# Statistical Theory and Modeling (ST2601)

## Discrete random variables

Mattias Villani

**Department of Statistics**
**Stockholm University**

mattiasvillani.com @matvil @matvil mattiasvillani

# Overview

- **Random variables recap**
- **Bernoulli**, **Geometric** and **Binomial** distributions
- **Negative binomial distribution**
- **Chebychev's inequality**

# Probabilities of events

- **Probabilities** for events $A$ and $B$ in a **sample space** $S$.

$$0 \leq \Pr(A) \leq 1$$

- **Complement rule**

$$\Pr(\underbrace{A^c}_{\text{not A}}) = 1 - \Pr(A)$$

- **Addition rule**

$$\Pr(\underbrace{A \cup B}_{\text{union}}) = \Pr(A) + \Pr(B) - \Pr(\underbrace{A \cap B}_{\text{intersection}})$$

- **Multiplication rule**

$$\Pr(A \cap B) = \underbrace{\Pr(A|B)}_{\text{conditional prob}} \cdot \Pr(B)$$

- Multiplication rule when $A$ and $B$ are **independent**

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(AB)$$

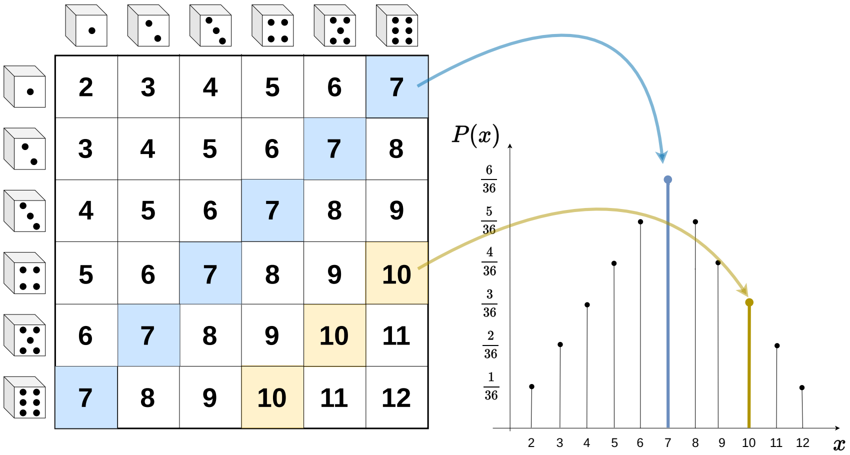
# Throwing two dice

| | ⚀ | ⚁ | ⚂ | ⚃ | ⚄ | ⚅ |
|---|---|---|---|---|---|---|
| ⚀ | 2 | 3 | 4 | 5 | 6 | 7 |
| ⚁ | 3 | 4 | 5 | 6 | 7 | 8 |
| ⚂ | 4 | 5 | 6 | 7 | 8 | 9 |
| ⚃ | 5 | 6 | 7 | 8 | 9 | 10 |
| ⚄ | 6 | 7 | 8 | 9 | 10 | 11 |
| ⚅ | 7 | 8 | 9 | 10 | 11 | 12 |

| | ⚀ | ⚁ | ⚂ | ⚃ | ⚄ | ⚅ |
|---|---|---|---|---|---|---|
| ⚀ | 2 | 3 | 4 | 5 | 6 | 7 |
| ⚁ | 3 | 4 | 5 | 6 | 7 | 8 |
| ⚂ | 4 | 5 | 6 | 7 | 8 | 9 |
| ⚃ | 5 | 6 | 7 | 8 | 9 | 10 |
| ⚄ | 6 | 7 | 8 | 9 | 10 | 11 |
| ⚅ | 7 | 8 | 9 | 10 | 11 | 12 |

| | ⚀ | ⚁ | ⚂ | ⚃ | ⚄ | ⚅ |
|---|---|---|---|---|---|---|
| ⚀ | 2 | 3 | 4 | 5 | 6 | 7 |
| ⚁ | 3 | 4 | 5 | 6 | 7 | 8 |
| ⚂ | 4 | 5 | 6 | 7 | 8 | 9 |
| ⚃ | 5 | 6 | 7 | 8 | 9 | 10 |
| ⚄ | 6 | 7 | 8 | 9 | 10 | 11 |
| ⚅ | 7 | 8 | 9 | 10 | 11 | 12 |

# Random variables and probability distributions

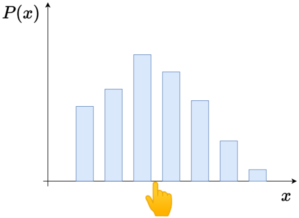


# Mean and variance

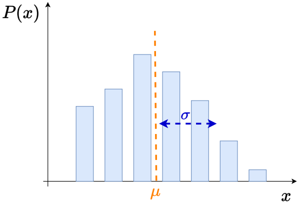
- **Discrete variable** with support $x \in \{x_1, x_2, \ldots, x_K\}$ and
$$p_k = \Pr(X = x_k)$$
- **Expected value** (**mean**) is the **center** of the distribution
$$\mathbb{E}(X) = \sum_{k=1}^{K} x_k \cdot p_k$$
- Alternative: **probability function** $p(x)$
$$\mathbb{E}(X) = \sum_{x} x \cdot p(x)$$
where the sum implicity is over all $x \in \{x_1, x_2, \ldots, x_K\}$.
- **Variance** measures the **spread** of the distribution
$$\sigma^2 = \mathbb{V}(X) = \mathbb{E}\left((X - \mu)^2\right) = \mathbb{E}(X^2) - \mu^2$$
- **Standard deviation** (same units as $X$)
$$\sigma = \mathbb{S}(X) = \sqrt{\mathbb{V}(X)}$$

# Mean and variance

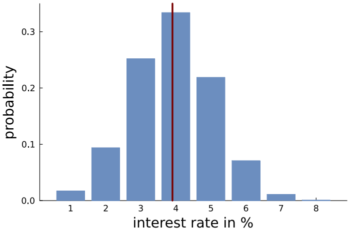
- The **mean** is where the probability distribution **balances**

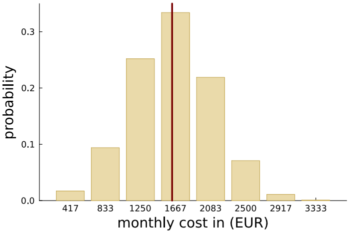



- The **standard deviation** measures the **spread around $\mu$**.



# Example: Taking a 500,000 Euro bank loan




- Mean interest rate

$$1 \cdot 0.017 + 2 \cdot 0.094 + \ldots + 8 \cdot 0.001 \approx 3.9\%$$

- **Mean monthly cost** for a $500000$ Euro loan:

  $\mathbb{E}(\text{cost}) = 417{\cdot}0.017{+}833{\cdot}0.094{+}\ldots{+}3333{\cdot}0.001 \approx 1626$ EUR

- **Variance monthly cost** (in Euro$^2$)

  $\mathbb{V}(\text{cost}) = (417{-}3252)^2{\cdot}0.017{+}\ldots{+}(3333{-}1626)^2{\cdot}0.001 \approx 241368$

- **Standard deviation monthly cost**

$$\mathbb{S}(\text{cost}) = \sqrt{241368} \approx 491 \text{ EUR}$$

# Law of the unconscious statistician

- Let $g(Y)$ be a **function of the random variable** $Y$.
- The function need **not** be one-to-one.
- Theorem 3.2 in the WMS book

$$\mathbb{E}\big(g(Y)\big) = \sum_{\text{all} y} g(y) \cdot p(y)$$

- This result allows us to compute the mean of the new random variable $g(Y)$ *without computing its probability distribution*.
- Unconscious, since we do it almost without thinking.

# Mean and variance of a linear transformation

**Mean and variance of a linear transformation**

**Shift with constant $c$**

$$\mathbb{E}(X + c) = \mathbb{E}(X) + c \qquad \mathbb{V}(X + c) = \mathbb{V}(X)$$

**Scaling with constant $a$**

$$\mathbb{E}(a \cdot X) = a \cdot \mathbb{E}(X) \qquad \mathbb{V}(a \cdot X) = a^2\mathbb{V}(X)$$

**Linear transformation**

$$\mathbb{E}(c + a \cdot X) = c + a \cdot \mathbb{E}(X) \qquad \mathbb{V}(c + a \cdot X) = a^2\mathbb{V}(X)$$

# Mean and variance of a sum

**Mean and variance of a sum of independent variables**

If $X$ and $Y$ are independent random variables, then

**Sum of two random variables**

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

$$\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(X)$$

**Linear transformation**

$$\mathbb{E}(a \cdot X + b \cdot Y) = a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y)$$

$$\mathbb{V}(a \cdot X + b \cdot Y) = a^2 \mathbb{V}(X) + b^2 \mathbb{V}(Y)$$

If $X_1, \ldots, X_n$ are independent random variables, then

**Sum of $n$ random variables**

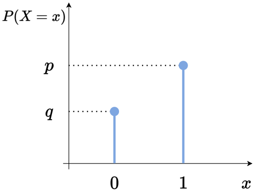
$$\mathbb{E}(X_1 + \ldots + X_n) = \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_n)$$

$$\mathbb{V}(X_1 + \ldots + X_n) = \mathbb{V}(X_1) + \ldots + \mathbb{V}(X_n)$$

# Bernoulli distribution

- Success/Failure. $X \in \{0, 1\}$
- $X \sim \text{Bernoulli}(p)$, where $p$ is success probability.
- **Probability function**

$$p(x) = \begin{cases} p & \text{for } x = 1 \\ q = 1 - p & \text{for } x = 0 \end{cases}$$




- **Mean** and **Variance**

$$\mathbb{E}(X) = p$$
$$\mathbb{V}(X) = pq$$

# Binomial distribution

- $X_1, X_2, \ldots, X_n \overset{\text{ind}}{\sim} \mathrm{Bernoulli}(p)$
- Then $Y = X_1 + X_2 + \ldots + X_n$ follows a binomial distribution

$$Y \sim \mathrm{Binomial}(n, p)$$

- **Mean** and **Variance**

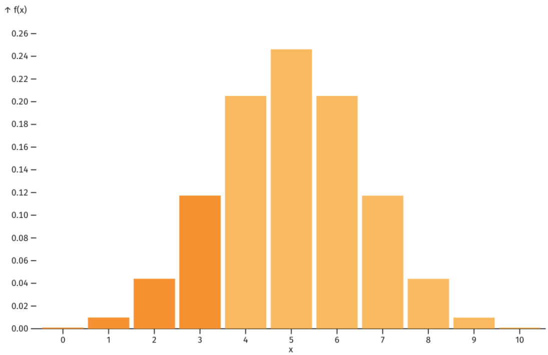
$$\mathbb{E}(X) = np$$
$$\mathbb{V}(X) = npq$$

- Proof: use that binomial $=$ sum of independent Bernoullis.
- **Probability function**

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

- Binomial does not care about the order, so $(0, 1, 1) = (1, 0, 1)$ etc. The **binomial coefficient** $\binom{n}{x}$ counts the number of ways we can order $x$ successes in $n$ trials.

# Binomial distribution - widget



# Geometric distribution - total number of trials

- Counts the number of Bernoulli trials **until first success**.
- $X \sim \mathrm{Geom}(p)$ where $X \in \{1, 2, \ldots\}$ and

$$p(x) = \Pr(\text{first success on trial } x) = \overbrace{\underbrace{q \cdot q \cdots q}_{x-1 \text{ failures}} \cdot \underbrace{p}_{\text{success}}}^{\text{multiply because indep}} = q^{x-1} \cdot p$$

- This definition is used in the course book.
- **Mean** and **Variance**

$$\mathbb{E}(X) = \frac{1}{p}$$
$$\mathbb{V}(X) = \frac{1-p}{p^2}$$

- Proof involves the geometric series $\sum_{k=1}^{\infty} q^k = \frac{q}{1-q}$.

# Geometric distribution - number of failures

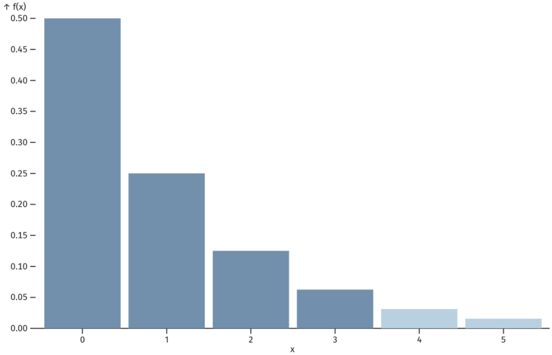
- This an **alternative definition** of the Geometric distribution.
- Counts the **number of failed** trials **before first success**.
- Wikipedia has both definitions side-by-side.
- R, my widget and the formula sheet uses the one on this slide.
- $X \sim \mathrm{Geom}(p)$ where $X \in \{0, 1, 2, \ldots\}$ and

$$p(x) = \Pr(x \text{ failures before first success}) = \overbrace{\underbrace{q \cdot q \cdots q}_{x \text{ failures}} \cdot \underbrace{p}_{\text{success}}}^{\text{multiply because indep}} = q^x \cdot p$$

- **Mean** and **Variance**

$$\mathbb{E}(X) = \frac{1-p}{p}$$
$$\mathbb{V}(X) = \frac{1-p}{p^2}$$

# Geometric distribution - widget



# Poisson distribution

- $X \sim \mathrm{Pois}(\lambda)$ where $X \in \{0, 1, 2, \ldots\}$

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

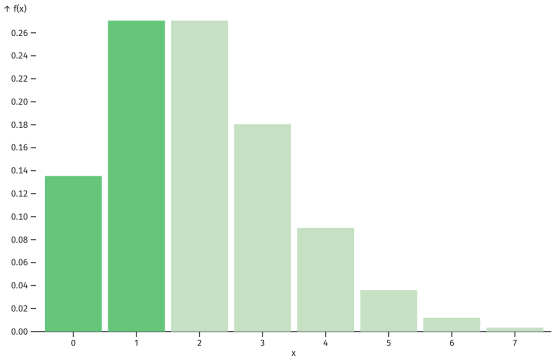
- Approximates $\mathrm{Bin}(n, p)$ distribution for large $n$ and small $p$.
- **Mean** and **Variance**

$$\mathbb{E}(X) = \lambda$$
$$\mathbb{V}(X) = \lambda$$

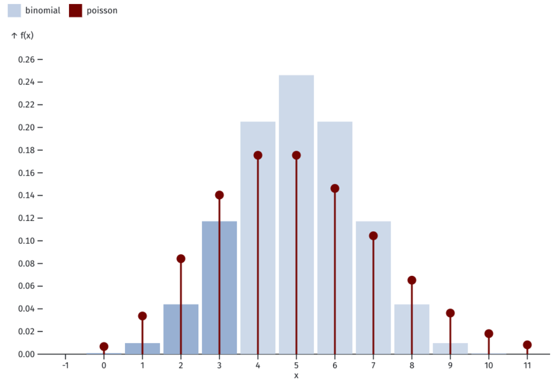
- **Mean = Variance**. Can be restrictive for real data.
- Proofs involve (see Taylor approximation in prequel if curious)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

# Poisson distribution - widget



# Poisson approximates Binomial - widget



# Negative binomial distribution

- $X$ = **total number of trials** until $r$ successes
- Total = failures + successes
- $X \sim \mathrm{NegBin}(r, p)$ where $X \in \{r, r+1, r+2, \ldots\}$

$$p(x) = \binom{x-1}{r-1} p^r q^{x-r}$$

- **Mean** and **Variance**

$$\mathbb{E}(X) = \frac{r}{p} \qquad \mathbb{V}(X) = \frac{r(1-p)}{p^2}$$

- Alternatively, count $X$ = **number of failures** before the $r$:th success. Then $X \in \{0, 1, 2, ...\}$ and

$$\mathbb{E}(X) = \frac{r(1-p)}{p} \qquad \mathbb{V}(X) = \frac{r(1-p)}{p^2}$$

This is used in R, see the help `?dnbinom`

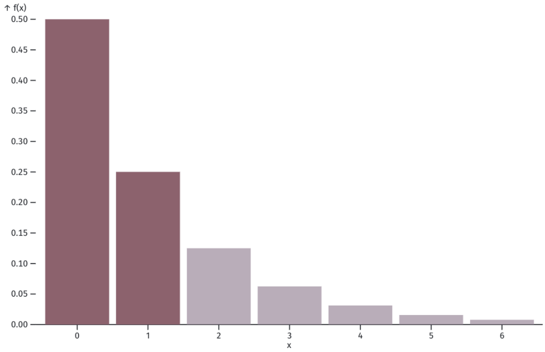
# Negative binomial - mean parameterization

- Parameters $p$ and $r$ come naturally from Bernoulli trials.
- When modeling data, more interpretable to use:
  - $X =$ **number of failures**, and
  - parameterization $\mathrm{NegBin}(r, \mu)$ with the **mean $\mu$ as an explicit parameter**.
- Set $p = \frac{r}{r+\mu}$. Then, $\mathbb{E}(X) = \mu$, so $\mu$ is really the mean.
- The variance is

$$\mathbb{V}(X) = \frac{r(1-p)}{p^2} = \frac{\mu}{p} = \frac{\mu}{\left(\frac{r}{r+\mu}\right)} = \mu\left(1 + \frac{\mu}{r}\right)$$

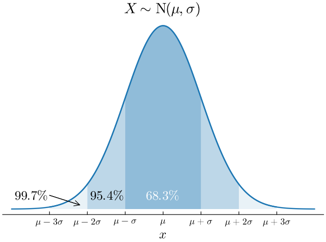
  so smaller $r$ gives larger variance.
- The parameter $r$ models **overdispersion** $\mathbb{V}(X) > \mathbb{E}(X)$.
  We can let $r$ be any positive real number, not just an integer.
- As $r \to \infty$, $\mathrm{NegBin}(r, \mu)$ becomes $\mathrm{Pois}(\mu)$.

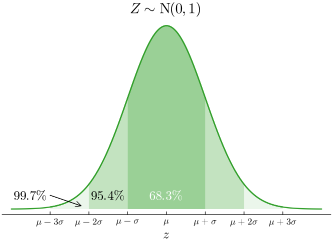
# Negative binomial distribution - widget



# Chebyshev's inequality

- Normal distribution **68-95-99.7% rule**




- Chebyshev: for **any** distribution with mean $\mu$ and variance $\sigma^2$

$$\Pr\big(|X-\mu| \geq k\sigma\big) \leq \frac{1}{k^2}$$

- Chebyshev's bound is usually not tight:
  - Normal: $\Pr\big(|X-\mu| \geq 2\sigma\big) \approx 0.0455$
  - Chebshev: $\Pr\big(|X-\mu| \geq 2\sigma\big) \leq \frac{1}{2^2} = 0.25$
- Useful for proofs, however.

# Chebyshev's inequality - widget

**True probability:** $\Pr\left(|Y-\mu| \geq 2.18\sigma\right) = 0.03949$

**Chebychev's bound:** $\Pr\left(|Y-\mu| \geq 2.18\sigma\right) \leq \frac{1}{2.18^2} = 0.21042$