estimating equation and asymptotics
Simple setting

Tianchen Qian

2020.12.22

1 Notation

- Data from each individual: $(L_t, X_t, I_t, A_t, Y_{t,1}, Y_{t,2}, \ldots Y_{t,120}) : 1 \leq t \leq T$.
- This is specific to Marianne's generative model that she uses in the simulation, where the IPTW weights $\prod_{m=1}^{119} \cdots$ are not needed.
- Constant randomization p, no missing outcome.
- $\vec{X}_t := (X_t, 1 - X_t)^T$
- $\beta_k := (\beta_{k0}, \beta_{k1})^T$

2 Modeling Assumptions

- **Causal Effect Model (Treatment effect on proximal outcome conditional on X_t).** For each of $k \in \{1, 2\}$, for all t and all m,

$$\log \frac{P\left\{Y_{t,m}(\bar{A}_{t-1}, 1) = k \mid I_t(\bar{A}_{t-1}) = 1, X_t(\bar{A}_{t-1})\right\}}{P\left\{Y_{t,m}(\bar{A}_{t-1}, 0) = k \mid I_t(\bar{A}_{t-1}) = 1, X_t(\bar{A}_{t-1})\right\}} = \vec{X}_t(\bar{A}_{t-1})^T \beta_k \equiv \beta_{0k}\left\{1 - X_t(\bar{A}_{t-1})\right\} + \beta_{1k} X_t(\bar{A}_{t-1}) \tag{1}$$

- **Working Model (Control part of proximal outcome conditional on L_t).** For each of $k \in \{1, 2\}$, for all t and all m,

$$\log P\left\{Y_{t,m}(\bar{A}_{t-1}, 0, \bar{0}_{m-1}) = k \mid I_t(\bar{A}_{t-1}) = 1, L_t(\bar{A}_{t-1})\right\} = L_t(\bar{A}_{t-1})^T \alpha_k$$

We will show that the estimator for β_k is consistent even if the working model is wrong.

3 Estimator for β

3.1 Estimating quation for (α, β)

Suppose L_t is q-dimensional. Define

$$D_t = \begin{bmatrix} L_t & 0_{q\times 1} \\ 0_{q\times 1} & L_t \\ (A_t - p)\vec{X}_t & 0_{2\times 1} \\ 0_{2\times 1} & (A_t - p)\vec{X}_t \end{bmatrix}_{(2q+4)\times 2},$$

and define the blipped-down residuals

$$R_{ktm}(\alpha, \beta) = e^{-A_t \vec{X}_t^T \beta_k} 1(Y_{t,m} = k) - e^{L_t^T \alpha_k}.$$

The estimating function for (α, β) is

$$U_1(O;\alpha,\beta) = \sum_{t=1}^{T}\sum_{m=1}^{120} I_t D_t \begin{bmatrix} R_{1tm}(\alpha,\beta) \\ R_{2tm}(\alpha,\beta) \end{bmatrix}. \tag{2}$$

Suppose $\hat{\alpha}, \hat{\beta}$ solves $\mathbb{P}_n U_1(O;\alpha,\beta) = 0$.

The final estimator is $\hat{\beta}$. We order the parameters as

$$\gamma = (\alpha^T, \beta^T)^T = (\alpha_1^T, \alpha_2^T, \beta_{10}, \beta_{11}, \beta_{20}, \beta_{21})^T.$$

The variance of $\hat{\beta}$ can be estimated by

$$\frac{1}{n} V_n^{-1} \Sigma_n V_n^{-1,T},$$

where

$$\Sigma_n = \mathbb{P}_n \left\{ U_1(O;\hat{\gamma}) U_1(O;\hat{\gamma})^T \right\}$$

and $V_n = \mathbb{P}_n \left\{ \frac{\partial U_1(O;\hat{\gamma})}{\partial \gamma^T} \right\}$ is given below, both are square matrices of dimension $(2q+4) \times (2q+4)$.

3.1.1 Form of $\frac{\partial U_1(O;\gamma)}{\partial \gamma^T}$

$$\frac{\partial U_1(O;\gamma)}{\partial \gamma^T} = \sum_{t=1}^{T}\sum_{m=1}^{120} I_t D_t \begin{bmatrix} -e^{L_t^T \alpha_1} L_t^T & 0_{1\times q} & -e^{-A_t \vec{X}_t^T \beta_1} 1(Y_{t,m}=1) A_t \vec{X}_t^T & 0_{1\times 2} \\ 0_{1\times q} & -e^{L_t^T \alpha_2} L_t^T & 0_{1\times 2} & -e^{-A_t \vec{X}_t^T \beta_2} 1(Y_{t,m}=2) A_t \vec{X}_t^T \end{bmatrix}$$