The Futurama Theorem.

A friendly introduction to permutations.

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The Royal Institution
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Mathematics and Statistics
LANCASTER UNIVERSITY

Permutations

In this class we are going to consider the theory of permutations, and use them to solve a problem posed in an episode of Futurama.

What is a permutation?

A permutation of a set X is a rearrangement of its elements.

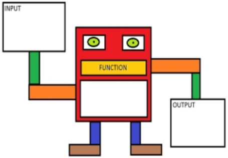
Example

Let X $=\{$Jack, Queen, King$\}$. Then there are six permutations:

Jack Queen King,	Queen King Jack,	King Jack Queen,
Queen Jack King,	King Queen Jack,	Jack King Queen.

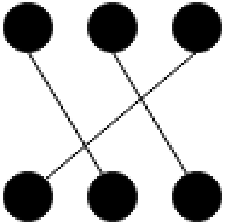
Another example

- Let the set be $X = \{1, 2, 3\}$.
- Define α be a permutation that takes $\alpha(1) \to 2, \alpha(2) \to 3, \alpha(3) \to 1$.
- We can think of α as a function machine.



Permutation Diagrams

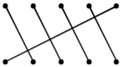
We can write this permutation down in a permutation diagram as shown.

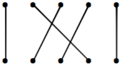


$\alpha(1) \to 2, \qquad \alpha(2) \to 3, \qquad \alpha(3) \to 1.$

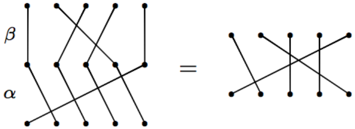
Lets try multiplying

If we write $\alpha \circ \beta$, this means apply the permutation β to our set, and then apply the permutation α to our set. We're using the convention working from right to left which might look a bit strange but you'll get used to it.
Example

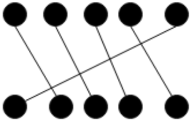
$\alpha =$ 

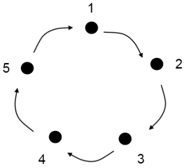
$\beta =$ 

$\alpha \circ \beta =$



Cycle Notation

Cycle notation allows us to write down permutation diagrams in a more efficient way. Lets think about this permutation $\alpha =$ 
We could write this in a cycle,



or in short hand $\alpha = (1\ 2\ 3\ 4\ 5)$.

Composition of cycles

We can write down compositions of cycles without the $\circ$ symbol (again mathematicians are lazy!)
So

$$(1\ 2\ 3) \circ (4\ 5) = (1\ 2\ 3)(4\ 5)$$

Example

$$(2\ 3\ 5)(1\ 5\ 4) = (1\ 2\ 3\ 5\ 4)$$

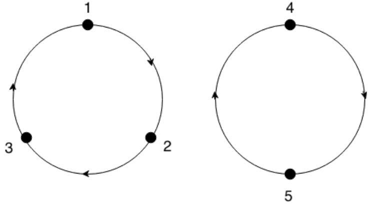
Reminder: We work from right to left.

Disjoint cycles

- Two cycles are said to be *disjoint* if they act on disjoint sets of symbols.
- In the examples on the previous slide the cycles $(1\ 2\ 3)$ & $(4\ 5)$ are disjoint, while the cycles $(1\ 5\ 4)$ & $(2\ 3\ 5)$ are not.
- Note that $(1\ 2\ 3)(4\ 5) = (4\ 5)(1\ 2\ 3)$. These cycles commute.
- This makes sense if we look at a diagram.

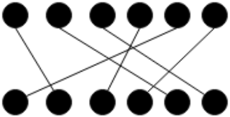
Disjoint cycles

Since the cycles (1 2 3) and (4 5) are disjoint, they act in a sense independently of one another so it doesn't matter which one you consider to be taking first.



It is very useful to be able to express a permutation as a product of disjoint cycles, because then its structure is immediately clear.

Disjoint cycles: Example

We write the permutation $\alpha =$ 
as a product of disjoint cycles.
Start with any number, say 1. Notice that

$$\alpha : 1 \mapsto 2, \quad 2 \mapsto 5, \quad 5 \mapsto 1.$$

Thus (1 2 5) is one of the cycles of which α is composed.
Next take any of the remaining numbers, say 3. Then

$$\alpha : 3 \mapsto 6, \quad 6 \mapsto 4, \quad 4 \mapsto 3.$$

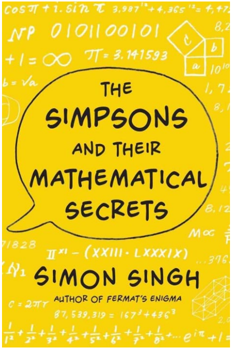
Hence $\alpha = (1\ 2\ 5)(3\ 6\ 4) = (3\ 6\ 4)(1\ 2\ 5)$.

Transpositions

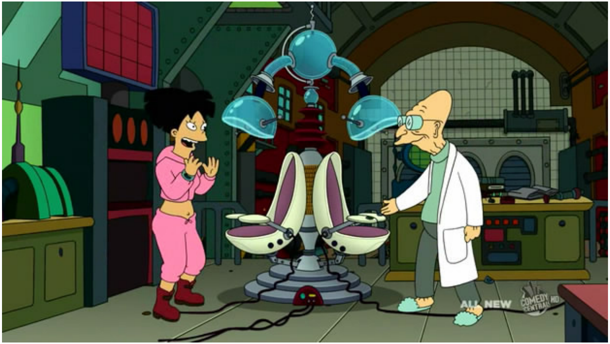
- A transposition is simply a permutation that only switches 2 elements of the set, and everything else stays the same.
- One example of a transposition is $(3\ 4)$.

Futurama

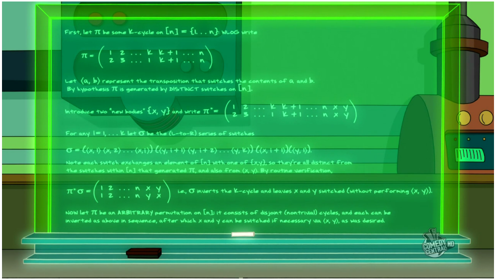

MATT GROENING


THE SIMPSONS AND THEIR MATHEMATICAL SECRETS
SIMON SINGH
AUTHOR OF FERMAT'S ENIGMA

The Prisoner of Bender (6x10)



Ken Keeler


First, let π be some k-cycle on $[n] = \{1 \dots n\}$: WLOG write
$\pi = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n \\ 2 & 3 & \dots & 1 & k+1 & \dots & n \end{pmatrix}$
Let $\langle a, b \rangle$ represent the transposition that switches the contents of a and b.
By hypothesis π is generated by DISTINCT switches on $[n]$.
Introduce two "new bodies" $\{x, y\}$ and write $\pi^* = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n & x & y \\ 2 & 3 & \dots & 1 & k+1 & \dots & n & x & y \end{pmatrix}$
For any $i = 1, \dots k$ let σ be the (L-to-R) series of switches
$\sigma = (\langle x, 1 \rangle \langle x, 2 \rangle \dots \langle x, i \rangle)(\langle y, i+1 \rangle \langle y, i+2 \rangle \dots \langle y, k \rangle)(\langle x, i+1 \rangle)(\langle y, 1 \rangle)$.
Note each switch exchanges an element of $[n]$ with one of $\{x, y\}$, so they're all distinct from
the switches within $[n]$ that generated π, and also from $\langle x, y \rangle$. By routine verification,
$\pi^* \sigma = \begin{pmatrix} 1 & 2 & \dots & n & x & y \\ 1 & 2 & \dots & n & y & x \end{pmatrix}$ i.e., σ inverts the k-cycle and leaves x and y switched (without performing $\langle x, y \rangle$).
NOW let π be an ARBITRARY permutation on $[n]$: it consists of disjoint (nontrivial) cycles, and each can be
inverted as above in sequence, after which x and y can be switched if necessary via $\langle x, y \rangle$, as was desired.
COMEDY CENTRAL HD

Mind swaps

- Amy $\leftrightarrow$ Professor
- Amy $\leftrightarrow$ Bender
- Leela $\leftrightarrow$ Professor
- Amy $\leftrightarrow$ Wash Bucket
- Fry $\leftrightarrow$ Zoidberg
- Emperor Nikolai $\leftrightarrow$ Wash Bucket
- Hermes $\leftrightarrow$ Leela

We can write this as a product of transpositions, working right to left.
(h l)(e w)(f z)(a w)(l p)(a b)(a p)
Task: Write this down in disjoint cycles.

Challenge.

- Apply the same swaps that happen in the episode.
- Try to swap the bodies around to get everyone back to the right place.
- Make sure someone in your group keeps track of the swaps
- Remember no pair of bodies can swap more than once.
- Write down the bodies that swap.

Fixing a 7-cycle

- Consider just the 7-cycle $(1\ 2\ 3\ 4\ 5\ 6\ 7)$.
- Introduce two new bodies that have not had their minds swapped, say x and y.
- To return all minds of back to the right bodies, apply the following sequence of transpositions:
- $(x\ 7)(y\ 1)(y\ 2)(y\ 3)(y\ 4)(y\ 5)(y\ 6)(y\ 7)(x\ 1)$.

Does this work?

- $(x\ 7)(y\ 1)(y\ 2)(y\ 3)(y\ 4)(y\ 5)(y\ 6)(y\ 7)(x\ 1)(1\ 2\ 3\ 4\ 5\ 6\ 7) = ?$
- Have we swapped the same two bodies more than once?

What if the cycle is really long?

- Consider the k-cycle $(1\ 2\ \dots\ k)$.
- Introduce two new bodies that have not had their minds swapped, say x and y.
- Apply the following transpositions:
- $(x\ k)(y\ 1)(y\ 2)\dots(y\ k-1)(y\ k)(x\ 1)$.

Why does it work ? $(x\ k)(y\ 1)(y\ 2)\ldots(y\ k-1)(y\ k)(x\ 1)$.

- First step is to apply $(y\ k)(x\ 1)$.
- $(y\ k)(x\ 1)(1\ 2\ \ldots\ k) = (1\ 2\ \ldots\ k-1\ y\ k\ x)$.
- Then $(y\ k-1)$ puts the mind of $k-1$ back where it belongs.
- And then $(y\ k-2)$ puts the mind of $k-2$ back where it belongs.
- Continue this until $(y\ 1)$ puts the mind of 1 back where it belongs.
- And finally, we swap x and k.
- When we finish x and y are still muddled but they have never swapped with each other!

Fixing products of disjoint cycles

- What do we do when we have multiple cycles to start with?
- Use x and y to fix each individual cycle.

Let's have a go

1. Choose two people to be the "X" and "Y" and label them.
2. Everyone else labels themselves and shuffles their brains.
3. Everyone (except X and Y) stand so the **person on your left holds your brain**.
4. Make your (disjoint) cycles into long lines.
5. Fix each cycle, one at a time.
   - X swaps with person with no-one on their **right hand side**.
   - Y swaps with person with no-one on their **left hand side**.
   - Y continues to swap with everybody people down the line (except X).
   - Swap the mind in X's body back where it belongs, into the body at the back of the line.
6. Once all cycles are fixed, swap the two helpers (if necessary).

Questions for the road.

- In which situations do we need to need to swap X and Y at the end?
- What is the minimum number of switches required?

More info at `www.lancs.ac.uk/~daviesr3/rimasterclass`