# Statistical Theory and Modeling (ST2601) Continuous random variables

#### Mattias Villani

Department of Statistics Stockholm University











### **Overview**

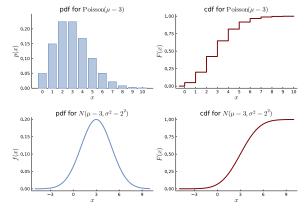
- **Continuous random variables**
- **Exponential distribution**
- Gamma distribution
- Chi2 distribution
- Beta distribution

### **Cumulative distribution function**

Cumulative distribution function (cdf) for a random variable X is

$$F(x) = \Pr(X \le x)$$
 for  $-\infty < x < \infty$ 

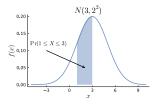
- Applies to both discrete and continuous random variables.
- The p-functions in R, for example ppois(4, lambda = 2)



# **Probability density function**

- The outcome of a **continuous random variable** can be any real number, but Pr(X = x) = 0 for all x!
- A probability density function (pdf) for random variable X satisfies
  - ▶  $f(x) \ge 0$  for all x,  $-\infty < x < \infty$

  - $\Pr(a \le X \le b) = \int_a^b f(x) dx$
- The d-functions in R. dnorm(-1, mu = 2, sd = 1).



# **Probability density function**

The pdf is the derivative of the cdf:

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)$$

■ The cdf is the integral of the pdf:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Fundamental theorem of calculus applied to pdf-cdf pair:

$$\Pr(a \le X \le b) = \int_a^b f(x) dx = \underbrace{F(b)}_{\Pr(X \le b)} - \underbrace{F(a)}_{\Pr(X \le a)}$$

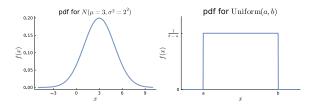
# **Examples of pdfs**

lacksquare The pdf of a Normal variable  $extit{X} \sim extit{N}(\mu, \sigma^2)$  variable

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

■ The pdf of a  $X \sim \text{Uniform}(a, b)$ 

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b\\ 0 & \text{otherwise} \end{cases}$$



# Example pdf and cdf

Let X be a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1\\ 0 & \text{otherwise} \end{cases}$$

■ Probability by integrating the pdf

$$\Pr(X \le 0.5) = \int_{-\infty}^{0.5} f(x) dx = \int_{0}^{0.5} 3x^{2} dx = \left[x^{3}\right]_{0}^{0.5} = 0.5^{3} = 0.125$$

The cdf is

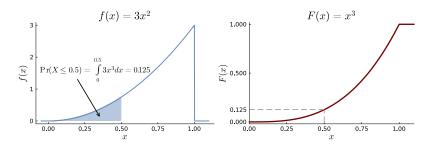
$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 3t^{2}dt = \left[t^{3}\right]_{0}^{x} = x^{3}$$

(note: the anti-derivative is  $x^3 + C$  for some C, but here the condition  $F(\infty) = F(1) = 1$  implies that C = 0).

Check:

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2 = f(x) \qquad \text{OK!}$$

# **Example pdf and cdf**



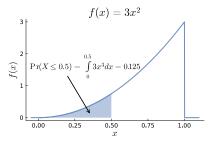
# **Expected value for continuous random variables**

Expected value

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Example:  $f(x) = 3x^2$ 

$$\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot 3x^2 dx = \int_{0}^{1} 3x^3 dx = \left[\frac{3}{4}x^4\right]_{0}^{1} = \frac{3}{4}$$



### Variance for continuous random variables

#### Variance

$$\mathbb{V}(X) = \mathbb{E}\left((X - \mu)^2\right) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx$$

or

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mu^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot 3x^2 dx = \int_{-\infty}^{\infty} 3x^4 dx = \left[\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}$$

So

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mu^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.0375$$

The standard deviation is

$$S(X) = \sqrt{V(X)} = \sqrt{0.0375} \approx 0.194.$$

# **Exponential distribution**

- $X \sim \operatorname{Expon}(\beta)$  with support  $X \in [0, \infty)$ .
- Parameters:  $\beta > 0$ .
- Data as life time, duration etc. Memoryless.
- Probability density function

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \quad \text{for } x \ge 0$$

Cumulative distribution function

$$F(x) = \int_0^x \frac{1}{\beta} e^{-\frac{t}{\beta}} dt = \left[ -e^{-\frac{t}{\beta}} \right]_0^x = -e^{-\frac{x}{\beta}} - (-1) = 1 - e^{-\frac{x}{\beta}}$$

Mean and variance

$$\mathbb{E}(X) = \beta \qquad \mathbb{V}(X) = \beta^2$$

$$\text{pdf for } \operatorname{Expon}(\beta = 1)$$

$$\frac{\mathbb{E}(X)}{0.50}$$

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### **Gamma distribution**

- $X \sim \operatorname{Gamma}(\alpha, \beta)$  with support  $X \in [0, \infty)$ .
- Parameters:  $\alpha > 0$ ,  $\beta > 0$ .
- Probability density function (scale parameteriztion)

$$f(x) = \frac{1}{\beta^{\alpha} \Gamma(\alpha)} x^{\alpha - 1} e^{-x/\beta}$$
 for  $x \ge 0$ 

- Expon( $\beta$ ) is the special case Gamma( $\alpha = 1, \beta$ ).
- Mean and variance

$$\mathbb{E}(X) = \alpha\beta \qquad \mathbb{V}(X) = \alpha\beta^2$$
 pdf for  $\mathrm{Gamma}(\alpha=2,\beta=1)$  cdf for  $\mathrm{Gamma}(\alpha=2,\beta=1)$  o.75 cdf for  $\mathrm{Gamma}(\alpha=2,\beta=1)$  o.25 cdf for  $\mathrm{Gamma}(\alpha=2,\beta=1)$  o.25 cdf for  $\mathrm{Gamma}(\alpha=2,\beta=1)$ 

### The Gamma function

Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Fundamental property

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

 $\blacksquare$  When x is an integer

$$\Gamma(x) = (x - 1)!$$

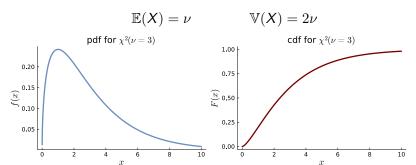
gamma(x) in R.

# **Chi-squared distribution**

- The special case  $Gamma(\alpha = \nu/2, \beta = 2)$  is the **Chi-squared**  $(\chi^2)$  distribution with  $\nu$  degrees of freedom.
- We write  $X \sim \chi^2_{\nu}$ .
- Important distribution only because of

$$X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} N(0, 1)$$
 then  $\sum_{i=1}^n X_i^2 \sim \chi_n^2$ 

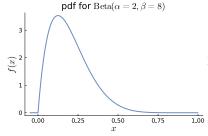
Mean and variance

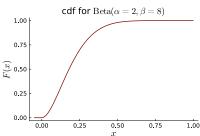


- $X \sim \text{Beta}(\alpha, \beta)$  with support  $X \in (0, 1)$ .
- Parameters:  $\alpha > 0$ ,  $\beta > 0$ .
- Data as proportions:
  - $X = \frac{\text{firm own capital}}{\text{firm total capital}}$
  - $\rightarrow$  X = %bleached coral.
- Probability density function

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha - 1} (1 - x)^{\beta - 1}$$
 for  $0 < x < 1$ 

 $\blacksquare$  B( $\alpha, \beta$ ) is the **Beta function**. beta(a, b) in R.



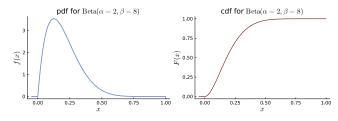


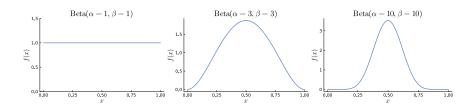
- $X \sim \text{Beta}(\alpha, \beta)$  with support  $X \in (0, 1)$ .
- Parameters:  $\alpha > 0$ ,  $\beta > 0$ .
- Data as **proportions**:  $X = \frac{\text{own capital}}{\text{total capital}}$  or X = %bleached coral.
- Probability density function

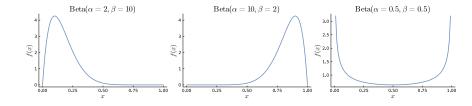
$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha - 1} (1 - x)^{\beta - 1}$$
 for  $0 < x < 1$ 

where  $B(\alpha, \beta)$  is the **Beta function** (beta(a, b) in R):

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$







 $\blacksquare X \sim \operatorname{Beta}(\alpha, \beta)$  then

$$\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}$$

$$\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

- When  $\alpha = \beta$ , the Beta distribution is symmetric around the mean.
- Larger  $\alpha$  and  $\beta$  gives a more concentrated distribution (smaller variance).

## Inference - parameter estimation

- Probability distributions have parameters:
  - ightharpoonup Exponential  $\beta$
  - ▶ Normal  $\mu$ ,  $\sigma^2$
  - $\blacktriangleright$  Beta  $\alpha$ ,  $\beta$
- We learn (estimate) such parameters from data.
- **Example:** X = proportion of crude oil converted to gasoline.
- **Fitting** a  $Beta(\alpha, \beta)$  distribution using maximum likelihood.
- Estimates:  $\hat{\alpha} = 2.504$  and  $\hat{\beta} = 10.233$ .

