

Statistical Theory and Modeling (ST2601)

Continuous random variables

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Overview

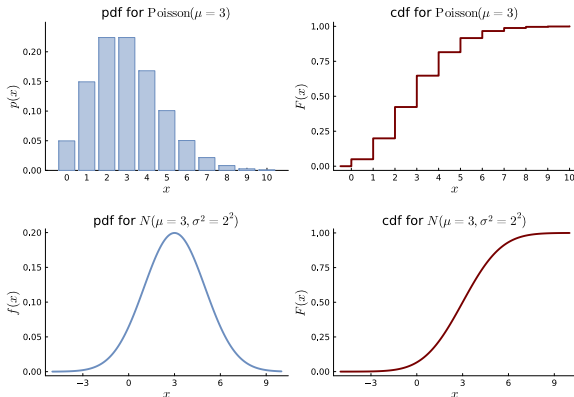
- Continuous random variables
- Exponential distribution
- Gamma distribution
- Chi2 distribution
- Beta distribution

Cumulative distribution function

- **Cumulative distribution function (cdf)** for a random variable X is

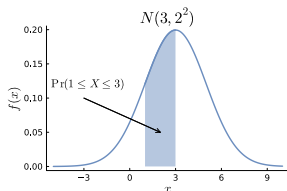
$$F(x) = \Pr(X \leq x) \quad \text{for } -\infty < x < \infty$$

- Applies to both discrete and continuous random variables.
- The p-functions in R, for example `ppois(4, lambda = 2)`



Probability density function

- The outcome of a **continuous random variable** can be any real number, but $\Pr(X = x) = 0$ for all x ! 🤖
- A **probability density function (pdf)** for random variable X satisfies
 - ▶ $f(x) \geq 0$ for all x , $-\infty < x < \infty$
 - ▶ $\int_{-\infty}^{\infty} f(x)dx = 1$
 - ▶ $\Pr(a \leq X \leq b) = \int_a^b f(x)dx$
- The d-functions in R. `dnorm(-1, mu = 2, sd = 1)`.



Probability density function

- The pdf is the derivative of the cdf:

$$f(x) = \frac{d}{dx} F(x)$$

- The cdf is the integral of the pdf:

$$F(x) = \int_{-\infty}^x f(t) dt$$

- Fundamental theorem of calculus applied to pdf-cdf pair:

$$\Pr(a \leq X \leq b) = \int_a^b f(x) dx = \underbrace{F(b)}_{\Pr(X \leq b)} - \underbrace{F(a)}_{\Pr(X \leq a)}$$

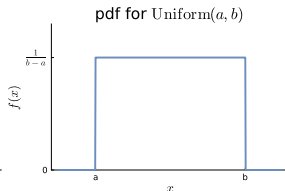
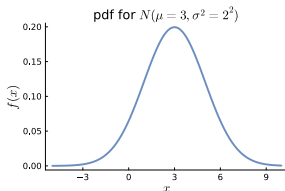
Examples of pdfs

- The pdf of a **Normal** variable $X \sim N(\mu, \sigma^2)$ variable

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right)$$

- The pdf of a $X \sim \text{Uniform}(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



Example pdf and cdf

- Let X be a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Probability by integrating the pdf

$$\Pr(X \leq 0.5) = \int_{-\infty}^{0.5} f(x) dx = \int_0^{0.5} 3x^2 dx = [x^3]_0^{0.5} = 0.5^3 = 0.125$$

- The cdf is

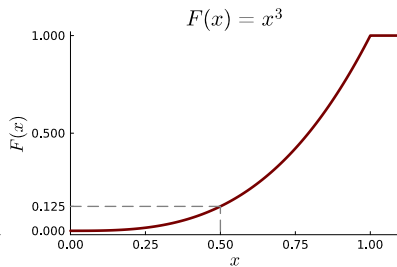
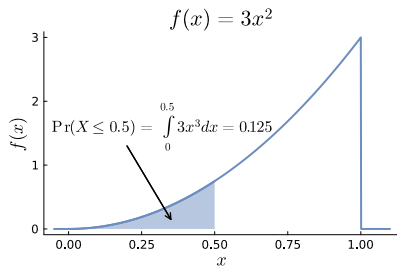
$$F(x) = \int_{-\infty}^x f(t) dt = \int_0^x 3t^2 dt = [t^3]_0^x = x^3$$

(note: the anti-derivative is $x^3 + C$ for some C , but here the condition $F(\infty) = F(1) = 1$ implies that $C = 0$).

- Check:

$$\frac{d}{dx} F(x) = \frac{d}{dx} x^3 = 3x^2 = f(x) \quad \text{OK!}$$

Example pdf and cdf



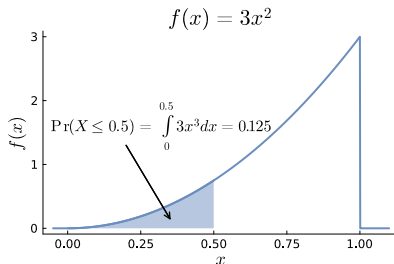
Expected value for continuous random variables

Expected value

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Example: $f(x) = 3x^2$

$$\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot 3x^2 dx = \int_0^1 3x^3 dx = \left[\frac{3}{4} x^4 \right]_0^1 = \frac{3}{4}$$



Variance for continuous random variables

■ Variance

$$\mathbb{V}(X) = \mathbb{E}((X - \mu)^2) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx$$

or

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mu^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot 3x^2 dx = \int_{-\infty}^{\infty} 3x^4 dx = \left[\frac{3}{5} x^5 \right]_0^1 = \frac{3}{5}$$

So

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mu^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.0375$$

■ The standard deviation is

$$\mathbb{S}(X) = \sqrt{\mathbb{V}(X)} = \sqrt{0.0375} \approx 0.194.$$

Exponential distribution

- $X \sim \text{Expon}(\beta)$ with support $X \in [0, \infty)$.
- Parameters: $\beta > 0$.
- Data as life time, duration etc. Memoryless.
- Probability density function

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \quad \text{for } x \geq 0$$

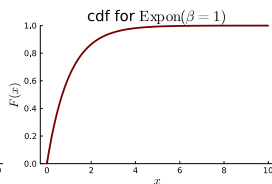
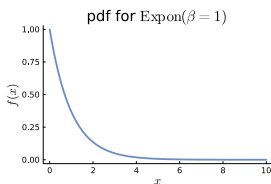
- Cumulative distribution function

$$F(x) = \int_0^x \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = \left[-e^{-\frac{t}{\beta}} \right]_0^x = -e^{-\frac{x}{\beta}} - (-1) = 1 - e^{-\frac{x}{\beta}}$$

- **Mean** and **variance**

$$\mathbb{E}(X) = \beta$$

$$\mathbb{V}(X) = \beta^2$$



Gamma distribution

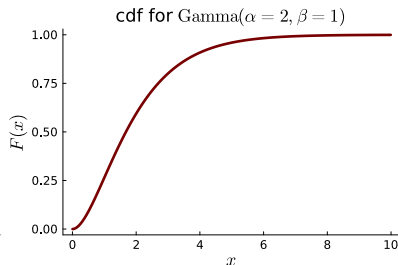
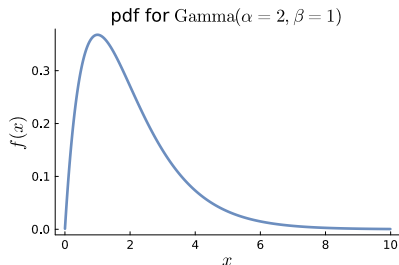
- $X \sim \text{Gamma}(\alpha, \beta)$ with support $X \in [0, \infty)$.
- Parameters: $\alpha > 0$, $\beta > 0$.
- Probability density function (**scale** parameterization)

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad \text{for } x \geq 0$$

- $\text{Expon}(\beta)$ is the special case $\text{Gamma}(\alpha = 1, \beta)$.
- **Mean** and **variance**

$$\mathbb{E}(X) = \alpha\beta$$

$$\mathbb{V}(X) = \alpha\beta^2$$



The Gamma function

■ Gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

■ Fundamental property

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

■ When x is an integer

$$\Gamma(x) = (x-1)!$$

■ `gamma(x)` in R.

Chi-squared distribution

- The special case $\text{Gamma}(\alpha = \nu/2, \beta = 2)$ is the **Chi-squared (χ^2) distribution** with ν degrees of freedom.
- We write $X \sim \chi_\nu^2$.
- Important distribution only because of

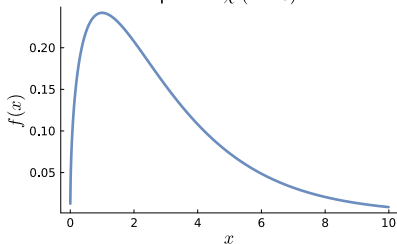
$$X_1, \dots, X_n \stackrel{\text{iid}}{\sim} N(0, 1) \quad \text{then} \quad \sum_{i=1}^n X_i^2 \sim \chi_n^2$$

- **Mean** and **variance**

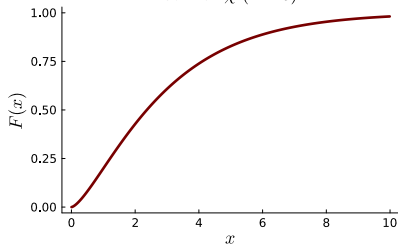
$$\mathbb{E}(X) = \nu$$

$$\mathbb{V}(X) = 2\nu$$

pdf for $\chi^2(\nu = 3)$



cdf for $\chi^2(\nu = 3)$



Beta distribution

■ $X \sim \text{Beta}(\alpha, \beta)$ with support $X \in (0, 1)$.

■ Parameters: $\alpha > 0$, $\beta > 0$.

■ Data as **proportions**:

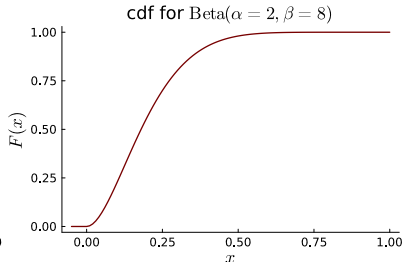
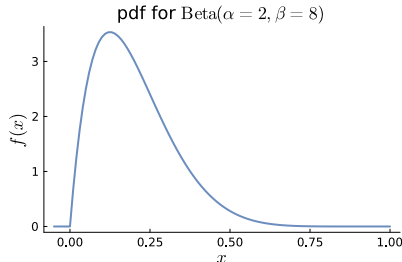
▶ $X = \frac{\text{firm own capital}}{\text{firm total capital}}$

▶ $X = \text{\%bleached coral}$.

■ Probability density function

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad \text{for } 0 < x < 1$$

■ $B(\alpha, \beta)$ is the **Beta function**. `beta(a, b)` in R.



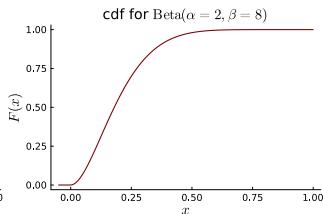
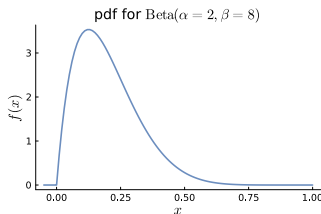
Beta distribution

- $X \sim \text{Beta}(\alpha, \beta)$ with support $X \in (0, 1)$.
- Parameters: $\alpha > 0$, $\beta > 0$.
- Data as **proportions**: $X = \frac{\text{own capital}}{\text{total capital}}$ or $X = \%\text{bleached coral}$.
- Probability density function

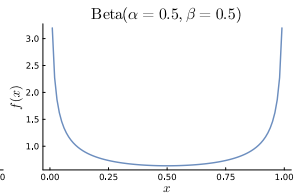
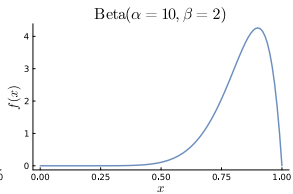
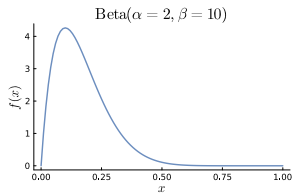
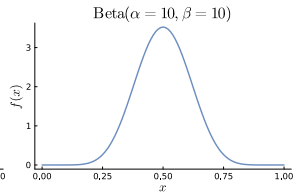
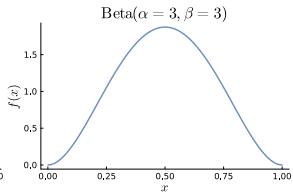
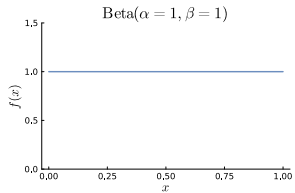
$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad \text{for } 0 < x < 1$$

where $B(\alpha, \beta)$ is the **Beta function** (beta(a, b) in R):

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

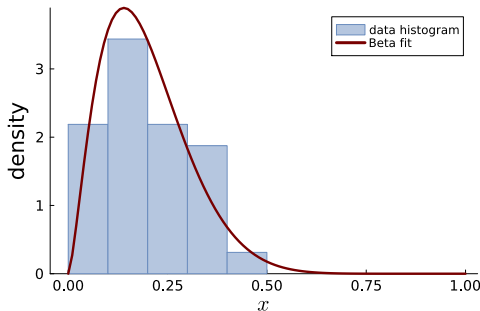


Beta distribution



Inference - parameter estimation

- Probability distributions have parameters:
 - ▶ Exponential β
 - ▶ Normal μ, σ^2
 - ▶ Beta α, β
- We learn (**estimate**) such parameters from data.
- Example: X = proportion of crude oil converted to gasoline.
- **Fitting** a $\text{Beta}(\alpha, \beta)$ distribution using **maximum likelihood**.
- Estimates: $\hat{\alpha} = 2.504$ and $\hat{\beta} = 10.233$.



Beta distribution

■ $X \sim \text{Beta}(\alpha, \beta)$ then

$$\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}$$

$$\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$