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Graded Homework 1- Exercise 1

1. As the probability density function of d is:

$$f(d;\lambda) = \lambda e^{-\lambda d}$$

To get a p^{th} population quantitle:

$$\int_0^x \lambda e^{-\lambda x} dx = p$$

$$-\frac{1}{\lambda} e^{-\lambda x} + \frac{1}{\lambda} e^0 = \frac{p}{\lambda}$$

So the p^{th} population quantitle is:

$$x = Q_D(p) = -\frac{\ln(1-p)}{\lambda}$$

2. From the expectation of exponential distribution:

$$E(D) = \frac{1}{\lambda}$$

$$Q_D(p) = -\ln(1-p)E(D)$$

The first empirical moment is:

$$\hat{\mu}_1 = \bar{D_n}$$

$$Q_D\hat{(p)}_1^{MM} = \varphi^{-1}(\hat{\mu}_1)$$

$$= -\ln(1-p)\bar{D_n}$$

3. From CLT we know that:

$$\bar{D_n} \sim N(\frac{1}{\lambda}, \frac{1}{n\lambda^2})$$

$$-\ln(1-p)\bar{D_n} \sim N(-\frac{\ln(1-p)}{\lambda}, \frac{(\ln(1-p))^2}{n\lambda^2})$$

$$\frac{-\ln(1-p)\bar{D_n} - -\frac{\ln(1-p)}{\lambda}}{\frac{-\ln(1-p)}{\sqrt{n}\lambda}} \sim N(0,1)$$

The margin should be:

$$Z_{1-\frac{\alpha}{2}} \frac{-\ln(1-p)}{\sqrt{n}\lambda}$$

So the confidence interval should be:

$$[-\ln(1-p)\bar{D_n} - Z_{1-\frac{\alpha}{2}} \frac{-\ln(1-p)}{\sqrt{n}\lambda}, -\ln(1-p)\bar{D_n} + Z_{1-\frac{\alpha}{2}} \frac{-\ln(1-p)}{\sqrt{n}\lambda}]$$

4. As $D_i \sim exp(\lambda)$, $\lambda D_i \sim exp(1)$, which can be written as:

$$\lambda D_i \sim gamma(1,1)$$

$$\sum_{i=1}^{n} \lambda D_i \sim gamma(n,1)$$

$$\frac{1}{n}\sum_{i=1}^{n} \lambda D_i \sim gamma(n,n)$$

so that:

$$\lambda \bar{D_n} \sim gamma(n,n)$$

As we know that $\lambda = \frac{\ln 2}{Q_p(0.5)}$, we could construct the interval:

$$gamma_{\frac{\alpha}{2}}(n,n) \leq \frac{\ln 2\bar{D_n}}{Q_p(0.5)} \leq gamma_{1-\frac{\alpha}{2}(n,n)}$$

Get that:

$$\frac{\ln 2\bar{D_n}}{gamma_{1-\frac{\alpha}{2}}(n,n)} \leq Q_p(\hat{0}.5) \leq \frac{\ln 2\bar{D_n}}{gamma_{\frac{\alpha}{2}}(n,n)}$$

So the exact confidence interval should be:

$$[\frac{\ln 2\bar{D_n}}{gamma_{1-\frac{\alpha}{2}}(n,n)}, \frac{\ln 2\bar{D_n}}{gamma_{\frac{\alpha}{2}}(n,n)}]$$