

Ri Masterclass: A friendly introduction to permutations.

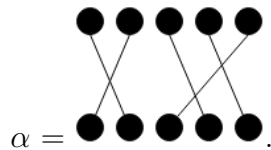
Solutions to Exercises

Question 1. Let α be defined as a permutation such that:

$$\begin{aligned}\alpha(1) &\mapsto 2, \\ \alpha(2) &\mapsto 1, \\ \alpha(3) &\mapsto 4, \\ \alpha(4) &\mapsto 5, \\ \alpha(5) &\mapsto 3.\end{aligned}$$

Draw the permutation α as a permutation diagram.

Solution:



Question 2. Let

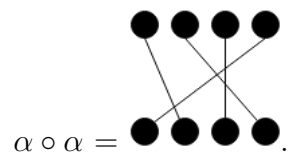
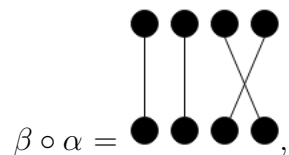
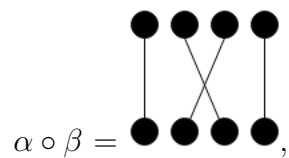


Find:

- (a) $\alpha \circ \beta$,
- (b) $\beta \circ \alpha$,
- (c) α^2 .

Reminder: Functions are multiplied from right to left.

Solution:



Question 3. Write down all possible permutation diagrams for the set $\{1\ 2\ 3\}$.

Solution:

There are six different permutations of the set $\{1\ 2\ 3\}$.

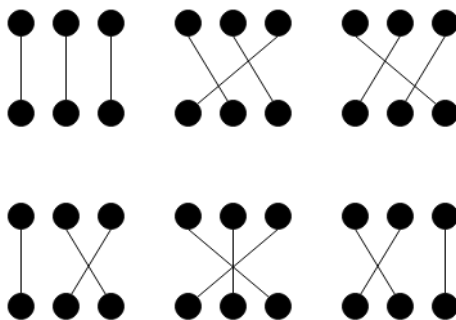


Figure 1: Permutations of the set $\{1\ 2\ 3\}$.

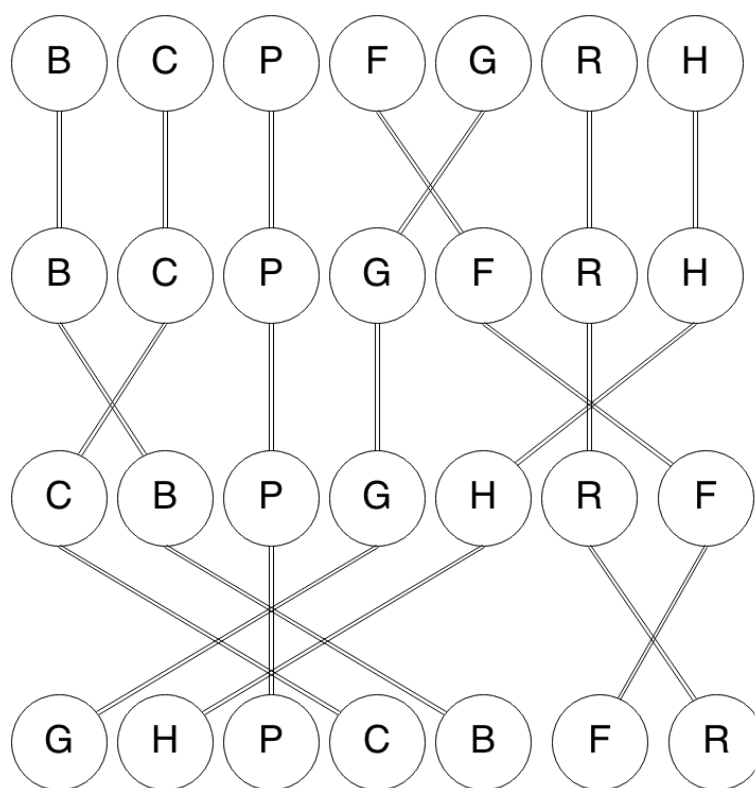
Question 4. Every year, Molly Weasley knits Christmas jumpers for her children: Bill, Charlie, Percy, Fred, George, Ron and family friend Harry Potter (excluding Ginny for simplicity). Each jumper is labelled with the first letter of the owners name.



- The children put on the correct jumpers in the morning.
 - At breakfast, Fred and George swap jumpers.
 - At lunch, Charlie swaps jumpers with Bill and Harry swaps jumpers with George.
 - At teatime, Bill swaps jumpers with Fred, Ron swaps jumpers with Harry and Charlie swaps jumpers with George.
- (a) Express these jumper swaps as a permutation diagram.
- (b) Write down the resulting permutation (who is wearing which jumper at the end of the day?)

Solution:

Bill is wearing George's jumper, Charlie is wearing Harry's jumper, Percy is wearing his own jumper, Fred is wearing Charlie's jumper, George is wearing Bill's jumper, Ron is wearing Fred's jumper and Harry is wearing Ron's jumper.



Question 5. A number of characters from “The Simpsons” decide to take part in a Secret Santa. Each character has to buy one gift for another character. Draw the connections in a circle and then write this permutation as a product of disjoint cycles.

Participant	Buys for
Homer	Bart
Marge	Mr Burns
Bart	Apu
Lisa	Homer
Maggie	Groundskeeper Willie
Mr Burns	Smithers
Smithers	Krusty the Clown
Ned	Marge
Apu	Lisa
Moe	Maggie
Krusty the Clown	Moe
Groundskeeper Willie	Ned

Solution:

If we number the characters so Homer = 1, Marge = 2, . . . , Groundskeeper Willie = 12, we can express the permutation as $\alpha = (1\ 3\ 9\ 4)(2\ 6\ 7\ 11\ 10\ 5\ 12\ 8)$.

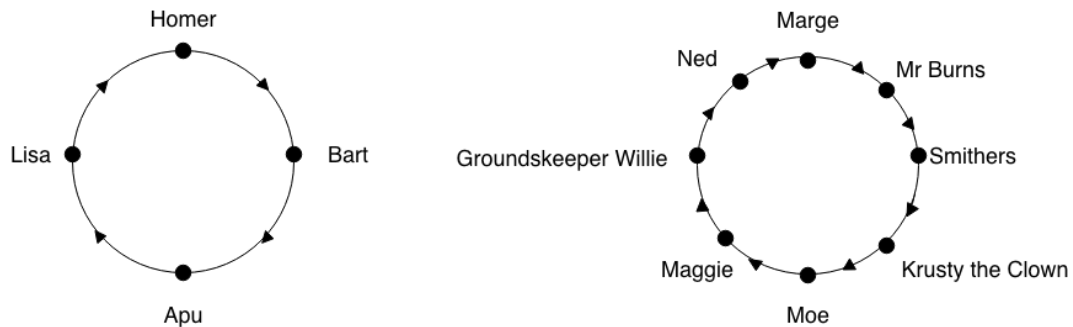
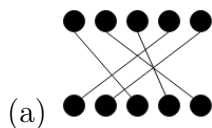


Figure 2: Disjoint cycles for the Simpson’s Secret Santa. The symbol \rightarrow means “Buys a gift for”.

Question 6. Express each of the following permutations as a single cycle or as the product of disjoint cycles.



- (b) $\alpha(1) \mapsto 5, \alpha(2) \mapsto 8, \alpha(3) \mapsto 7, \alpha(4) \mapsto 1, \alpha(5) \mapsto 3, \alpha(6) \mapsto 6, \alpha(7) \mapsto 2, \alpha(8) \mapsto 4.$
- (c) $(1\ 2\ 5)(2\ 3\ 6)$
- (d) $(3\ 5\ 6)(1\ 6)(2\ 3\ 4)$

Reminder: Apply cycles from right to left.

Solution:

- (a) $(1\ 3\ 4)(2\ 5)$
- (b) $(1\ 5\ 3\ 7\ 2\ 8\ 4)$
- (c) $(1\ 2\ 3\ 6\ 5)$
- (d) $(1\ 3\ 4\ 2\ 5\ 6)$

Question 7. Express $\alpha = (1\ 3\ 4)(2\ 5)$ as a product of transpositions. Can we always represent a cycle as a product of transpositions? Can you prove your answer?

Solution:

$$\alpha = (1\ 3)(3\ 4)(2\ 5).$$

Proof.

$$(a_1\ a_2\ \dots\ a_k) = (a_1\ a_k)(a_1\ a_{k-1})\dots(a_1\ a_3)(a_1\ a_2).$$

□