Software Package for Inference after Pooling

How to Interface with the RL Algorithm

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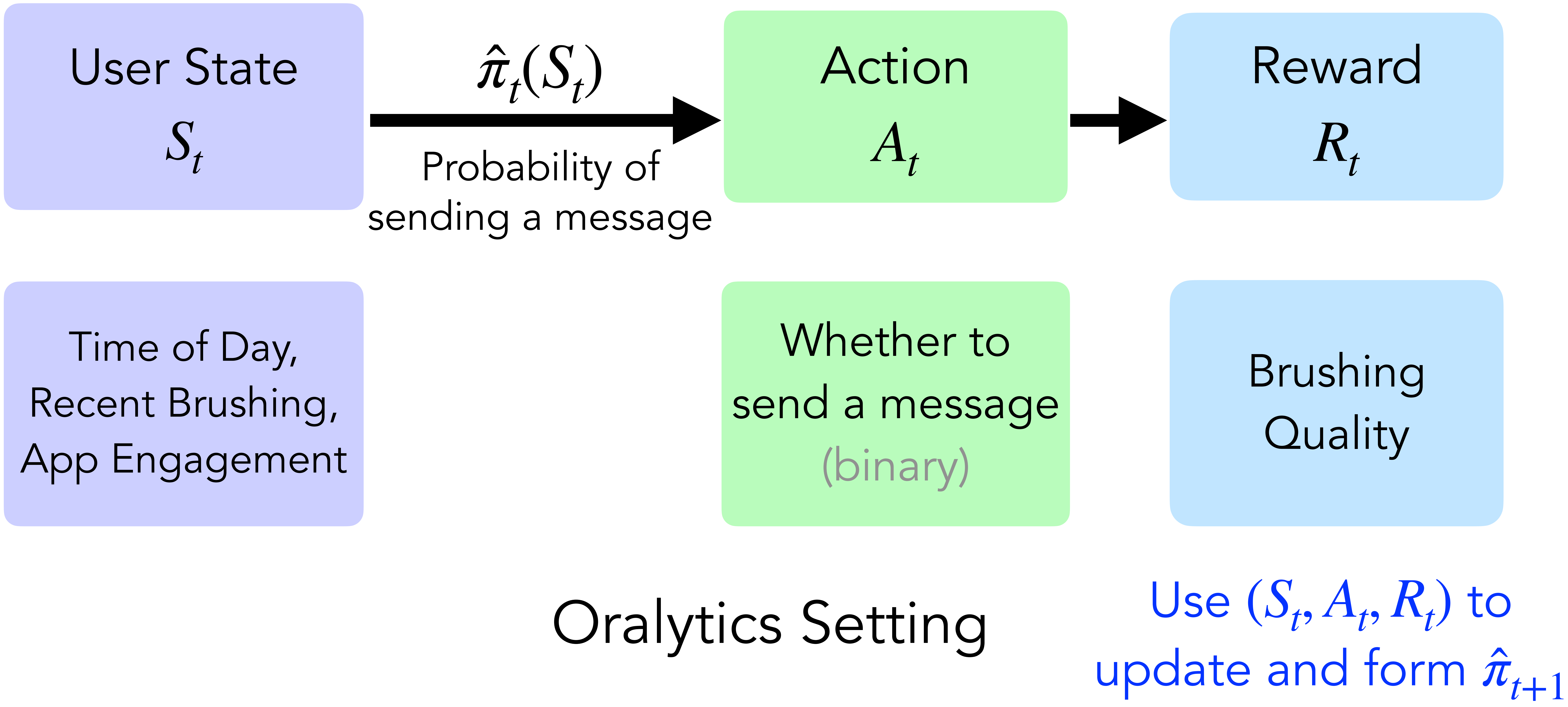
Motivation and Challenges Due to Using Pooling RL Algorithms

Online Reinforcement Learning (RL)



MRT Study Data

- **Total Decision Times:** T
- **Number of users:** N
- **Data Collected After Study:** For each user $i \in [1: N]$,

$$\underbrace{\left(S_{i,1}, A_{i,1}, R_{i,1}\right)}_{D_{i,1}} \qquad \underbrace{\left(S_{i,2}, A_{i,2}, R_{i,2}\right)}_{D_{i,2}} \qquad \cdots \qquad \underbrace{\left(S_{i,T}, A_{i,T}, R_{i,T}\right)}_{D_{i,T}}$$

- **User History:** $H_{i,t} = \left\{D_{i,1}, D_{i,2}, D_{i,3}, \ldots, D_{i,t}\right\}$

$D_{i,t} \triangleq (S_{i,t}, A_{i,t}, R_{i,t})$

Individual RL Algorithms

$\longrightarrow$ Algorithm Update

$\longrightarrow$ Data Collection

$$
\begin{array}{lcccccccccccc}
\textbf{User } 1 & D_{1,1} & \longrightarrow & \hat{\pi}_{1,2} & \longrightarrow & D_{1,2} & \longrightarrow & \hat{\pi}_{1,3} & \longrightarrow & \dots & \longrightarrow & \hat{\pi}_{1,T} & \longrightarrow & D_{1,T} \\
\textbf{User } 2 & D_{2,1} & \longrightarrow & \hat{\pi}_{2,2} & \longrightarrow & D_{2,2} & \longrightarrow & \hat{\pi}_{2,3} & \longrightarrow & \dots & \longrightarrow & \hat{\pi}_{2,T} & \longrightarrow & D_{2,T} \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\textbf{User } N & D_{N,1} & \longrightarrow & \hat{\pi}_{N,2} & \longrightarrow & D_{N,2} & \longrightarrow & \hat{\pi}_{N,3} & \longrightarrow & \dots & \longrightarrow & \hat{\pi}_{N,T} & \longrightarrow & D_{N,T}
\end{array}
$$

Dependence Within a User

User states/rewards can be dependent over time

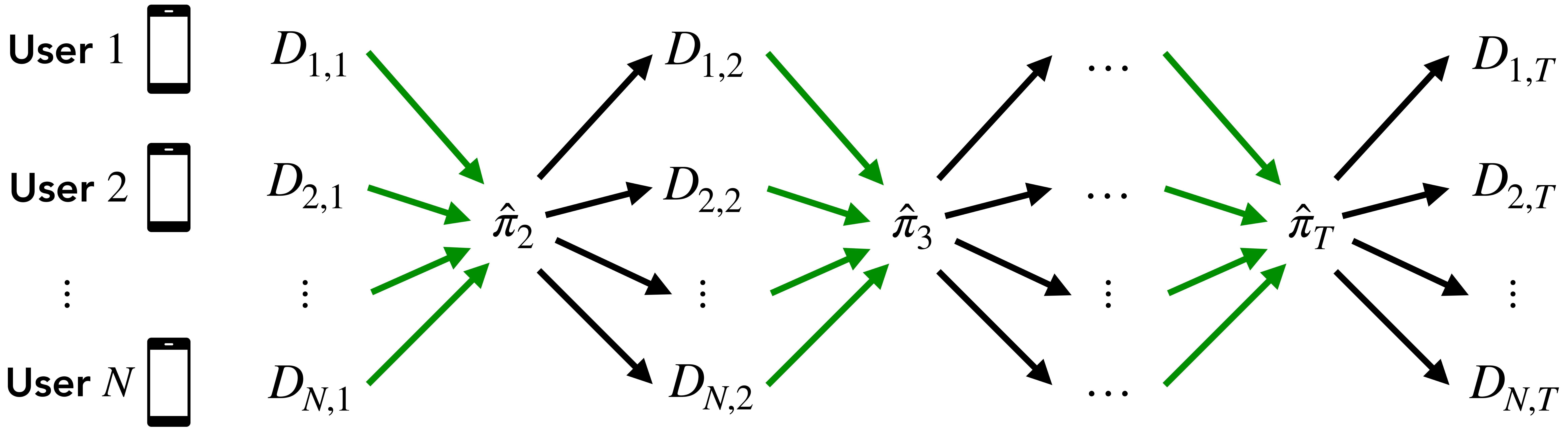
Limitations

Rewards are noisy and few decision times per user → slow learning

$D_{i,t} \triangleq (S_{i,t}, A_{i,t}, R_{i,t})$

Pooling RL Algorithm

Algorithm Update
Data Collection



Dependence Within a User

User states/rewards can be dependent over time

Dependence Between Users

Due to use of pooling algorithm

Inference After Pooling Approach Overview

Pooling Algorithm

- At each decision time t, use **all past user data** to form a statistic $\hat{\beta}_{t-1}^{(N)}$ (e.g. estimator for parameters in a reward model)
- $\hat{\pi}_t(s) = \pi_t\left(s; \hat{\beta}_{t-1}^{(N)}\right)$ is pooled policy at time t
- Select action $A_{i,t} \Big| S_{i,t}, H_{1:n,t-1} \sim \text{Bernoulli}\left(\hat{\pi}_t(S_{i,t})\right)$

What is $\hat{\beta}_t$ in general?

$$\phi_t(H_t; \beta) = (R_t - \beta_a^\top f(S_t) - \beta_b^\top A_t g(S_t)) \begin{bmatrix} f(S_t) \\ A_t g(S_t) \end{bmatrix}$$

$\hat{\beta}_t \in \mathbb{R}^{p_\beta}$ is the solution to an estimating equation for

$$0 = \frac{1}{N} \sum_{i=1}^{N} \phi_t \left(H_{i,t}; \hat{\beta}_{1:t} \right) \in \mathbb{R}^{p_\beta}$$

e.g. minimizer of a loss function, least squares, MLEs, etc.

Theory applies when: $\hat{\beta}_t^{(N)} \to \beta_t^\star$ as $N \to \infty$

where $0 = \mathbb{E}_\star \left[\phi_t \left(H_{i,t}; \beta_{1:t}^\star \right) \right] \in \mathbb{R}^{p_\beta}$

Inference Approach

- **Inferential Goal:**

$$0 = \mathbb{E}_\star \left[\psi \left(H_{i,T}; \theta^\star, \beta^\star_{1:T-1} \right) \right] \in \mathbb{R}^{p_\theta}$$

- **Estimator:**

$$0 = \frac{1}{N} \sum_{i=1}^{N} \psi \left(H_{i,T}; \hat{\theta}, \hat{\beta}_{1:T-1} \right) \in \mathbb{R}^{p_\theta}$$

e.g. minimizer of a loss function, least squares, MLEs, etc.

Asymptotic Normality Result (T = 3 Case)

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\beta}_2 - \beta_2^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(0, B^{-1}\Sigma(B^{-1})^\top\right)$$

$$\Sigma = \mathbb{E}_\star\left[\begin{pmatrix}\phi_1(H_{i,1};\beta_1^\star) \\ \phi_2(H_{i,2};\beta_{1:2}^\star) \\ \psi(H_{i,3};\theta^\star,\beta_{1:2}^\star)\end{pmatrix}^{\otimes 2}\right] = \mathbb{E}_\star\left[\begin{pmatrix}\phi_{i,1}(\beta_1^\star) \\ \phi_{i,2}(\beta_{1:2}^\star) \\ \psi_i(\theta^\star,\beta_{1:2}^\star)\end{pmatrix}^{\otimes 2}\right]$$

Asymptotic Normality Result (T = 3 Case)

$$\sqrt{n}\begin{pmatrix}\hat{\beta}_1 - \beta_1^\star \\ \hat{\beta}_2 - \beta_2^\star \\ \hat{\theta} - \theta^\star\end{pmatrix} \xrightarrow{D} \mathcal{N}\left(B^{-1}\Sigma(B^{-1})^\top\right)$$

$$B = \frac{\partial}{\partial(\beta_1, \beta_2, \theta)}\mathbb{E}_\star\left[\begin{matrix}\phi_{i,1}(\beta_1) \\ W_{i,2}^\star(\beta_1)\phi_{i,2}(\beta_{1:2}) \\ W_{i,2}^\star(\beta_1)W_{i,3}^\star(\beta_2)\psi_i(\theta, \beta_{1:2})\end{matrix}\right]\Bigg|_{\beta_1^\star, \beta_2^\star, \theta^\star}$$

$$W_{i,t}^\star(\beta_{t-1}) = \left(\frac{\pi_t(S_{i,t}; \beta_{t-1})}{\pi_t(S_t; \beta_{t-1}^\star)}\right)^{A_{i,t}}\left(\frac{1 - \pi_t(S_{i,t}; \beta_{t-1})}{1 - \pi_t(S_t; \beta_{t-1}^\star)}\right)^{1-A_{i,t}}$$

Rewriting the "bread" part B

$$\mathbb{E}_\star \begin{bmatrix} \frac{\partial}{\partial \beta_1}\phi_1(\beta_1) & 0 & 0 \\ \left[\frac{\partial}{\partial \beta_1} W_2^\star(\beta_1)\right]\phi_2 + \frac{\partial}{\partial \beta_1}\phi_2(\beta_{1:2}) & \frac{\partial}{\partial \beta_2}\phi_2(\beta_{1:2}) & 0 \\ \left[\frac{\partial}{\partial \beta_1} W_2^\star(\beta_1)\right]\psi + \frac{\partial}{\partial \beta_1}\psi(\beta_{1:2},\theta) & \left[\frac{\partial}{\partial \beta_2} W_3^\star(\beta_2)\right]\psi + \frac{\partial}{\partial \beta_2}\psi(\beta_{1:2},\theta) & \frac{\partial}{\partial \theta}\psi(\beta_{1:2},\theta) \end{bmatrix} \Bigg|_{\beta_{1:2}^\star, \theta^\star}$$

$$W_t^\star(\beta_{t-1}) = \left(\frac{\pi_t(S_t;\beta_{t-1})}{\pi_t(S_t;\beta_{t-1}^\star)}\right)^{A_t} \left(\frac{1-\pi_t(S_t;\beta_{t-1})}{1-\pi_t(S_t;\beta_{t-1}^\star)}\right)^{1-A_t}$$

Software and Computational Challenges

Estimating the Matrix B

$$\frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix} \frac{\partial}{\partial\beta_1}\phi_{i,1} & 0 & 0 \\ \left[\frac{\partial}{\partial\beta_1}W_{i,2}(\beta_1)\right]\phi_{i,2}+\frac{\partial}{\partial\beta_1}\phi_{i,2} & \frac{\partial}{\partial\beta_2}\phi_{i,2} & 0 \\ \left[\frac{\partial}{\partial\beta_1}W_{i,2}(\beta_1)\right]\psi_i+\frac{\partial}{\partial\beta_1}\psi_i & \left[\frac{\partial}{\partial\beta_2}W_{i,3}(\beta_2)\right]\psi_i+\frac{\partial}{\partial\beta_2}\psi_i & \frac{\partial}{\partial\theta}\psi_i \end{bmatrix}\Bigg|_{\hat\beta_{1:2},\hat\theta}$$

$$W_{i,t}(\beta_{t-1}) = \left(\frac{\pi_t(S_{i,t};\beta_{t-1})}{\pi_t(S_{i,t};\hat\beta_{t-1})}\right)^{A_{i,t}}\left(\frac{1-\pi_t(S_{i,t};\beta_{t-1})}{1-\pi_t(S_{i,t};\hat\beta_{t-1})}\right)^{1-A_{i,t}}$$

$$\frac{\partial}{\partial\beta_{t-1}}\pi_t(S_{i,t};\beta_{t-1})$$

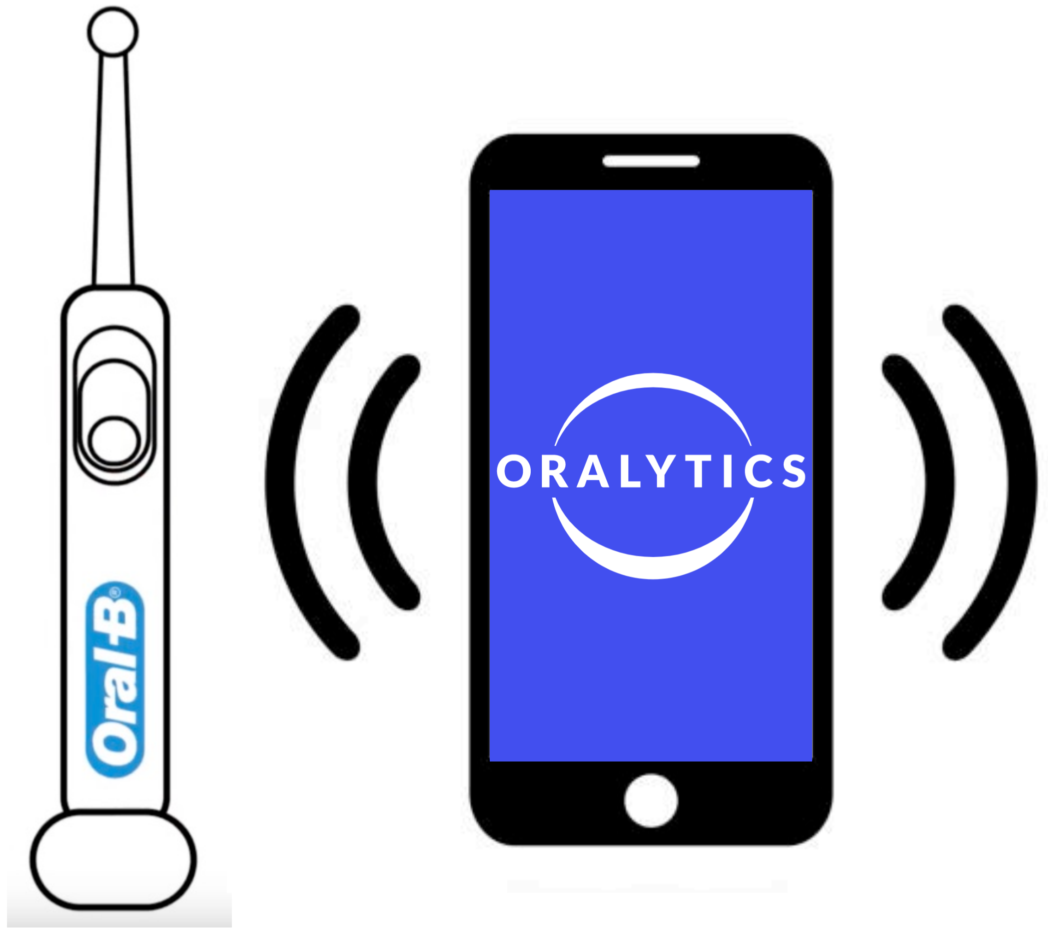
Considerations

1. **Quality of Variance Estimator**
    - Accuracy and Numerical Stability
2. **Computational Considerations**
    - How long to compute the standard errors from a run? (simulate thousands of runs)
3. **RL Algorithm Designer Experience**
    - Want to allow for flexibility in algorithm design
    - Want to make it easy and automatic for RL algorithm to provide necessary statistics
4. **Data Analyzer Experience**
    - Want to allow for a variety of analyses
    - Want to allow data analyzer to have little to no knowledge of RL algorithm to use

Approaches (go to following google doc):

https://docs.google.com/document/d/1NVAiaqv5fNhPUtkUMv1pPY55jO4jUDx0hdlKrJPy84Q/edit?usp=sharing

Backup Slides



UNIVERSITY OF MICHIGAN

UCLA School of Dentistry

Digital Oral Health Coaching

Challenge: Learning what interventions to deliver—and when

Minimize:
User Burden

Maximize:
User Benefit

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Minimize:
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Online Reinforcement Learning (RL)

Maximize:
User Benefit

Digital Intervention Study Design Objectives

Within-Study Personalization

Maximize User Benefit

- Send messages at opportune moments

Use Online RL Algorithms

$$\mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

After-Study Analyses

Digital Intervention Study Design Objectives

Within-Study Personalization

Maximize User Benefit

- Send messages at opportune moments

Use Online RL Algorithms

$$\mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

After-Study Analyses

Evaluate the Intervention

- Understand heterogeneity across user types and user states

Infer Treatment Effects

$$\mathbb{E}\left[R_t \mid S_t, A_t = 1\right] - \mathbb{E}\left[R_t \mid S_t, A_t = 0\right]$$

Digital Intervention Study Design Objectives

Within-Study Personalization

Maximize User Benefit

- Send messages at opportune moments

Use Online RL Algorithms

$$\mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

After-Study Analyses

Confidence Intervals Critical for

- Replicable science
- Publishing and sharing results

Infer Treatment Effects

$$\mathbb{E}\left[R_t \mid S_t, A_t = 1\right] - \mathbb{E}\left[R_t \mid S_t, A_t = 0\right]$$