First we recall the formula given in **?**.

The normalized Laplacian has the following property. For every $f \in \mathbb{R}^n$ we have

$$f' L_{\text{symm}} f = \frac{1}{2} \sum_{i,j=1}^{n} w_{ij} \left(\frac{f_i}{\sqrt{d_i}} - \frac{f_j}{\sqrt{d_j}} \right)^2 . \tag{0.0.1}$$

We show that the second smallest eigenvector u of L is linked to the second smallest eigenvector $\tilde{u}$ of an enlarged Laplacian $\tilde{L}$ where points are repeated. Infact this will relationship will hold for all eigenvectors, but here we show the proof for the second smallest eigenvectors only.

In order to show this we need to show First we will show that

1. $\|u\| = 1$
2. $u \perp D^{1/2}\mathbb{1}$
3. $u^\top L u$
4. Part 4

Part 1

$$\begin{aligned} \|u\|^2 &= \sum_{i=1}^{n} u_i^2 \\ &= \sum_{i=1}^{k} n_i \frac{\tilde{u}_i^2}{n_i} \\ &= \sum_{i=1}^{k} \tilde{u}_i^2 = 1 \end{aligned}$$

since $\tilde{u}$ is an eigenvector. Therefore $\|u\| = 1$.

Part2

$$
\begin{aligned}
\sum_{i=1}^{n} u_i D_{ii}^{1/2} &= \sum_{i=1}^{k} n_i \frac{\tilde{u}_i}{\sqrt{n_i}} \frac{\tilde{D}_{ii}^{1/2}}{\sqrt{n_i}} \\
&= \sum_{i=1}^{k} \tilde{u}_i \tilde{D}_{ii}^{1/2} = 0
\end{aligned}
$$

since $\tilde{u} \perp \tilde{D}^{1/2}\mathbb{1}$, therefore $u \perp D^{1/2}\mathbb{1}$.

Part 3

The first part arises from Von Luxberg.

$$
\begin{aligned}
\tilde{u}^\top \tilde{L} \tilde{u} &= \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \left(\frac{\tilde{u}_i}{\sqrt{\tilde{D}_{ii}}} - \frac{\tilde{u}_j}{\sqrt{\tilde{D}_{jj}}} \right)^2 \tilde{A}_{ij} \\
&= \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \left(\frac{\tilde{u}_i/\sqrt{n_i}}{\sqrt{D_{ii}}} - \frac{\tilde{u}_j/\sqrt{n_j}}{\sqrt{D_{jj}}} \right)^2 n_i n_j A'_{ij}
\end{aligned}
$$

which we can see is equal to $u'Lu$.

Part4

if $\exists\, v \perp D^{1/2}\mathbb{1}$ with $\|v\| = 1$ and $v^\top L v < u^\top L u$, then $\exists\, \tilde{v}$ with $\tilde{v}^\top \perp \tilde{D}^{1/2}\mathbb{1}$ and $\tilde{v}^T \tilde{L} \tilde{v} < \tilde{u}^\top \tilde{L} \tilde{u}$

which contradicts the fact that $\tilde{u}$ is the second smallest eigenvector of $\tilde{L}$.

Therefore u is the second smallest eigenvector of L.