Fault-Tolerant and Local Spanners Revisited

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Abstract

For a set of points $P \in \mathbb{R}^d$, and a family of regions $\mathcal{F}$, a geometric local t-spanner of P, is a sparse graph G over P, that remains a t-spanner even when restricted to a region $\mathscr{r} \in \mathcal{F}$. That is, for any region $\mathscr{r} \in \mathcal{F}$, the subgraph restricted to $\mathscr{r}$, denoted by $G \cap \mathscr{r}$, is still a t-spanner. Here $G \cap \mathscr{r}$ is the subgraph of G induced on $P \cap \mathscr{r}$. A weak $\mathcal{F}$-local spanner of P provides the same guarantee for some smaller subregion of $\mathscr{r}$.

In this paper, we present algorithms for the construction of local of points in $\mathbb{R}^2$ with respect to several families of convex regions. This includes an improvement of the known construction for axis parallel squares, and a construction for disks that does not require Steiner points, along with a matching lower bound. The last result settles an open problem raised by Abam and Borouny.

1. Introduction

Euclidean graph. For a set P of points in $\mathbb{R}^d$, the *Euclidean graph* $G_P = \left(P, \binom{P}{2}\right)$ of P is an undirected graph. Here, an edge $pq \in E$ is associated with the segment pq, and its weight is the (Euclidean) length of the segment.

t-spanners. Let $G = (P, E)$ and $I = (P, E')$ be two graphs over the same set of vertices (usually I is a subgraph of G). Consider two vertices $p, q \in P$, and parameter $t \geq 1$. A path π between p and q in I, is a *t-path*, if the length of π in I is at most $t\,\mathsf{d}_G(p,q)$, where $\mathsf{d}_G(p,q)$ is the length of the shortest path between p and q in G. The graph I is a *t-spanner* of G if there is a t-path in I, for any $p, q \in P$.

Thus, for $P \subseteq \mathbb{R}^d$, a graph G over P is a *t-spanner* if it is a t-spanner of the euclidean graph G_P.

Local spanners. Given a set of points P in $\mathbb{R}^d$ and a parameter $t > 1$, the problem of constructing *t-spanners* for P is well studied [NS07]. Recently, Abam and Borouny [AB21] introduced the problem of designing such spanners, that remains spanner when cropped to regions from a prespecified set $\mathcal{F}$. Formally, such a graph $G = (P, E)$ has weights on the edges that are their euclidean length. Informally, cropping (or restricting) G to a region $\mathscr{r} \in \mathcal{F}$, is the subgraph induced on $P \cap \mathscr{r}$. We require that this graph $I = G \cap \mathscr{r}$ is a t-spanner for all $p, q \in P \cap \mathscr{r}$ – that is, $\mathsf{d}_I(p,q) \leq t\,\|pq\|$, where $\mathsf{d}_I(p,q)$ denotes the shortest path length in I between p and q. This property can be interpreted as fault-tolerance, where the faults (or attacks) are complements of the regions of $\mathcal{F}$.

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Fault-tolerant and local spanners. An $\mathcal{F}$*-fault-tolerant spanner* for $P \subseteq \mathbb{R}^d$, is a graph $G = (P, E)$, such that for any region $\mathscr{r}$ (i.e., the "attack"), the graph $G - \mathscr{r}$ is a t-spanner of $G_P - \mathscr{r}$. Surprisingly, as shown by Abam *et al.* [AdBFG09], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

In the same spirit, a graph $G = (P, E)$ is an $\mathcal{F}$*-local spanner* for P if for any region $\mathscr{r} \in \mathcal{F}$, we have that $G \cap \mathscr{r}$ is a t-spanner of $G_P \cap \mathscr{r}$. The notion of local-spanners was defined by Abam and Borouny [AB21] who showed how to construct such spanners for axis-parallel squares and vertical slabs. They also showed how to construct such spanners for disks, if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

Related work

Geometric spanners have been widely studied, see [NS07]. Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size k of vertices and edges has failed. Levcopoulos *et al.* [LNS02] showed the existence of k-vertex/edges fault tolerant spanners for a set of points P in some metric space. Their spanner had $O(kn \log n)$ edges, and weight, i.e. sum of edge weights, bounded by $f(k) \cdot wt(MST(P))$ for some function f. Lukovszki [Luk99] later achieved a similar construction, improving the number of edges to $O(kn)$ and was able to prove that the result is asymptotically tight.

Abam *et al.* [AdBFG09] introduced region-fault tolerant spanners, where the faults were not arbitrary sets of vertices and/or edges, but instead all of the points and geometric edges (segments between points) intersecting some region. They showed several results, including a construction of convex regions fault tolerant t-spanners of size $\mathcal{O}_\varepsilon(n \log n)$. More recently, Abam and Borouny [AB21] introduced the concept of local spanners, and showed constructions of local t-spanners of size $\mathcal{O}_\varepsilon(n \operatorname{polylog} n)$ for axis-parallel squares and vertical slabs, and also showed that constructions using $O(n)$ Steiner points for the same cases as well as for the case of disk local spanners.

Our results

Disks. In Section 3 we present a construction of spanners, which surprisingly, is not only fault-tolerant for convex regions, but it also a local spanner for disks. This resolves the aforementioned open problem from Abam and Borouny [AB21]. Our construction is a variant of the original construction of Abam *et al.* [AdBFG09]. For a parameter $\varepsilon > 0$ the construction of a $(1+\varepsilon)$-local spanner for disks takes $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ time, and the resulted spanner is of size $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$, where Φ is the spread of the point set. We also provide a lower bound showing that this logarithmic dependency on Φ cannot be avoided.

In Section **??** we extend this construction to scaled and translated copies (*homothets*) of a convex shape $\mathcal{C}$.

Squares. In Section 4 we show a construction similar to that of Section 3, but prove that for the case of axis parallel square local spanners we are able to produce a spanner of size $\mathcal{O}(\varepsilon^{-3} n \log n)$, that is, independent of the spread of the point set.

Triangles. In Section 3.4 we give a construction of local spanners for the family $\mathcal{F}$ of homothets of a given triangle $\triangle$, and get a spanner of size $O\left((\alpha\varepsilon)^{-1} n\right)$ in $O\left((\alpha\varepsilon)^{-1} n\right) \log n$ time, where α is the smallest angle in $\triangle$. We also show that if we allow α to be arbitrarily small there exists a set of points that requires $O(n^2)$ edges for some triangles.

Region	# edges	Paper	New # edges	Location in paper
		Local $(1+\varepsilon)$-spanners		
Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AdBFG09]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[AB21]	$\mathcal{O}(\varepsilon^{-3} n \log n)$	Theorem 4.3
Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AB21]		
Disks+Steiner points	$\mathcal{O}_\varepsilon(n)$	[AB21]		
Disks			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 3.6
			$\Omega(n \log \Phi)$	Lemma 3.10
Homothets convex shape			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem **??**
α-fat triangles			$\mathcal{O}((\alpha\varepsilon)^{-1} n)$	Theorem 3.16
		δ-weak local $(1+\varepsilon)$-spanners		
Bounded convex shape			$\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2})n\big)$	Lemma 5.3
		$(1-\delta)$-local $(1+\varepsilon)$-spanners		
Axis-parallel rectangles			$\mathcal{O}\big((\varepsilon^{-2} + \delta^{-2})n \log^2 n\big)$	Theorem 6.6

Figure 1.1: Known and new results. The notation $\mathcal{O}_\varepsilon$ hides polynomial dependency on ε which is not specified in the original work.

FILL IN. For the following families of regions, it is possible to show that even after constraining parameter such as fatness/aspect ratio, a local spanner with respect to the set of regions might require a quadratic number of edges. We therefore present constructions of weak spanners for those families, where the mathematical definition of "weak" changes between the two section, even though the two definitions are very closely related. In both cases, the size of the spanner and the construction time both depend on the "weakness" of the requested spanner, parameterized by $\delta \in (0, 1)$.

Convex regions. In Section 5 we construct weak local spanners for homothets of a given convex region $\mathcal{C}$ with a bounded aspect ratio. We are able to show an algorithm for constructing a δ-weak $\mathcal{C}$-local $(1+\varepsilon)$-spanner of size $\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2})n\big)$ in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time.

Rectangles. In Section 6 we describe a new pair decomposition data structure, the *Quadrant Separated Pair Decomposition* (QSPD), and use it to construct a weak local spanner for axis parallel rectangles.We get a δ-weak local $(1+\varepsilon)$-spanner for the axis parallel rectangles, with size $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ time.

See Figure 1.1 for a summary of known results and comparisons to the results of this paper.

2. Preliminaries

Residual graphs. Let $\mathcal{F}$ be a family of regions in the plane. For a fault region $\mathscr{r} \in \mathcal{F}$ and a geometric graph G on a point set P, let $G - \mathscr{r}$ be the residual graph after removing from it all the points of P in

$\mathcal{r}$. Similarly, let $G \cap \mathcal{r}$ denote the graph restricted to $\mathcal{r}$. Formally, let

$$G - \mathcal{r} = \big(P \setminus \mathcal{r}, \{uv \in E \mid uv \cap \mathrm{int}(\mathcal{r}) = \emptyset\}\big) \qquad \text{and} \qquad G \cap \mathcal{r} = \big(P \cap \mathcal{r}, \{uv \in E \mid uv \subseteq \mathcal{r}\}\big).$$

where $\mathrm{int}(\mathcal{r})$ denotes the interior of $\mathcal{r}$.

2.1. On various pair decompositions

For sets X, Y, let $X \otimes Y = \{\{x, y\} \mid x \in X, y \in Y, x \neq y\}$ be the set of all the (unordered) pairs of points formed by the sets X and Y.

Definition 2.1 (Pair decomposition). For a point set P, a ***pair decomposition*** of P is a set of pairs

$$\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\},$$

such that (I) $X_i, Y_i \subseteq P$ for every i, (II) $X_i \cap Y_i = \emptyset$ for every i, and (III) $\bigcup_{i=1}^{s} X_i \otimes Y_i = P \otimes P$. Its ***weight*** is $\omega(\mathcal{W}) = \sum_{i=1}^{s}(|X_i| + |Y_i|)$.

The ***closest pair*** distance of a set of points $P \subseteq \mathbb{R}^d$, is $\mathrm{cp}(P) = \min_{p,q \in P, p \neq q} \|pq\|$. The ***diameter*** of P is $\mathrm{diam}(P) = \max_{p,q \in P} \|pq\|$. The ***spread*** of P is $\Phi(P) = \mathrm{diam}(P)/\mathrm{cp}(P)$, which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD can be quadratic, if the spread is bounded, the weight is near linear. For $X, Y \subseteq \mathbb{R}^d$, let $\mathsf{d}(X, Y) = \min_{p \in X, q \in Y} \|pq\|$ be the ***distance*** between the two sets.

Definition 2.2. Two sets $X, Y \subseteq \mathbb{R}^d$ are

$$\begin{array}{rlcl} & 1/\varepsilon\text{-}\textbf{\textit{well-separated}} & \text{if} & \max(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \varepsilon\, \mathsf{d}(X, Y), \\ \text{and} & 1/\varepsilon\text{-}\textbf{\textit{semi-separated}} & \text{if} & \mathbf{min}(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \varepsilon\, \mathsf{d}(X, Y). \end{array}$$

For a point set P, a ***well-separated pair decomposition*** (**WSPD**) of P with parameter $1/\varepsilon$ is a pair decomposition of P with a set of pairs $\mathcal{W} = \big\{\{B_1, C_1\}, \ldots, \{B_s, C_s\}\big\}$, such that, for all i, the sets B_i and C_i are $(1/\varepsilon)$-separated. The notion of $1/\varepsilon$-SSPD (aka ***semi-separated pairs decomposition***) is defined analogously.

Lemma 2.3 ([AH12]). *Let P be a set of n points in $\mathbb{R}^d$, with spread $\Phi = \Phi(P)$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-WSPD $\mathcal{W}$ for P of total weight $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}(\varepsilon^{-d} \log \Phi)$ pairs.*

Theorem 2.4 ([AH12, Har11]). *Let P be a set of n points in $\mathbb{R}^d$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-SSPD for P of total weight $\mathcal{O}(n\varepsilon^{-d} \log n)$. The number of pairs in the SSPD is $\mathcal{O}(n\varepsilon^{-d})$, and the computation time is $\mathcal{O}(n\varepsilon^{-d} \log n)$.*

The proof of the following two lemmas is straightforward, and are delegated to the appendix.

Lemma 2.5 (Proof in Appendix A.1). *Given an α-SSPD $\mathcal{W}$ of a set P of n points in $\mathbb{R}^d$ and a parameter $\beta \geq 2$, one can refine $\mathcal{W}$ into an $\alpha\beta$-SSPD $\mathcal{W}'$, such that that $|\mathcal{W}'| = O(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\beta^d)$.*

Definition 2.6. An ***ε-double-wedge*** is a region between two lines, where the angle between the two lines is at most ε.

Tow points sets X and Y that each lie on their own cone of a shared ε-double-wedge are ***ε-angularly separated***.

Lemma 2.7 (Proof in Appendix A.2). *Given a ε^{-1}-SSPD $\mathcal{W}$ of n points in the plane, one can refine it, into a ε^{-1}-SSPD $\mathcal{W}'$, such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε-double-wedge $\times_\Xi$, such that X and Y are contained in the two different faces of the double wedge $\times_\Xi$. We have that $|\mathcal{W}'| = O(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\varepsilon)$. The construction time is proportional to the weight of $\mathcal{W}'$.*

3. Local spanners of homothets of convex region

3.1. Delaunay triangulation for homothets

Let $\mathcal{C}$ be a bounded convex and closed region in the plane (e.g., a disk). A ***homothet*** of $\mathcal{C}$ is a scaled and translated copy of $\mathcal{C}$. A point set P is ***general position*** for $\mathcal{C}$, if no four points of P lie on the boundary of a homothet of $\mathcal{C}$, and no three points are colinear.

Definition 3.1 ([CDI85]). Given $\mathcal{C}$ as above, and a point set P in general position for $\mathcal{C}$, the ***$\mathcal{C}$-Delaunay triangulation*** of P is the graph formed by edges between any two points $p, q \in P$ such that there exist a homothet of $\mathcal{C}$ that contains only p and q and no other point of P.

Theorem 3.2 ([CDI85]). *For any convex shape $\mathcal{C}$ and a set of points P, $\mathcal{D}_\mathcal{C}(P)$ can be computed in $O(n \log n)$ time. Furthermore, the triangulation $\mathcal{D}_\mathcal{C}(P)$ has $O(n)$ edges, vertices, and faces.*

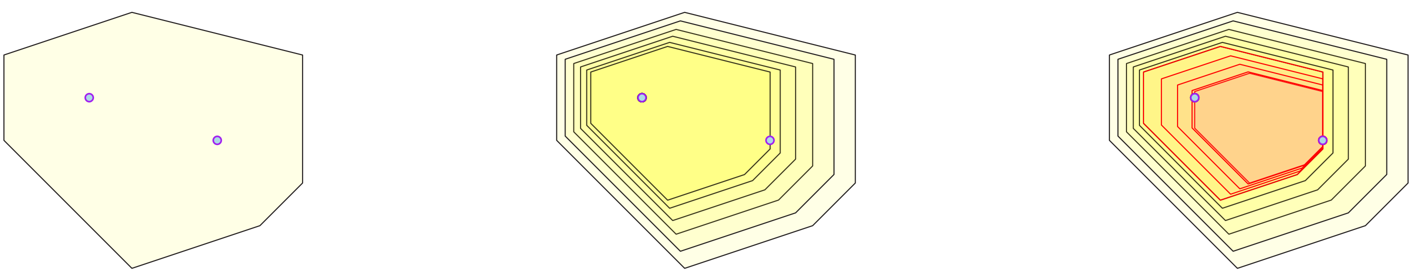


Figure 3.1: Shrinking of homothets so two points becomes on the boundary of the homothet.

Lemma 3.3. *Let $\mathcal{C}$ be a convex bounded body, and let P be a set of points in general position for $\mathcal{C}$. Then, if C is a homothet of $\mathcal{C}$ that contains two points $p.q \in \mathcal{C} \cap P$, then there exists a homothet $C' \subseteq C$ of $\mathcal{C}$ such that $p, q \in \partial C'$.*

Proof: The idea is to apply a shrinking process of C, as illustrated in Figure 3.1. Consider the mapping $f_{\beta,v} : u \to \beta(x - v) + v$. It is a scaling of the plane around v by a factor of β. Let β' be the minimum value of β such that $C_1 = f_{\beta,p}(C)$ contains q (i.e., we shrink C around p till q becomes a boundary point). Next, shrink C' around q, till p becomes a boundary point – formally, let β'' be the minimum value of β such that $C' = f_{\beta,q}(C_1)$ contains p. Since $C' \subseteq C_1 \subseteq C$, and $p, q \in \partial C'$, the claim follows. ■

The following standard claim, usually stated about the standard Delaunay triangulations, also holds for homothets.

Claim 3.4 (Proof in Appendix A.3). *Let $\mathcal{C}$ be a bounded close convex shape. Given a set of points $P \subseteq \mathbb{R}^2$ in general position for $\mathcal{C}$, let $\mathcal{D} = \mathcal{D}_\mathcal{C}(P)$ be the $\mathcal{C}$-Delaunay triangulation of P. For any homothet C of $\mathcal{C}$, we have that $\mathcal{D} \cap C$ is connected.*

3.2. The construction

The input is a set P of n points in the plane (in general position) with spread $\Phi = \Phi(P)$, and a parameter $\varepsilon \in (0, 1)$. We have a convex body $\mathcal{C}$ that defines the "unit" ball. The task is to construct a local spanner for any homothet of $\mathcal{C}$.

The algorithm computes a $1/\vartheta$-WSPD $\mathcal{W}$ of P using the algorithm of Lemma 2.3, where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the $\mathcal{C}$-Delaunay triangulation $\mathcal{D}_\Xi = \mathcal{DT}(X \cup Y)$. The algorithm adds all the edges in $\mathcal{D}_\Xi \cap (X \otimes Y)$ to the computed graph G.

3.2.1. Analysis

Size. For each pair $\Xi = \{X, Y\}$ in the WSPD, its $\mathcal{C}$-Delaunay triangulation contains at most $\mathcal{O}(|X| + |Y|)$ edges. As such, the number of edges in the resulting graph is bounded by $\sum_{\{X,Y\} \in \mathcal{W}} O\bigl(|X| + |Y|\bigr) = O(\omega(\mathcal{W})) = O\bigl(\frac{n \log \Phi}{\vartheta^2}\bigr)$, by Lemma 2.3.

Construction time. The construction time is bounded by $\sum_{\{X,Y\} \in \mathcal{W}} O\bigl((|X| + |Y|) \log(|X| + |Y|)\bigr) = O(\omega(\mathcal{W}) \log n) = O\bigl(\frac{n \log \Phi \log n}{\vartheta^2}\bigr)$,

Lemma 3.5 (Local spanner property). *For $P, \mathcal{C}, \varepsilon$ as above, let G be the graph constructed above for the point set P. Then, for any homothet C of $\mathcal{C}$ and any two points $x, y \in P \cap C$, we have that $G \cap C$ has a $(1 + \varepsilon)$-path between x and y. That is, G is a $(1 + \varepsilon)$-local spanner for homothets of $\mathcal{C}$.*

Proof: Fix a homothet C of $\mathcal{C}$, and consider two points $p, q \in P \cap C$. The proof is by induction on the distance between p and q (or more precisely, the rank of their distance among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and $y \in Y$.

If $xy \in \mathcal{D}_\Xi$ then the claim holds. So assume this is not the case. By the connectivity of $\mathcal{D}_\Xi \cap C$, see Claim 3.4, there must be points $x' \in X \cap C$, $y' \in Y \cap C$, such that $x'y' \in E(\mathcal{D}_\Xi)$. As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property, we have that

$$\max(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \vartheta \, \mathsf{d}(X, Y) \leq \vartheta \ell,$$

where $\ell = \|xy\|$. In particular, $\|x'x\| \leq \vartheta \ell$ and $\|y'y\| \leq \vartheta \ell$. As such, by induction, we have $\mathsf{d}_G(x, x') \leq (1 + \varepsilon) \|xx'\| \leq (1 + \varepsilon) \vartheta \ell$ and $\mathsf{d}_G(y, y') \leq (1 + \varepsilon) \|yy'\| \leq (1 + \varepsilon) \vartheta \ell$. Furthermore, $\|x'y'\| \leq (1 + 2\vartheta)\ell$. As $x'y' \in E(G)$, we have

$$\begin{aligned} \mathsf{d}_G(x, y) &\leq \mathsf{d}_G(x, x') + \|x'y'\| + \mathsf{d}_G(y', y) \leq (1 + \varepsilon)\vartheta \ell + (1 + 2\vartheta)\ell + (1 + \varepsilon)\vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta)\ell \\ &= (1 + 6\vartheta)\ell \leq (1 + \varepsilon) \|xy\|, \end{aligned}$$

if $\vartheta \leq \varepsilon/6$. ■

The result.

Theorem 3.6. *Let $\mathcal{C}$ be a bounded convex shape in the plane, let P be a given set of n points in the plane (in general position), and let $\varepsilon \in (0, 1/2)$ be a parameter. The above algorithm constructs a local $(1 + \varepsilon)$-spanner G for homothets of $\mathcal{C}$. The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any homothet C of $\mathcal{C}$, and any two points $p, q \in P \cap C$, we have a $(1 + \varepsilon)$-path in $G \cap C$.*

3.2.2. Applications and comments

The following defines a "visibility" graph when we are restricted to a region R, where two points are visible if there is a witness homothet contained in R having both points on its boundary.

Definition 3.7. Let $\mathcal{C}$ be a bounded convex shape in the plane. Given a region R in the plane and a point set P, consider two points $p, q \in P$. The edge pq is *safe* in R, if there is a homothet C of $\mathcal{C}$, such that $p, q \in C \subseteq R$. The ***safe graph*** for P and R, denoted by $\mathcal{G}(P, R)$, is the graph formed by all the safe edges in P for R. Note, that this graph might have a quadratic number of edges in the worst case.

Observe that $\mathcal{G}(P, \mathbb{R}^2)$ is a clique. Surprisingly, the spanner graph when restricted to region R, is a spanner for

Corollary 3.8. *Let $\mathcal{C}$ be a bounded convex body, P be a set of n points in the plane, $\varepsilon \in (0,1)$ be a parameter, and let G be a local $(1+\varepsilon)$-spanner of P for homothets of $\mathcal{C}$. Let G be the above spanner constructed for P and $\mathcal{C}$.*

Consider a region R in the plane, and the associated graph $H = \mathcal{G}(P, R)$, we have that $G \cap R$ is a $(1+\varepsilon)$-spanner for H. Formally, for any two points $p, q \in P \cap R$, we have that $\mathsf{d}_{G\cap R}(p,q) \leq (1+\varepsilon)\mathsf{d}_H(p,q)$.

In particular, for any convex region D, the graph $G - D$ is a $(1+\varepsilon)$-spanner for $\mathcal{G}(P, \mathbb{R}^2) - D$.

Proof: Consider the shortest path $\pi = u_1 u_2 \ldots u_k$ between p and q realizing $\mathsf{d}_H(p,q)$. Every edge $e_i = u_i u_{i+1}$ has a homothet C_i such that $u_i, u_{i+1} \in C_i \subseteq R$. As such, there is a $(1+\varepsilon)$-path between u_i and u_{i+1} in $G \cap C_i \subseteq G \cap R$. Concatenating these paths directly yields the desired result.

The second claim follows by observing that the complement of D is the union of halfspaces, and halfspaces can be considered to be "infinite" homothets of $\mathcal{C}$. As such, the above argument applies verbatim. ∎

Remark 3.9. The above implies that local spanners for homothets are also robust to convex region faults.

3.3. Lower bounds

3.3.1. A lower bound for local spanner for disks

The result of Theorem 3.6 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). We next show a lower bound proving that this dependency is unavoidable, even in the case of disks.

Some intuition. A natural way is to try and emulate the construction of Abam *et al.* [AdBFG09] and use a SSPD instead of a WSPD. The total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε-separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are of different sides of the double wedge. The problem is that for the local disk $\bigcirc$, it might be that the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \bigcirc$ is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 3.2.

Lemma 3.10. *For $\varepsilon = 1/4$, and parameters n and $\Phi \geq 1$, there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $O(n\Phi)$, such that any local $(1+\varepsilon)$-spanner of P for disks, must have $\Omega(n \log \Phi)$ edges.*

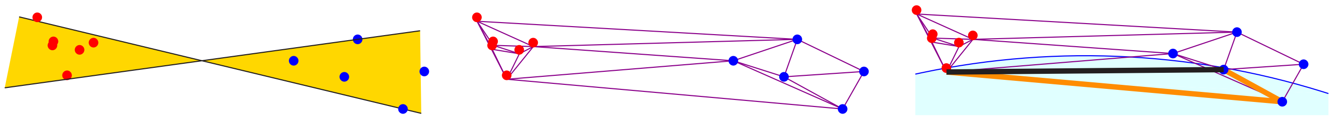


Figure 3.2: A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

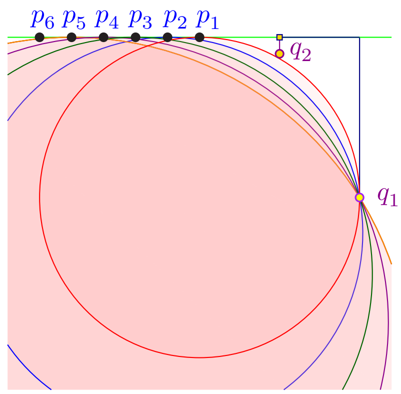


Figure 3.3: The set of disks D_1, and the construction of q_2.

Proof: Let $p_i = (-i, 0)$, for $i = 1, \ldots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$ and $q_1 = (n2^M, -1)$. For a point p on the x-axis, and a point q below the x-axis, and to the right of p, let $\bigcirc_{\downarrow}^{p}(q)$ be the disk whose boundary passes through p and q, and its center has the same x-coordinate as p.

In the jth iteration, for $j = 2, \ldots, M-1$, Let $x_j = n2^{M-j+1} = x(q_{j-1})/2$, and let $y_j < 0$ be the maximum y-coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the disks of $D_1 \cup \cdots \cup D_j$. Let $q_j = (x_j, 0.99y_j)$. Consider the set of disks

$$D_j = \left\{ \bigcirc_{\downarrow}^{p_i}\left(q_{j-1}\right) \;\middle|\; i = 1, \ldots, n \right\},$$

see Figure 3.3.

Clearly, the point q_j lies outside all the disks of $D_1 \cup \ldots \cup D_j$. The construction now continues to the next value of j. Let $P = \{p_1, \ldots, p_n, q_2, \ldots, q_M\}$. We have that $|P| = n + M - 1$.

The minimum distance between any points in the construction is 1 (i.e., $\|p_1p_2\|$). Indeed $x(q_M) = 2n$ and thus $\|q_Mp_1\| \geq 2n$. The diameter of P is $\|p_1q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = O(n\Phi)$.

For any i and j, consider the disk $\bigcirc_{\downarrow}^{p_i}\left(q_j\right)$. This disk does not contain any point of $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ since its interior lies below the x-axis. By construction it does not contain any point $q_{j+1}, \ldots, q_{M-1}$. This disk potentially contains the points $q_{j-1}, \ldots, q_1$, but observe that for any index $k \in [\![j-1]\!]$, we have that

$$\|p_iq_k\| = \sqrt{\left(i + n2^{M-k+1}\right)^2 + \left(y(q_j)\right)^2},$$

which implies that $n2^{M-k+1} \leq \|p_iq_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$\frac{\|p_iq_k\|}{\|p_iq_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since $j \in [\![M-1]\!]$. Namely, the shortest path in G between p_i and q_j, can not use any of the points $q_1, \ldots q_{j-1}$. As such, the graph G must contain the edge p_iq_j. This implies that $|E(G)| \geq n(M-1)$, which implies the claim. ■

3.3.2. A lower bound for triangles

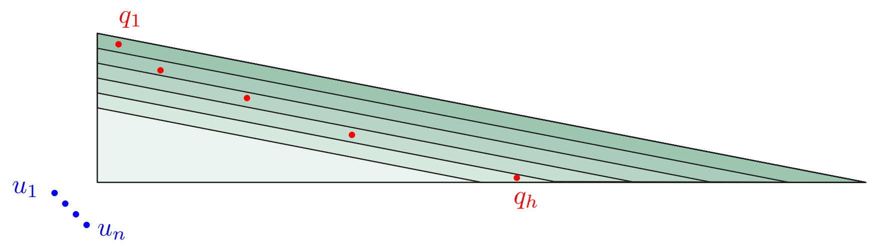


Figure 3.4: An Illustration of the construction of Lemma 3.11.

Lemma 3.11. *For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a set P of $n + O(\log \Phi)$ points, with spread $O(\Phi n)$, and a triangle $\triangle$, such that any $\triangle$-local $3/2$-spanner of P requires $\Omega(n \log \Phi)$ edges.*

Proof: Let $h = \lceil \log_2 \Phi \rceil$. Let $\triangle$ be the triangle formed by the points $(0,0)$, $(0,1)$ and $(8\Phi h, 0)$. The hypotenuse of this triangle lies on the line $\ell \equiv \frac{1}{8\Phi h}x + y = 1$, and let $v = \left(\frac{1}{8\Phi h}, 1\right)$ be the vector orthogonal to this line.

For $i \in [\![h]\!]$ and $j \in [\![n]\!]$, let

$$q_i = \left(2^{i+1}, 1 - i/h\right) \qquad \text{and} \qquad u_j = \left(\tfrac{j}{n} - 1, -\tfrac{j}{n}\right),$$

and let $P = \{q_1, \ldots, q_h, u_1, \ldots, u_n\}$, see Figure 3.4. Observe that $\mathrm{cp}(P) = \|u_1 u_2\| = \sqrt{2}/n$, and as such have that $\Phi(P) = n \cdot \mathrm{diam}(P)/\sqrt{2} \le n(4\Phi + 2n) \le 8\Phi n$, as $\Phi \ge n$. Observe that

$$\left\langle q_{i+1} - q_i, v \right\rangle = \left\langle (2^{i+1}, -\tfrac{1}{h}), \left(\tfrac{1}{4\Phi h}, 1\right) \right\rangle \le \tfrac{4\Phi}{8\Phi h} - \tfrac{1}{h} < 0.$$

That is, the points $q_1, \ldots, q_i$ are in increasing distance from ℓ.

Let $\triangle_{i,j}$ be the homothet of $\triangle$, that has its bottom left corner at u_j, and its hypotenuse passes through q_i. By the above, $P(i,j) = \triangle_{i,j} \cap P = \{u_j, q_i, q_{i+1}, \ldots q_h\}$. Any $(1+\varepsilon)$-spanner for $P(i,j)$ must contain the edge $u_j q_i$. Indeed, we have, for any k, that $2^{k+1} \le \|u_j q_k\| \le 2^{k+1} + 3$. As such, any path on a graph induced on $P(i,j)$ from u_j to q_i that uses (say) a midpoint q_k, for $k > i$, must have dilation at least

$$\frac{\|u_j q_k\| + \|q_k q_i\|}{\|u_j q_i\|} \ge \frac{2^{k+1} + 2^k}{2^{i+1} + 3} \ge \frac{3 \cdot 2^{i+1}}{(1 + 3/4)2^{i+1}} = \frac{12}{7} > \frac{3}{2}.$$

Thus, any $\triangle$-local $3/2$-spanner for homothets of $\triangle$, must contain the edge $q_i u_j$, for any $i \in [\![h]\!]$ and $j \in [\![n]\!]$. Thus, such a spanner must have $\Omega(n \log \Phi)$ edges, as claimed. ■

3.4. Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex shape are costly, if we are given a triangle $\triangle$ with the single constraint that $\triangle$ is not too "thin", then one can construct a $\triangle$-local spanner with a number of edges that does not depend on the spread of the points. See Figure 3.4 for an illustration of a construction showing that dependency if "thin" triangles are allowed.

Definition 3.12. A triangle $\triangle$ is ***α-fat*** if the smallest angle in $\triangle$ is at least α.

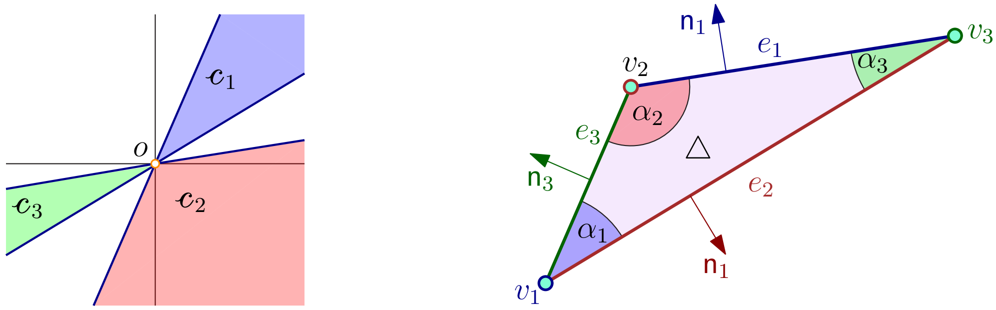


Figure 3.5: For the triangle $\triangle$ with angles α_1, α_2, and α_3 we create the cones c_1, c_2, and c_3.

3.4.1. Construction

The input is a set P of n points in the plane, an α-fat triangle $\triangle$, and an approximation parameter $\varepsilon \in (0, 1)$. Let v_i denote the ith vertex of $\triangle$, α_i be the adjacent angle, and let e_i denote the facing edges, for $i \in [\![3]\!]$. Let $\boldsymbol{c}_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$ denote the cone with an apex at the origin induced by the ith vertex of $\triangle$. Let d_i be the outer normal of $\triangle$ orthogonal to e_i. See Figure 3.5 for an illustration. Let $\mathcal{C}_i$ be a minimum partition of $\boldsymbol{c}_i$ into cones each with angle in the range $[\beta/2, \beta]$, where $\beta = \varepsilon\alpha/c_2$, and $c_2 > 1$ is a constant to be determined shortly. For each point $p \in P$, and a cone $\boldsymbol{c} \in \mathcal{C}_i$, let $\mathsf{nn}_i(p, \boldsymbol{c})$ be the first point in $(P - p) \cap (p + \boldsymbol{c})$ ordered by the direction d_i (it is the "nearest-neighbor" to p in $p + \boldsymbol{c}$).

The construction. Let G be the graph over P formed by connecting all points $p \in P$ to $\mathsf{nn}_i(p, \boldsymbol{c})$, for all $i \in [\![3]\!]$ and $\boldsymbol{c} \in \mathcal{C}_i$.

3.4.2. Analysis

Lemma 3.13. *Let $p \in P$, $\boldsymbol{c} \in \mathcal{C}_i$, and $u = \mathsf{nn}_i(p, \boldsymbol{c})$, and let q be a point in $(P \cap (p + \boldsymbol{c})) \setminus \{p, u\}$. We have that $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ and $\|qu\| \leq \|pq\|$.*

Proof: Consider the triangle Δpqu and denote the angles at p, q, and u by $\sphericalangle p, \sphericalangle q$, and $\sphericalangle u$ respectively. Since the angle of $\boldsymbol{c}$ is smaller than 60 degrees, we have that $\|qu\| \leq \max\{\|pu\|, \|pq\|\}$.

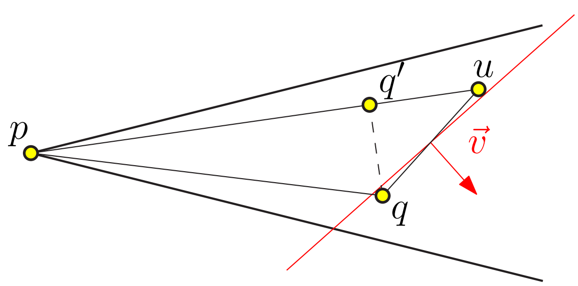


Figure 3.6

Consider the case that $\|pq\| \leq \|pu\|$, illustrated in Figure 3.6. Observe that $\sphericalangle u \leq \sphericalangle q$. As such $\sphericalangle u \leq \pi/2$. Furthermore, $\sphericalangle u \geq \alpha \gg \varepsilon\alpha/c_2 = \beta \geq \sphericalangle p$. Similarly, $\sphericalangle q \in [\alpha, \pi - \alpha]$. By the 1-Lipshitz of sin, and as $\sin x \approx x$, for small x, and for c_2 sufficiently large. we have that

$$\sin(\sphericalangle q + \sphericalangle p) \in [1 - \varepsilon/4, 1 + \varepsilon/4] \sin \sphericalangle q \qquad \text{and} \qquad \sin \sphericalangle p \leq (\varepsilon/4) \sin \sphericalangle u.$$

As such, by the law of sines, we have that $\frac{\|qu\|}{\sin \sphericalangle p} = \frac{\|pq\|}{\sin \sphericalangle u} = \frac{\|pu\|}{\sin \sphericalangle q}$. This implies that

$$\|pu\| + (1+\varepsilon)\|qu\| = \left(\frac{\sin \sphericalangle q}{\sin \sphericalangle u} + (1+\varepsilon)\frac{\sin \sphericalangle p}{\sin \sphericalangle u}\right)\|pq\|.$$

Observe, by the above that

$$\frac{\sin \sphericalangle q}{\sin \sphericalangle u} + (1+\varepsilon)\frac{\sin \sphericalangle p}{\sin \sphericalangle u} \le \frac{\sin \sphericalangle q}{\sin(\sphericalangle p + \sphericalangle q)} + (1+\varepsilon)\frac{\varepsilon}{4} \le \frac{\sin \sphericalangle q}{(1-\varepsilon/4)\sin(\sphericalangle q)} + (1+\varepsilon)\frac{\varepsilon}{4} \le 1+\varepsilon.$$

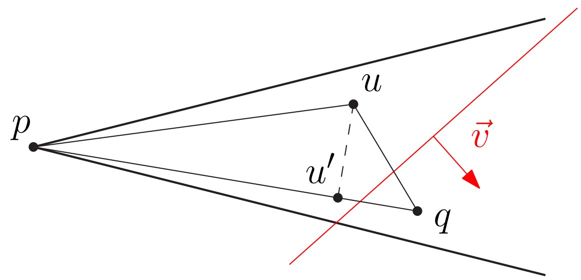


Figure 3.7

The other possibility is that $\|pq\| > \|pu\|$, illustrated in Figure 3.7. Let u' be the projection of u to pq. Observe that

$$\|uu'\| = \|pu'\| \tan \sphericalangle p \le 2\beta \|pu'\| \le (\varepsilon/8)\|pu'\|.$$

Observe that $\cos \sphericalangle p \ge 1 - (\sphericalangle p)^2/2 \ge 1 - \varepsilon^2/8$ as $\sphericalangle p$ is an angle smaller than (say) $\varepsilon/16$. As such $1/\cos \sphericalangle p \le 1 + \varepsilon^2/4$. This implies that $\|pu\| \le \|pu'\|/\cos \sphericalangle p \le (1+\varepsilon^2/4)\|pu'\|$. We thus have that

$$\begin{aligned}\tau = \|pu\| + (1+\varepsilon)\|qu\| &\le (1+\varepsilon^2/4)\|pu'\| + (1+\varepsilon)\Big(\|uu'\| + \|u'q\|\Big)\\ &\le \big(1+\varepsilon^2/4 + (1+\varepsilon)\varepsilon/8\big)\|pu'\| + (1+\varepsilon)\|u'q\| \le (1+\varepsilon)\|pq\|.\end{aligned}$$ ■

Lemma 3.14. *Let $\triangle$ be a triangle that contains two points p, q. Then, there is a homothet $\triangle' \subseteq \triangle$ of $\triangle$, such that,one of these points is a vertex of $\triangle'$, and the other point lies on a facing edge of $\triangle'$.*

Proof: This follows by the same shrinking argument as Lemma 3.3, and is illustrated in Figure 3.8. ■

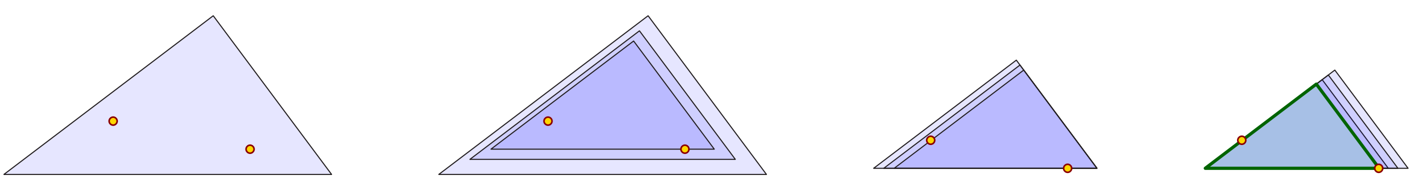


Figure 3.8

Local spanner property.

Lemma 3.15. *Let $\triangle'$ be a homothet of $\triangle$, for any two points $p, q \in P \cap \triangle'$, we have a $(1+\varepsilon)$-path in $G' = G \cap \triangle'$.*

Proof: Consider the closest pair $p, q \in P \cap \triangle$. They must be connected directly in G', as otherwise there is a point $u \in P' = P \cap \triangle'$ in the cone containing the segment pq, such that $pu \in E(G')$. But then, by Lemma 3.13, we have $\|pu\| + (1+\varepsilon)\|qu\| \leq (1+\varepsilon)\|pq\|$, which implies that either pu or qu are the closest pairs, which is a contradiction.

For any other pair $p, q \in PS'$ we have from Lemma 3.14 that there exists a homothet $\triangle'' \subseteq \triangle'$ with one of the two points, say p, at a vertex, and the other on the opposite edge. We therefore have a cone c with apex at p such that $q \in c \cap \triangle''$. If pq is an edge in G then we are done. Otherwise, we have a vertex $u \in c$ such that pu is an edge in G, and by Lemma 3.13 we have $\|qu\| \leq \|pq\|$. Thus, by induction, we have that there exists a $(1+\varepsilon)$ path between u and q in G. Lemma 3.13 now implies that $\|pu\| + (1+\varepsilon)\|qu\| \leq (1+\varepsilon)\|pq\|$. Thus, there is a $(1+\varepsilon)$ path between p and q in G', as desired. ∎

Size and running time.

Theorem 3.16. *Let P be a set of n points in the plane, and let $\varepsilon \in (0,1)$ be an approximation parameter. The above algorithm computes a local $(1+\varepsilon)$-spanner G for homothets of an α-fat triangle $\triangle$. The construction time is $O\left(\frac{1}{\alpha\varepsilon} n \log n\right)$, and the spanner G has $O\big(n/(\alpha\varepsilon)\big)$ edges.*

Proof: The local-spanning property is proven in Lemma 3.15, and we are only left with bounding the size and the running time of the algorithm. The bound on the size is immediate from the construction, as every point p is the apex of $O\left(\frac{2\pi}{\varepsilon\alpha}\right)$ cones, each giving rise to a single edge incident to p. The construction time is bounded by the construction time for a θ-graph with cone size $\alpha\varepsilon$, which is $O((\alpha\varepsilon)^{-1} n \log n)$ [Cla87]. ∎

3.5. A local spanner for nice polygons

3.5.1. A good jump for narrow trapezoids

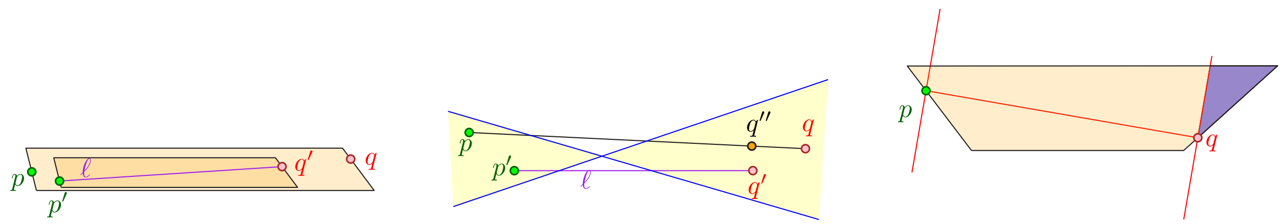
As a reminder, a trapezoids is a quadrilateral with two parallel edges, known as its *bases*. The other two edges are its legs. For $\varepsilon \in (0, 1/4)$, a trapezoid T is ***ε-narrow*** if the length of each of its legs is at most $\varepsilon \mathrm{diam}(T)$.

Lemma 3.17. *Let $\varepsilon \in (0,1)$ be some parameter, and $\vartheta = \varepsilon/16$. Let X, Y be two points sets that are ϑ-semi separated and ϑ-angularly separated (see Definition 2.6), and let T be ϑ-narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on the two legs of T. Then, one can compute a homothet $T' \subseteq T$ of T, such that:*

(I) There are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T-Delaunay triangulation of $X \cup Y$.

(II) We have that $(1+\varepsilon)\|pp'\| + \|p'q'\| + (1+\varepsilon)\|q'q\| \leq (1+\varepsilon)\|pq\|$.

Proof: Let $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(X \cup Y)$. Claim 3.4 implies that $\mathcal{D} \cap T$ is connected. Thus, there is a path in $\mathcal{D} \cap T$ between p and q, and thus, there must be an edge $p'q'$ along this path with $p' \in X$ and $q' \in Y$. This implies part (I).



(i) (ii) (iii)

Figure 3.9

Let $\ell = \|p'q'\|$. Assume for concreteness that $\|pp'\| \le \operatorname{diam}(X) \le \vartheta \mathsf{d}(X,Y) \le \vartheta\ell \le \vartheta d$, where $d = \operatorname{diam}(T)$. Let q'' be the closest point on pq to q'.

We first consider the case that $q'' \in \operatorname{int}(pq)$. We have that

$$\|pq''\| \ge \big(\|p'q'\| - \|pp'\|\big)\cos\angle p'pq' \ge (1-\vartheta)\ell(1-\vartheta^2/2) \ge (1-2\vartheta)\ell.$$

Similar argumentation implies that $\|pq''\| \le (1+\vartheta)\ell$. As such, we have

$$\|q'q''\| \le (1+\vartheta)\ell \sin\angle p'pq' \le 2\vartheta\ell.$$

Thus, we have that

$$\|qq'\| \le \|qq''\| + \|q''q'\| \le \|pq\| - \|pq''\| + 2\vartheta\ell \le \|pq\| - (1-2\vartheta)\ell + 2\vartheta\ell \le \|pq\| - \ell.$$

Thus, we have that

$$\begin{aligned}(1+\varepsilon)\|pp'\| + \|p'q'\| + (1+\varepsilon)\|q'q\| &\le (1+\varepsilon)\vartheta\ell + \ell + (1+\varepsilon)\big(\|pq\| - \ell\big)\\ &= (1+\varepsilon)\|pq\| + (1+\varepsilon)\vartheta\ell + \ell - (1+\varepsilon)\ell \le (1+\varepsilon)\|pq\|,\end{aligned}$$

for $\vartheta \le \varepsilon/2$. Which establish the claim in this case.

The case that $q'' = p$ is impossible, because of the angular separation property. Thus, the only remaining possibility is that $q'' = q$. This however implies that q' must be in the triangle of all the points of the trapezoids that their nearest point on pq is q. The diameter of this triangle is bounded by the length of the leg of the trapezoid, which is bounded by $\vartheta\, d$. Namely, we have $\|qq'\| \le \vartheta d$. Similarly, we have $(1-2\vartheta)d \le \|pq\| \le (1+2\vartheta)d$. Since We $\|pp'\|, \|qq'\| \le \vartheta d$, it follows that

$$(1-4\vartheta)d \le \ell \le (1+4\vartheta)d.$$

As such, for $\vartheta \le \varepsilon/16$ and $\varepsilon \le 1$, we have

$$\begin{aligned}(1+\varepsilon)\|pp'\| + \ell + (1+\varepsilon)\|q'q\| &\le 4\vartheta d + (1+4\vartheta)d = (1+8\vartheta)(1+4\vartheta)(1-2\vartheta)d\\ &\le (1+8\vartheta)(1+4\vartheta)\|pq\| \le (1+16\vartheta)\|pq\| \le (1+\varepsilon)\|pq\|.\end{aligned}$$

■

3.5.2. A local spanner for nice polygons

A convex polygon C is t-***nice*** polygon, if the outer angle at any vertex of the polygon is at least π/t, and the ratio of length of any two edges of C is bounded by some absolute constant.

Lemma 3.18. *Given a t-nice polygon, and a parameter ϑ, one can cover it by a set $\mathcal{T}$ of $O((t/\varepsilon)^2)$ ϑ-narrow trapezoids, such that for any two points $p, q \in \partial C$ that belong to two edges of C that are not adjacent, there exists A narrow trapezoid $T \in \mathcal{T}$ such that p, q are located on the two short legs of T.*

4. A local spanner for axis parallel squares

One can modify the above construction for axis-parallel squares, and get a local spanner without dependency on the spread.

4.0.1. Construction

The input is a point set P of n points in the plane, and an approximation parameter $\varepsilon \in (0, 1/2)$. We assume that the input point set P is in general position. Specifically, no two points of P share a coordinate value, or appear in opposing corners of an axis-parallel square – this can be ensured by slightly perturbing the points if necessary (or symbolic perturbation). Let $\mathcal{b}_\infty = [-1, 1]^2$ be the unit "ball" under the L_∞ norm.

Let $\vartheta = \varepsilon/20$. The algorithm computes a $1/\vartheta$-SSPD $\mathcal{W}$, using the algorithm of Theorem 2.4. By using the algorithm of Lemma 2.7, and increasing the weight and number of pairs by a factor of $\mathcal{O}(1/\vartheta)$, one can assume that every pair $\{X, Y\} \in \mathcal{W}$ is not only semi-separated, but also that there is an associated double wedge of angle $\leq \vartheta$ containing X and Y in opposing wedges, using the algorithm of Lemma 2.7. The algorithm now computes $\mathcal{b}_\infty$-Delaunay triangulation, see Definition 3.1, for each such pair, and adds the edges of the triangulation to the resulting graph G.

4.0.2. Analysis

Size and running time. Computing the SSPD takes $\mathcal{O}\big(n\vartheta^{-2}\log n\big)$ time, and the refinement takes $\mathcal{O}\big(n\vartheta^{-3}\log n\big)$ time (which is also the weight of the resulting SSPD). The number of edges of each L_∞-Delaunay triangulation for a pair is proportional to its weight, which implies that the total number of edges in the resulting graph G is $\mathcal{O}\big(\vartheta^{-3}n\log n\big)$. Computing all these Delaunay triangulations takes $\mathcal{O}\big(\vartheta^{-3}n\log^2 n\big)$ time.

Shrinking squares. We need the following lemma about the shrinking of axis-parallel squares. Observe that this property definitely does not hold for disks, as illustrated in Figure 3.2.

Lemma 4.1. *(A) Let $\mathfrak{s}$ be an axis parallel square in the plane, and let p, q be two arbitrary points in $\mathfrak{s}$. Then, there is a square $\mathfrak{t} \subseteq \mathfrak{s}$ that contains p and q on its boundary.*

(B) Let $\mathfrak{s}$ be as before, and let X, Y be two point sets in the plane, such that $X' = X \cap \mathfrak{s} \neq \emptyset$ and $Y' = Y \cap \mathfrak{s} \neq \emptyset$. Let $x \in X, y \in Y$ be the two points realizing $\mathsf{d}_\infty(X', Y') = \min_{p \in X', q \in Y'} \|p - q\|_\infty$. Then, there is a square $\mathfrak{t} \subseteq \mathfrak{s}$ that contains x and y on its boundary, and $\mathfrak{t}$ does not contain any other point of $X \cup Y$.

Proof: (A) Start shrinking $\mathfrak{s}$ around its center till it contains one of the points (say p is on its boundary. Next, move the center of the square towards p till the boundary of the continuously shrinking square passes through q. If p and q lie on adjacent edges, then continue the shrinking process by moving the center towards the common corner of the shared edges – this process stops when one of the points is on the corner of the square. Clearly, the resulting square $\mathfrak{t}$ is the desired square, see Figure 4.1.

(B) Let $r = \mathsf{d}_\infty(X', Y')$. By (A), there is a square $\mathfrak{t} \subseteq \mathfrak{s}$ having x and y on apposing sides. As such, the side length of $\mathfrak{t}$ is r. Assume for contradiction, that there is some other point $x' \in X \cap \mathfrak{t}$. By our general position assumption, x' is in the interior of $\mathfrak{t}$, and in particular, $\|x' - y\|_\infty < r$, which is a contradiction to the choice of x and y. ■

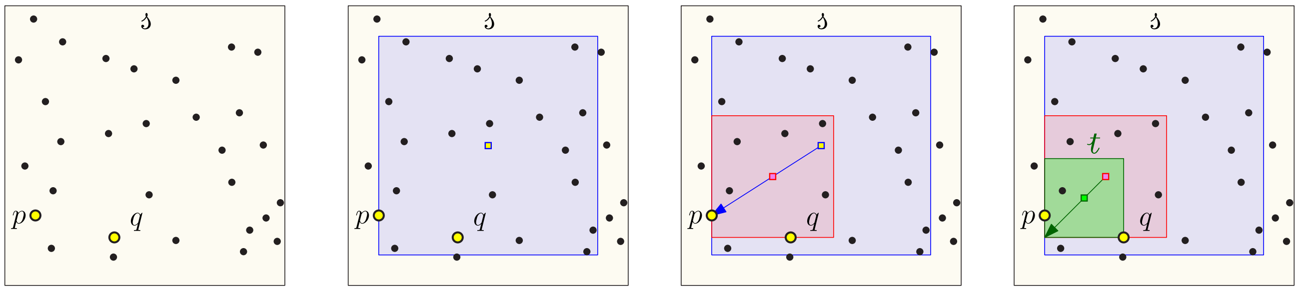


Figure 4.1

Local spanner property.

Lemma 4.2. *For any axis parallel square $\mathscr{s}$ in the plane, and any two points $p, q \in P \cap \mathscr{s}$, we have a $(1+\varepsilon)$-path in $G \cap \mathscr{s}$.*

Proof: We prove the existence of a $(1+\varepsilon)$-path between every pair $x, y \in P \cap \mathscr{s}$ of points, by induction over the rank of $\|xy\|$. The base case is simple, as the pair $(\{x\}, \{y\})$ is a pair in $\mathcal{W}$, and xy is thus an edge in G. Now, consider two points $x, y \in P \cap \mathscr{s}$, where $\mathscr{s}$ is some arbitrary square. There exists a pair $\Xi = \{X, Y\} \in \mathcal{W}$ such that $x \in X$ and $y \in Y$, and this pair is ϑ^{-1}-semi separated and is also separated by a double wedge of angle $\leq \vartheta$. See Figure 4.2. Furthermore, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$.

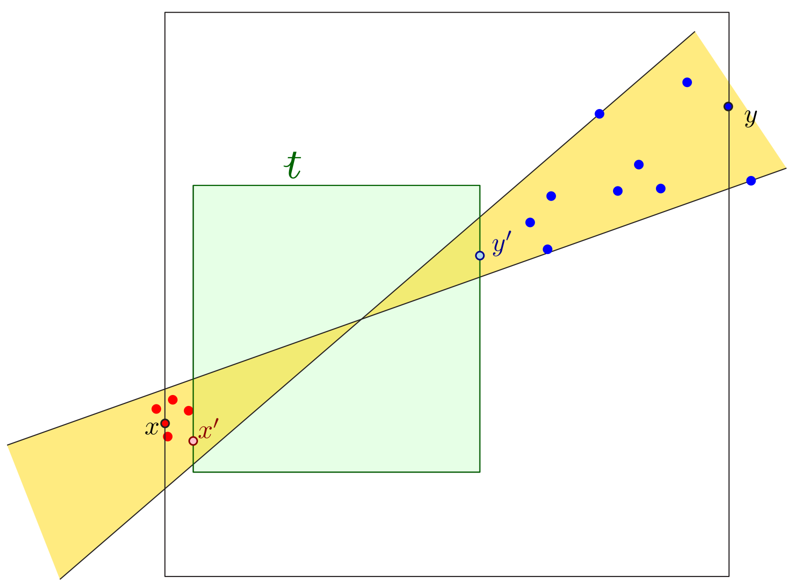


Figure 4.2: A square region $\mathscr{t}$ and two double-wedge semi-separated point sets X (red) and Y (blue. Notice that while x' and y' are the closest pair using L_∞, that is not necessarily true for the Euclidean distance.

Let $X' = X \cap \mathscr{s}$ and $Y' = Y \cap \mathscr{s}$, and consider the two points $x' \in X'$ and $y' \in Y'$ realizing $r = \mathsf{d}_\infty(X', Y')$. By Lemma 4.1 there exists a square $\mathscr{t}$ containing x', y' on its boundary (on two apposing edges), such that $\mathscr{t} \subseteq \mathscr{s}$, and $\mathscr{t}$ contains no other points $X \cup Y$. By construction, we have that $x'y'$ is in the $\mathscr{b}_\infty$-Delaunay triangulation of Ξ, and thus $x'y' \in G$. Since $\|xx'\| \ll \|xy\|$ we have that by induction $\mathsf{d}_G(x, x') \leq (1+\varepsilon)\|xx'\|$.

Let $\ell = \|x'y'\|$. Due to the semi-separation property and since $\mathrm{diam}(X) < \mathrm{diam}(Y)$,

$$\|xx'\| \leq \operatorname{diam}(X) \leq \vartheta \|XY\| \leq \vartheta\sqrt{2} \cdot \mathsf{d}_\infty(X, Y) \leq 2\vartheta\ell.$$

Thus, we have that

$$\mathsf{d}_G(x, x') \leq (1+\varepsilon) \|xx'\| \leq (1+\varepsilon)2\vartheta\ell \leq 4\vartheta\ell.$$

By the triangle inequality, we have

$$(1 - 2\vartheta)\ell \leq \|x'y'\| - \|xx'\| \leq \|xy'\| \leq (1 + 2\vartheta)\ell.$$

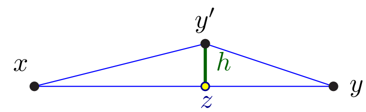


Figure 4.3: An illustration of the relative positions of x, y, y', and z. The angle of the separating double-wedge guarantees that $\angle y'xy$ is small.

Consider the triangle $\triangle xy'y$, and observe that by the double-wedge property $\alpha = \angle y'xy \leq \vartheta$. Let z be the projection of y' to xy, and let

$$h = \|y'z\| = \|xy'\| \sin\alpha \leq \|xy'\| \sin\vartheta \leq \|xy'\| \vartheta \leq \vartheta(1 + 2\vartheta)\ell \leq 2\vartheta\ell,$$

as $\vartheta \in (0, 1/10)$, the monotonicity of sin in this range, and as $\sin\vartheta \leq \vartheta$.

We have that $\|xz\| \leq \|xy'\| \leq (1 + 2\vartheta)\ell$. Similarly, we have

$$\|xz\| = \|xy'\| \cos\alpha \geq (1 - \alpha^2/2) \|xy'\| \geq (1 - \vartheta^2/2)(1 - 2\vartheta)\ell \geq (1 - 3\vartheta)\ell.$$

By the triangle inequality, we have that

$$\|y'y\| \geq \|xy\| - \|y'x\| \geq \|xy\| - (1 + 2\vartheta)\ell.$$

As for an upper bound, we have

$$\begin{aligned}\|y'y\| &\leq \|zy\| + h \leq \|xy\| - \|xz\| + 2\vartheta\ell \leq \|xy\| - (1 - 3\vartheta)\ell + 2\vartheta\ell \\ &= \|xy\| - (1 - 5\vartheta)\ell < \|xy\|.\end{aligned}$$

As such, by induction $\mathsf{d}_G(y', y) \leq (1 + \varepsilon) \|y'y\|$.

We thus have that

$$\begin{aligned}\mathsf{d}_G(x, y) &\leq \mathsf{d}_G(x, x') + \|x'y'\| + \mathsf{d}_G(y', y) \leq 4\vartheta\ell + \ell + (1 + \varepsilon) \|y'y\| \\ &\leq (1 + 4\vartheta)\ell + (1 + \varepsilon)\big(\|xy\| - (1 - 5\vartheta)\ell\big) \\ &= \big[1 + 4\vartheta - (1 + \varepsilon)(1 - 5\vartheta)\big]\ell + (1 + \varepsilon) \|xy\| \\ &\leq (1 + \varepsilon) \|xy\|,\end{aligned}$$

for $\vartheta \leq \varepsilon/20$, as $1 + 4\vartheta - (1 + \varepsilon)(1 - 5\vartheta) \leq 1 + \varepsilon/5 - (1 + \varepsilon)(1 - \varepsilon/4) = \varepsilon/5 - (3/4)\varepsilon + \varepsilon^2/4 < 0$, as $\varepsilon < 1$. ■

Theorem 4.3. *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be an approximation parameter. The above algorithm computes a local $(1+\varepsilon)$-spanner G for axis parallel squares. The construction time is $\mathcal{O}\big(\varepsilon^{-3} n \log^2 n\big)$, and the spanner G has $\mathcal{O}(\varepsilon^{-3} n \log n)$ edges.*

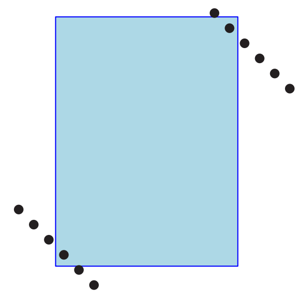


Figure 5.1: There are quadratic number of pairs of points that has to be connected in any local spanner for axis parallel rectangles. Indeed, for any point in the top diagonal and bottom diagonal, there is an axis parallel rectangle that contains only these two points. This holds even if we restrict ourselves to fat rectangles of similar size.

5. Weak local spanners for convex regions with bounded aspect ratio

We would like to build local spanners (of subquadratic size) for axis-parallel rectangles, but as Figure 5.1 shows, there is no hope of achieving this. As such, we need to somewhat change the requirements, and instead describe a weak local spanner for this case. The exact meaning of the weakness of the spanner, parameterized by δ is given below.

Definition 5.1. Given a convex region C, let

$$C_{\boxminus\delta} = \left\{ p \in C \;\middle|\; \mathsf{d}\big(p, \mathbb{R}^2 \setminus C\big) \geq \delta \cdot \mathrm{diam}(C) \right\}.$$

Formally, $C_{\boxminus\delta}$ is the Minkowski difference of C with a disk of radius $\delta \cdot \mathrm{diam}(C)$.

Definition 5.2. Consider a (bounded) set C in the plane. Let $r_{\mathrm{in}}(C)$ be the radius of the largest disk contained inside C. Similarly, $R_{\mathrm{out}}(C)$ is the smallest radius of a disk containing C.

The ***aspect ratio*** of a region C in the plane is $\mathsf{ar}(C) = R_{\mathrm{out}}(C)/r_{\mathrm{in}}(C)$. Given a family $\mathcal{F}$ or regions in the plane, its *aspect ratio* is $\mathsf{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \mathsf{ar}(C)$.

Note, that if a convex region C has bounded aspect ratio, then $C_{\boxminus\delta}$ is similar to the result of scaling C by a factor of $1 - O(\delta)$. On the other hand, if C is long and skinny, say is has width smaller than $2\delta \cdot \mathrm{diam}(C)$, then $C_{\boxminus\delta}$ is empty.

Lemma 5.3. *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0, 1)$. One can construct a graph G over P, in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2}) n \log n)$ time, and with $\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2}) n\big)$ edges, such that for any (bounded) convex C in the plane, we have that for any two points $p, q \in P \cap C_{\boxminus\delta}$ the graph $C \cap P$ has $(1 + \varepsilon)$-path between p and q.*

Proof: Let $\vartheta = \min(\varepsilon, \delta^2)$. Construct, in $\mathcal{O}(\vartheta^{-1} n \log n)$ time, a standard $(1 + \vartheta)$-spanner G for P using $\mathcal{O}(\vartheta^{-1} n)$ edges [AMS99].

So, consider any body $C \in \mathcal{F}$, and any two vertices $p, q \in P \cap C'$, where $C' = C_{\boxminus\delta}$. Let $\ell = \|pq\|$, let π be the shortest path between p and q in G, and let $\mathcal{E}$ be the locus of all points u, such that $\|pu\| + \|uq\| \leq (1 + \vartheta)\ell$. The region $\mathcal{E}$ is an ellipse that contains π. The furthest point from the segment pq in this ellipse is realized by the co-vertex of the ellipse. Formally, it is one of the two intersection

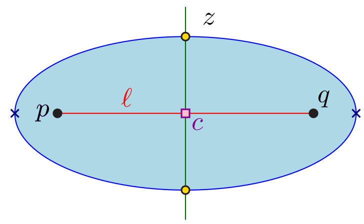


Figure 5.2

points of the boundary of the ellipse with the line orthogonal to $\overline{pq}$ that passes through the middle point c of this segment, see Figure 5.2. Let z be one of these points.

We have that $\|pz\| = (1+\vartheta)\ell/2$. Setting $h = \|zc\|$, we have that

$$h = \sqrt{\|pz\|^2 - \|pc\|^2} = \frac{\ell}{2}\sqrt{(1+\vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2+\vartheta)}}{2}\ell \le \sqrt{\vartheta}\ell \le \sqrt{\vartheta} \cdot \mathrm{diam}(C).$$

as $\ell \le \mathrm{diam}(C') \le \mathrm{diam}(C)$.

For any point $x \in C'$, we have that $\mathsf{d}(x, \mathbb{R}^2 \setminus C) \ge \delta \cdot \mathrm{diam}(C)$. As such, to ensure that $\pi \subseteq \mathcal{E} \subseteq C$, we need that $\delta \cdot \mathrm{diam}(C) \ge h$, which holds if $\delta \cdot \mathrm{diam}(C) \ge \sqrt{\vartheta} \cdot \mathrm{diam}(C)$. This in turn holds if $\vartheta \le \delta^2$. Namely, we have the desired properties if $\vartheta = \min(\varepsilon, \delta^2)$. ■

6. Weak local spanners for axis-parallel rectangles

6.1. Quadrant separated pair decomposition

For points $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ in $\mathbb{R}^d$, let $p \prec q$ denotes that q ***dominates*** p coordinate-wise. That is $p_i < q_i$, for all i. More generally, let $p <_i q$ denote that $p_i < q_i$. For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $\forall x \in X, y \in Y \quad x <_i y$. In particular X and Y are *i-coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is ***quadrant-separated***, if X and Y are i-coordinate separated, for $i = 1, \ldots, d$.

A ***quadrant-separated pair decomposition*** of a point set $P \subseteq \mathbb{R}^d$, is a pair decomposition (see Definition 2.1) $\mathcal{W} = \bigl\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\bigr\}$ of P, such that $\{X_i, Y_i\}$ are quadrant-separated for all i.

Lemma 6.1. *Given a set P of n points in $\mathbb{R}$, one can compute, in $\mathcal{O}(n \log n)$ time, a* QSPD *of P with $\mathcal{O}(n)$ pairs, and of total weight $\mathcal{O}(n \log n)$.*

Proof: If P is a singleton then there is nothing to do. If $P = \{p, q\}$, then the decomposition is the pair formed by the two singleton points.

Otherwise, let x be the median of P, such that $P_{\le x} = \{p \in P \mid p \le x\}$ contains exactly $\lceil n/2 \rceil$ points, and $P_{>x} = P \setminus P_{\le x}$ contains $\lfloor n/2 \rfloor$ points. Construct the pair $\Xi = \{P_{\le x}, P_{>x}\}$, and compute recursively a QSPDs $\mathcal{Q}_{\le x}$ and $\mathcal{Q}_{>x}$ for $P_{\le x}$ and $P_{>x}$, respectively. The desired QSPD is $\mathcal{Q}_{\le x} \cup \mathcal{Q}_{>x} \cup \{\Xi\}$. The bounds on the size and weight of the desired QSPD are immediate. ■

Lemma 6.2. *Given a set P of n points in $\mathbb{R}^d$, one can compute, in $\mathcal{O}(n \log^d n)$ time, a* QSPD *of P with $\mathcal{O}(n \log^{d-1} n)$ pairs, and of total weight $\mathcal{O}(n \log^d n)$.*

Proof: The construction algorithm is recursive on the dimensions, using the algorithm of Lemma 6.1 in one dimension.

The algorithm computes a value α_d that partitions the values of the points' dth coordinates roughly equally (and is distinct from all of them), and let h be a hyperplane parallel to the first $d-1$ coordinate axes, and having value α_d in the dth coordinate.

Let $P_\uparrow$ and $P_\downarrow$ be the subset of points of P that are above and below h, respectively. The algorithm recursively computes QSPDs $\mathcal{Q}_\uparrow$ and $\mathcal{Q}_\downarrow$ for $P_\uparrow$ and $P_\downarrow$, respectively. Next, the algorithm projects the points of P on h, let P' be the resulting $d-1$ dimensional point set (after we ignore the dth coordinate), and recursively computes a QSPD $\mathcal{Q}'$ for P'.

For a point set $X' \subseteq P'$, let $\mathrm{lift}(X')$ be the subset of points of P whose projection on h is X'. The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\mathrm{lift}(X') \cap P_\uparrow, \mathrm{lift}(Y') \cap P_\downarrow\},\ \{\mathrm{lift}(X') \cap P_\downarrow, \mathrm{lift}(Y') \cap P_\uparrow\} \;\middle|\; \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is $\widehat{\mathcal{Q}} \cup \mathcal{Q}_\uparrow \cup \mathcal{Q}_\downarrow$.

To observe that this is indeed a QSPD, observe that all the pairs in $\mathcal{Q}_\uparrow$, $\mathcal{Q}_\downarrow$ are quadrant separated by induction. As for pairs in $\widehat{\mathcal{Q}}$, they are quadrant separated in the first $d-1$ coordinates by induction on the dimension, and separated in the d coordinate since one side of the pair comes from $P_\uparrow$, and the other side from $P_\downarrow$.

As for coverage, consider any pair of points $p, q \in P$, and observe that the claim holds by induction if they are both in $P_\uparrow$ or $P_\downarrow$. As such, assume that $p \in P_\uparrow$ and $q \ni P_\downarrow$. But then there is a pair $\{X', Y'\} \in \mathcal{Q}'$ that separates the two projected points in h, and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates p and q as desired.

The number pairs in the decomposition is $N(n,d) = 2N(n,d-1) + 2N(n/2,d)$ with $N(n,1) = O(n)$. The solution to this recurrence is $N(n,d) = O(n \log^{d-1} n)$. The total weight of the decomposition is $W(n,d) = 2W(n,d-1) + 2W(n/2,d)$ with $W(n,1) = O(n \log n)$. The solution to this recurrence is $W(n,d) = O(n \log^d n)$. Clearly, this also bounds the construction time. ∎

6.2. Weak local spanner for axis parallel rectangles

For a parameter $\delta \in (0,1)$, and an interval $I = [b,c]$, let $(1-\delta)I = [t-(1-\delta)r, t+(1-\delta)r]$ be the shrinking of I by a factor of $1-\delta$, where $t = (b+c)/2$, and $r = (c-b)/2$.

Let $\mathcal{R}$ be the set of all axis parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$, with $R = I \times J$, let $(1-\delta)R = (1-\delta)I \times (1-\delta)J$ denote the rectangle resulting from shrinking R by a factor of $1-\delta$.

Definition 6.3. Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0,1)$, a graph G is a *$(1-\delta)$-local $(1+\varepsilon)$-spanner* for rectangles, if for any axis-parallel rectangle R, we have that $G \cap R$ is a $(1+\varepsilon)$-spanner for all the points in $(1-\delta)R \cap P$.

Observe that rectangles in $\mathcal{R}$ might be quite "skinny", so the previous notion of shrinkage used before are not useful in this case.

6.2.1. Construction for a single quadrant separated pair

Consider a pair $\Xi = \{X, Y\}$ in a QSPD of P. The set X is quadrant-separated from Y. That is, there is a point c_Ξ, such that X and Y are contained in two opposing quadrants in the partition of the plane formed by the vertical and horizontal line through c_Ξ.

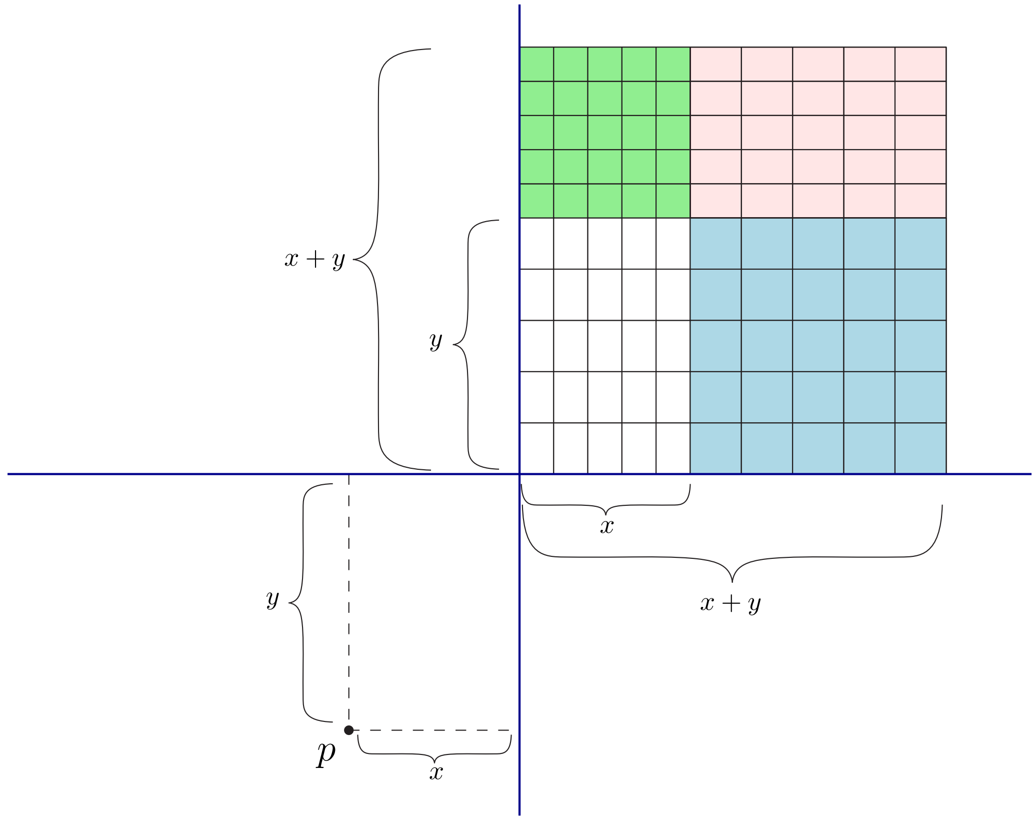


Figure 6.1: The construction of the grid $\mathsf{K}(p, \Xi)$ for a point $p = (-x, -y)$ and a pair Ξ.

For simplicity of exposition, assume that $c_\Xi = (0, 0)$, and $X \prec (0, 0) \prec Y$. That is, the points of X are in the negative quadrant, and the points of Y are in the positive quadrant.

Consider a point $p = (-x, -y) \in X$. Its *set of clients* in Y, is

$$\mathsf{C}(p, Y) = \left\{ q \in Y \mid \|q - c_\Xi\|_1 \leq \|p - c_\Xi\|_1 \right\}.$$

We construct a non-uniform grid $\mathsf{K}(p, \Xi)$ in the square $[0, x + y]^2$. To this end, we first partition it into four subrectangles

$B_{\nwarrow} = [0, x] \times [y, x + y]$	$B_{\nearrow} = [x, x + y] \times [y, x + y]$
$B_{\swarrow} = [0, x] \times [0, y]$	$B_{\searrow} = [x, x + y] \times [0, y]$.

Let $\tau \geq 4/\varepsilon + 4/\delta$ be an integer number. We partition each of these rectangles into a $\tau \times \tau$ grid, where each cell is a copy of the rectangle scaled by a factor of $1/\tau$. See Figure 6.1. This grid has $\mathcal{O}(\tau^2)$ cells. For a cell C in this grid, let $Y \cap \mathsf{C}$ be the points of Y contained in it. We connect p to the left-most and bottom-most points in $Y \cap \mathsf{C}$. This process generates two edges in the constructed graph for each grid cell, and $\mathcal{O}(\tau^2)$ edges overall.

The algorithm repeats this construction for all the points $p \in X$, and does the symmetric construction for all the points of Y.

6.2.2. The construction algorithm

The algorithm computes a QSPD $\mathcal{W}$ of P. For each pair $\Xi \in \mathcal{W}$, the algorithm generates edges for Ξ using the algorithm of Section 6.2.1 and adds them to the generated spanner G.

Figure 6.2

6.2.3. Correctness

For a rectangle R, let $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists (x', y) \in R\}$ be its expansion into a horizontal slab. Restricted to a rectangle R', the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure 6.2. Similarly, we denote $\updownarrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists (x, y') \in R\}$.

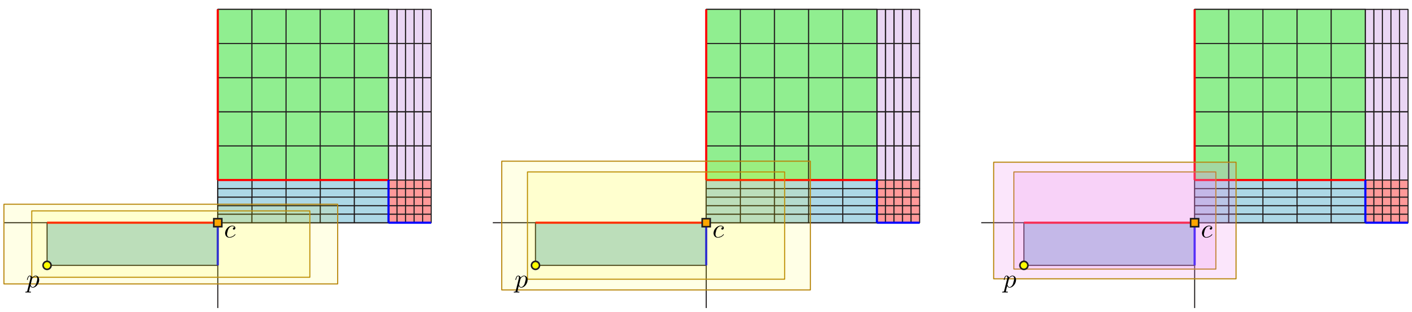


Figure 6.3

Lemma 6.4. *Assume that $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above construction, and a point $p = (-x, -y) \in X$, and its associated grid $\mathsf{K} = \mathsf{K}(p, \Xi)$. Consider any axis parallel rectangle R, such that $p \in (1-\delta)R = I \times J$, and $(1-\delta)R$ intersects a cell $\mathsf{C} \in \mathsf{K}$. We have that:*

(I) If $\mathsf{C} \subseteq (1-\delta)R$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.
(II) $\mathrm{diam}(\mathsf{C}) \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.
(III) If $x \geq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.
(IV) If $x \leq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\nwarrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.
(V) If $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$, then $(1-\delta)^{-1}\big(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\big) \subseteq R$.
(VI) If $x \leq y$ and $\mathsf{C} \subseteq R_{\searrow}$, then $(1-\delta)^{-1}\Big(\updownarrow\big((1-\delta)R\big) \cap \mathsf{C}\Big) \subseteq R$.

Proof: (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.
(II) We have that $\mathrm{diam}(\mathsf{C}) \leq (x+y)/\tau \leq \|p\|_1/\tau \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.
(III) The width, denoted $\mathsf{wd}(\cdot)$, of $(1-\delta)R$ is at least x, as it contains both p and the origin. As such,

$$\big(\mathsf{wd}(R) - \mathsf{wd}((1-\delta)R)\big)/2 \geq 2(x/\tau) \geq 2\mathsf{wd}(\mathsf{C}).$$

That is, the width of the "expanded" rectangle R is enough to cover C, and a grid cell adjacent to it to the right.

A similar argument about the height shows that R covers the region immediately above C – in particular, the vertical distance from C to the top boundary of R is at least the height of C. This implies that the expanded cell $(1-\delta)^{-1}\mathsf{C}$ is contained in R, as claimed, as $\delta < 1/2$.

(V) We decompose the claim to the two dimensions of the region. Let $B = \left(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\right)$. Observe that containment in the x-axis follows by arguing as in (III). As for the y-interval of B, observe that it is contained in the y-interval of $(1-\delta)R$, which implies that when expanded by $(1-\delta)^{-1}$, it would be contained in the y-interval of R. Combining the two implies the result. ■

Lemma 6.5. *For any axis-parallel rectangle R, and any two points $p, q \in (1-\delta)R \cap P$, there exists a $(1+\varepsilon)$-path between p and q in G.*

Proof: The proof is the spirit of the "standard" recursive proof for spanners, and is done by induction over the size of R (i.e. area, width, or height). Let $\Xi = \{X, Y\} \in \mathcal{W}$ be the pair in the QSPD that separates p and q, let c be the separation point of the pair, and assume for the simplicity of exposition that $p \in X$, $X \prec c \prec Y$, and $c = (0,0)$. Furthermore, assume that $\|p\|_1 \geq \|q\|_1$.

Let $p = (-x, -y)$, and let C be the grid cell of $\mathsf{K}(p, \Xi)$ that contains q. If $\mathsf{C} \subseteq (1-\delta)R$, then $(1-\delta)^{-1}\mathsf{C} \subseteq R$ by Lemma 6.4 (I). As such, let u be the leftmost point in $\mathsf{C} \cap P$. Both $q, u \in (1-\delta)^{-1}\mathsf{C}$, and by induction, there is an $(1+\varepsilon)$-path π between them in G (note that the induction applies to the two points, and the "expanded" rectangle $(1-\delta)^{-1}\mathsf{C}$). Since pu is an edge of G, prefixing π by this edge results in an $(1+\varepsilon)$-path, as $\|qu\| \leq (\varepsilon/4)\|pq\|$, by Lemma 6.4 (II) (verifying this requires some standard calculations which we omit).

Otherwise, one need to apply the same argument using the appropriate case of Lemma 6.4. So assume that $x \geq y$ (the case that $y \geq x$ is handled symmetrically). If $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$, then (III) implies that $(1-\delta)^{-1}\mathsf{C} \subseteq R$. Which implies that induction applies, and the claim holds.

The remaining case is that $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$. Let $D = \overleftrightarrow{(1-\delta)R} \cap \mathsf{C}$. By (V), we have $(1-\delta)^{-1}\left(D\right) \subseteq R$. Namely, $q \in (1-\delta)R \cap \mathsf{C} \subseteq D$, and let u be the lowest point in $\mathsf{C} \cap P$. By construction $pu \in E(G)$, $q, u \in D$, $(1-\delta)^{-1}D \subseteq R$. As such, we can apply induction to q, u, and $(1-\delta)^{-1}D$, and conclude that $\mathsf{d}_G(q, u) \leq (1+\varepsilon)\|qu\|$. Plugging this into the regular machinery implies the claim. ■

Theorem 6.6. *Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0,1)$ be parameters. The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ edges. The graph G is a $(1-\delta)$-local $(1+\varepsilon)$-spanner for axis parallel rectangles. Formally, for any axis-parallel rectangle R, we have that $R \cap P$ is an $(1+\varepsilon)$-spanner for all the points of $\left((1-\delta)R\right) \cap P$.*

Proof: Computing the QSPD $\mathcal{W}$ takes $\mathcal{O}(n \log^2 n)$ time. For each pair $\{X, Y\}$ in the decomposition with $m = |X| + |Y|$ points, we need to compute the lowest and leftmost points in $(X \cup Y) \cap \mathsf{C}$, for each cell in the constructed grid. This can readily be done using orthogonal range trees in $\mathcal{O}(\log^2 n)$ time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is $\mathcal{O}(\omega(\mathcal{W})\tau^2) = O\left((1/\delta^2 + 1/\varepsilon^2)n \log^2 n\right)$.

The desired local spanner property is provided by Lemma 6.5. ■

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A. Some missing proofs

A.1. Proof of Lemma 2.5

Restatement of Lemma 2.5. *Given an α-SSPD $\mathcal{W}$ of a set P of n points in $\mathbb{R}^d$ and a parameter $\beta \geq 2$, one can refine $\mathcal{W}$ into an $\alpha\beta$-SSPD $\mathcal{W}'$, such that that $|\mathcal{W}'| = O(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\beta^d)$.*

Proof: The algorithm scans the pairs of $\mathcal{W}$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$. Let $\mathfrak{s}$ be the smallest axis-parallel cube containing X, and denote its sidelength by r. Let $r' = r/\left\lceil \sqrt{d}\beta \right\rceil$. Partition $\mathfrak{s}$ into a grid of cubes of sidelength r', and let T_Ξ be the resulting set of squares. The algorithm now add the set pairs

$$\{\{X \cap t, Y\} \mid t \in T_\Xi\}$$

to the output SSPD. Clearly, the resulting set is now $\alpha\beta$-semi separated, as we chopped the smaller part of each pair into β smaller portions. ■

A.2. Proof of Lemma 2.7

Restatement of Lemma 2.7. *Given a ε^{-1}-SSPD $\mathcal{W}$ of n points in the plane, one can refine it, into a ε^{-1}-SSPD $\mathcal{W}'$, such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε-double-wedge $\times_\Xi$, such that X and Y are contained in the two different faces of the double wedge $\times_\Xi$. We have that $|\mathcal{W}'| = O(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\varepsilon)$. The construction time is proportional to the weight of $\mathcal{W}'$.*

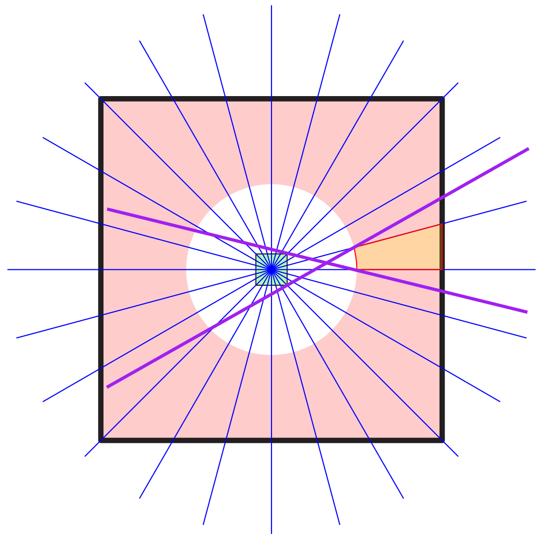

Figure A.1: An illustration of refining the pairs in a SSPD into pairs in opposite parts of an ε-double-wedge. X is contained in the green square $\square$, while Y is contained in the red square, and the white gap between them is a result of the separation property. The set of cones with the apex at the center of $\square$ give us the desired partition as demonstrated by the purple double-wedge.

Proof: By using Lemma 2.5, we can assume that $\mathcal{W}$ is (say) $(10/\varepsilon)$-separated. Now, the algorithm scans the pairs of $\mathcal{W}$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$. Let $\square$ be the smallest axis-parallel square containing X, centered at point o. Partition the plane around o, by drawing around it $\mathcal{O}(1/\varepsilon)$ lines with the angle between any two consecutive lines being at most (say) $\varepsilon/4$, see Figure A.1. This partitions the plane into a set of cones $\mathfrak{C}$. For a cone $C \in \mathfrak{C}$, we show that there exists an ε-double-wedge that contains X in one side, and $Y \cap C$ in the other.

To see that, take the double-wedge formed by the cross tangents between $\mathsf{ch}(X)$ and $\mathsf{ch}(Y \cap C)$, where $\mathsf{ch}(X)$ denotes the convex-hull of X. Assume w.l.o.g that $\square$ has side length 1, and let c be a cone of angle $\varepsilon/4$ with apex o, whose angular bisector is a horizontal ray in the positive direction of the x axis. See figure Figure A.2 for an illustration.

We would like to find a vertical segment s such that all points of Y lie to its right, with one endpoint on the upper line of c, and the other on the lower line of c. Using the segments' height and distance from the right side of $\square$ we will be able to get a bound on the angle of the cross tangents. We first find a segment s with all points of Y to its right. A trivial bound on that distance is given by the segment from, say, the lower left corner of $\square$, denoted p, of length $10/\varepsilon$ with its right endpoint on the upper line of c, denote this point by q. This is due to the $10/\varepsilon$ separation property of the SSPD. We know that this segment creates an angle of less than $\pi/4$ with the x-axis, since o is the center of $\square$, and lies on the ray with apex p that creates a $\pi/4$ angle with the x-axis. We therefore get that the x-coordinate difference between $\square$ and q is at most $10/\varepsilon \cdot \cos\frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$. So let s' be a vertical segment between the upper and lower rays of c, with x-coordinate distance of $6/\varepsilon - \frac{1}{2}$ from $\square$ (in order to make calculations easier). We get that s' is of length $2 \cdot \frac{6}{\varepsilon}\tan\frac{\varepsilon}{8}$. Finally, we take s to be a vertical segment of length $\frac{12}{\varepsilon}\tan\frac{\varepsilon}{8}$, with its center on the x-axis at a distance of $5/\varepsilon + \frac{1}{2}$ away from o. The angle of the x-axis and the segment between the lower end of the right side of $\square$ and the upper end of s is now given by:

$$\arctan\left(\frac{\frac{6}{\varepsilon}\tan\frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5}\tan\frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) = \Theta(\varepsilon) \qquad \blacksquare$$

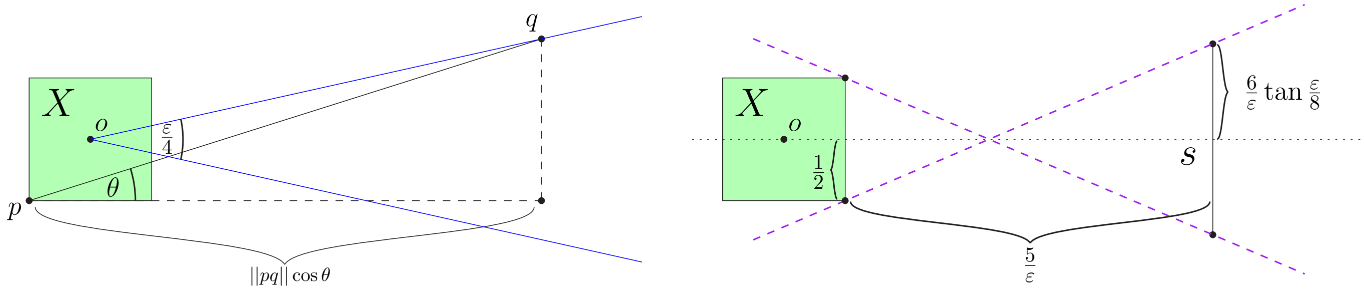

Figure A.2: An illustration of the proof for Lemma 2.7

A.3. Proof of Claim 3.4

Restatement of Claim 3.4. *Let $\mathcal{C}$ be a bounded close convex shape. Given a set of points $P \subseteq \mathbb{R}^2$ in general position for $\mathcal{C}$, let $\mathcal{D} = \mathcal{D}_\mathcal{C}(P)$ be the $\mathcal{C}$-Delaunay triangulation of P. For any homothet C of $\mathcal{C}$, we have that $\mathcal{D} \cap C$ is connected.*

Proof: We first prove that for any homothet C with two points $p, q \in P$ on its boundary, there is a path between p and q in $\mathcal{D} \cap C$. The proof is by induction over the number m of points of P in the interior of C. If $m = 0$ then C contains no points of P in its interior, and thus pq is an edge of the Delaunay triangulation, as C testifies.

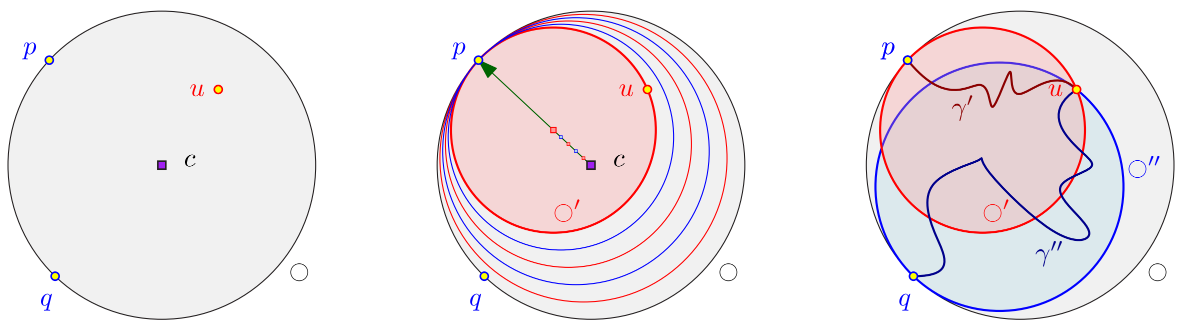

Figure A.3

Otherwise, let $u \in P$ be a point in the interior of C. We move the center c of C in the direction of p, shrinking C in the process, so that the radius the disk is $\|cp\|$, until we get a disk $C' \subseteq C$ such that u is on the boundary of C', see Figure A.3. Observe that p and u are on the boundary of the new disk, and $|\mathrm{int}(C') \cap P| < |\mathrm{int}(C) \cap P|$. Thus, by induction, there is a path γ' between p and u in $\mathcal{D} \cap C' \subseteq \mathcal{D} \cap C$. Similarly, there must be a path γ'' between u and q, and concatenating the two paths results in a path between p and q in $\mathcal{D} \cap C$.

Back to the original claim. For any two points $p, q \in C \cap P$ one can get a disk $C' \subseteq C$ that contains p and q on its boundary. Indeed, shrink the radius of C till, say, p is on the boundary, and then move the center of the disk towards p while shrinking the size of the disk to maintain p on the boundary, until q is also on the boundary of the shrunken disk. ■