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Graded Homework 1 - Exercise 3

1. $\gamma = E[{R_1}^3]$. Based on the moment generating function, we could get:

$$\phi'''(0) = E[{R_1}^3]$$

Since $R_1 \sim N(\mu, \sigma^2)$, we could get the MGF:

$$\phi(t) = e^{t\mu + 0.5\sigma^2 t^2}$$

So,

$$\phi'''(t) = (2\sigma^4 t + 2\mu\sigma^2)e^{t\mu+0.5\sigma^2t^2} + (\mu^2 + \sigma^4 t^2 + 2\mu\sigma^2 t + \sigma^2)(\mu + \sigma^2 t)e^{t\mu+0.5\sigma^2t^2}$$

Plug in $t = 0$ we could get that,

$$\phi'''(0) = \mu^3 + 3\mu\sigma^2$$

So, $\gamma = E[{R_1}^3] = \mu^3 + 3\mu\sigma^2$.

2. (a) By CLT, we have

$$\sqrt{n}(\bar{R} - \mu) \xrightarrow[n\to\infty]{D} N(0, \sigma^2)$$

By δ-method, we have

$$\sqrt{n}((\bar{R})^3 - \mu^3) \xrightarrow[n\to\infty]{D} N(0, \sigma^2 \cdot (3\mu^2)^2) = N(0, 9\mu^4\sigma^2)$$

Therefore, $E[\hat{\gamma}] = E[(\bar{R})^3] = \mu^3$,
$bias(\hat{\gamma}) = E[\hat{\gamma}] - \gamma = \mu^3 - \mu^3 - 3\mu\sigma^2 = -3\mu\sigma^2$.

(b) $\hat{\gamma}$ is not consistent.
Since $\hat{\gamma} = (\bar{R})^3$, as $n \to \infty$, due to WLLN and Continuous Mapping Theorem, $\hat{\gamma} \to \mu^3$.
While $\gamma = \mu^3 + 3\mu\sigma^2$. So, this estimator is not consistent.

3. We claim that $\hat{\gamma}' = (\bar{R})^3 + 3\bar{R}S^2$ is an unbiased estimator, where $S = \frac{1}{n-1}\sum(R_i - \bar{R})^2$ and $E[S^2] = \sigma^2$.

Since R_i is $i.i.d.$ normal, $\bar{R}$ and S^2 are independent. That is, $E[\bar{R}S^2] = E[\bar{R}] \cdot E[S^2] = \mu\sigma^2$.

Therefore, $E[(\bar{R})^3 + 3\bar{R}S^2] = E[(\bar{R})^3] + 3E[\bar{R}] \cdot E[S^2] = \mu^3 + 3\mu\sigma^2 = \gamma$.

Thus we have proven that $\hat{\gamma}'$ is unbiased.

If we want to estimate μ^3, we can use $(\frac{1}{n}\sum_{i=1}^n R_i)^3$. Actually, this estimator is just $\hat{\gamma}$ in point 2, and we have already shown that $E[\hat{\gamma}] = \mu^3$, so it is an unbiased estimator of μ^3.

4. (a) We want to calculate $E[\tilde{\gamma}] - \gamma$.

$$E[\tilde{\gamma}] = E[\frac{1}{n}\sum_{i=1}^{n} {R_i}^3] = \frac{1}{n}\sum_{i=1}^{n} E[{R_i}^3]$$

Since $E[{R_i}^3] = E[{R_1}^3]$, we have $E[\tilde{\gamma}] = \gamma$. So, the bias of this estimator is

$$E[\tilde{\gamma}] - \gamma = 0$$

(b) By WLLN, we have

$$\tilde{\gamma} = \frac{1}{n}\sum R_i^3 \xrightarrow[n\to\infty]{P} E[R_1^3] = \gamma$$

Therefore, $\tilde{\gamma}$ is consistent.

5. Using the Rao-Blackwell Theorem, now we have an unbiased estimator

$$\hat{\gamma}' = (\bar{R})^3 + 3\bar{R}S^2$$

According to the slides, the minimal sufficient statistic for normal distribution is that

$$T(X) = (\sum_{i=1}^{n} X_i, \sum_{i=1}^{n}(X_i - \bar{X}_i)^2) \propto (\bar{R}, S^2)$$

Then we could get a new unbiased estimator $\mathring{\gamma} = E[\tilde{\gamma}|T(X)]$.

$$\begin{aligned}
\mathring{\gamma} &= E[\hat{\gamma}'|T(X)] \\
&= E[(\bar{R})^3 + 3\bar{R}S^2|\bar{R}, S^2] \\
&= (\bar{R})^3 + 3\bar{R}S^2 \\
&= \hat{\gamma}'
\end{aligned}$$

That is, the unbiased estimator $\hat{\gamma}'$ has already had the minimum variance.