let S_n be cauchy. cauchy sequences are bounded. So we have a convergent sub-sequence for it (S_{n_k}) $\rightarrow$ $\therefore$ sub-sequence is convergent.

$|S_m - S_{n_k}| < \frac{\epsilon}{2} \quad \forall m, n_k \ge N_1$

$|S_{n_k} - l| < \frac{\epsilon}{2} \quad \forall n_k \ge N_2.$

$N = \max\{N_1, N_2\}$

$|S_n - l| = |S_n - S_{n_k} + S_{n_k} - l|$

$\leq |S_n - S_{n_k}| + |S_{n_k} - l|$

$|S_n - l| < \epsilon \quad \forall n \geq N$

$\therefore S_n$ is converging //.

* if a sequence is cauchy → it is also convergent.
↳ another way to prove convergence.

Q) $\left(\frac{(-1)^n}{n}\right)$ is convergent. Prove.

proof)
prove it is cauchy: $|S_n - S_m| < \epsilon \quad \forall n, m \geq N$

consider $|S_n - S_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$.

$\frac{1}{2}$ $\frac{1}{3}$ $\leq$ $\frac{1}{2} + \frac{1}{3}$.

let $m \geq n$

$\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{2}{n}$. → previous.

$\leq k|t_n| \quad t_n \to 0$

$\therefore$

$\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| < \epsilon$ //

Q) Cesaro Average

- Let $\{x_n\}$ be a given convergent ~~series~~ sequence converging to l.

~~S.t.~~ P.t. $y_n = \dfrac{x_1 + x_2 + \dots + x_n}{n}$ also converges to l.

need to prove $|y_n - l| < \epsilon \quad \forall\, n \geq$ some $K(\epsilon) \in \mathbb{N}$.

$\because$ x_n converges, $\{x_n\}$ is bounded

$|x_n| \leq M$.

given: $|x_n - l| < \epsilon \quad \forall\, n \geq N_0$.

consider $|y_n - l| = \left| \dfrac{x_1 + x_2 + \dots + x_n}{n} - l \right|$

$= \left| \dfrac{x_1 + x_2 + \dots + x_n}{n} - \dfrac{nl}{n} \right|$

$= \left| \dfrac{(x_1 - l) + (x_2 - l) + \dots + (x_n - l)}{n} \right|$

~~$\leq |x_1 - l| + |x_2 - l| + \dots + |x_n - l|$~~

$\leq \left| \dfrac{(x_1 - l) + (x_2 - l) + \dots + (x_{n_0 - 1} - l)}{n} \right| + \left| \dfrac{(x_{n_0} - l) + (x_{n_0+1} - l) + \dots + (x_n - l)}{n} \right|$

$\left| \dfrac{(x_1 - l) + \dots + (x_{n_0 - 1} - l)}{n} \right| < \epsilon \quad \forall\, n \geq N_1$, ~~$\forall\, n \geq N_1 < \epsilon + \dfrac{|(x_{n_0} - l) + \dots + (x_n - l)|}{n}$~~

converges to 0.

$\forall\, n \geq N$, $N = \max\{N_1, N_0\}$

$|y_n - l| < \epsilon + \dfrac{|(x_{n_0} - l) + (x_{n_0+1} - l) + \dots + (x_n - l)|}{n} \quad \forall\, n \geq N$

$N = \max\{N_0, N_1\}$

$< \epsilon + \dfrac{|x_{n_0} - l| + |x_{n_0+1} - l| + \dots + |x_n - l|}{n} < \epsilon + \dfrac{(n - N_0 + 1)\epsilon}{n}$

$= 6!$

$\rightarrow |y_n - \ell| < \epsilon' \quad \forall\, n \geq N$

ϵ' is arbitrary.

Q) $x_n = \dfrac{n^2}{\sqrt{n^6+1}} + \dfrac{n^2}{\sqrt{n^6+2}} + \dots + \dfrac{n^2}{\sqrt{n^6+n}}$

$\dfrac{n^2}{\sqrt{n^6+1}} \rightarrow$ biggest, $\dfrac{n^2}{\sqrt{n^6+n}} \rightarrow$ smallest.

Find if x_n is converging or not.

$u_n \leq x_n \leq t_n$

can we define these in a way in which we can easily converge u_n, t_n.

$\cancel{\dfrac{n^2}{\sqrt{n^6}} \leq x_n < \dfrac{n^2}{\sqrt{2n^6}}}$

$\cancel{u_n = \sum_{i=1}^{n} \dfrac{n^2}{\sqrt{n^6+n}}}, \quad \cancel{t_n = \sum_{i=1}^{n} \dfrac{n^2}{\sqrt{n^6+1}}}$

$\sum_{i=1}^{n} \dfrac{n^2}{\sqrt{n^6+n}} \leq x_n \leq \sum_{i=1}^{n} \dfrac{n^2}{\sqrt{n^6+1}}$

$\dfrac{n^3}{\sqrt{n^6+1}} \leq x_n \leq \dfrac{n^3}{\sqrt{n^6+n}}$

$\dfrac{1}{\sqrt{1+\frac{1}{n^6}}} \leq x_n \leq \dfrac{1}{\sqrt{1+\frac{1}{n^5}}}$ $\Rightarrow x_n$ converges to 1.

Q) $x_n = \dfrac{[\alpha] + [2\alpha] + \cdots + [n\alpha]}{n^2}$, $[\cdot]$ = Greater integer function.

$$\frac{\cancel{n}[\alpha]}{n^{\cancel{2}}} \le x_n \le \frac{\cancel{n}[n\alpha]}{n^{\cancel{2}}}$$

$$\alpha - 1 < [\alpha] \le \alpha$$

$$2\alpha - 1 < [2\alpha] \le 2\alpha$$

$$\cdots$$

$$n\alpha - 1 < [n\alpha] \le n\alpha$$

$$\frac{(1+2+\cdots+n)\alpha - n}{n^2} < \cancel{\frac{x_n}{n^2}} \le \frac{\frac{n(n+1)}{2}\alpha}{n^2}$$

$$\frac{\frac{\cancel{n}(n+1)}{2}\alpha - n}{n^{\cancel{2}}} < x_n \le \frac{(1+\frac{1}{n})\alpha}{2}$$

n ∈ ... conv...

Q) $\lim_{n\to\infty} (2\sqrt[n]{x} - 1)^n = x^2$; $x > 1$

$|(2\cdot\sqrt[n]{x} - 1)^n - x^2| < \epsilon$, $\forall n \ge N$ to be shown,

consider $|(2\cdot\sqrt[n]{x} - 1)^n| = |(2\sqrt[n]{x} - 1)|^n$

$$|2\sqrt[n]{x} - 1| \le |2\sqrt[n]{x}| + 1$$

$$|2\sqrt[n]{x} - 1|^n \le (|2\cdot\sqrt[n]{x}| + 1)^n$$

$0 \leq (\sqrt[n]{x}-1)^2 = (\sqrt[n]{x})^2 - 2\sqrt[n]{x}+1.$

$(2\cdot\sqrt[n]{x}-1) \leq (\sqrt[n]{x})^{\frac{2}{n}}$ $\quad (x^{\frac{1}{n}})^2 = x^{\frac{2}{n}}$

$(2\cdot\sqrt[n]{x}-1)^n \leq x^2$

consider $(2\cdot\sqrt[n]{x}-1)^n$

$= x^2\left(\frac{2}{\sqrt[n]{x}} - \frac{1}{\sqrt[n]{x^2}}\right)^n$

$= x^2\left(1-\left(1-\frac{1}{\sqrt[n]{x}}\right)^2\right)^n$

$\geq 1-n\left(1-\frac{1}{\sqrt[n]{x}}\right)^2.$

$= \cancel{(1-h)}^n$

we know ~~consider~~ $(1-h)^n \geq 1-nh$

$h \geq 0.$

$x = (\sqrt[n]{x}-1+1)^n$ iff $x>1.$

$\geq 1+n(\sqrt[n]{x}-1)$

$x > n(\sqrt[n]{x}-1).$

$(\sqrt[n]{x}-1)^2 < \frac{x^2}{n^2}.$

$(2\cdot\sqrt[n]{x}-1)^2 > \frac{x^2(1-x^2)}{n\cdot\sqrt[n]{x^2}}.$

Chapter 4: Infinite Series

- ~~conver~~ divergent sequence $\rightarrow$ divergent series

  convergent sequence $\not\rightarrow$ convergent series

- $S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots\cdots$

$S = \sum_{n=1}^{\infty} u_n \; ; \; u_n = \frac{1}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right),$

$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$S_n = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1$. $\therefore$ S is a convergent series.

- e.g. G.P. sum if $r < 1$ is converging.

Cauchy's Principle of Convergence

- A necessary and sufficient condition for a series $S_n = \sum u_n$ to be convergent is that; for a given $\epsilon > 0$ $(\epsilon \in \mathbb{R}^+)$

$\exists\, m \in \mathbb{N}$ s.t. $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon \;\; \forall n \geq m$.

for every $p \in \mathbb{N}$.

1) take ϵ.
2) find $m \in \mathbb{N}$ s.t. $|u_{n+1} + \cdots + u_{n+p}| < \epsilon \;\; \forall n \geq m$

$\forall p \in \mathbb{N}$,

① **Corollary:**

A necessary condition for a series $S_n = \sum u_n$ to be conv. is that the corresponding sequence must converge to 0 ($\lim_{n\to\infty} u_n = 0$).

Proof:

Let $S_n = \sum u_n$ be a converging series.

i.e. $|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon \quad \forall n \geq m \quad \forall p \in \mathbb{N}$.

take $p = 1$

$|u_{n+1}| < \epsilon \quad \forall n \geq m$.

$|u_{n+1} - 0| < \epsilon \quad \forall n+1 \geq m+1$

$\therefore$ ~~u_{n+1}~~

Counter-Example.

- $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ $\rightarrow$ diverges $\quad u_n = \frac{1}{n}$.

- $|S_{n+p} - S_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p}\right|$

$|S_{2n} - S_n| = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p}$. let $p = n$.

$\geq \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n}$.

$|S_{2n} - S_n| \geq \frac{1}{2}$ $\rightarrow$ fails cauchy $\therefore$ not convergent.

If we choose $\epsilon < \frac{1}{2}$, the convergence inequality will never satisfy.

• Series of positive terms $S_n = \sum u_n$ where $u_n \in \mathbb{R}^+$.

Theorem:

A series of positive terms is convergent iff the sequence $\{S_n\}$ of partial sums is ~~convergent~~, bounded above.

$S_n = u_1 + u_2 + \dots + u_n$

$|S_{n+p} - S_n| = |\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p}| < \epsilon \ \forall n \geq m$

put $p = 1$

$\{S_{n+1} - S_n\} = \{\frac{1}{n+1}\} \to 0$

$|S_{n+1} - S_n| < \epsilon \quad \forall n \geq m$, $\quad S_{n+1} > S_n \ \forall n \in \mathbb{N}$

$\{S_n\} \to$ monotonically increasing sequence.

$\{S_n\}$ can ~~only~~ converge iff it is bounded above B//.

① Comparison Tests

① Let $\sum u_n$ and $\sum v_n$ are two series and $\exists\, m \in \mathbb{N}$, s.t.

$u_n \leq k v_n \ \forall n \geq m$, $k > 0$ Then $\sum u_n$ and $\sum v_n$ converge together.

↳ not necessarily the same limit as v_n

② Limit Form:

$\sum u_n$ and $\sum v_n$ ....

and $\lim_{n \to \infty} \frac{u_n}{v_n} = l > 0$

$\sum u_n$ and $\sum v_n$ converge/diverge together.

Proof;

$\left|\frac{U_n}{V_n} - l\right| < \epsilon \quad \forall\, n \geq m$

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$l - \epsilon < \frac{U_n}{V_n} < l+\epsilon$

$l+\epsilon = k > 0, \quad U_n < kV_n.$

Theorem : p-series test.

The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ converges for $p > 1$ and diverges otherwise.

Proof;

i) $p \geq 0$

$$\underbrace{1}_{V_1} + \underbrace{\left(\frac{1}{2^p} + \frac{1}{3^p}\right)}_{V_2} + \overbrace{\left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)}^{V_3} + \underbrace{\left(\frac{1}{8^p} + \dots + \frac{1}{15^p}\right)}_{V_4} + \dots$$

$V_2 = \frac{1}{2^p} + \frac{1}{3^p}, \quad V_2 \leq \frac{1}{2^{p-1}}$

$V_3 \leq \frac{1}{2^{2(p-1)}}$

$\vdots$

$V_n \leq \frac{1}{2^{(n-1)(p-1)}}$

$V_n \leq \left(\frac{1}{2^{p-1}}\right)^{n-1}$ if $p > 1$, V_n is conv.

$V_n \leq 1 \cdot W_n \rightarrow$ comparison test from here)

Chapter 5: Limits

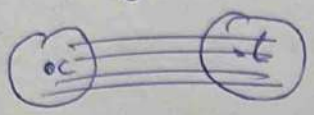
→ all nbhds have at least one element in D

- Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. Let c be a limit-point of D

A real number l is said to be a limit of f, iff

$|f(x) - l| < \epsilon \quad \forall\, x \in N'(c, \delta) \cap D$

$\Downarrow$



means if l is the limit, all points in the nbhd of c are mapped only to the points in the nbhd of l.

$l - \epsilon < f(x) < l + \epsilon$
$\forall x \in N'(c, \delta) \cap D$.

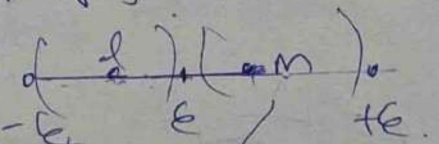
$\lim_{x \to c} f(x) = l$.

Theorem: Uniqueness of limit

Proof by contradiction;

(l, m) be a limit of f at c (limit point of D).

$m > l$.



$\epsilon = \frac{m - l}{2}$.

Then $(l - \epsilon, l + \epsilon) \cap (m - \epsilon, m + \epsilon) = \phi$

$l - \epsilon < f(x) < l + \epsilon \quad \forall x \in N'(c, \delta_1) \cap D$.

$m - \epsilon < f(x) < m + \epsilon \quad \forall x \in N'(c, \delta_2) \cap D$.

$\delta = \min(\delta_1, \delta_2)$

Then

$l - \epsilon < f(x) < l + \epsilon \;\; \forall x \in N'(c, \delta) \cap D$
$m - \epsilon < f(x) < m + \epsilon \;\; \forall x \in N'(c, \delta) \cap D$
} maps x to disjoint sets

one-many mapping → not allowed.

contradiction ∴ $l = m$

Sequential Criteria

• Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ and let c be a limit point of D.

then $\lim_{x \to c} f(x) = l$ iff for every $\{x_n\} \subseteq D - \{c\}$

which converges to c, the $\{f(x_n)\}$ converges to l.

• Proof; let $\lim_{x \to c} f(x) = l$.

i) $l - \epsilon < f(x) < l + \epsilon$. $\forall x \in N'(c, \delta) \cap D$.

$\{x_n\}$ converges to c

$\therefore c - \delta < x_n < c + \delta \;\; \forall n \geq k \in \mathbb{N}$.

$l - \epsilon < f(x_n) < l + \epsilon$ by def. of limit $\forall n \geq k$.

$|f(x_n) - l| < \epsilon \;\; \forall n \geq k$

$\therefore \{f(x_n)\}$ converges to l.

~~let $\{f(x_n)\}$ converge to l.~~

let $\{x_n\}$ $\in D - \{c\}$ converge to c

$|x_n - c| < \delta \;\; \forall n \geq k; k \in \mathbb{N}$.

$c - \delta < x_n < c + \delta$.

$\because$ all x_n converge to some c

$\therefore$ all $f(x_n)$ must also lie in the nbhd of some l

$\Rightarrow \{f(x_n)\}$ converges to some l.

$|f(x_n) - l| < \epsilon \;\; \forall n \geq k$.

$l - \epsilon < f(x_n) < l + \epsilon$. but l not be a limit of $f(x)$ at c.

~~$\Rightarrow f(x_n)$~~

ii) $\{x_n\} \subseteq D - \{c\}$.

$\exists$ a nbhd V of l s.t. for every nbhd W of c, $\exists$

$x_n \in [W - \{c\}] \cap D$ for which $f(x_n) \notin V$.

$W_1 = N(c,1)$ $x_1 \in N'(c,1) \cap D$ s.t. $f(x_1) \notin V$.

$W_2 = N(c, \frac{1}{2})$ $x_2 \in N'(c, \frac{1}{2}) \cap D$ s.t. $f(x_2) \notin V$.

$W_n = N(c, \frac{1}{n})$ $x_n \in N'(c, \frac{1}{n}) \cap D$ s.t. $f(x_n) \notin V$.

So,
We have $\{x_n\}$ in D s.t. $\lim_{n \to \infty} x_n = c$.

but $f(x_n)$ does not belong in V, which is the nbhd around l.

Therefore, $\{f(x_n)\}$ does not converge at l,

contradiction

$\lim_{x \to c} f(x) = l$.

(7.1.3 Mapa)

<u>Theorem:</u> Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ $c \in D'$ $c \in D'$

If f has a limit $l \in \mathbb{R}$ at c then f is bounded on $N(c) \cap D$.

- Theorem :

$\lim_{x \to c} f(x) = l$, then

1) ~~f(x)~~ if $l > 0$ then $f(x)$ is > 0 $\forall x$, ~~$\in N'(c,\delta) \cap D$.~~
$\exists \delta$ s.t. $x \in N'(c,\delta) \cap D$.

2) if $l < 0$ then $f(x) < 0$ $\forall x \in N'(c,\delta) \cap D$.

Proof: ①
take $\epsilon > 0$ s.t. $l - \epsilon > 0$.
$l - \epsilon < f(x) < l + \epsilon$. $\forall x \in N'(c,\delta) \cap D$ by definition.

$0 < f(x) < l + \epsilon$. □// Q.E.D.

② $l < 0$.
take $\epsilon > 0$ s.t. $l + \epsilon < 0$.
$l - \epsilon < f(x) < l + \epsilon$ $\forall x \in N'(c,\delta) \cap D$.

$l - \epsilon < f(x) < 0$ □// Q.E.D.

Algebra of Limits;
Theorem
if $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$

1) $\lim_{x \to c} (f(x) + g(x)) = l + m$

2) $\lim_{x \to c} f(x) g(x) = lm$

3) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{l}{m}$ iff $g(x) \neq 0$.

①
$|f(x)-l| < \frac{\epsilon}{2} \ \forall x \in N'(c,\delta_1) \cap D$

$|g(x)-m| < \frac{\epsilon}{2} \ \forall x \in N'(c,\delta_2) \cap D$

$\delta = \min(\delta_1, \delta_2)$

then ~~the~~

$|f(x)-l+g(x)-m| \le |f(x)-l| + |g(x)-m|$

$< \epsilon \ \forall x \in N(c,\delta) \cap D.$

$2^n > n$

consider 2^n

$2^n = (1+1)^n$

$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

$2^n = 1 + n$

$2^n > n.$

②. $|f(x)-l| < \frac{\epsilon}{2} \ \forall x \in N'(c,\delta_1) \cap D$

$|g(x)-l| < \frac{\epsilon}{2} \ \forall x \in N'(c,\delta_2) \cap D.$

$|mf(x) - lg(x)|$ $\qquad \delta = \min\{\delta_1, \delta_2\}.$

$= |f(x)m - lm - g(x)l + lm|$

$= |m(f(x)-l) + l(m-g(x))| \le m|f(x)-l| + l|g(x)-m|.$

$M = \max\{m, l\}.$

$|mf(x) - lg(x)| \le M(\epsilon)$

$|f(x)g(x) - lm| = |f(x)g(x) - lg(x) + lg(x) - lm|$

$|f(x)g(x) - lm| \le |g(x)||f(x)-l| + l|g(x)-m|$

$k = \max\{|g(x)|, l\}$ ∵ $g(x)$ is bounded $\forall x \in N'(c,\delta) \cap D$

$|f(x)g(x) - lm| < k(|f(x)-l| + |g(x)-m|).$

$|f(x)g(x) - lm| < \epsilon. \ \forall x \in N'(c,\delta) \cap D.$

Sandwhich theorem

Let $D \in \mathbb{R}$ and $f\ D \to \mathbb{R}$ $c \in D$.

if $f(x) \leq g(x) \leq h(x) \quad \forall x \in D - \{c\}$.

then if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l \in \mathbb{R}$ then $\lim_{x \to c} g(x) = l$.

$l - \epsilon < f(x) < l + \epsilon \quad \forall x \in N'(c, \delta_1) \cap D$

$l - \epsilon < h(x) < l + \epsilon \quad \forall x \in N'(c, \delta_2) \cap D$

$\delta = \min\{\delta_1, \delta_2\}$

$l - \epsilon < f(x) \leq g(x) \leq h(x) < l + \epsilon \quad \forall x \in N'(c, \delta) \cap D$.

$\Rightarrow l - \epsilon < g(x) < l + \epsilon \quad \forall x \in N'(c, \delta) \cap D$

$\Rightarrow \lim_{x \to c} g(x) = l$.

$f(x) = x$

$g(x) = x^2$

$f(x)g(x) = x^3$.

$\lim_{x \to 3} f(x) = 3 = l$

$\lim_{x \to 3} g(x) = 9 = m$

$\lim_{x \to 3} x^3 = 27$

lm

Cauchy Criterion

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$

$c \in D'$

Let if $\lim_{x \to c} f(x) = l$ then $|f(x') - f(x'')| < \epsilon$

$\forall (x', x'') \in N'(c, \delta) \cap D$.

Q) $\lim_{x \to 4} \frac{4-x}{2-\sqrt{x}}$

$\epsilon > 0$

$|f(x) - l| < \epsilon \quad \forall x \in N'(c, \delta) \cap D$.

$l - \epsilon < f(x) < l + \epsilon \quad \forall x \in N'(c, \delta) \cap D. \qquad c - \delta < x < c + \delta - \{c\}$

$\Rightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ (generally)

• $\lim_{x \to 4} \frac{4-x}{2-\sqrt{x}} = \lim_{x \to 4} \frac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})} = \lim_{x \to 4} (2+\sqrt{x})$.

$\epsilon > 0, \delta = 2\epsilon$.

$\left| \frac{4-x}{2-\sqrt{x}} - 4 \right| = |2 + \sqrt{x} - 4| = |\sqrt{x} - 2| = \left| \frac{x-4}{\sqrt{x}+2} \right| \leq \frac{|x-4|}{2}$

$\left| \frac{4-x}{2-\sqrt{x}} - 4 \right| < \epsilon$

$\Uparrow$

$\left| \frac{x-4}{\sqrt{x}+2} \right| < \epsilon$

$4 - \delta < x < 4 + \delta$.

$-\delta < x - 4 < \delta$.

$0 < |x-4| < \delta$

$0 < \frac{|x-4|}{2} < \frac{\delta}{2}$

$\Leftarrow \frac{|x-4|}{2} < \epsilon$,

Q) $\lim_{x \to 0} x \sin(\frac{1}{x})$

$|x \sin(\frac{1}{x}) - 0| < \epsilon \quad \forall x \in N'(0, \delta) \cap D$

$-\delta < x < \delta - \{0\}. \quad \delta = \epsilon$

$0 < |x| < \delta$.

$|x \sin(\frac{1}{x})| \leq |x| < \delta = \epsilon$.

Q) $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ q \neq 0. \\ 0 & \text{if } x \neq \frac{p}{q}.\ x \in \mathbb{R}. \end{cases}$

~~0.001~~. ~~$x = \frac{1}{1000} = \frac{p}{q}$~~

PT:

$\lim_{x \to a} f(x) = 0 \ \forall\ a \in (0,1)$.

if $l = 0$, then $|f(x) - 0| < \epsilon \ \forall\ x \in N'(a, \delta) \cap D$ should be satisfied.

$\epsilon > 0$.

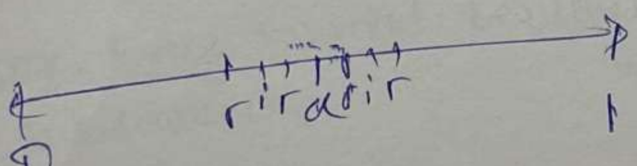
$a - \delta < x < a + \delta - \{a\}$.

$-\delta < x - a < \delta$

~~x is rational~~:

$A_\epsilon = \{\frac{p}{q} \in \mathbb{Q} \cap [0,1]\};\ \frac{1}{q} \geq \epsilon\}$ $0 < |x - a| < \delta$

$\delta = \min\{|a - \frac{p}{q}|;\ \frac{p}{q} \in A_\epsilon \text{ and } \frac{p}{q} \neq a\}$.



$\delta > 0$

$x \in [0,1]$. If x is irrational

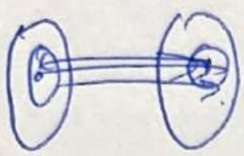
$|f(x) - 0| = 0 < \epsilon$. //

if $x = \frac{p}{q}$, $|x - a| < \delta$, then $\frac{p}{q} \notin A_\epsilon$

• limit

$\lim_{x \to c} f(x) = L$

this means $f(x)$ has limit L at $x=c$.



$\forall x \in N'(c,\delta)$ $\delta > 0$.

$f(x) \in N'(L,\epsilon)$. $\epsilon > 0$.

$\delta(\epsilon)$.

⇒ in this order only.

given.

i.e) for every $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t. whenever $|x-c| < \delta$ we have $|f(x)-L| < \epsilon$.

no matter how small given ϵ is, we can always find a $\delta(\epsilon)$ s.t. whenever we are within δ distance of x, $f(x)$ will be within ϵ distance of L. ⇒ used to show limit exist and find it.

• B-W in seq: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

every $\{x_n\}$ converging to c, the sequence $\{f(x_n)\}$ converges to L. (alternative def. of limits). } sequential criterion?

↳ used to show L does not exist.

↳ find a sequence that converges to c but $\{f(x_n)\}$ does not converge to L.

• e.g. $f(x) = \sin(\frac{1}{x})$, $x \neq 0$.

$\lim_{x \to 0} f(x) = ?$

i) Let $\{x_n\} = \frac{1}{n\pi}$. $\lim_{n \to \infty} x_n = 0$.

$\{f(x_n)\} = \{\sin(n\pi)\}$ $\lim_{n \to \infty} f(x_n) = 0$.

ii) Let $\{y_n\} = \frac{1}{(2n+1)\frac{\pi}{2}}$. $\lim_{n \to \infty} y_n = 0$.

$\{f(y_n)\} = \{\sin((2n+1)\frac{\pi}{2})\}$ $\lim_{n \to \infty} f(y_n) \Rightarrow$ divergent
$+1, -1, +1, -1 \ldots$

$\Rightarrow f(x)$ does not have a lt at $x = 0$ // converge at $x = 0$.

• e.g. $f(x) = x\sin(\frac{1}{x})$, $x \neq 0$

$\lim_{x \to 0} f(x) = 0$. ($\sin\frac{1}{x}$ is bounded).

use $\epsilon - \delta$ def. of limits to prove this?

* Show: $\forall \epsilon > 0$, $|x\sin\frac{1}{x} - 0| < \epsilon$.

$\delta = \epsilon$.

$|x - 0| < \delta = \epsilon$.

$|x| < \epsilon$.

$|x\sin\frac{1}{x}| = |x||\sin\frac{1}{x}|$

$\leq |x| < \delta = \epsilon$.

$\therefore |x\sin\frac{1}{x}| < \epsilon$ $\square$//.

- $h: \mathbb{R} \to \mathbb{R}$

$h(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$

dedicean-cuts } what is a
ordered field } $\mathbb{R}$?

what about $x = p$; $p \in \mathbb{N}$ → it does not converge
if

$\lim_{x \to p} h(x) = ?$

let $\{x_n\} = \frac{1}{n} + p$ $\lim_{n \to \infty} x_n = 0$

$f(x_n) = 1.$

let $\{y_n\} = \frac{\sqrt{2}}{n} + p$ $\lim_{n \to \infty} y_n = 0$

$f(y_n) = 0.$

$\{x_n\}, \{y_n\}$ are 2 seq. that converge to p.

→ $h(x)$ does not converge at $x = 0$

$\{x_n\}$ $\{y_n\}$ are chosen based on how ~~the~~ h was defined.

- in limit, the limit point may not be inside the set, but for continuity the lt point is always inside the set.

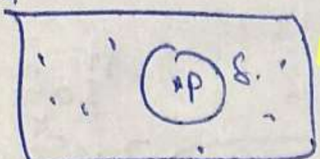
- A set can be both open and closed. (clopen sets).
  ↳ not opposites
  $\neg$ open $\not\Rightarrow$ closed.
  e.g. $\mathbb{R}$ is a clopen set, $\emptyset$ is clopen?

- $(5, 7]$ is neither open nor closed.

- $\lim_{x \to p} f(x) = f(p)$: def. of continuity for limit points
  ↳ def if p is an open point?

- isolated point: [redacted]

A



$B(P,\delta) \cap A = \{P\} \iff P$ is isolated.

if P is an isolated point,

$\forall x \in A,\ d(x,P) < \delta \Rightarrow x = P.$

$d(f(x), f(P)) < \epsilon.$

$d(f(P), f(P)) = 0 < \epsilon$

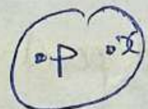
$\therefore$ we don't have to worry about continuity for isolated points,

this is diff. to $|f(x) - f(P)| < \epsilon$:

made of open sets | balls

Topological Spaces.

Metric Spaces	Topological Spaces
$d : A \times A \to \mathbb{R}.$	- more general version of metric spaces.
$d \equiv \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. euclidean distance	- nbhds and open balls.
can also define ficticiously	↳ if two objects are "close"
$d \equiv \sin x - \sin y$.	if within ϵ dist.
$\lvert x - P \rvert < \delta \equiv d(P, x) < \delta$ metric	if x is within of the open ball centered at P.
* euclidean. $\Updownarrow$	used when we cannot associate the dist. function to some objects
$x \in B(P,\delta)$ topological.	e.g. groups, fields, rings.



- If $\lim_{x \to P} f(x) = f(P)$. can you express this using open balls?

~~$\forall x \in B(P,\delta),\ f(x) \in B(f(P), \epsilon)\ \forall \epsilon > 0$~~

$\delta, \epsilon > 0.$

if this is true, it means f is continuous at $x = P$.

$\forall \epsilon > 0,\ \exists \delta > 0$ s.t.

$x \in B(P,\delta) \Rightarrow f(x) \in B(f(P), \epsilon)$.

Chapter 6: Continuity

- can draw a function without lifting pen
- Formally: $f: X \to Y$.

  f is continuous if for every open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$. is an open set. (in X)

  ↳ most general def. of continuity.

  $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$

*need to choose an open set in Y, find its pre-images in X and the set of the pre-images should be open in X.

Proof:

i) $\forall \epsilon > 0, \exists \delta > 0$ s.t. $x \in B(p, \delta) \iff |x - p| < \delta$

$\Downarrow$

$f(x) \in B(f(p), \epsilon)$ — ① (by def. of cont. $\lim_{x \to p} f(x) = f(p)$)

$\Rightarrow$ $f(B(p,\delta)) \subseteq B(f(p), \epsilon)$. = ① (by def. of continuity).

LT def only

i) assume f is cont., prove $f^{-1}(U) \subseteq X$ is open.

- if $f^{-1}(U)$ is empty then it is open ✓.

if $f^{-1}(U) \neq \emptyset$: $\exists\, p \in f^{-1}(U)$,

$\Rightarrow f(p) \in U$, U is an open set.

$\Rightarrow f(p)$ is an interior point of U by def.

$\Rightarrow \exists \epsilon > 0$ s.t. $B(f(p), \epsilon) \subseteq U$.

from ①: $f(B(p,\delta)) \subseteq B(f(p), \epsilon) \subseteq U$.

$B(p, \delta) \subseteq f^{-1}(U)$. $\therefore f^{-1}(U)$ is an open set.

$\because p$ is arbitrary.

ii) let $p \in X$

$f(p) \in Y$.

ii) assume $f^{-1}(U) \subseteq X$ is open. prove f is continuous.

take $B(f(p), \epsilon)$ open in Y.

$f^{-1}(B(f(p), \epsilon))$ is open in X from (i).

$p \in f^{-1}(B(f(p), \epsilon))$ in X

$\therefore$ is open set & p is interior point.

$\exists B(p, \delta) \subseteq f^{-1}(B(f(p), \epsilon))$

$f(B(p, \delta)) \subseteq B(f(p), \epsilon)$

closed interval $\not\to$ bounded interval

Continuous functions on compact intervals:

• Closed and bounded interval.

Thm: Take a function that is continuous on this interval. It is also closed and bounded, and the bound is attained

$f(x) = \frac{1}{x}$ $(0, 1]$ X not allowed to take this.

Proof:

Let I be an interval that is closed and bounded. compact

Let f be a continuous function $f: I \to \mathbb{R}$, then:

i) f is bounded $\longrightarrow$ $\exists M$ s.t. $\forall x \in I$, $|f(x)| \le M$.

ii) its bounds are attainable in I.

i) Proof by contradiction: Let f be unbounded.

then $\neg(\exists M\ \forall x\in I$ s.t. $|f(x)| \le M)$ → meaning of unbounded!

$\forall M\ \exists x_n \in I$ s.t. $|f(x_n)| > M$

by B.W. Th^m.

This creates a sequence $\{x_n\}$ lying in I

I is bounded → $\{x_n\}$ is bounded ⇒ $\exists$ a convergent $\{x_{n_k}\}$

By B.W. $\exists$ a sub-sequence $X = \{x_{n_r}\} \to x^*$.

$\because I$ is closed it contains all its limit pts ⇒ $x^* \in I$.

$\because f$ is continuous ⇒ $f(x_{n_r}) \to f(x^*)$ → $f(x_{n_r}) \ge n_r \ge r$.

every convergent seq. is bounded

$\because f(x_{n_r}) \to f(x^*)$, $\{f(x_{n_r})\}$ is bounded.

$\therefore$ contradiction //

I, bounded: used to s.t. we pass to a convg. sub-seq. *

I, closed: lt. point $\in$ set. *

continuous: $\{x_{n_k}\} \to x^*$ then $f(x_{n_k}) \to f(x^*)$ *

ii) $|f(x)| \le M$. want to show $\exists\, x^* \in I$ s.t. $|f(x^*)| = M$.

$s^* = \sup(f)$ s.t. $s^* = \exists\, x^*$ s.t. $f(x^*) = s^*$.

$s_* = \inf(f)$

$\exists x_n \in I$ s.t $s^* - \frac{1}{n} < f(x_n) \le s^*$ → chose n, can choose some $x_n \in I$ for which $f(x_n) > s^* - \frac{1}{n}$.

this generates $\{x_n\} \subseteq I$.

since I is bounded, $\{x_n\}$ is bounded.

by B.W., $\exists \{x_{n_k}\} \to x^*$ convergent seq.

$\{x_n\} \subset I$. $\because I$ is closed, $x^* \in I$.

$\because f$ is continuous, $f(x_{n_r}) \to f(x^*)$ — (1)

by squeeze thm:

$$s^* - \frac{1}{n_r} < f(x_{n_r}) \leq s^*.$$

$$\lim_{n_r\to\infty} s^* - \frac{1}{n_r} = \lim_{n_r\to\infty} s^* = s^* \quad \therefore \lim_{n_r\to\infty} f(x_{n_r}) = s^*.$$

but also by cont. $\lim_{n\to\infty} f(x_{n_r}) = f(x^*)$

$\therefore f(x^*) = s^*$.

↳ can use this proof to prove other things.

- The sum of two continuous functions is continuous.

- <u>Properties of Continuous Functions</u>

1) If f, g are continuous, then $f+g$ is also continuous
2) fg is also continuous
3) $\frac{f}{g}$ is also continuous provided that $g \neq 0$ (at that point)

↳ can prove all using sequential def. of continuity.

seq or epsilon-delta?

4) composition of f, g | g, f is also continuous.

↳ open-set def.

$f: X \to Y$, $g: Y \to Z$ are continuous.

$h(x) = g(f(x))$ is continuous. $h: X \to Z$.

why?

$\forall$ open sets $U_f \subseteq Y$, $f^{-1}(U_f)$ is an open set in X.

$\forall$ open sets $U_g \subseteq Z$, $g^{-1}(U_g)$ is an open set in Y.

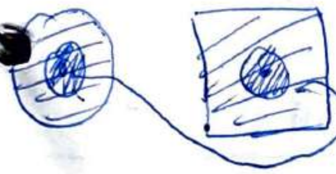
$\therefore \forall$ open sets $U \subseteq Z$, $f^{-1}(g^{-1}(U_g))$ is an open set in X. D//.

Homeomorphism

- $f: X \to Y$ is a homeomorphism if f is a continuous bijection s.t. the $f^{-1}: Y \to X$ is also continuous.

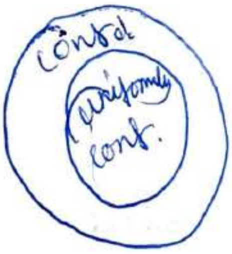
homeomorphism: i) continuous f
ii) bijective f
iii) inverse is continuous.

- two spaces are homeomorphic if $\exists$ a homeomorphism b/w them.



these two spaces are homeomorphic.
→ topologically equivalent.

Uniform Continuity



- consider two metric spaces (X, d) (Y, ρ); d, ρ are metrics.

$f: X \to Y$.

Let $A \subseteq X$.
f is said to be uniformly continuous on A if for every $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall x, y \in A$, whenever $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$.

δ here works $\forall x, y$.

~~this δ depends on the x, y we choose.~~

$\delta(\epsilon)$ only.

e.g.) $f: \mathbb{R} \to \mathbb{R}$
$f(x) = 2x$ → uniformly continuous.

let $\epsilon > 0$, choose $\frac{\epsilon}{2} = \delta$

show whenever $|x - y| < \delta$, $|f(x) - f(y)| = |2x - 2y|$
$= 2|x - y| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2}$

$\therefore |f(x) - f(y)| < \epsilon$ ∎

e.g.) $f(x) = x^2$; $f: \mathbb{R} \to \mathbb{R}$.

let $\epsilon > 0$. $|x-y| < \delta \to |x^2 - y^2| < \epsilon$

$|x^2 - y^2| = |x-y|\,|x+y|$ → cannot have $\delta(\epsilon)$ $\forall x, y$.

$< \delta \cdot \frac{\epsilon}{\delta}$ ϵ

∴ not uniformly cont.

$|x+y| < \frac{\epsilon}{\delta}$.

$|y - x + 2x| < \frac{\epsilon}{\delta}$

$|x-y| < \delta$.

$|y+x| \le |y-x| + 2|x| < \frac{\epsilon}{\delta}$,

$\delta < |x|$. || let $\delta < 1$

$< \delta + 2|x|$ $\quad 1 + 2|x| < \frac{\epsilon}{\delta}$

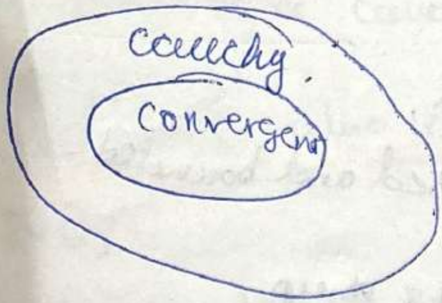
$3|x| < \frac{\epsilon}{\delta}$. $\quad \delta < \frac{\epsilon}{1+2|x|} \Rightarrow \delta = \min\left(1, \frac{\epsilon}{1+2|x|}\right)$ //

$\delta < \frac{\epsilon}{3|x|}$.

$\delta = \min\left(|x|, \frac{\epsilon}{3|x|}\right)$ → δ is dependent on x, ∴ not uniformly continuous.

* $f: [1,2] \to \mathbb{R}$ $f(x) = x^2$ is uniformly cont.
 ↳ can loop over $[1,2]$ find all δ's, take max δ

this will then work for all x,y E A whenver d(x,y)<delta



~~over~~

A cauchy is conv.
iff the metric space in which
it is defined is complete

$\Downarrow$

$3, 3.1, 3.14, 3.142, 3.1428 \ldots$

converges to π.

in $\mathbb{Q}$, it is not convergent

$\therefore \mathbb{Q}$ is not a complete metric space.

... every cauchy converges and it converges to a point in that space.

Location of Roots

- $I = [a, b]$.. $f : I \to \mathbb{R}$ is cont.

if $f(a) < 0 < f(b)$, then $\exists\, c \in I$ s.t. $f(c) = 0$

* Intermediate Value Theorem;

Let I be an interval, let f be cont in I
if $a \in I$, $b \in I$, and let k be s.t.
$f(a) < k < f(b)$, then $\exists\, c \in I$ s.t. $f(c) = k$.

Let $g(x) = f(x) - k$.

$g(a) = f(a) - k$, $g(b) = f(b) - k$.

$\because f(a) < k < f(b)$

$f(a) - k < 0 < f(b) - k$

$g(a) < 0 < g(b) \Rightarrow \exists\, c \in I$ s.t. $g(c) = 0$.

$\Rightarrow f(c) - k = 0$

$\Rightarrow f(c) = k$ //

• Let I be a compact interval closed and bounded for now

let f be a continuous func $f : I \to \mathbb{R}$,

Then $f(I)$ is a compact interval $\to$ closed and bounded.

proof;

$f(I) = \{f(x) \mid x \in I\}$. $\to$ attains inf, sup.

$\because$ cont., attains all values b/w.

let $m = \inf f(I)$
$M = \sup f(I)$.

we will s.t. $f(I) = [m, M]$

$f(I) \subseteq [m, M]$ is known.

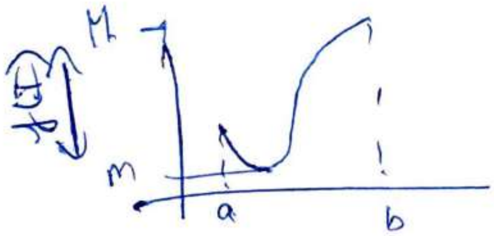
let K be s.t. $m < K < M$.

by IVT; $\exists\, c \in I$ s.t. $f(c) = K \in f(I)$.

$\Rightarrow \forall K \in [m, M]$, $K \in f(I)$.

$\Rightarrow [m, M] \subseteq f(I)$

$\therefore [m, M] = f(I)$ $\square$//.



$f(x) \in [m, M]$ not $[f(a), f(b)]$.

• Let $\{x_n\}$ be a cauchy. Let f be cont.

is $\{f(x_n)\}$ also cauchy?

→ NO;

$f: X \to Y$. Let $\{x_n\}$ be cauchy.

$X = (0,1)$, $Y = \mathbb{R}$.

$f(x) = \frac{1}{x}$, $x_n = \frac{1}{n}$.

$f(x_n) = n$; not cauchy.

* Let $\{x_n\}$ be cauchy. Let f be uniformly continuous.

$\{f(x_n)\}$ is cauchy!

* proof; ? unclear

$\forall \epsilon' > 0$, $\exists \delta$ s.t $\forall x_m, x_n \in X$ with $d(x_m, x_n) < \delta$,

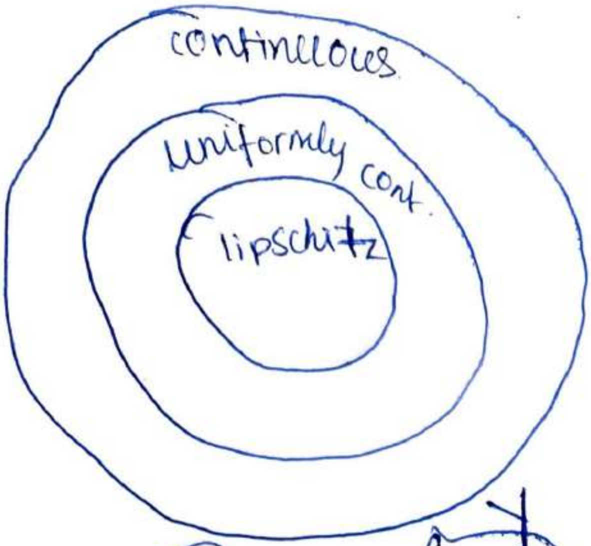
then $\rho(f(x_m), f(x_n)) < \epsilon'$. (uniform cont.)

$\forall \epsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t $\forall m, n \geq N_0$
$d(x_n, x_m) < \epsilon$. (cauchy).

Fix $\epsilon' > 0$, $\exists \delta(\epsilon') \Rightarrow d(x_m, x_n) < \delta$.

choose $\epsilon = \delta$.

$\{f(x_n)\}$ cauchy



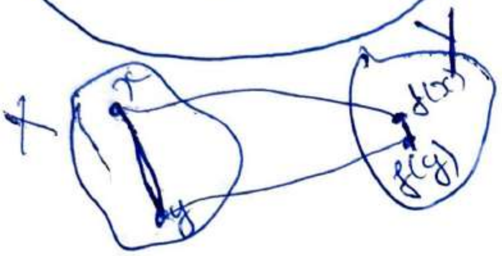
f is lipshitz cont. if $\exists\, \alpha > 0$ s.t. $\forall\, x, y$

$\rho(f(x), f(y)) \leq \alpha\, d(x, y)$.

when $\alpha < 1$, f is called a contraction.

$\alpha = 1$, f is isometric

$\alpha > 1$,



Uniform Continuity Th^m

• Let I be a compact interval. Let $f : I \to \mathbb{R}$ be continuous.

Then f is uniformly continuous on I.

__Proof:__ Assume f is not uniformly continuous. $\neg(\forall \epsilon > 0, \exists \delta > 0 \ldots)$

i) $\exists\, \epsilon_0 > 0$ ~~$\forall$~~, and

$\exists\, (x_n), (y_n)$ s.t. whenever $|x_n - y_n| < \frac{1}{n}$

we have $|f(x_n) - f(y_n)| \geq \epsilon_0 \quad \forall n$. (not uniformly cont.).

"~~∃~~ 2 points which are close, but after $f(x)$ is applied ~~f~~ they become far apart."

ii) I is bounded $\to$ (x_n) is bounded

by BW Th^m, $\exists$ (x_{n_k}) which converges to z

$\because$ I is closed, $z \in I$. (for any seq., I contains all its lt points)

$|\cancel{x_{n_r} - z}|$

$\cancel{|x_n - y_n| \geq \Gamma}$

$\cancel{|x_{n_r} - z| = |x_{n_r} \therefore v_n}$

$$|v_{n_r} - z| \leq |x_{n_r} - z| + |x_{n_r} - v_{n_r}|$$
$$< 0 + \frac{1}{n}$$

$|v_{n_r} - z| < \epsilon \Rightarrow v_{n_r} \to z$

by continuity of f, $f(x_{n_r}) \to f(z)$, $f(v_{n_r}) \to f(z)$ } contradiction.

for large enough n, their differences $\to 0$,

but by assumption it is lower bounded.

Th^m: Continuity Extension

A function is uniformly continuous on (a,b) iff it can be defined on the end-points, a and b, s.t. the extended function is continuous on $[a,b]$

why? density?

since any continuous function on compact [a,b] is also uniformly continuous on [a,b]

i) backward $\rightarrow$ from prev. thm.

ii) Suppose f is uniformly cont. on (a,b)

$\exists (x_n) \in (a,b)$ s.t. $x_n \to a$. $\Rightarrow (x_n)$ is cauchy.

uniform continuous map of cauchy is also cauchy.

$f((x_n))$ is cauchy, in a complete metric space $f(x_n) \to l$ convergent too.

need to show f is continuous at a:
-> we can show for all {Xn}->a, f(Xn) converges to a unique L. set f(a) to this L.
this is what we are doing here.

suppose $\exists (y_n) \to$ s.t. $y_n \to a$.

$(x_n - y_n) \to 0$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(y_n) - f(x_n) + f(x_n) = \lim_{n\to\infty} f(x_n) = l.$$

$\lim$ exists $\therefore f(a) = l$.

Similarly for b, done $\mathbb{B}$//.

• on any dense subset, if f is ~~continu~~ uniformly continuous in it, then f is uniformly ~~extended~~ cont all over the set.

Th^m:

(X, d), (Y, ρ).

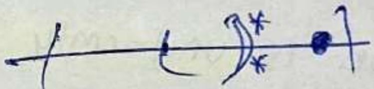
Let ~~$A \subseteq X$~~ A be a dense subset

$f : A \to Y$ is uniformly cont.

Then f can be extended

Proof:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ \underline{\quad L \quad} & \text{if } x \in X - A \end{cases}$$



$\in A$

let $x \in X - A$, $\exists \{x_n\} \longrightarrow x$

(x_n) is cauchy. $f(x_n)$ is cauchy. $f(x_n)$ is convergent

$f(x_n) \to L$ by prev. argument.

$\forall \epsilon > 0$, $\exists \delta > 0$ $\forall x, y$ whenever $d(x, y) < \delta$, $\rho(g(x), g(y)) < \epsilon$.

Take 2 points $x, x' \in X$. s.t. $d(x, x') < \delta$.

$\exists$ (x_n) (x_n') s.t. $(x_n) \to x$ $\quad \because A$ is a

$(x_n') \to x'$ dense subset.

$d(x_n, x_n') \longrightarrow d(x, x') < \delta$

$$\left.\begin{array}{l} f(x_n) \to y \\ f(x_n') \to y' \end{array}\right\} \text{show } \rho(y, y') < \epsilon_0.$$

$\rho(y, y') \leq \rho(y, f(x_n))$

$\delta(y, y') \leq \delta(y, f(x_n'))$.

$\leq \delta(f(x_n), f(x_n')) < \epsilon$.

$< \delta(f(x_n'), y') \to 0$.

every cauchy is convergent and converges to a point in Y.

Redo of Proof:

(X, d), (Y, δ). Y is complete.

let $A \subseteq X$ is a dense subset of X. $f(x)$

$f: A \to Y$ is uniformly continuous

Then f can be extended to X and this extension is uniformly cont in X.

let $g(x)$ be the extension of f on X.

$$g(x) = \begin{cases} f(x) ; & x \in A. \\ ?? ; & x \in X - A. \end{cases}$$

ans is: take x. take all sequences $\{x_n\}$ converging to x. apply f. find lt point. $g(x) =$ this lt pt.

we see that L is well defined and unique.

all sequences in A, but converge to x outside A.

Let $x \in X - A$, $\exists$ $\underline{\{x_n\} \in A}$ s.t. $x_n \to x$.

$\because \{x_n\}$ is convergent $\Rightarrow \{x_n\}$ is cauchy.

$f(x_n)$ is cauchy. $\Rightarrow f(x_n) \to L_x$ $\because Y$ is complete.

now show L_x is unique. contradiction proof;

$(y_n) \to x$.

$f(y_n) \to L'$ not L_x. proof next.

$d(x_n, y_n) \to 0$ $\because$ both $\{x_n\}, \{y_n\}$ converge to same point.

$\rho(f(x_n), f(y_n)) \to 0$ $\Rightarrow$ $f(x_n) = f(y_n) = L_x$.

↳ by continuity.

$\therefore g = \begin{cases} f(x) & ; x \in A \\ L_x & ; x \in X \setminus A \end{cases}$ $\to$ now we defined g.

$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(y_n) - f(x_n) + f(x_n)$.

denseness means we abuse the fact that every point in the larger set can be approximated by a seq. in the smaller set.

we have to show g is uniformly cont. on X:

→ $\forall \epsilon > 0, \exists \delta > 0$ s.t $\forall x, x' \in \underline{X}$.
if $d(x, x') < \delta$, then $\rho(g(x), g(x')) < \epsilon$.

consider $\underline{x, x' \in X}$ s.t. $d(x, x') < \delta$

consider $\{x_n\} \to x$, $\{x_n'\} \to x'$. by $\underline{\text{denseness of } A}$
$\{x_n\}, \{x_n'\} \in A$, $x, x' \in X$.

$\{x_n\}$ $\{x_n'\}$ is cauchy. $(\because g(x) = f(x), \forall x \in A)$
is not this route.

$\therefore g(x_n), g(x_n')$ are cauchy.

$\because Y$ is complete $f(x_n)$ $f(x_n')$ are convergent. ~~$\forall x \in A$~~.

$\left.\begin{array}{l} f(x_n) \to y \\ f(x_n') \to y' \end{array}\right\}$ def. of $g(x)$

need to show $\rho(y, y') < \epsilon$.

$$\rho(y,y') \le \underbrace{\rho(y, f(x_n))}_{\to 0} + \underbrace{\rho(f(x_n), f(x'_n))}_{< \epsilon} + \underbrace{\rho(f(x'_n), y')}_{\to 0}$$

↳ Δle inequality twice

$\Rightarrow g$ is uniformly continuous on X.

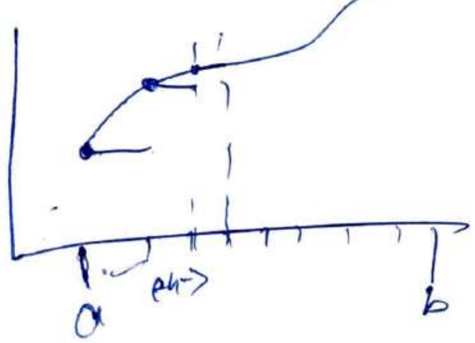
Approximations : Any cont. f. can be approximated to some other f.

• Step function - piecewise constant.

Thm: $\forall \epsilon > 0$, $f: I \to \mathbb{R}$ is continuous on compact I.
$\exists S_\epsilon(\,)$ s.t. $|f(x) - S_\epsilon(x)| < \epsilon$.

Proof:
f is uniformly continuous too $\because$ (I is compact)
$\forall \epsilon > 0$, $\exists \delta > 0$ s.t., $x, y \in I$
if $d(x,y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$.
if $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. by def,

Whenever $d(x,y) < \delta$.



choose

ensures that max $d(x,y)$ b/w any two x,y in I_i is at most h which is less than δ.

$h = \frac{b-a}{m}$, $m \in \mathbb{N}$

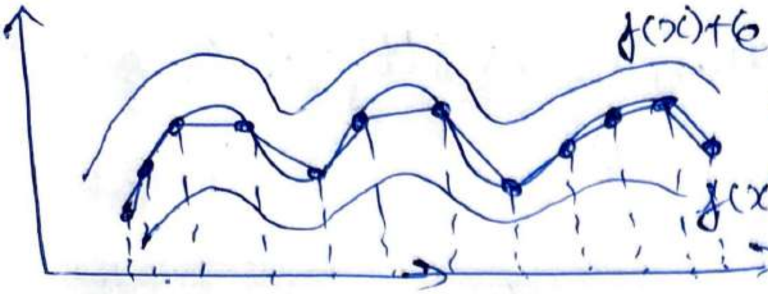
$S_\epsilon(x) = f(a+kh)$ for $x \in I_k$.

$I_1 = [a, a+h)$

$I_2 = [a+h, a+2h)$

$I_k = [a+(k-1)h, a+kh)$

take it starting point of I_k).



Th^m: can approximate any continuous $f: I \to \mathbb{R}$ (I is compact) using piece-wise linear functions.

Proof:

Since f is continuous on $I \Rightarrow f$ is uniformly continuous on I.

$\therefore \forall \epsilon > 0, \exists \delta > 0\ (\delta(\epsilon))$ s.t. $\forall x, y \in I$ s.t. if $|x-y| < \delta$,

$|f(x) - f(y)| < \epsilon$ by def. of uniform continuity.

$I_1 = [a, a+h)$

$I_2 = [a+h, a+2h)$

$I_3 = [a+2h, a+3h)$

$\dots$

$I_k = [a+(k-1)h, a+kh)$

in each sub-interval $\lambda f(a) + (1-\lambda) f(b)$ eq^n. $0 \le \lambda \le 1$.

choose m s.t. $h = \dfrac{b-a}{m} < \delta$.

$m \in \mathbb{N}$.

$\Rightarrow$ any two points in one sub-interval dist b/w them is $< \delta$.

~~then $|f(x) - f(y)|$ must~~ be $< \epsilon$.

Show $\left| f(x) - \left(\lambda f(a+(k-1)h) + (1-\lambda) f(a+kh) \right) \right| < \epsilon$.

use inequality.

• Weierstrass Theorem, Bernstein polynomial.

↳ any cont. f can be approximated by a polynomial.

Monotone Sequences

• very powerful

• $X = (x_n)$ be a sequence of real numbers. We say that

X is increasing if: $x_1 \le x_2 \le x_3 \le x_4 \le \dots$ ($x_n \le x_{n+1}$ $\forall n \in \mathbb{N}$)

X is decreasing if: $x_1 \ge x_2 \ge x_3 \ge x_4 \ge \dots$ ($x_n \ge x_{n+1}$ $\forall n \in \mathbb{N}$)

• X is monotone if it is either ↑ or ↓.

• Th^m: Monotone Convergence Th^m;

A monotone sequence is convergent iff it is bounded.

convergence def. + play around

Proof;

i) forward ez. convergent ⇒ bounded

don't need to use the monotonic properties

a) if X is ↑, then $x_n \to x^*$ where x^* is the supremum

b) if X is ↓, then $x_n \to x_*$ where x_* is the inf. (x_n).

ii) A monotone bounded sequence is convergent; (↑ case)

⇒ sup. exists let it be x^*.

~~$\exists n \in \mathbb{N}$ s.t. $x^* - \frac{1}{n} < x_n < x^*$~~

let $\epsilon > 0$. then $\exists K$ s.t.

$$\boxed{x^* - \epsilon} \le x_K \le \boxed{x_n} \le x^* \boxed{< x^* + \epsilon}$$

$\forall n \ge K$

(↑).

$\Rightarrow |x_n - x^*| \le \epsilon$

$\Rightarrow x_n \to x^*$

$n \ge K$.

Thm. Any sequence has a monotone sub-seq. ?

- x_k is a "peak" if $x_k \geq x_n$ $\forall k \geq n$ $n \geq k$?

larger than all terms to its right.

a) Infinitely many peaks;

$x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq \dots\dots$ monotonically $\downarrow$ sub-seq.

b) Finitely many peaks;

$x_{m_1}, x_{m_2}, \dots, x_{m_r}$

$s_1 = m_r + 1$ not a peak.

$\exists\, s_2$ s.t. $x_{s_2} > x_{s_1}$

s_2 is not a peak.

$\exists\, s_3$ s.t. $x_{s_3} > x_{s_2}$

s_3 is not a peak

$\exists\, s_4$ s.t. $x_{s_4} > x_{s_3}$

$\therefore\ x_{s_1} < x_{s_2} < x_{s_3} < x_{s_4} \dots\dots$

monotonically $\uparrow$ sub-seq.

- Bolzano-Weierstrass Theorem;

"Every bounded sequence has a convergent subsequence."

1) every sequence has a monotone subsequence.

2) any monotone, bounded sequence is convergent. (MCT)

$\therefore$ proved //.

-

Accumulation Point / cluster point / lt pt of a set	Limit of a sequence
• x is an accumulation point if for every $\epsilon > 0$, for every $n \in \mathbb{N}$, there exists $n_0 \geq n$ s.t. $\lvert x_{n_0} - x \rvert < \epsilon$.	• $\lim_{n \to \infty} x_n = x \Rightarrow x_n \to x$ $\Rightarrow \forall \epsilon > 0, \exists N_0$ s.t. $\lvert x_n - x \rvert < \epsilon$ $\forall n \geq N_0$ (def. of lt).
↳ don't require all terms after n.	↳ generates a "partition" of a sequence $(n \geq N \mid n < N)$.
↳ choose 1 index n, $\exists$ at least 1 other index $n_0 \geq n$ for which x_{n_0} is ϵ distance from x.	

- $a_n = (-1)^n \dfrac{n}{n+1} = \left\{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}, \ldots \right\}$.

↳ does not converge to a single value ~~for this sequence~~.

$\therefore$ lt of this seq does not exist.

↳ what are the accumulation points of this sequence?

-1 and $+1$.

↳ No matter which index you give me and ϵ you give me, $\exists$ always at least 1 term further down the sequence which is in ϵ distance of ± 1.

↳ odd terms approach -1

even terms approach $+1$.

- What's an accumulation point of a set?

↳ P is a lt pt / cluster point / accumulation pt of a set S if every deleted ϵ nbhd of P contains at least one point in S.

- Recall lt point of a set ↳ same!

Fix $\epsilon > 0$

- If x is an accumulation point of a set S, there exist infinitely elements of S that are within ϵ distance of x.

Proof:

For contradiction, assume there are finitely many elements of S in the ϵ distance of x.

$N'(x,\epsilon) \cap S \neq \emptyset$.

let $B =$ this set $N'(x,\epsilon) \cap S$.

$B = \{a_1, a_2, a_3, \ldots, a_k\}$ (finite)

$\therefore |x-a_1| = \delta_1,\ |x-a_2| = \delta_2,\ |x-a_3| = \delta_3, \ldots |x-a_k| < \delta_k$

where $\delta_i > 0$ and $\delta_i < \epsilon$.

Choose $\delta' = \min\{\delta_1, \delta_2, \delta_3, \ldots, \delta_k\}$; $\delta' > 0$.

$\Rightarrow |x-a_1| > \delta',\ |x-a_2| > \delta', \ldots, |x-a_k| > \delta'$.

$\Rightarrow$ suppose $\exists\, z$ s.t. $|x-z| < \delta' < \epsilon$. $\Rightarrow z \notin \{a_1, a_2, \ldots, a_k\}$

$\Rightarrow z \notin S$. $\Rightarrow x$ is not an accumulation point.

contradiction

$\Rightarrow a_i \notin N'(P,\delta')$; $i = 1, 2, \ldots, k$.

It follows that $N'(P,\delta') \cap S = \emptyset$ which disallows p to be an accumulation point.

• lim sup and lim inf.

• $a_N^+ = \sup(a_n)_{n=N}^{\infty}$

$a_n = 1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots\ldots$

$a_1^+ = 1.1$
$a_2^+ = 1.001$
$a_3^+ = 1.001$

$a_N^+ \Rightarrow$ supremum of the rest of the sequence from N onwards.

$\inf(a_N^+) = 1.$

$\lim\limits_{n\to\infty} \sup a_n = 1.$
$= \inf(a_N^+).$

* lim sup:
take new sup sequence, take inf. of this, is lim sup of a_n.

* lim inf a_n:
take new inf sequence, take sup of this, its lim inf of a_n.

• $a_N^- = \inf(a_n)_{n=N}^{\infty}$

$a_1^- = -1.01$
$a_2^- = -1.01$
$a_3^- = -1.0001 \ldots$

$\sup(a_N^-) = -1$
$\therefore \lim\limits_{n\to\infty} \inf a_n = -1$
$= \sup(a_N^-).$

• $\lim\limits_{n\to\infty} \sup(a_n) = \inf. \sup\limits_{K \geq N}(a_K)$

$\lim\limits_{n\to\infty} \inf a_n = \sup \inf\limits_{K \geq N} a_K$

• Relate limsup liminf to the limit of a sequence.

• **Thm:** A bounded sequence (x_n) converges to L iff

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L.$$

↳ a sequence converges iff limsup and liminf coincide.

↳ Recall:

$x_n^+ = \sup_{k\geq N} x_k$ $\quad x_n^+ \to L \Rightarrow x_n^+$ converges to its inf. (non-increasing + bounded)

$x_n^- = \inf_{k\geq N} x_k$ $\quad x_n^- \to L. \Rightarrow$ (non-decreasing + bounded) x_n^- converges to its sup.

Proof:

i) Suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l$;

$$x_n^- \leq x_n \leq x_n^+$$

use squeeze theorem.

$$\lim_{n\to\infty} x_n^- = \lim_{n\to\infty} x_n^+ = L \Rightarrow \lim_{n\to\infty} x_n = l \;//$$

ii) let $x_n \to L$ means $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$

$$|x_n - L| < \epsilon.$$

$L - \epsilon < x_n < L + \epsilon$, holds for $n \geq N_0$.

$$L-\epsilon \leq \underbrace{\inf_{k\geq N} x_k}_{x_k^-} \leq \underbrace{\sup_{k\geq N} x_k}_{x_k^+} \leq L+\epsilon, \quad \forall n \geq N_0.$$

$\Rightarrow |x_n^+ - L| < \epsilon \Rightarrow x_n^+ \to L$

$|x_n^- - L| < \epsilon \Rightarrow x_n^- \to L$

$\therefore$ limsup $\to l$

liminf $\to l$.

Properties of lim inf and lim sup

- Two bounded sequences $(a_n), (b_n)$

i) $\limsup_{n\to\infty} (a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

ii) $\liminf_{n\to\infty} (a_n+b_n) \ge \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.

Why?

$\Rightarrow \left.\inf_{n\ge K} a_n + \inf_{n\ge K} b_n \le a_j + b_j \le \sup_{n\ge K} a_n + \sup_{n\ge K} b_n \right\}$

$j \ge K$

- follows from definition.

- a_n, b_n are bounded sequences.

if $\lim_{n\to\infty} b_n = b$, then $b_n \to b$

$\limsup(a_n+b_n) = \limsup a_n + b$.

Proof:

- show $\le$ and $\ge$ then equal.

if $b_n \to l$
$\limsup b = \liminf b = l$.

i) $\limsup(a_n+b_n) \le \limsup a_n + \limsup b_n$

$\Rightarrow \limsup_{n\to\infty}(a_n+b_n) \le \limsup a_n + b$ — ①.

ii) $a_n = \underbrace{(a_n+b_n)}_{x_n} - \underbrace{b_n}_{y_n}$.

~~limsup (|an+bn)~~

$\limsup(x_n+y_n) \le \limsup x_n + \limsup y_n$.

$\Rightarrow \limsup(a_n) \le \limsup(a_n+b) \cancel{+\lim} - b$.

$\Rightarrow \limsup(a_n+b) \ge \limsup(a_n) + b$ — ②

from ① and ② $\Rightarrow \limsup(a_n+b) = \limsup a_n + b$ //.

- (a_n) (b_n) be two bounded sequences.

$b_n \to b$, $b \geq 0$, then

$\limsup_{n\to\infty}(a_n b_n) = b \limsup(a_n)$.

i) S_n bounded, $t_n \to 0$ $\Big\}$ $b = 0$ case
$S_n t_n \to 0$.

ii) $a_n b_n = \underbrace{a_n b}_{x_n} + \underbrace{(b_n - b) a_n}_{y_n}$, $y_n \to 0$.

$\limsup a_n b_n = \limsup a_n \cdot b + 0$.

$\limsup a_n b_n = b \cdot \limsup a_n$ $\because b > 0$, won't affect $\limsup a_n$.

Real analysis
Quiz 2 (Fall 2024)
Duration: 1 hour

Question 1: (8 marks) Define what it means for a function to be uniformly continuous on a set.

+8

Question 2: (9 marks) Give examples, with justification, of each of the following.

1. A bounded sequence (x_n) for which $\lim_{n\to\infty} \sup x_n \neq \lim_{n\to\infty} \inf x_n$.
2. A function $f : [0,1] \to \mathbb{R}$ which is discontinuous at each $x \in [0,1]$.
3. A continuous function which is not uniformly continuous.

Question 3: (8 marks)
Show the following statements:

1. A bounded monotone sequence is convergent.
2. Every sequence has a monotone subsequence.

$\exists N \in \mathbb{N}, \forall n \geq N, \forall \epsilon > 0$ $|c_n - c| < \epsilon$

Question 4 (6 marks)
Let us say that a sequence $(c_n)_{n=1}^{\infty}$ of real numbers "*cervonges* to c" (where $c \in \mathbb{R}$) if and only if there is an $N \in \mathbb{N}$ such that, for all $n > N$ and all $\epsilon > 0$, we have $|c_n - c| < \epsilon$.

1. If a sequence (c_n) cervonges to c, does (c_n) converge to c? Explain, and if not, give an example.
2. If a sequence (c_n) converges to c, does (c_n) cervonge to c? Explain, and if not, give an example.

Question 5: (9 marks) $\Rightarrow$ X is not complete?
Let X be a metric space such that $X \subseteq Y$, where Y is a complete metric space. Let (x_n) be a Cauchy sequence in X such that (x_n) contains a convergent subsequence in X. Then (x_n) converges in X. $\rightarrow$ show last sentence

Question 6: (10 marks)
Let Z be a metric space and let Y be a dense subset of Z. Suppose that every Cauchy sequence in Y converges in Z. Prove that Z is complete.

RA Quiz 2 Sol^n

Q1) ez claps

Q2) a) ez inclass eg
b) lebeg function.
c) x^2 | $\frac{1}{x}$ etc.

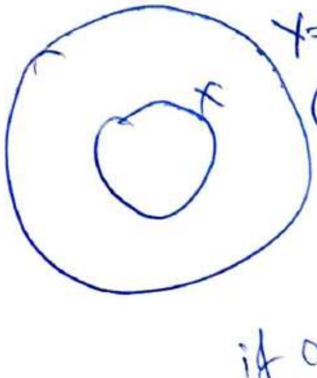
Q3 a) bounded → sup, inf ez.
b) peaks

Q4 a) ez!!!
b)

**
Q5) $X \subseteq Y$, Y is complete
(a_n) is cauchy in X.
$\exists$ subsequence of (a_n) converging in X.

To show:- a_n converges in X,

• complete means every cauchy ~~has~~ is convergent.

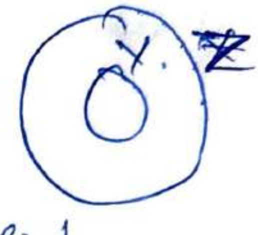


Y=complete.
(a_n) cauchy in $X \Rightarrow \overset{(a_n)}{\text{cauchy}}$ in Y
$\Rightarrow a_n \to l$ in Y $\because$ y is complete.
$\Rightarrow$ ~~bec~~ a_n is convergent to l.
if $a_n \to l$, all $\{a_{n_k}\} \to l$ only.
$\{a_{n_k}\} \to x$ in X
$\Rightarrow l = x$ in X Q.E.D //.

Q6) Z, Y dense subset of Z. Every cauchy seq. in Y_Z converges in Z. Then Z is complete;

let (Z_n) be cauchy in Z (if $Z_n \to Z$ in Z we are done)

$\because$ Y is dense, $\exists$ $(y_n) \in Y$ s.t $|Z_n - Y_n| < \frac{1}{n}$



we can approximate Z_n with Y_n.

$\because$ $Y_A \subset Z_A$
Y is dense

now we show Y is cauchy

$\hookrightarrow$ $d(y_m, y_n) \leq \underbrace{d(y_m, Z_m)}_{\frac{1}{m}} + \underbrace{d(Z_m, Z_n)}_{<\frac{\epsilon}{2}} + \underbrace{d(Z_n, y_n)}_{\frac{1}{n}}$

everything in Z (big set) can be approximated by something in Y (small set).

choose m, n large enough

$\therefore$ $\{Y_n\}$ is cauchy in Y $\Rightarrow$ it converges in Z (given).

$\because$ $\{Y_n\}$ is an approximation of $\{Z_n\}$,

$\{Z_n\}$ is also convergent.

* " A is a dense subset of X iff A is a closure of X "

" A is dense $\Leftrightarrow$ $\bar{A} = X$. "

include lt points

*

Important Theorems in Analysis

i) Cantor's Intersection Theorem } strong theorems in Analysis.
ii) Baire Category Theorem
↳ basic + fundamental, uses dense properties and cantor's intersection theorem. to prove.

huge applications in functional analysis.

Cantor's Intersection Theorem

every cauchy has a lt.

- (X, d) be a complete metric space
- let $\{F_n\}$ be a decreasing family of non-empty closed sets such that the diameter$(F_n) \to 0$ as $n \to \infty$.

Then;

$\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

* $\{F_n\}$ is a family of sets

decreasing family $\Rightarrow F_1 \supseteq F_2 \supseteq F_3 \ldots$

* $dia(F_n) = \sup_{x,y \in F_n} d(x,y) \to$ maximum distance b/w any two points in the set is the diameter.

Proof:

- easy to show $F = \bigcap_{n=1}^{\infty} F_n$ cannot have more than 1 element

$F \subseteq F_n \ \forall n$ ∵ intersection.

$dia(F) \leq dia(F_n) \ \forall n$

but $dia(F_n) \to 0 \Rightarrow dia(F) \to 0$

if it had more than 1, $\exists\, d(x,y) > 0$ finite $\neq 0$.

∴ F cannot contain more than 1 element // Q.E.D.

• Show that F has at least 1 element;
↳ uses all assumptions;

i) each F_n is non-empty, ∴ we can do the following randomly)
let $x_n \in F_n$ (one element from each F_n

$x_1 \in F_1$
$x_2 \in F_2 \cap F_1$
$x_3 \in F_3 \cap F_2 \cap F_1$

ii) $x_{n+1} \in F_{n+1} \subseteq F_n$ (decreasing family)

$x_m \in F_n$ whenever $m \geq n$. $\{x_1, x_2, x_3, x_4, x_5, \ldots\}$

iii) claim $\{x_n\}$ is a cauchy sequence.

$d(x_m, x_n) \leq dia(F_n) \to 0$ ∴ cauchy.

$x_m, x_n \in F_n$. let $m > n$. choose large ns s.t. $dia(F_n) \to 0$
for m, n large enough $(m > n)$.

iv) ∵ $\{x_n\}$ is cauchy in a complete metric space X, it converges to some point in X. $x \in X$. ($x \in F_i$ or $x \notin F_i$, don't know).

$(x_{n+1}, x_{n+2}, x_{n+3}, \ldots\ldots)$ is a subsequence of $\{x_n\}$.

$(x_{n+1}, x_{n+2}, x_{n+3}, \ldots) \subseteq F_n$.

$(x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$ (∵ if $\{x_n\} \to x$, any $\{x_{n_k}\} \to x$ too).

↓

lt is x for this

v) ∵ F_n is closed $x \in F_n$. ⇒ $x \in$ all F_i before n (dec. family)

$\Rightarrow x \in \bigcap_{i=1}^{\infty} F_n$.

$x \in$ all F_i higher than n
as sub-seq. of $(x_{n+1}, x_{n+2}, \ldots x_{n+3})$
is contained in F_i.

• A is a dense subset of X if $\bar{A} = X$

$\bar{A} = A +$ lt. points / cluster / accumulation points of A.

e.g. $(0,1)$ is dense in $[0,1]$.

(Q) • $\mathbb{N}$ in $\mathbb{Z}$ dense or not?
• odds $(-1,1,3)$ in $\mathbb{Z}$

(no balls in $\mathbb{Z}, \mathbb{Q}$)

* A is nowhere dense if $\text{int}(\bar{A}) = \phi$.

$\mathbb{N} = \{1,2,3,\ldots\}$.

$\bar{\mathbb{N}} = \mathbb{N} \rightarrow$ a sequence of naturals can only tend to a natural (if it does) $\therefore \bar{\mathbb{N}} = \mathbb{N}$ only).

$\text{int}(\bar{\mathbb{N}}) = \text{int}(\mathbb{N}) = \phi$.

• every non-empty open set contains a ball U s.t. $U \cap A = \phi$

no points whose ϵ nbhds are completely contained in $\mathbb{N}$.

$\therefore \mathbb{N}$ is nowhere dense in $\mathbb{R}$.

" no N b/w N_1, N_2 "

• $\mathbb{Q}$ is dense in $\mathbb{R}$.
↳ take a ball in $\mathbb{R}$, will 100% contain some $\mathbb{Q}$s.

Baire's category Theorem

• A set is of "category 1" if it can be expressed as a countable union of nowhere dense sets, and

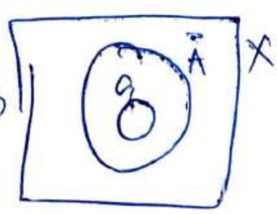
"category 2" if not

Recap:

* A is nowhere dense in X → ~~there is~~ $\bar{A}$ does not contain an open set in its interior.

⇕

for every non-empty open set $G \in X$, $(X \setminus \bar{A}) \cap G \neq \phi$.



but G cannot be contained in $\bar{A}$.

⇕

every non-empty open set contains an open ball disjoint from A.

Baire Category Theorem

- A complete metric space is always of the second category.

Let X be a complete metric space

Let $\{A_n\}$ be a countable family of nowhere dense sets,

To show $\exists\ x \in X$ s.t $x \notin A_n\ \forall n$. ($x \notin A_1, x \notin A_2, \ldots$) — spond.

A_1 is nowhere dense in X; Let Y_1 be an open set in X. — rish.

$\Rightarrow \exists$ an open ball $U(a_1, r_1)$ in Y s.t. $U(a_1, r_1) \cap A_1 = \emptyset$. — net.

(open ball in net.)

$F_1 = B(a_1, \frac{r_1}{2})$ closed ball inside $U(a_1, r_1)$, $F_1 \cap A_1 = \emptyset$.

open set $\Rightarrow$ $\mathrm{int}(F_1) \neq \emptyset$, A_2 is nowhere dense.

$\exists\ U_2(a_2, r_2) \subseteq \mathrm{int}(F_1)$ s.t. $U_2(a_2, r_2) \cap A_2 = \emptyset$

$B(a_2, \frac{r_2}{2}) \subseteq U(a_2, r_2)$

- take an open set inside F_2.

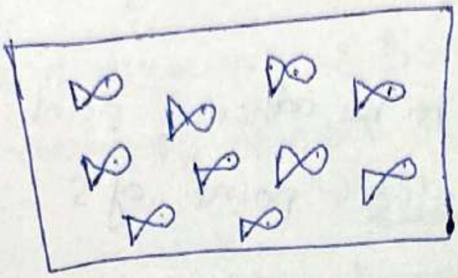
$U(a_2, r_2) \cap A_2 = \emptyset$.

$\Rightarrow$ decreasing family of non-empty closed sets.

$x \in \bigcap F_i$ but $F_i \cap A_i = \emptyset$

$\therefore x \notin A_n\ \forall n$.

• A set $D \subseteq X$ is said to be dense in X if for every open set $U \subseteq X$, we have $U \cap D \neq \emptyset$.



• no matter of the size of the net thrown / area where it is thrown, we will always catch some (part of) fish.

D = fish, X = Fish + pond.

U = fishermen's net.

• $\mathbb{Q}$ is dense in reals ✓

⇔ no matter which open interval we take, we will always find some rational in the interval.

• A set $D \subseteq X$ is said to be dense in X if $\bar{D} = X$ (closure of D wrt X is equivalent to X). $Cl_X(D) = \bar{D} = X$.

• Adherent Point:- Accumulation / limit / cluster point Isolated Point.

Interior point Boundary point.

• Accumulation point of a set S: x^* is said to be accumulation point of S iff every <u>deleted</u> neighbourhood contains a point of S.

• Adherent Point of S: x^* is said to be adherent point of S iff every neighbourhood contains a point of S.

* every accumulation pt. is an adherent point, but converse is not true.

e.g.) $S = \{0\} \cup [1,2]$.

~~is S~~ What is "0" for S. → adherent point, isolated, boundary (3).

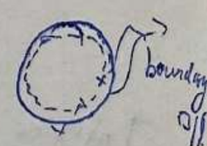
NOT an accumulation pt:

↳ Adherent: Yes because for every nbhd $N(0,\epsilon), \epsilon > 0$, $0 \in N(0,\epsilon)$
$\Rightarrow N(0,\epsilon) \cap S$ ~~has~~ $\neq \phi$, has 0 100%. ∴ Yes, adherent.

↳ Isolated: $\exists$ a nbhd of that point (deleted) which does not contain any other point of S. ✓.

↳ Not Accumulation: Many deleted nbhds of 0 don't contain a point of S

↳ Boundary Point: every nbhd of that point contains at least one point in S ∧ at least one point not in S.



* Adherent → Accumulation
  → Isolated.

{1} {2} } Boundary, Adherent, Accumulation

{1.5}. Accum, Adherent, Interior pt.

- Can a point be a boundary pt. but not an adherent point?
- The following are eqv!

i) A set $D \subseteq X$ is dense in X if $\bar{D} = X$. ($D \cup$ accums.)

ii) For every non-empty open set $U \subseteq X$, we have $U \cap D \neq \emptyset$. (fish pond.)

~~iii) $\bar{D} = X$~~

iii) Every $x \in X$ is an adherent point of D.

fish + pond best!

- (iii) $\rightarrow$ (ii): Let $x \in X$, x is an adherent point of D (accum / iso)

$\Rightarrow$ ~~$\forall N(x, \epsilon)\ (\epsilon > 0)$~~

~~$\forall \epsilon > 0,\ N(x, \epsilon) \cap D \neq \emptyset$.~~

$x \in \bar{D}$

$\Rightarrow X \subseteq \bar{D} \subseteq X \Rightarrow \bar{D} = X$

- Outside the set: Adherent points (possible), accumulation/cluster

	Adherent	Accumulation	Isolated	Interior	Boundary
capacity to be outside? inside?	✓	✓	✗	✗	✓

(Isolated, Interior) $\hookrightarrow$ point must be inside the set.

- Accumulation and isolated points have exact opposite def^{n}s.
  $\hookrightarrow$ Accumulation $\Rightarrow$ adherent. e.g. $(1,2)$, the point $\{1\}$ is an accumulation, hence adherent point.

e.g. $\{0\} \cup (1,2) \Rightarrow$ classify for this.

- dense → approximate ~~a~~ a point in X by a sequence of points in D converging to it. always do this, whenever you see dense.

  ↳ something holds on the dense subset, approximate every point in the big set using dense subset, ∴ holds on the big set too.

- $(2,3]$ is neither open nor closed

  ϕ, $(-\infty,\infty)$ open and closed → why? ~~c~~ complementality holds. open by def. (closed by complementality from def.

  $(2,3)$ open not closed

  $[2,3]$ closed. not open.

  closed: a set that contains all its limit/cluster/accumulation points. i.e. $S=\bar{S}$.

* how would we talk of open sets wrt $\mathbb{N}$? how do we choose the ϵ for the nbhd? usually $\epsilon \in \mathbb{R}$.

* (X,τ) τ is a topological space if

i) $\emptyset$ and $X \in \tau$

ii) $\bigcup_{A_i \in \tau} A_i \in \tau$

iii) $\bigcap_{\text{finite}} A_i \in \tau$

τ = open sets of X. is a topological space

• Discrete topology:

(IN, discrete) → cannot use ball | open sense.

$X = \{1, 2, 3\}$.

$P(X) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \emptyset\}$.

~~X is an open~~ P~~(X) = Z is an open set~~ ? ??

• a singleton pt $\{1\}$ is closed in (IR, standard) but open in (IN, discrete).

discrete topology of X is $P(X)$ basically here.

• nowhere dense ~~o~~ no matter how small / large or where we throw the net, we can always find a ball inside the net which does not intersect with the fish.

• seq, continuity: ε-δ + sequential, uniform continuity, continuous functions on compact sets, BW theorem, complete metric space, cauchy sequence, *dense subsets (fish pond), using dense subsets for extensions of properties, approximation of continuous functions, adherent accum iso boundary, cantor's intersection thm, Baire dense + nowhere dense v/s not dense, category thm.

- Not dense: $\exists$ an open set $\neq \emptyset$, s.t. $S \cap D = \emptyset$

nowhere dense: $\forall$ open sets $S \neq \emptyset$, $\exists$ an open ball $B \subseteq S$ s.t.

$B \cap D = \emptyset$.

Banach Contraction Mapping Principle

$[d(f(x), f(y)) \leq \alpha d(x,y)$
contraction $\alpha \lneq 1]$.

- Let (X,d) be a complete metric space. Let $f: X \to X$ be a contraction mapping. Then f has a unique fixed point

$\to$ a fixed point is : take a point, apply f, we get the same point itself.

$\Rightarrow$ x_1 $\quad y_1$ $\longrightarrow$ $f(x_1)$ $f(y_1)$.

keep decreasing $d(x_1, y_1)$, $d(f(x_1), f(y_1))$ keeps decreasing
hope is that as x_i y_i get very very close, eventually we get one x^* which maps to itself?

Proof; (easier to show cannot have more than 1 fixed point)

i) let x_0, y_0 be two fixed points.

$\to f(x_0) = x_0, \; f(y_0) = y_0$.

$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq \alpha d(x_0, y_0)$

given $\alpha < 1 \Rightarrow d(x_0, y_0) = 0 \Rightarrow x_0 = y_0$ □// (i)

only used contraction properties

ii) f has at least one fixed point:

consider $x_0 \in X$

$x_1 = f(x_0)$

$x_2 = f(x_1)$

$x_3 = f(x_2)$

...

$x_{n+1} = f(x_n)$

consider $\{x_n\} = \{x_0, x_1, x_2, \ldots\}$.
if $\{x_n\}$ is convergent then lt pt of $\{x_n\}$ is a fixed point.

$x_n \to x^* \Rightarrow f(x_n) \to f(x^*)$

$x_{n+1} \to x^*$

~~f(x_{n+1})~~

$f(x_n) = x_{n+1} \Rightarrow f(x_n) \to x^*$.

$\Rightarrow f(x^*) = x^*$

need to show it is convergent.
$\hookrightarrow$ show $\{x_n\}$ is cauchy, use completeness of X.

$\cancel{d(x_{n+1}, x_n)}$

Analyse

$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \le \alpha\, d(x_n, x_{n-1})$

$\le \alpha^2 d(x_{n-1}, x_{n-2})$

$\vdots$

$\le \alpha^{n-1} d(x_2, x_1)$

keep on applying the contraction

Δle inequality:
$d(x,z) \le d(x,y) + d(y,z)$

$0 < \alpha < 1$.

if n is large enough, $d(x_{n+1}, x_n) \to 0$.

distance between consecutive elements $\to 0$ for large n,

not cauchy yet :

→ use Δle inequality.

by Δle inequality:
100 to 105
(100 to 101) (101 to 102) . . .
. . . (104 to 105)

$m = n+k \quad (m, n)$.

$d(x_{n+k}, x_n) \le d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + \dots + d(x_{n+k}, x_{n+k-1})$

$d(x_{n+k}, x_n) \le \alpha^{n-1} d(x_2, x_1) + \alpha^n d(x_2, x_1) + \alpha^{n+1} d(x_2, x_1) + \dots + \alpha^{n+k-2} d(x_2, x_1)$

$d(x_{n+k}, x_n) \le \alpha^{n-1} d(x_2, x_1) \left[1 + \alpha + \alpha^2 + \dots + \alpha^{k-1}\right]$

$0 \because \alpha \in (0,1)$ for large enough n.

$\therefore d(x_m, x_n) \to 0 \quad \therefore \{x_n\}$ is cauchy,
as $m, n \to \infty$.

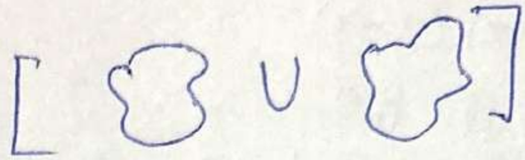
- ~~$f(y) - f(y)$~~ f is continuous, $\exists z$ s.t.
$f(y) - f(x) = f'(z)(y-x)$
Some mean val. thm.
and if $|f'(z)| < 1 \Rightarrow f$ is a contraction mapping.
$\Downarrow$
$\exists$ unique x^* s.t. $f(x^*) = x^*$. by Banach's Contraction Principle.

- Complete metric space: every cauchy is convergent.

Let (X,d) be a metric space. Then the completion (X^*, d^*) along with an isometry $\phi : X \to X^*$ s.t. $\phi(X)$ is dense in X^*.

$X \subseteq X^*$,
$(\mathbb{Q}, d)$ euclidean, $(\mathbb{R}, d)$ is its completion.
$\phi : \mathbb{Q} \to \mathbb{R}$
$\phi(q) = q$. is an isometry and $\phi(\mathbb{Q}) = \mathbb{Q}$ and
$\mathbb{Q}$ is dense in $\mathbb{R}$.

Connectedness

- Say we have $S =$  in $\mathbb{R}^2$.

  Is S connected? No.

- one-piece set  connected set.

- A set $A \subseteq X$ is disconnected if $\exists$ two open sets

  G_1, G_2 s.t.

  i) $G_1 \cap G_2 = \emptyset$ disjoint

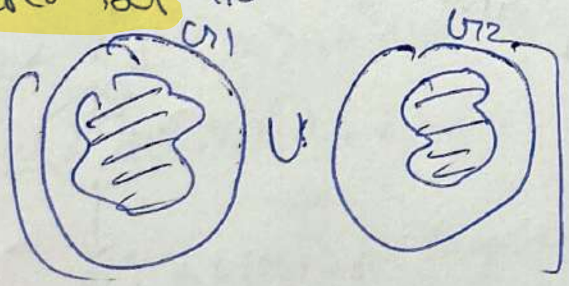
  ii) $A \subseteq G_1 \cup G_2$

  iii) $G_1 \cap A \neq \emptyset$

  iv) $G_2 \cap A \neq \emptyset$.

  imp.!!

- A connected set is one that is not disconnected.

$A =$ 

is disconnected (obeys above conditions).

* The empty set is always connected
Singleton set is always connected } not disconnected.

* The connected sets on $\mathbb{R}$ are intervals only.

$x, y \in A$, $z \notin A$ } ~~not an interval~~ $\rightarrow$ disconnected.

$x \quad z \quad y$ two piece $\rightarrow$ not connected.

$G_1 = (-\infty, z)$, $G_2 = (z, \infty)$ } disconnected.

$\underline{\text{Th}^m}$: if $A \subseteq X$ is connected, f is continuous, $f(A)$ is connected.

<u>Proof:</u>

* to show something is ~~NOT~~ connected, assume contradiction

Let $f(A)$ be disconnected

$\Rightarrow \exists$ two non-empty open sets G_1, G_2 s.t.

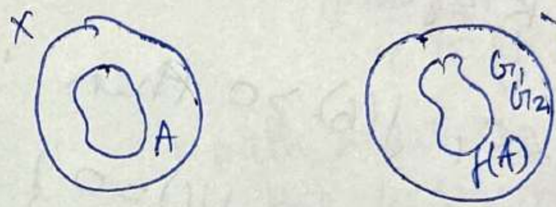
i) $G_1 \cap G_2 = \emptyset$

ii) $f(A) \subseteq G_1 \cup G_2$

iii) $G_1 \cap f(A) \neq \emptyset$

iv) $G_2 \cap f(A) \neq \emptyset$.

then we can show the same for A: (being disc.)



$\because G_1$ is an open set $\subseteq Y$, $\because f$ is continuous (open-set def).

$f^{-1}(G_1)$ is open in X.

$f^{-1}(G_2)$ is open in X.

i) $f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset$.

if not then $x \in f^{-1}(G_1)$, $x \in f^{-1}(G_2)$

$f(x) \in G_1$, $f(x) \in G_2$

$\therefore f(x) \in G_1 \cap G_2$ but no because $G_1 \cap G_2 = \emptyset$.

ii) $A \subseteq f^{-1}(G_1) \cup f^{-1}(G_2)$ → from (ii above). → can prove

$\because A \subseteq f^{-1}(G_1 \cup G_2) = f^{-1}(G_1) \cup f^{-1}(G_2)$

iii) $A \cap f^{-1}(G_1) \neq \emptyset$

iv) $A \cap f^{-1}(G_2) \neq \emptyset$.

$\Rightarrow$ A is disconnected. contradiction.

→ trivial ($\because G_1, G_2$ contain $f(A)$ and together $G_1 \cap G_2 = \emptyset$).

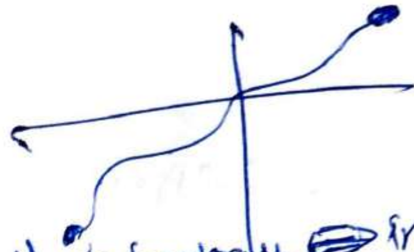
→ from iii, iv above.

if $x \in f(A) \cap G_1$, $x \in A \cap f^{-1}(G_1)$ ☐ // Q.E.D.

• The only connected sets of $\mathbb{R}$ are intervals

The empty set and singleton set are always connected.

• This gives us the IVT. in calculus.



Thm: $f: X \to \mathbb{R}$ is continuous. (open balls in topology $\Leftrightarrow$ intervals in 1D $\mathbb{R}$ line).

$f(a) \le f(b)$

$f(a) \le K \le f(b)$ then $\exists\, c \in X$ s.t. $f(c) = K$ //

if $f(a) < 0$, $f(b) > 0$ then $\because$ $f(a) < 0 < f(b)$

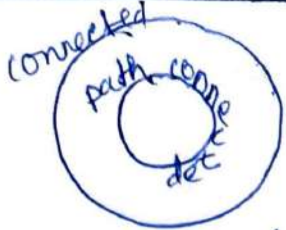
$\exists\, c \in X$ s.t. $f(c) = 0$ } ~~can find roots~~

can establish that the function definitely has a root by IVT.

→ let $A \subseteq X$. $f(A)$ is connected. $f(A) \subseteq \mathbb{R}$ and the only connected sets in $\mathbb{R}$ are intervals so $f(A)$ is an interval $\boxdot$ // (this is what IVT says).

↳ f takes all values b/w $f(a)$ and $f(b)$ by def. of intervals?

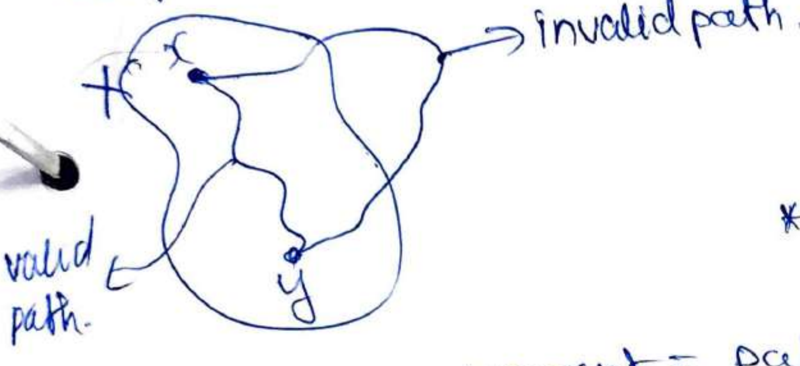
Path Connectedness



- stronger than connected.
- A path connected ~~set~~ $\Rightarrow$ Any two points in the set are connected by a path that lies in the set.

What is a path? $\rightarrow$ a continuous map ~~that is~~ $\phi : [0,1] \rightarrow X$ s.t. $\phi(0) = x$, $\phi(1) = y$.

~~always~~



ϕ is a path from x to y.

* path must lie inside the set.

- a map of a line segment $\equiv$ path.
- it seems like every connected (one-piece) set is also path connected, but this is not the case.

$\phi : (t, t^2)$. $\phi(t) = t^2$. (e.g.)

? Topologist Sine Curve ?

$y(x) = \sin\left(\frac{1}{x}\right) \cup \{(0,0)\}$

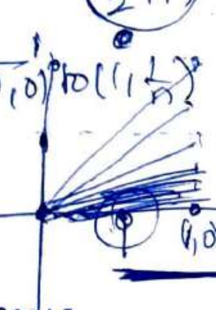
y is a connected set but not path connected.

↳ any open ball at origin intersects curve $\therefore$ connected.
↳ not disconnected ($\because$ every ball of (0,0) has a point of the curve)

↳ no path from (0,0) to curve.
(curve never touches the origin).

? Infinite Broom ?

- all line segments from (0,0) to $(1, \frac{1}{n})$ $\cup$ $(\frac{1}{2}, 1)$



connected but not path connected.
↳ every open set always intersect?

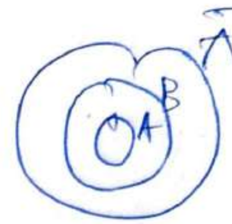
($\cup$ line $\frac{1}{2}$ to 1) never intersects the brooms.

• **Thm:** If A is connected and $A \subseteq B \subseteq \bar{A}$; then B is connected.

Assume B is not connected to show A is not connected.

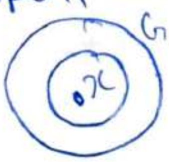
⇒ for B:
$G_1 \cap G_2 \neq \emptyset$
$B \subseteq G_1 \cup G_2$
$B \cap G_1 \neq \emptyset$
$B \cap G_2 \neq \emptyset$

for A:
$G_1 \cap G_2 = \emptyset$
⇒ $A \subseteq G_1 \cup G_2$.



Let $x \in B \cap G_1$

G_1 is open, $x \in G_1$ ⇒ x is an int. pt
$U(x,r) \subseteq G_1$



? claim that $U(x,r) \cap A \neq \emptyset$. ⇒ Similarly for G_2.
∴ A is not connected //.

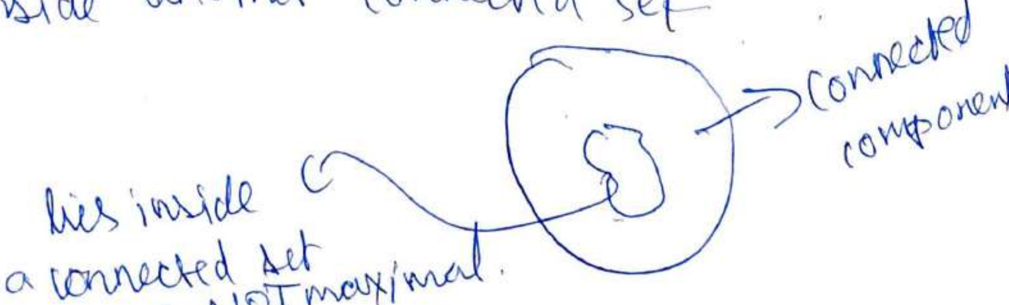
$y \in U(x,r) \cap A$
$y \in G_1 \cap A$.

Q.E.D.

• If a set is connected, its closure is also connected //.

• Any metric space is a union of connected components.

• connected component is a maximally connected set.
↳ there's nothing larger than this set that is also connected and contains this set.
↳ cannot lie inside another connected set



* Every metric space is a disjoint union of connected components.

* If we have two connected components, they must be the same or ~~not~~ disjoint

  if they have an intersection, take the union → we get a larger connected set → means they are not connected components.

* ~~Co~~ Disconnected sets :

* Totally disconnected sets :

- The only connected components are singleton sets,
  ↳ only singletons are connected.

- Any discrete metric space is totally disconnected.
  Q) What about $\mathbb{N}$ in standard topology?

More On Compact Sets

- closed and bounded. ↳ in euclidean / standard topology. Heine-Borel property

- More generally; For a compact set
"Every open cover has a finite sub cover."

- What is a cover?
↳ imagine SH1. Bring a small cloth (each) s.t. all cloths cover the floor, there may be overlap/excess, then the set of all cloths is ~~an~~ a cover.

More formally:
$S \subseteq X$, Then $\{S_\alpha\}_{\alpha \in A}$ is an open cover of S if for all. $S_\alpha, \alpha \in A$ open sets and $S \subseteq \bigcup_{\alpha \in A} \{S_\alpha\}$.

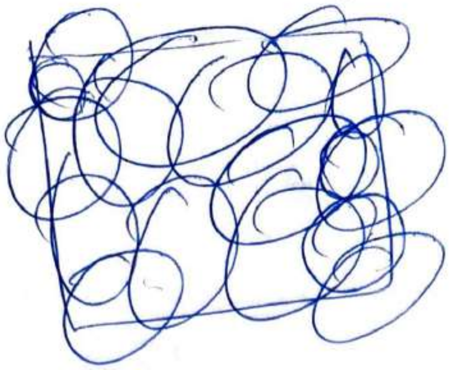
A is some set.

- A family $\{S_\beta\}_{\beta \in B}$ is a sub cover of (S_α) if

i) $B \subseteq A$
ii) ~~$\bigcup_{\beta \in B} S_\beta$~~ $S \subseteq \bigcup_{\beta \in B} S_\beta$.

- B is finite (finite sub cover).
↳ number of 'cloths' is finite but each cloth can be ∞.

e.g.)



We have covered the space with a finite no. of open balls but each ball can have ∞ points. $\therefore$ ~~set of all open balls?~~

$\Rightarrow$ ~~B~~ finite cover

eg.) $S = [0,1]$.

fix $\epsilon > 0$.

$S_\alpha = (\alpha - \epsilon, \alpha + \epsilon)$ $\alpha \in A = \{x \mid 0 \le x \le 1\}$.

S_β is a subcover of S_α if $S_\beta = (\beta - \epsilon, \beta + \epsilon)$ $\beta \in B = \{x \in \mathbb{Q} \mid 0 \le x \le 1\}$

$\because B \subseteq A$. ∞ subcover not finite.

let $\epsilon = 0.2$,

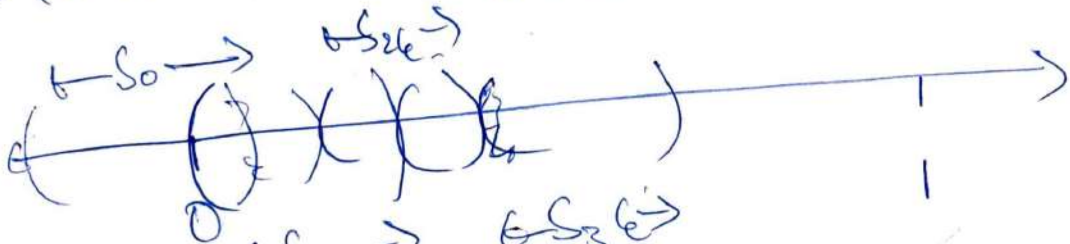
$S_0 = (-0.2, 0.2)$ $\quad S_{4\epsilon} = (0.6, 1)$

$S_\epsilon = (0, 0.4)$ $\quad S_{5\epsilon} = (0.8, 1.2)$

$S_{2\epsilon} = (0.2, 0.6)$ $\quad$ using these 6 sets

$S_{3\epsilon} = (0.4, 0.8)$ $\quad$ we can cover $[0,1]$.

$S_{4\epsilon}$



this is a finite subcover for S_α:
but we need to show that every cover has a finite subcover. easier to show not compact.

• Not compact → find a cover s.t. ∄ finite sub-cover.

construct a cover s.t. after deleting an element ~~so~~ no other subcover can cover the set ?

↳ each point in S is covered by exactly one cover, then we can delete one element and no longer cover S.

$S = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$. $S \subseteq X$.

↳ not closed. ∴ not compact,

$\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ 1

(·) (·) (·) (·)

X

∵ they are isolated points ∃ B s.t. $B'(p,\varepsilon) \cap S = \emptyset$.

all these balls union covers S, but if we remove any one ball, it ~~is~~ no longer a cover.

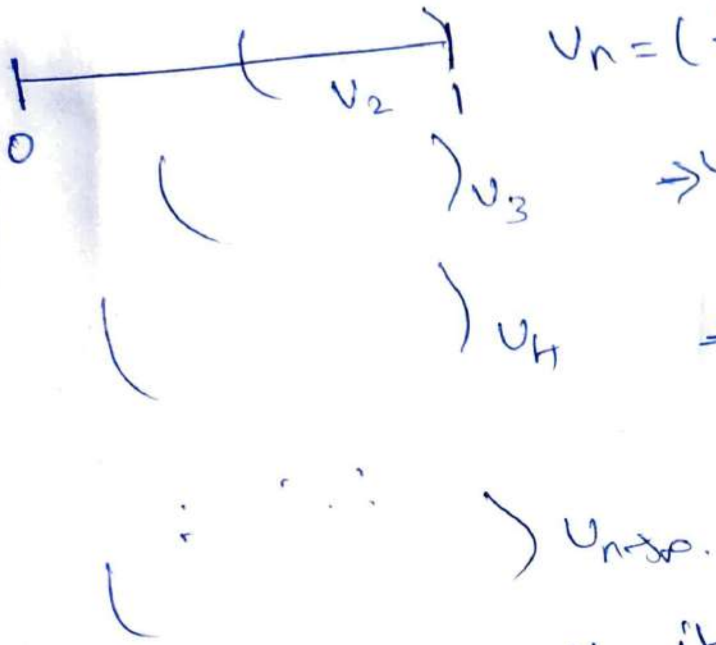
↳/any one element from the family.

* just construct a cover which has no subcover[finite].

$(-\infty, \infty)$ not compact.

if every element of S is contained in only one element of the family, then by deletion of one element, ∄ any finite subcover of S.

• $(0,1)$ is not compact.



$U_n = \left(\frac{1}{n}, 1\right)$

$\Rightarrow U_2 \cup U_3 \cup \ldots U_n$ is a cover.

$\Rightarrow$ if we take only finitely many of these, we can never cover $(0,1)$.

$\therefore$ for this cover U_n, it does not have a finite subcover $\therefore$ $(0,1)$ is NOT COMPACT.

*

Convergence of Functions

- $f_n : E \to \mathbb{R}$.

  $\{f_n\}$

  if $\{f_n(x)\}$ converges $\forall x \in E$,

  $f : E \to \mathbb{R}$

  $f(x) = \lim_{n\to\infty} f_n(x)$

→ many functions $\{f_1, f_2, f_3, \ldots\}$.

these functions converge to f **point-wise**.

- seq. of functions that converge to a function.

- $\{\cancel{f_n(x)}\}$

Pointwise Convergence	Uniform Convergence.
$\boxed{\forall x \in E,}\ \forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $\lvert f_n(x) - f(x) \rvert < \epsilon$. ($\lim_{n\to\infty} f_n(x) = f(x)$)	$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $\lvert f_n(x) - f(x) \rvert < \epsilon$. $\boxed{\forall x \in E.}$
N_0 depends on x, ϵ.	N_0 depends on ϵ only. Same N_0 works $\forall x \in E$.

- $f_n(x) = x^n$.

  $f_1(x) = x$

  $f_2(x) = x^2$

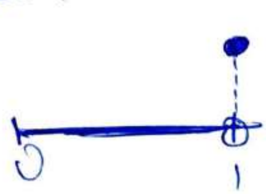
  $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$

$\{f_n\} \to f$ pointwise.

$f(x) = \lim_{n\to\infty} f_n(x)$.

is true ✓.

- limit of pointwise convergent functions need not be cont



- not uniformly convergent ;
  contradiction.

let $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$

$|f_n(x) - f(x)| < \epsilon, \forall x \in E$.

let $\epsilon = \frac{1}{2}, N_0, x = \left(\frac{3}{4}\right)^{\frac{1}{N_0}}$ ($x > \epsilon$ needed)

$n = N_0$, $f(x) = 0$ by def.

$\Rightarrow f_n(x) = f_{N_0}(x) = x^{N_0} = \left(\frac{3}{4}\right)^{\frac{N_0}{N_0}} = \frac{3}{4}$.

$|f_n(x) - f(x)| = |\frac{3}{4} - 0| \nless \frac{1}{2}$ contradiction

//Q.E.D.

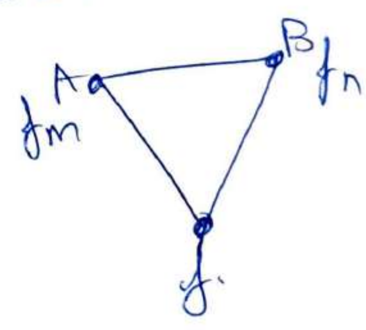
Q) is the limit of (uniformly convergent functions) also uniformly convergent.

<u>Cauchy Criterion For Uniform Convergence</u>

- $\{f_n\}$ satisfies Cauchy criteria if $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t.
  $\forall n, m \geq N_0$, $|f_n(x) - f_m(x)| < \epsilon$, $\forall x \in E$.

$\Updownarrow$

$\{f_n\} \to f$ uniformly.



<u>Proof:</u>

a) if $\{f_n\} \to f$ converges uniformly
$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$.

$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$

$\Rightarrow |f_n(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

b) let it satisfy cauchy criterion :

$\Rightarrow \forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n, m \geq N_0, |f_n(x) - f_m(x)| < \epsilon$.

$\{f_n(x)\}$ $\rightarrow$ consider images.

$\{f_n(x)\}$ is cauchy in $\mathbb{R}$.

$\Rightarrow \{f_n(x)\} \longrightarrow f$.

$\therefore f : E \rightarrow \mathbb{R}$
s.t. $f(x) = \lim_{n\to\infty} f_n(x) \rightarrow f(x)$ is well defined.

$|f_m(x) - f_n(x)| < \epsilon$.

Fix n, let $m \to \infty$.

~~$|f(x) - f_n$~~ $|f_n(x) - f(x)| < \epsilon$ $\square$ // Q.E.D.

$\forall x \in E$.