Ri Masterclass

Exercises

Question 1. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Find:

(a) $\alpha \circ \beta$,

(b) $\beta \circ \alpha$,

(c) α^2.

Reminder: Functions are multiplied from right to left.

Solution:

(a)

$$\alpha \circ \beta = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{array} \right).$$

(b)

$$\beta \circ \alpha = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{array} \right).$$

(c)

$$\alpha^2 = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{array} \right).$$

Question 2. Write down all possible permutations of the set $\{1\ 2\ 3\}$. You can either write them down as permutation diagrams or in bracket form (or both!)

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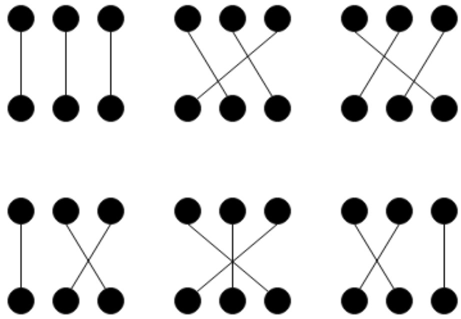

Figure 1: Permutations of the set $\{1\ 2\ 3\}$.

Solution:
There are six different permutations of the set $\{1\ 2\ 3\}$.

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}; \qquad \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}; \qquad \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix};$$

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}; \qquad \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}; \qquad \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}.$$

Question 3. Every year, Molly Weasley knits Christmas jumpers for her children; Bill, Charlie, Percy, Fred, George, Ron and family friend Harry Potter (excluding Ginny for simplicity). Each jumper is labelled with the first letter of the owners name.

- The children put on the correct jumpers in the morning.
- At breakfast, Fred and George swap jumpers.
- At lunch, Charlie swaps jumpers with Bill and Harry swaps jumpers with George

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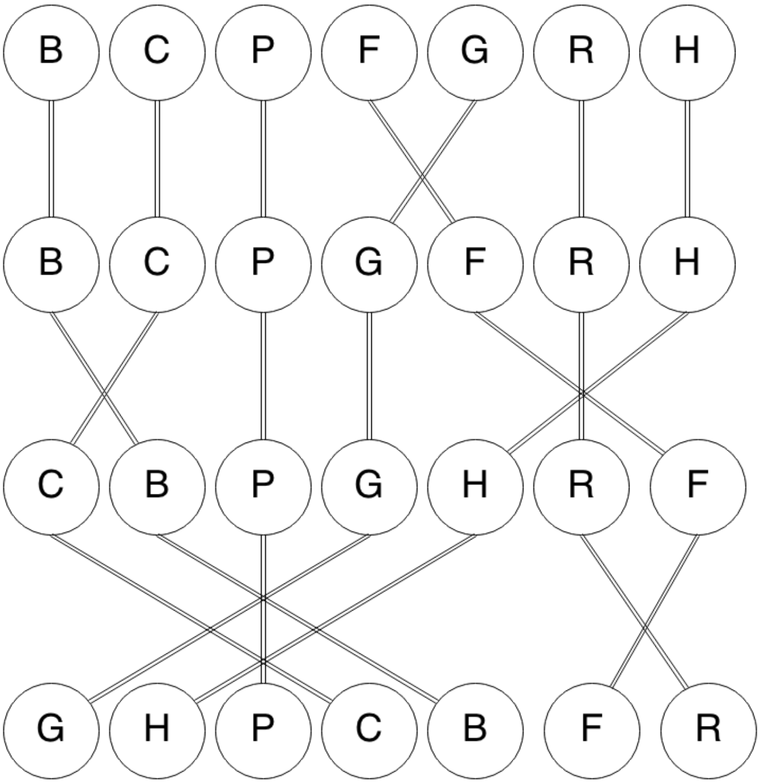
- At teatime, Bill swaps jumpers with Fred, Ron swaps jumpers with Harry and Charlie swaps jumpers with George.

(a) Express these jumper swaps as a permutation diagram.

(b) Write down the resulting permutation (i.e. who is wearing which jumper at the end of the day?)

Solution:



So, Bill is wearing George's jumper, Charlie is wearing Harry's jumper, Percy is wearing his own jumper, Fred is wearing Charlie's jumper, George is wearing Bill's jumper, Ron is wearing Fred's jumper and Harry is wearing Ron's jumper. This can be neated summarised in the permutation below.

$$\text{Jumper Swap} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 3 & 2 & 1 & 4 & 6 \end{pmatrix}$$

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Question 4. A number of characters from "The Simpson's" decide to take part in a Secret Santa. Each character has to buy one gift for another character. Write down the permutation as the product of disjoint cycles.

Participant	Buys for
Homer	Bart
Marge	Mr Burns
Bart	Apu
Lisa	Homer
Maggie	Groundskeeper Willie
Mr Burns	Smithers
Smithers	Krusty the Clown
Ned	Marge
Apu	Lisa
Moe	Maggie
Krusty the Clown	Moe
Groundskeeper Willie	Ned

Solution:

If we number the characters so Homer $= 1$, Marge $= 2$, $\ldots$, Groundskeeper Willie $= 12$, we can express the permutation as

$$\text{Secret Santa} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 6 & 9 & 1 & 12 & 7 & 11 & 2 & 4 & 5 & 10 & 8 \end{pmatrix}$$

If we split this up, we get two disjoint cycles.

$$\text{Secret Santa} = (1\ 3\ 9\ 4)(2\ 6\ 7\ 11\ 10\ 5\ 12\ 8).$$

Question 5. Express each of the following permutations as a single cycle or as the product of disjoint cycles, (omitting 1-cycles).

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$

(c) $(1\ 2\ 5)(2\ 3\ 6)$

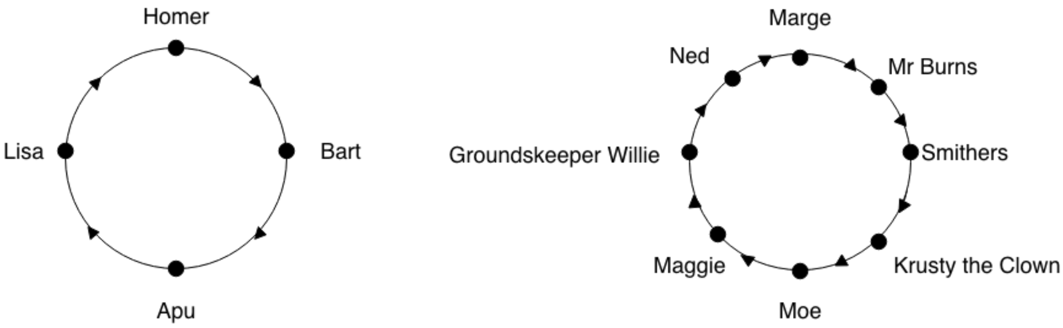

Figure 2: Disjoint cycles for the Simpson's Secret Santa. $\rightarrow$ means "Buys a gift for".

(d) (3 5 6)(1 6)(2 3 4)

Reminder: Apply cycles from right to left.

Solution:

(a) (1 3 4)(2 5)

(b) (1 5 3 7 2 8 4)

(c) (1 2 3 6 5)

(d) (1 3 4 2 5 6)

Question 6. (Extension) Express $\alpha = (1\ 3\ 4)(2\ 5)$ as a product of transpositions. Can we always represent a cycle as a product of disjoint cycles? Can you prove you answer?

Solution:

$$\alpha = (1\ 3)(3\ 4)(2\ 5).$$

Proof.

$$(a_1\ a_2\ \ldots a_k\ ) = (a_1\ a_k)(a_1\ a_{k-1}) \ldots (a_1\ a_3)(a_1\ a_2).$$

$\square$