# Chapter 1

# Losses

A loss is a "penalty" score to reduce when training an algorithm on data. It is usually call **objective function** to optimize.

Classifying means assigning a label to an observation:
x $\rightarrow$ y.

Such a function is named a **classifier**. To create such classifier, we ussually create models with parameters to define:

$$f_w : x \rightarrow y$$

The process of defining the optimal parameters $w$ given past observations X and their known labels Y is named **training**. The objective of the training is obviously to maximize the **likelihood**

$$likelihood(w) = P_w(y|w)$$

Since the logarithm is monotonous, it is equivalent to **minimize the negative log-likelihood**

$$\mathscr{L}(w) = -\ln P_w(y|x)$$

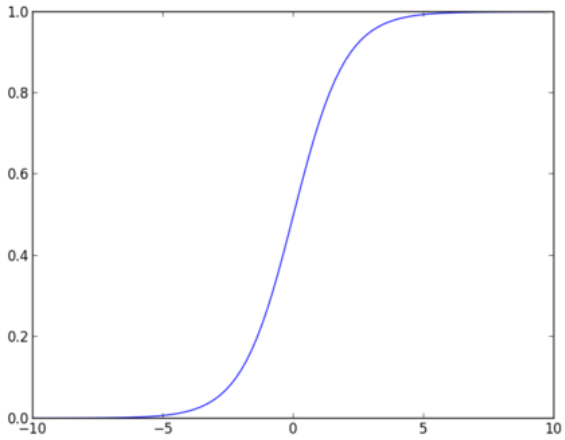
The reasons for taking the negative of the logarithm of the likelihood are :

- it is more convinient to work with the log, because the log-likelihood of statistically independant observations will simply be the sum of the log-likelihood of each observation
- we usually prefer to write the **objective function** as a **cost function** to minimize.

## 1.1 Binomial probabilities - log loss / logistic loss / cross-entropy loss

Binomial means 2 classes, which usually are 0 or 1. Each class has a probability $p$ and *1-p* (sums to 1). When using a network, we try to get 0 and 1 as values,

that's why we add a **sigmoid function or logistic function** that saturates as a last layer.




$$f : x \to \frac{1}{1 + e^{-x}}$$

Then, once the estimated probability to get 1 is $\hat{p}$, then it is easy to see that the negative log likelihood can be written

$$\mathscr{L} = -y \log \hat{p} - (1 - y) \log(1 - \hat{p})$$

which is also the **cross-entropy**

$$\text{crossentropy}(p, q) = E_p[-\log q] = -\sum_x p(x) \log q(x) = -\frac{1}{N} \sum_{n=1}^{N} \log q(x_n)$$

## 1.2 Multinomial probabilities / multi-class classification : multinomial logistic loss / cross entropy loss / log loss

It is a problem where we have $k$ classes or categories, and only one valid for each example. The target values are still binary but represented as a vector y that will be defined by the following, if the example x is of class c:

$$y_i = \begin{cases} 0, & \text{if } i \neq c \\ 1, & \text{otherwise} \end{cases}$$

If {pi} is the probability of each class, then it is a multinomial distribution and

$$\sum_i p_i = 1$$

The equivalent to the sigmoid functoin in multi-dimensional space is the **softmax function or logistic function or normalized exponential function** to produce such a distribution from any input vector $z$ :

$$z \to \left\{ \frac{\exp z_i}{\sum_k \exp^{z_k}} \right\}_i$$

The error is also best described by cross-entropy :

$$\mathscr{L} = -\sum_{i=0}^{k} y_i \ln \hat{p}_i$$

Cross-entropy is designed to deal with error on probabilities. For example, ln(0,01) will be a lot stronger error signal than ln(0.1) and encourage to resolve errors. In some cases, the logarithm is bounded to avoid extreme punishments.

Last, the combined softmax and cross-entropy has a very simple and stable derivative.

## 1.3 Multi-label classification

There is a variant for multi-label classification, in this case multiple $y_i$ can have a value set to 1. For example, "car", "automobile", "motor vehicle" are three labels that can be applied to a same image of a car. On the image of a truck, you'll only have "motor vehicle" active for example. In this case, the softmax function will not apply, we usually add a sigmoïd layer before the cross-entropy layer to ensure stable gradient estimation :

$$t \to \frac{1}{1 + e^{-t}}$$

The cross-entropy will look like :

$$\mathscr{L} = -\sum_{i=0}^{k} y_i \ln \hat{p}_i + (1 - y_i) \ln(1 - \hat{p}_i)$$

## 1.4 Absolute value loss / L1 loss

The absolute value los is the L1-norm of the error :

$$\mathscr{L}_1 = \sum_i |\hat{y}_i - y_i| = \|\hat{y} - y\|_1$$

Minimizing the absolute value loss means predicting the (conditional) median of $y$. Variants can handle other quantiles. 0/1 loss for classification is a special case. Note that the L1 norm is not differentable in 0, and it is possible to use a smooth L1:

$$|d|_{\text{smooth}} == \begin{cases} 0.5d^2, & \text{if } |d| \leq 1 \\ |d| - 0.5, & \text{otherwise} \end{cases}$$

## 1.5 L2 Loss

When predictions are scalars or metrics we usually use the **square error or euclidean loss** which is the L2-norm of the error:

$$\mathscr{L}_2 = \sum_i (\hat{y}_i - y_i)^2 = \|\hat{y} - y\|_2^2$$

Minimising the squared error is equivalent to predicting the (conditional) mean of y. Due to the gradient being flat at the extremes for a sigmoid function, we do not use a sigmoid activation with a squared error loss because convergence will be slow if some neurons saturate on the wrong side.

A squared error is often used with a rectified linear unit. The L2 norm penalizes large errors more strongly and therefore is very sensitive to outliers. To avoid this, we usually use the squared root version :

$$\mathscr{L} = \|\hat{y} - y\|_2$$