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Graded Homework 1 - Exercise 3

1. $\gamma = E[{R_1}^3]$. Based on the moment generating function, we could get:

$$\phi'''(0) = E[{R_1}^3]$$

Since $R_1 \sim N(\mu, \sigma^2)$, we could get the MGF:

$$\phi(t) = e^{t\mu + 0.5\sigma^2 t^2}$$

So,

$$\phi'''(t) = (2\sigma^4 t + 2\mu\sigma^2)e^{t\mu+0.5\sigma^2t^2} + (\mu^2 + \sigma^4 t^2 + 2\mu\sigma^2 t + \sigma^2)(\mu + \sigma^2 t)e^{t\mu+0.5\sigma^2t^2}$$

Plug in $t = 0$ we could get that,

$$\phi'''(0) = \mu^3 + 3\mu\sigma^2$$

So, $\gamma = E[{R_1}^3] = \mu^3 + 3\mu\sigma^2$.

2. (a) $\hat{\gamma} = (\frac{1}{n}\sum_{i=1}^n R_i)^3 = (\bar{R})^3$. We want to calculate the bias of this estimator, $E[\hat{\gamma}] - \gamma$. Since $R_1 \sim N(\mu, \sigma^2)$, based on the CLT, we could get that:

$$\bar{R} \sim N(\mu, \frac{\sigma^2}{n})$$

Now we have a question similar to the above one.

$$E[\bar{R}^3] = \Phi'''(0)$$

$$\Phi(t) = e^{t\mu + \frac{1}{2n}\sigma^2 t^2}$$

So,

$$\Phi'''(0) = \mu^3 + \frac{3\mu\sigma^2}{n} = E[(\bar{R})^3]$$

then,

$$E[\hat{\gamma}] - \gamma = E[(\bar{R})^3] - E[{R_1}^3] = \mu^3 + \frac{3\mu\sigma^2}{n} - (\mu^3 + 3\mu\sigma^2)$$

So, the bias of this estimator is that

$$E[\hat{\gamma}] - \gamma = 3(\frac{1}{n} - 1)\mu\sigma^2$$

(b) $\hat{\gamma}$ is not consistent.
Since $\hat{\gamma} = (\bar{R})^3$, as $n \to \infty$, due to WLLN and Continuous Mapping Theorem, $\hat{\gamma} \to \mu^3$. While $\gamma = \mu^3 + 3\mu\sigma^2$. So, this estimator is not consistent.

3. We claim that $U(R) = (\bar{R})^3 + 3(1-\frac{1}{n})\bar{R}S^2$ is an unbiased estimator for γ, where $S = \frac{1}{n-1}\sum(R_i - \bar{R})^2$ and $E[S^2] = \sigma^2$.

   We have already shown that $E[(\bar{R})^3] = \mu^3 + \frac{3\mu\sigma^2}{n}$.

   Since R_i is *i.i.d.* normal, $\bar{R}$ and S^2 are independent. That is, $E[\bar{R}S^2] = E[\bar{R}] \cdot E[S^2] = \mu\sigma^2$.

   Therefore, we have

$$E[(\bar{R})^3 + 3(1-\frac{1}{n})\bar{R}S^2] = E[(\bar{R})^3] + 3(1-\frac{1}{n})E[\bar{R}] \cdot E[S^2] = \mu^3 + 3\mu\sigma^2 = \gamma$$

   Thus we have proven that $U(R)$ is an unbiased estimator for γ.

4. (a) We want to calculate $E[\tilde{\gamma}] - \gamma$.

$$E[\tilde{\gamma}] = E[\frac{1}{n}\sum_{i=1}^{n} {R_i}^3] = \frac{1}{n}\sum_{i=1}^{n} E[{R_i}^3]$$

   Since $E[{R_i}^3] = E[{R_1}^3]$ for any i, we have $E[\tilde{\gamma}] = E[R_1^3] = \gamma$. So, the bias of this estimator is

$$E[\tilde{\gamma}] - \gamma = 0$$

   (b) By WLLN, we have

$$\tilde{\gamma} = \frac{1}{n}\sum R_i^3 \xrightarrow[n\to\infty]{P} E[R_1^3] = \gamma$$

   Therefore, $\tilde{\gamma}$ is consistent.

5. It is proven in point 3 that $U(R) = (\bar{R})^3 + 3(1-\frac{1}{n})\bar{R}S^2$ is an unbiased estimator for γ, where $S = \frac{1}{n-1}\sum(R_i - \bar{R})^2$ and $E[S^2] = \sigma^2$.

   According to the slides, the minimal sufficient statistic for normal distribution is that

$$T(R) = \left(\sum_{i=1}^{n} R_i, \sum_{i=1}^{n}(R_i - \bar{R}_i)^2\right) \propto (\bar{R}, S^2)$$

   Using the Rao-Blackwell Theorem, we can get a new unbiased estimator $\mathring{\gamma} = E[U(R)|T(R)]$.

$$\begin{aligned}\mathring{\gamma} &= E[U(R)|T(R)] \\ &= E\left[(\bar{R})^3 + 3(1-\frac{1}{n})\bar{R}S^2|\bar{R}, S^2\right] \\ &= (\bar{R})^3 + 3(1-\frac{1}{n})\bar{R}S^2\end{aligned}$$

   That is, the unbiased estimator $U(R)$ has already had the minimum variance.