6. 정규분포

6.1

(1) $\mu=0,\ \sigma=1:\ P(X\geq 1.3)=0.0968$

(2) $\mu=0,\ \sigma=1:\ P(-1.2\leq X\leq 2)=P(X\leq 2)-P(X\leq -1.2)=0.9772-0.1151=0.8621$

(3) $\mu=0,\ \sigma=1:\ P(|X|\geq 1)=P(X\geq 1)+P(X\leq -1)=2P(X\geq 1)=2\times 0.1587=0.3174$

(4) $\mu=0,\ \sigma=1:\ P(X\geq z)=0.1357\ \Rightarrow\ z=1.1$

(5) $\mu=0,\ \sigma=1:\ P(z<X<1.9)=P(X<1.9)-P(X\leq z)=0.9713-P(X\leq z)=0.4713$

$\Rightarrow\ P(Z\leq z)=0.5\ \Rightarrow\ z=0.0$

(6) $\mu=0,\ \sigma=1:\ P(|X|\geq z)=2P(X\geq z)=0.1212$

$\Rightarrow\ P(X\geq z)=0.0606\ \Rightarrow\ z=1.55$

(7) $\sigma=1:\ P(X>1.5)=0.3085\ \Rightarrow\ P(X-\mu>1.5-\mu)=P(Z>1.5-\mu)=0.3085$

$\Rightarrow\ 1.5-\mu=0.5\ \Rightarrow\ \mu=1$

(8) $\mu=0:\ P(|X|<1.5)=0.2358$

$$\Rightarrow\ P\left(\frac{|X|}{\sigma}<\frac{1.5}{\sigma}\right)=P\left(|Z|<\frac{1.5}{\sigma}\right)=P\left(-\frac{1.5}{\sigma}<Z<\frac{1.5}{\sigma}\right)=1-2P\left(Z\geq\frac{1.5}{\sigma}\right)$$

$$\Rightarrow\ P(Z\geq\frac{1.5}{\sigma})=0.3821\ \Rightarrow\ \frac{1.5}{\sigma}=0.3\ \Rightarrow\ \sigma=5$$

(9) $\mu=5,\ \sigma=10,\ P(11.7\leq X<21.7)=\alpha$

$$\begin{aligned}P(11.7\leq X<21.7)&=P(0.67\leq Z<1.67)=P(Z\leq 1.67)-P(Z\leq 0.67)\\&=0.9525-0.7486=0.2559\end{aligned}$$

(10) $\mu=5,\ \sigma=10,\ P(5\leq X<z)=0.1443$

$$P(5\leq X<z)=P(0\leq Z<\frac{z-5}{10})=0.1443$$

$$\Rightarrow\ P(Z<\frac{z-5}{10})=0.6443\ \Rightarrow\ z=5+10\times 0.37=8.7$$

(11) $\mu=5,\ \sigma=10,\ P(-14.6<X<z)=0.95$

$$P(-14.6<X<z)=P(-1.96<Z<\frac{z-5}{10})=P(Z<\frac{z-5}{10})-0.025=0.95$$

$$\Rightarrow\ P(Z<\frac{z-5}{10})=0.975\ \Rightarrow\ z=5+1.96\times 10=24.6$$

(12) $\sigma=10,\ P(X<13)=0.8413$인 μ?

$$P(X<13)=P(Z<\frac{13-\mu}{10})=0.8413\ \Rightarrow\ \frac{13-\mu}{10}=1\ \Rightarrow\ \mu=3$$

(13) $\mu=5$일 때, $P(X<3.5)=0.3821$를 만족하는 σ로 수정

$$P(X<3.5)=P(Z<\frac{3.5-5}{\sigma})=0.3821\ \Rightarrow\ \frac{3.5-5}{\sigma}=-0.3\ \Rightarrow\ \sigma=5$$

(14) $\mu=5$일 때, $P(-16.5<X<26.5)=0.7108$을 만족하는 σ는?

$$\begin{aligned}P(-16.5<X<26.5)&=P(\frac{-16.5-5}{\sigma}<Z<\frac{26.5-5}{\sigma})\\&=P(-\frac{21.5}{\sigma}<Z<\frac{21.5}{\sigma})=P(|Z|<\frac{21.5}{\sigma})=0.7108\end{aligned}$$

$$\Rightarrow\ 2P(0\leq Z<\frac{21.5}{\sigma})=0.7108\ \Rightarrow\ P(0\leq Z<\frac{21.5}{\sigma})=0.3554$$

$\Rightarrow\ P(Z \le \frac{21.5}{\sigma}) = 0.8554 \ \Rightarrow\ \frac{21.5}{\sigma} = 1.06$

6.2 $X_1 \sim N(\mu_1, \sigma^2)$, $X_2 \sim N(\mu_2, \sigma^2)$

(1) $X_1 - X_2 \sim N(\mu_1 - \mu_2, 2\sigma^2 - 2Cov(X_1, X_2))$

(2) $Cov(X_1 + X_2, X_1 - X_2) = Cov(X_1, X_1) - Cov(X_1, X_2) + Cov(X_2, X_1) - Cov(X_2, X_2) = 0$

(3) $X_1 + X_2$, $X_1 - X_2$는 정규분포를 따르고 공분산이 0이므로 독립

6.3 빵의 열량: $X_1 \sim N(200, 12^2)$, 우유의 열량: $X_2 \sim N(85, 9^2)$

(1) $X_1 + X_2 \sim N(285, 15^2)\ \Rightarrow\ p = P(X_1 + X_2 \ge 300) = P(Z \ge 1.0) = 0.1587$

$$\binom{7}{1} 0.1587^1 (1 - 0.1587)^6 = 0.3939$$

(2) 평균: $np = 7 \times 0.1587 = 1.1049$, 표준편차: $\sqrt{np(1-p)} = \sqrt{7 \times 0.1587 \times 0.8413} = 0.9667$

6.4 $X \sim N(30, 4^2)$

(1) $P(X \ge 36) = P(Z \ge 1.5) = 0.0668$

(2) $P(24 \le X \le 36) = P(-1.5 \le Z \le 1.5) = 1 - 2P(Z \ge 1.5) = 0.8664$

(3) $P(X \ge x) = 0.05\ \Rightarrow\ x = 30 + 1.645 \times 4 = 36.58$

6.5 $X \sim N(30, 5^2)$

(1) $P(X \ge 32.55) = P(Z \ge 0.51) = 0.3050$

(2) ① $p = 0.3050\ \Rightarrow\ \binom{4}{2} p^2 (1-p)^2 = \binom{4}{2} 0.3050^2 (1 - 0.3050)^2 = 0.2696$

② A: $X \sim N(30, 5^2)$, B: $Y \sim N(30, 5^2)$ 라고 하면 $X - Y > 0$ 일 확률 계산

$X - Y \sim N(0, 50)$ 이므로 $P(X - Y > 0) = P(Z > 0) = 0.5$

6.6 $X \sim N(10, 0.1^2)$

(1) ① $p = P(X \ge 10.1) = P(Z \ge 1) = 0.1587$

② $\binom{5}{1} p^1 (1-p)^4 = \binom{5}{1} 0.1587 \times 0.8413^4 = 0.3975$

③ 평균: $np = 10 \times 0.1587 = 1.587$, 분산: $np(1-p) = 10 \times 0.1587 \times 0.8413 = 1.3351$

(2) $P(X \le x) = 0.05\ \Rightarrow\ x = 10 - 1.645 \times 0.1 = 0.9836$

(3) $Y = X_1 + \cdots + X_4 \sim N(40, 0.2^2)\ \Rightarrow\ P(Y \ge 40.1) = P(Z \ge 0.5) = 0.3085$

6.7 $X \sim N(186, 54^2)$, 문제수정: 대기시간이 267초 이상인 고객이 5%이상이 되면

(1) 문제수정: 최소 267초 이상 $P(X \ge 267) = P(Z \ge \frac{267 - 186}{54}) = P(Z \ge 1.5) = 0.0668$

(2) 대기시간이 267초 이상일 확률이 6.68%이므로 상담원을 추가할 필요

(3) $P(X \ge x) = 0.05 = P(Z \ge \frac{x - 186}{54})\ \Rightarrow\ \frac{x - 186}{54} = 1.645\ \Rightarrow\ x = 186 + 54 \times 1.645 = 274.83$

(4) 1명당 평균 1초 단축되기 때문에 k명을 추가하면 $X \sim N(186-k, 54^2)$가 되고
$P(X \geq 267) \leq 0.05$를 만족하는 k를 찾아야 함

$$P(Z \geq \frac{267-(186-k)}{54}) = P(Z \geq \frac{81+k}{54}) \;\Rightarrow\; k \geq -81+54\times 1.645 = 7.83$$

8명을 추가로 고용해야 함

(3)의 결과를 이용하여 274.83-267 = 7.83인 것을 사용해도 됨

6.8 $X \sim N(10, 25^2)$

(1) $P(X \geq 20) = P\left(Z \geq \frac{20-10}{25}\right) = P(Z \geq 0.4) = 0.3446$

(2) $P(X \leq 10) = P\left(Z \leq \frac{0-10}{25}\right) = P(Z \leq -0.4) = 0.3446$

(3) $P(-10 \leq X \leq 10) = P(-0.8 \leq Z \leq 0) = 0.5 - 0.2119 = 0.2881$

(4) $P(X \geq 60) = P(Z \geq 2) = 0.0228$

(5) $X \sim N(-5, 25^2) \;\Rightarrow\; P(X \geq 0) = P(Z \geq 0.2) = 0.4207$

6.9 $X \sim N(20, 15^2)$

(1) $P(X \geq 10) = P\left(Z \geq \frac{10-20}{15}\right) = P(Z \geq -0.667) = 0.7475$

(2) ① $P(X \leq 0) = P\left(Z \leq \frac{0-20}{15}\right) = P(Z \leq -1.333) = 0.0918$

② $\binom{10}{1} 0.0918^1 0.9082^9 = 0.3859$

③ 평균 = 10×0.0918=0.918

(3) $P(X \geq 60) = P\left(Z \geq \frac{60-20}{15}\right) = P(Z \geq 2.667) = 0.0039$

6.10 $X_1 \sim N(22, 4^2)$, $X_2 \sim N(3, 1^2) \;\Rightarrow\; X_1 + X_2 \sim N(25, 17)$

(1) $P(X_1 > 30) = P\left(\frac{X_1 - 22}{4} > \frac{30-22}{4}\right) = P(Z > 2) = 0.0228$

(2) $P(X_1 + X_2 < 30) = P\left(Z < \frac{30-25}{\sqrt{17}}\right) = P(Z < 1.212) = 0.8869$

(3) 보강 전 위험에 빠질 확률은 (2)에 의해 $P(Z \geq 1.212) = 1 - 0.8869 = 0.1131$
보강 후: $X_1 + X_2 \sim N(23, 17)$

$\Rightarrow$ $P(X_1 + X_2 \geq 30) = P\left(Z \geq \frac{30-23}{\sqrt{17}}\right) = P(Z \geq 1.70) = 0.0446$: 6.85%P 줄어듬

(4) 상관계수 $\rho_{X_1 X_2} = \frac{\sigma_{X_1 X_2}}{\sigma_{X_1}\sigma_{X_2}} = \frac{\sigma_{X_1 X_2}}{4 \times 1} = -0.5 \;\Rightarrow\; Cov(X_1, X_2) = -2$

$\Rightarrow$ $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2) = 16 + 1 - 4 = 13$

$\Rightarrow$ $P(X_1 + X_2 \geq 30) = P\left(Z \geq \frac{30-25}{\sqrt{13}}\right) = P(Z \geq 1.387) = 0.0827$