

# Local Spanners Revisited

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## Abstract

For a set of points  $P \subseteq \mathbb{R}^2$ , and a family of regions  $\mathcal{F}$ , a *local  $t$ -spanner* of  $P$ , is a sparse graph  $G$  over  $P$ , that remains a  $t$ -spanner even when restricted to a region  $r \in \mathcal{F}$ . That is, for any region  $r \in \mathcal{F}$ , the subgraph restricted to  $r$ , denoted by  $G \cap r$ , is still a  $t$ -spanner for all the points of  $r \cap P$ . Here  $G \cap r$  is the subgraph of  $G$  induced on  $P \cap r$ .

We present algorithms for the construction of local spanners with respect to several families of convex regions. In particular, we construct a near linear size local spanner for homothets of a convex region. Unfortunately, the number of edges in the resulting graph depends logarithmically on the spread of the input point set. We prove that this dependency can not be removed, thus settling an open problem raised by Abam and Borouny. We also show improved constructions of local spanners for fat triangles, and regular  $k$ -gons. In particular, this improves over the known construction for axis parallel squares.

Finally, we study a somewhat weaker notion of local spanner where one allows to shrink the region a “bit”. We show that any spanner is a weak local spanner if one is allowed to shrink the region by some fraction of its diameter. Surprisingly, we show a near linear size construction of a weak spanner for axis-parallel rectangles, where the shrinkage is *multiplicative*.

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## 1 Introduction

For a set  $P$  of points in  $\mathbb{R}^d$ , the *Euclidean graph*  $G_P = (P, \binom{P}{2})$  of  $P$  is an undirected graph. Here, an edge  $pq \in E$  is associated with the segment  $pq$ , and its weight is the (Euclidean) length of the segment. Let  $G = (P, E)$  and  $I = (P, E')$  be two graphs over the same set of vertices (usually  $I$  is a subgraph of  $G$ ). Consider two vertices  $p, q \in P$ , and parameter  $t \geq 1$ . A path  $\pi$  between  $p$  and  $q$  in  $I$ , is a  *$t$ -path*, if the length of  $\pi$  in  $I$  is at most  $t d_G(p, q)$ , where  $d_G(p, q)$  is the length of the shortest path between  $p$  and  $q$  in  $G$ . The graph  $I$  is a  *$t$ -spanner* of  $G$  if there is a  $t$ -path in  $I$ , for any  $p, q \in P$ . Thus, for a set of points  $P \subseteq \mathbb{R}^d$ , a graph  $G$  over  $P$  is a  *$t$ -spanner* if it is a  $t$ -spanner of the euclidean graph  $G_P$ . There is a lot of work on building geometric spanners, see [NS07] and references there in.

## Fault-tolerant spanners

An  *$\mathcal{F}$ -fault-tolerant spanner* for  $P \subseteq \mathbb{R}^d$ , is a graph  $G = (P, E)$ , such that for any region  $r$  (i.e., the “attack”), the graph  $G - r$  is a  $t$ -spanner of  $G_P - r$ . Surprisingly, as shown by Abam *et al.* [AdBFG09], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size  $k$  of vertices and edges has failed. Levkopoulos *et al.* [LNS02]

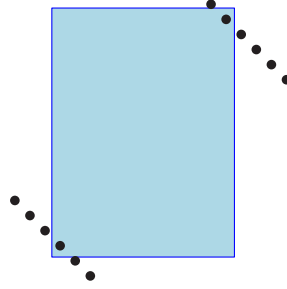
showed the existence of  $k$ -vertex/edges fault tolerant spanners for a set of points  $P$  in some metric space. Their spanner had  $\mathcal{O}(kn \log n)$  edges, and weight, i.e. sum of edge weights, bounded by  $f(k) \cdot wt(MST(P))$  for some function  $f$ . Lukovszki [Luk99] later achieved a similar construction, improving the number of edges to  $\mathcal{O}(kn)$ , and was able to prove that the result is asymptotically tight.

### Local spanners

Recently, Abam and Borouny [AB21] introduced the notion of local spanners. For a family of regions  $\mathcal{F}$ , a graph  $G = (P, E)$  is a *local  $t$ -spanner* for  $\mathcal{F}$ , if for any  $r \in \mathcal{F}$ , the subgraph of  $G$  induced on  $P \cap r$  is a  $t$ -spanner. Specifically, this induced subgraph  $G \cap r$  contains a  $t$ -path between any  $p, q \in P \cap r$  (note, that we keep an edge in the subgraph only if both its endpoints are in  $r$ ).

Abam and Borouny [AB21] showed how to construct such spanners for axis-parallel squares and vertical slabs. In this work, we are further extending their results. They also showed how to construct such spanners for disks, if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

To appreciate the difficulty in constructing local spanner, observe that unlike regular spanners, the construction has to take into account many different scenarios as far as which points are available to be used in the spanner. As a concrete example, a local spanner for axis-parallel rectangle requires quadratic number of edges, see Figure 1.1.



**Figure 1.1** For any point in the top diagonal and bottom diagonal, there is a fat axis parallel rectangle that contains only these two points. Thus, a local spanner requires quadratic size in this case.

Namely, regular spanners can rely on using midpoints in their path under the assurance that they are always there. For local spanners this is significantly harder as natural midpoints might “disappear”. Intuitively, a local spanner construction needs to use midpoints that are guaranteed to be present judging only from the source and destination points of the path.

### A good jump is hard to find

Most constructions for spanners can be viewed as searching for a way to build a path from the source to the destination by finding a “good” jump. Either by finding a way to move locally from the source to a near by point in the right direction, as done in the  $\theta$ -graph construction, or alternatively, by finding an edge in the spanner from the neighborhood of the source to the neighborhood of the destination, as done in the spanner constructions using well-separated pairs decomposition (WSPD). Usually, one argues inductively that the spanner must have (sufficiently short) paths from the source to the start of the jump, and

| Region  | # edges                                  | Paper     | New # edges   | Location in paper |
|---|--|-----------|---|-------------------|
| Local $(1 + \varepsilon)$ -spanners                 |  |           |   |                   |
| Halfplanes  | $\mathcal{O}(\varepsilon^{-2} n \log n)$ | [AdBFG09] |   |                   |
| Axis-parallel squares                               | $\mathcal{O}_\varepsilon(n \log^6 n)$    | [AB21]    | $\mathcal{O}(\varepsilon^{-3} n \log n)$                  | Remark 31         |
| Vertical slabs                                      | $\mathcal{O}(\varepsilon^{-2} n \log n)$ | [AB21]    |   |                   |
| Disks+Steiner points                                | $\mathcal{O}_\varepsilon(n)$             | [AB21]    |   |                   |
| Disks   |  |           | $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$               | Theorem 17        |
|   |  |           | $\Omega(n \log \Phi)$                                     | Lemma 21          |
| Homothets convex shape                              |  |           | $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$               | Theorem 17        |
| Homothets $\alpha$ -fat triangles                   |  |           | $\mathcal{O}((\alpha\varepsilon)^{-1} n)$                 | Theorem 27        |
| Homothets triangles                                 |  |           | $\Omega(n \log \Phi)$                                     | Lemma 22          |
| $\delta$ -weak local $(1 + \varepsilon)$ -spanners  |  |           |   |                   |
| Bounded convex shape                                |  |           | $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$          | Lemma 11          |
| $(1 - \delta)$ -local $(1 + \varepsilon)$ -spanners |  |           |   |                   |
| Axis-parallel rectangles                            |  |           | $\mathcal{O}((\varepsilon^{-2} + \delta^{-2})n \log^2 n)$ | Theorem 37        |

■ **Figure 1.2** Known and new results. The notation  $\mathcal{O}_\varepsilon$  hides polynomial dependency on  $\varepsilon$  which is not specified in the original work.

from the end of the jump to the destination. Then, combining these implies that the resulting new path is short. These ideas guide our constructions as well. However, the availability of specific edges depends on the query region, making the search for a good jump significantly more challenging. The constructions have to guarantee that there are many edges available, and at least one of them is useful as a jump.

## Our results

### Almost local spanners

We start by showing that regular geometric spanners are local spanners if one is required provide the spanner guarantee only to shrunken region. Namely, if  $G$  is a  $(1 + \varepsilon)$ -spanner of  $P$ , then for any convex region  $\mathcal{C}$ , the graph  $G \cap \mathcal{C}$  is a spanner for  $\mathcal{C}' \cap P$ , where  $\mathcal{C}'$  is the set of all points in  $\mathcal{C}$  that are in distance at least  $\varepsilon \cdot \text{diam}(\mathcal{C})$  from its boundary.

### Homothets

A *homothet* of a convex region  $\mathcal{C}$ , is a translated and scaled copy of  $\mathcal{C}$ . In Section 3 we present a construction of spanners, which surprisingly, is not only fault-tolerant for all convex regions, but it is also a local spanner for homothets of a prespecified convex region. This in particular works for disks, and resolves the aforementioned open problem from Abam and Borouny [AB21]. Our construction is somewhat similar to the original construction of Abam *et al.* [AdBFG09]. For a parameter  $\varepsilon > 0$  the construction of a  $(1 + \varepsilon)$ -local spanner for homothets takes  $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$  time, and the resulted spanner is of size  $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ , where  $\Phi$  is the spread of the input point set  $P$ , and  $n = |P|$ . We also provide a lower bound showing that this logarithmic dependency on  $\Phi$  cannot be avoided.

104 The dependency on the spread  $\Phi$  in the above construction is somewhat disappointing.  
 105 However, the lower bound constructions, provided in [Section 3.3](#), show that this is unavoidable  
 106 for disks or homothets of triangles.

107 Thus, the natural question is what are the cases where one can avoid the “curse of the  
 108 spread” – that is, cases where one can construct local spanners of size independent of the  
 109 spread of the input point set.

### 110 The basic building block: $\mathcal{C}$ -Delaunay triangulation

111 A key ingredient in the above construction, is the concept of Delaunay triangulation induced  
 112 by homothets of a convex body. Intuitively, one replaces the unit disk (of the standard  
 113  $L_2$ -norm) by the provided convex region. It is well known ([CDI85]) that such diagrams  
 114 exist, have linear complexity in the plane, and can be computed quickly. In [Section 3.1](#) we  
 115 review these results, and restate the well-known property that the  $\mathcal{C}$ -Delaunay triangulation  
 116 is connected when restricted to a homothet of  $\mathcal{C}$ . By computing these triangulations for  
 117 carefully chosen subsets of the input point set, we get the results stated above.

118 Specifically, we use well-separated and semi-separated decompositions to compute these  
 119 subsets.

### 120 Fat triangles

121 In [Section 3.4](#) we give a construction of local spanners for the family  $\mathcal{F}$  of homothets of a  
 122 given triangle  $\triangle$ , and get a spanner of size  $\mathcal{O}((\alpha\varepsilon)^{-1}n)$  in  $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$  time, where  $\alpha$   
 123 is the smallest angle in  $\triangle$ . This construction is a careful adaptation of the  $\theta$ -graph spanner  
 124 construction to the given triangle, and it is significantly more technically challenging than  
 125 the original construction.

### 126 $k$ -regular polygons

127 It seems natural that if one can handle fat triangles, then homothets of  $k$ -regular polygons  
 128 should readily follow by a simple decomposition of the polygon into fat triangles. Maybe  
 129 surprisingly, this is not the case – a critical configuration might involve two points that are  
 130 on the interior of the edges of a homothet of the input polygon. We overcome this by first  
 131 showing that sufficiently narrow trapezoids, provide us with a good jump somewhere inside  
 132 the trapezoid, assuming one compute the Delaunay triangulation induced by the trapezoid,  
 133 and that the source and destination lies on the two legs of the trapezoid. Next, we show that  
 134 such a polygon can be covered by a small number of narrow trapezoids and fat triangles.  
 135 By building appropriate graphs for each trapezoid/triangle in the collection, we prove that  
 136 the resulting graph is a spanner for homothets of the given  $k$ -regular polygon, and that the  
 137 resulting graph size has no dependency on the spread. Of course, the size does depend on  $k$ .  
 138 See [Section 3.5](#) for details, and [Theorem 30](#) for the precise result.

### 139 Quadrant separated pair decomposition (QSPD)

140 In [Appendix A.1](#), we describe a novel pair-decomposition. Specifically, the QSPD breaks  
 141 the input point set  $P$  into pairs, such that for any pair  $\{X, Y\}$  we have the property that  
 142 there is a translated axis system, such that  $X$  and  $Y$  belong to two antipodal quadrants.  
 143 In  $d$  dimensions there is such a decomposition with  $\mathcal{O}(n \log^{d-1} n)$  pairs, and total weight  
 144  $\mathcal{O}(n \log^d n)$ . A somewhat similar idea was used by Abam and Borouny [AB21] for the  $d = 1$   
 145 case. We believe this decomposition might be useful and is of independent interest.

## Multiplicative weak local spanner for rectangles

In [Appendix A.2](#), we use QSPDs to construct a weak local spanner for axis parallel rectangles. Here, the constructed graph  $G$  over  $P$ , has the property that for any axis-parallel rectangle  $R$ , the graph  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for all the points of  $((1 - \varepsilon)R) \cap P$ , where  $(1 - \varepsilon)R$  is the scaling of the rectangle by  $1 - \varepsilon$  around its center. Importantly, this works for narrow rectangles when this form of multiplicative shrinking is still meaningful (unlike the diameter based shrinking mentioned above). Contrast this with the lower bound of  $\Omega(n^2)$ , illustrated in [Figure 1.1](#), on the size of local spanner if one does not shrink the rectangles. See [Theorem 37](#) for details of the precise result.

See [Figure 1.2](#) for a summary of known results and comparisons to the results of this paper.

## 2 Preliminaries

### Residual graphs

Let  $\mathcal{F}$  be a family of regions in the plane. For a fault region  $r \in \mathcal{F}$  and a geometric graph  $G$  on a point set  $P$ , let  $G - r$  be the residual graph after removing from it all the points of  $P$  in  $r$ . Similarly, let  $G \cap r$  denote the graph restricted to  $r$ . Formally, let

$$G - r = (P \setminus r, \{uv \in E \mid uv \cap \text{int}(r) = \emptyset\}) \quad \text{and} \quad G \cap r = (P \cap r, \{uv \in E \mid uv \subseteq r\}).$$

where  $\text{int}(r)$  denotes the interior of  $r$ .

### 2.1 On various pair decompositions

For sets  $X, Y$ , let  $X \otimes Y = \{\{x, y\} \mid x \in X, y \in Y, x \neq y\}$  be the set of all the (unordered) pairs of points formed by the sets  $X$  and  $Y$ .

► **Definition 1** (Pair decomposition). *For a point set  $P$ , a **pair decomposition** of  $P$  is a set of pairs*

$$\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\},$$

such that (I)  $X_i, Y_i \subseteq P$  for every  $i$ , (II)  $X_i \cap Y_i = \emptyset$  for every  $i$ , and (III)  $\bigcup_{i=1}^s X_i \otimes Y_i = P \otimes P$ . Its **weight** is  $\omega(\mathcal{W}) = \sum_{i=1}^s (|X_i| + |Y_i|)$ .

The **closest pair** distance of a set of points  $P \subseteq \mathbb{R}^d$ , is  $\text{cp}(P) = \min_{p, q \in P, p \neq q} \|pq\|$ . The **diameter** of  $P$  is  $\text{diam}(P) = \max_{p, q \in P} \|pq\|$ . The **spread** of  $P$  is  $\Phi(P) = \text{diam}(P) / \text{cp}(P)$ , which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD (defined below) can be quadratic, if the spread is bounded, the weight is near linear. For  $X, Y \subseteq \mathbb{R}^d$ , let  $\text{d}(X, Y) = \min_{p \in X, q \in Y} \|pq\|$  be the **distance** between the two sets.

► **Definition 2.** *Two sets  $X, Y \subseteq \mathbb{R}^d$  are*

$$\text{1/}\varepsilon\text{-well-separated} \quad \text{if} \quad \max(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \text{d}(X, Y),$$

$$\text{and} \quad \text{1/}\varepsilon\text{-semi-separated} \quad \text{if} \quad \min(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \text{d}(X, Y).$$

For a point set  $P$ , a **well-separated pair decomposition (WSPD)** of  $P$  with parameter  $1/\varepsilon$  is a pair decomposition of  $P$  with a set of pairs  $\mathcal{W} = \{\{B_1, C_1\}, \dots, \{B_s, C_s\}\}$ , such that, for all  $i$ , the sets  $B_i$  and  $C_i$  are  $(1/\varepsilon)$ -separated. The notion of  $1/\varepsilon$ -SSPD (aka **semi-separated pairs decomposition**) is defined analogously.

► **Lemma 3** ([AH12]). Let  $P$  be a set of  $n$  points in  $\mathbb{R}^d$ , with spread  $\Phi = \Phi(P)$ , and let  $\varepsilon > 0$  be a parameter. Then, one can compute a  $(1/\varepsilon)$ -WSPD  $\mathcal{W}$  for  $P$  of total weight  $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$ . Furthermore, any point of  $P$  participates in at most  $\mathcal{O}(\varepsilon^{-d} \log \Phi)$  pairs.

► **Theorem 4** ([AH12, Har11]). Let  $P$  be a set of  $n$  points in  $\mathbb{R}^d$ , and let  $\varepsilon > 0$  be a parameter. Then, one can compute a  $(1/\varepsilon)$ -SSPD for  $P$  of total weight  $\mathcal{O}(n\varepsilon^{-d} \log n)$ . The number of pairs in the SSPD is  $\mathcal{O}(n\varepsilon^{-d})$ , and the computation time is  $\mathcal{O}(n\varepsilon^{-d} \log n)$ .

The proof of the following two lemmas is straightforward, and are delegated to the appendix.

► **Lemma 5** (Proof in Appendix B.1). Given an  $\alpha$ -SSPD  $\mathcal{W}$  of a set  $P$  of  $n$  points in  $\mathbb{R}^d$  and a parameter  $\beta \geq 2$ , one can refine  $\mathcal{W}$  into an  $\alpha\beta$ -SSPD  $\mathcal{W}'$ , such that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$ .

► **Definition 6.** An  $\varepsilon$ -double-wedge is a region between two lines, where the angle between the two lines is at most  $\varepsilon$ .

Two points sets  $X$  and  $Y$  that each lie on their own cone of a shared  $\varepsilon$ -double-wedge are  $\varepsilon$ -angularly separated.

► **Lemma 7** (Proof in Appendix B.2). Given a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}$  of  $n$  points in the plane, one can refine it, into a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}'$ , such that each pair  $\Xi = \{X, Y\} \in \mathcal{W}'$  is contained in a  $\varepsilon$ -double-wedge  $\times_\Xi$ , such that  $X$  and  $Y$  are contained in the two different faces of the double wedge  $\times_\Xi$ . We have that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$ . The construction time is proportional to the weight of  $\mathcal{W}'$ .

► **Corollary 8.** Let  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon > 0$  be a parameter. Then, one can compute a  $(1/\varepsilon)$ -SSPD for  $P$  such that every pair is  $\varepsilon$ -angularly separated. The total weight of the SSPD is  $\mathcal{O}(n\varepsilon^{-3} \log n)$ . The number of pairs in the SSPD is  $\mathcal{O}(n\varepsilon^{-3})$ , and the computation time is  $\mathcal{O}(n\varepsilon^{-3} \log n)$ .

## 2.2 Weak local spanners for fat convex regions

► **Definition 9.** Given a convex region  $C$ , let

$$C_{\Box\delta} = \{p \in C \mid d(p, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)\}.$$

Formally,  $C_{\Box\delta}$  is the Minkowski difference of  $C$  with a disk of radius  $\delta \cdot \text{diam}(C)$ .

► **Definition 10.** Consider a (bounded) set  $C$  in the plane. Let  $r_{\text{in}}(C)$  be the radius of the largest disk contained inside  $C$ . Similarly,  $R_{\text{out}}(C)$  is the smallest radius of a disk containing  $C$ .

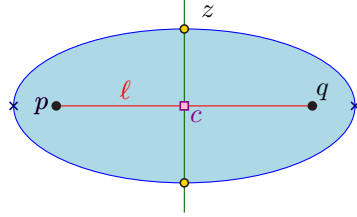
The **aspect ratio** of a region  $C$  in the plane is  $\text{ar}(C) = R_{\text{out}}(C)/r_{\text{in}}(C)$ . Given a family  $\mathcal{F}$  of regions in the plane, its aspect ratio is  $\text{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \text{ar}(C)$ .

Note, that if a convex region  $C$  has bounded aspect ratio, then  $C_{\Box\delta}$  is similar to the result of scaling  $C$  by a factor of  $1 - \mathcal{O}(\delta)$ . On the other hand, if  $C$  is long and skinny then this region is much smaller. Specifically, if  $C$  has width smaller than  $2\delta \cdot \text{diam}(C)$ , then  $C_{\Box\delta}$  is empty.

► **Lemma 11.** Given a set  $P$  of  $n$  points in the plane, and parameters  $\delta, \varepsilon \in (0, 1)$ . One can construct a graph  $G$  over  $P$ , in  $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$  time, and with  $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$  edges, such that for any (bounded) convex  $C$  in the plane, we have that for any two points  $p, q \in P \cap C_{\Box\delta}$  the graph  $C \cap P$  has  $(1 + \varepsilon)$ -path between  $p$  and  $q$ .

**Proof.** Let  $\vartheta = \min(\varepsilon, \delta^2)$ . Construct, in  $\mathcal{O}(\vartheta^{-1}n \log n)$  time, a standard  $(1 + \vartheta)$ -spanner  $G$  for  $P$  using  $\mathcal{O}(\vartheta^{-1}n)$  edges [AMS99].

So, consider any body  $C \in \mathcal{F}$ , and any two vertices  $p, q \in P \cap C'$ , where  $C' = C_{\square\delta}$ . Let  $\ell = \|pq\|$ , let  $\pi$  be the shortest path between  $p$  and  $q$  in  $G$ , and let  $\mathcal{E}$  be the locus of all points  $u$ , such that  $\|pu\| + \|uq\| \leq (1 + \vartheta)\ell$ . The region  $\mathcal{E}$  is an ellipse that contains  $\pi$ . The furthest point from the segment  $pq$  in this ellipse is realized by the co-vertex of the ellipse. Formally, it is one of the two intersection points of the boundary of the ellipse with the line orthogonal to  $pq$  that passes through the middle point  $c$  of this segment, see Figure 2.1. Let  $z$  be one of these points.



**Figure 2.1** An illustration of the settings in the proof of Lemma 11

We have that  $\|pz\| = (1 + \vartheta)\ell/2$ . Setting  $h = \|zc\|$ , we have that

$$h = \sqrt{\|pz\|^2 - \|pc\|^2} = \frac{\ell}{2} \sqrt{(1 + \vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2 + \vartheta)}}{2} \ell \leq \sqrt{\vartheta} \ell \leq \sqrt{\vartheta} \cdot \text{diam}(C).$$

as  $\ell \leq \text{diam}(C') \leq \text{diam}(C)$ .

For any point  $x \in C'$ , we have that  $d(x, \mathbb{R}^2 \setminus C') \geq \delta \cdot \text{diam}(C)$ . As such, to ensure that  $\pi \subseteq \mathcal{E} \subseteq C$ , we need that  $\delta \cdot \text{diam}(C) \geq h$ , which holds if  $\delta \cdot \text{diam}(C) \geq \sqrt{\vartheta} \cdot \text{diam}(C)$ . This in turn holds if  $\vartheta \leq \delta^2$ . Namely, we have the desired properties if  $\vartheta = \min(\varepsilon, \delta^2)$ . ◀

### 3 Local spanners of homothets of convex region

Let  $\mathcal{C}$  be a bounded convex and closed region in the plane (e.g., a disk). A *homothet* of  $\mathcal{C}$  is a scaled and translated copy of  $\mathcal{C}$ . A point set  $P$  is in *general position* with respect to  $\mathcal{C}$ , if no four points of  $P$  lie on the boundary of a homothet of  $\mathcal{C}$ , and no three points are colinear.

A graph  $G = (P, E)$  is a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner for  $P$  if for any homothet  $\mathcal{r}$  of  $\mathcal{C}$ , we have that  $G \cap \mathcal{r}$  is a  $t$ -spanner of  $G_P \cap \mathcal{r}$ .

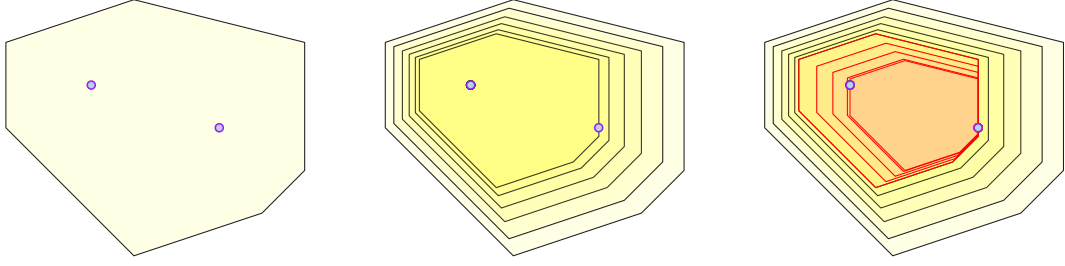
#### 3.1 Delaunay triangulation for homothets

► **Definition 12** ([CDI85]). Given  $\mathcal{C}$  as above, and a point set  $P$  in general position for  $\mathcal{C}$ , the  $\mathcal{C}$ -Delaunay triangulation of  $P$ , denoted  $\mathcal{D}_{\mathcal{C}}(P)$ , is the graph formed by edges between any two points  $p, q \in P$  such that there exist a homothet of  $\mathcal{C}$  that contains only  $p$  and  $q$  and no other point of  $P$ .

► **Theorem 13** ([CDI85]). For any convex shape  $\mathcal{C}$  and a set of points  $P$ ,  $\mathcal{D}_{\mathcal{C}}(P)$  can be computed in  $\mathcal{O}(n \log n)$  time. Furthermore, the triangulation  $\mathcal{D}_{\mathcal{C}}(P)$  has  $\mathcal{O}(n)$  edges, vertices, and faces.

► **Lemma 14.** Let  $\mathcal{C}$  be a convex bounded body, and let  $P$  be a set of points in general position for  $\mathcal{C}$ . Then, if  $C$  is a homothet of  $\mathcal{C}$  that contains two points  $p, q \in C \cap P$ , then there exists a homothet  $C' \subseteq C$  of  $\mathcal{C}$  such that  $p, q \in \partial C'$ .





256 ■ **Figure 3.1** Shrinking of homothets so two points becomes on the boundary of the homothet.

260 **Proof.** The idea is to apply a shrinking process of  $C$ , as illustrated in Figure 3.1. Consider  
 261 the mapping  $f_{\beta,v} : x \mapsto \beta(x - v) + v$ . It is a scaling of the plane around  $v$  by a factor of  
 262  $\beta$ . Let  $\beta'$  be the minimum value of  $\beta$  such that  $C_1 = f_{\beta,p}(C)$  contains  $q$  (i.e., we shrink  
 263  $C$  around  $p$  till  $q$  becomes a boundary point). Next, shrink  $C'$  around  $q$ , till  $p$  becomes  
 264 a boundary point – formally, let  $\beta''$  be the minimum value of  $\beta$  such that  $C' = f_{\beta,q}(C_1)$   
 265 contains  $p$ . Since  $C' \subseteq C_1 \subseteq C$ , and  $p, q \in \partial C'$ , the claim follows. ◀

266 The following standard claim, usually stated about the standard Delaunay triangulations,  
 267 also holds for homothets.

268 ▷ **Claim 15** (Proof in Appendix B.3). Let  $\mathcal{C}$  be a bounded close convex shape. Given a set of  
 269 points  $P \subseteq \mathbb{R}^2$  in general position for  $\mathcal{C}$ , let  $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$  be the  $\mathcal{C}$ -Delaunay triangulation of  
 270  $P$ . For any homothet  $C$  of  $\mathcal{C}$ , we have that  $\mathcal{D} \cap C$  is connected.

## 271 3.2 The generic construction

272 The input is a set  $P$  of  $n$  points in the plane (in general position) with spread  $\Phi = \Phi(P)$ ,  
 273 and a parameter  $\varepsilon \in (0, 1)$ . We have a convex body  $\mathcal{C}$  that defines the “unit” ball. The task  
 274 is to construct a local spanner for any homothet of  $\mathcal{C}$ .

275 The algorithm computes a  $(1/\vartheta)$ -WSPD  $\mathcal{W}$  of  $P$  using the algorithm of Lemma 3,  
 276 where  $\vartheta = \varepsilon/6$ . For each pair  $\Xi = \{X, Y\} \in \mathcal{W}$ , the algorithm computes the  $\mathcal{C}$ -Delaunay  
 277 triangulation  $\mathcal{D}_{\Xi} = \mathcal{D}_{\mathcal{C}}(X \cup Y)$ . The algorithm adds all the edges in  $\mathcal{D}_{\Xi} \cap (X \otimes Y)$  to the  
 278 computed graph  $G$ .

### 279 3.2.1 Analysis

#### 280 Size

281 For each pair  $\Xi = \{X, Y\}$  in the WSPD, its  $\mathcal{C}$ -Delaunay triangulation contains at most  
 282  $\mathcal{O}(|X| + |Y|)$  edges. As such, the number of edges in the resulting graph is bounded by  
 283  $\sum_{\{X,Y\} \in \mathcal{W}} \mathcal{O}(|X| + |Y|) = \mathcal{O}(\omega(\mathcal{W})) = \mathcal{O}\left(\frac{n \log \Phi}{\vartheta^2}\right)$ , by Lemma 3.

#### 284 Construction time

285 The construction time is bounded by  $\sum_{\{X,Y\} \in \mathcal{W}} \mathcal{O}((|X| + |Y|) \log(|X| + |Y|)) = \mathcal{O}(\omega(\mathcal{W}) \log n) =$   
 286  $\mathcal{O}\left(\frac{n \log \Phi \log n}{\vartheta^2}\right)$ ,

287 ► **Lemma 16** (Local spanner property). For  $P, \mathcal{C}, \varepsilon$  as above, let  $G$  be the graph constructed  
 288 above for the point set  $P$ . Then, for any homothet  $C$  of  $\mathcal{C}$  and any two points  $x, y \in P \cap C$ ,



we have that  $G \cap C$  has a  $(1 + \varepsilon)$ -path between  $x$  and  $y$ . That is,  $G$  is a  $(1 + \varepsilon)$ -local spanner for homothets of  $\mathcal{C}$ .

**Proof.** Fix a homothet  $C$  of  $\mathcal{C}$ , and consider two points  $p, q \in P \cap C$ . The proof is by induction on the distance between  $p$  and  $q$  (or more precisely, the rank of their distance among the  $\binom{n}{2}$  pairwise distances). Consider the pair  $\Xi = \{X, Y\}$  such that  $x \in X$  and  $y \in Y$ .

If  $xy \in \mathcal{D}_\Xi$  then the claim holds. So assume this is not the case. By the connectivity of  $\mathcal{D}_\Xi \cap C$ , see [Claim 15](#), there must be points  $x' \in X \cap C$ ,  $y' \in Y \cap C$ , such that  $x'y' \in E(\mathcal{D}_\Xi)$ . As such, by construction, we have that  $x'y' \in E(G)$ . Furthermore, by the separation property, we have that

$$\max(\text{diam}(X), \text{diam}(Y)) \leq \vartheta \text{d}(X, Y) \leq \vartheta \ell,$$

where  $\ell = \|xy\|$ . In particular,  $\|x'x\| \leq \vartheta \ell$  and  $\|y'y\| \leq \vartheta \ell$ . As such, by induction, we have  $\text{d}_G(x, x') \leq (1 + \varepsilon) \|xx'\| \leq (1 + \varepsilon) \vartheta \ell$  and  $\text{d}_G(y, y') \leq (1 + \varepsilon) \|yy'\| \leq (1 + \varepsilon) \vartheta \ell$ . Furthermore,  $\|x'y'\| \leq (1 + 2\vartheta)\ell$ . As  $x'y' \in E(G)$ , we have

$$\begin{aligned} \text{d}_G(x, y) &\leq \text{d}_G(x, x') + \|x'y'\| + \text{d}_G(y', y) \leq (1 + \varepsilon) \vartheta \ell + (1 + 2\vartheta)\ell + (1 + \varepsilon) \vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta)\ell \\ &= (1 + 6\vartheta)\ell \leq (1 + \varepsilon) \|xy\|, \end{aligned}$$

if  $\vartheta \leq \varepsilon/6$ . ◀

### The result

► **Theorem 17.** *Let  $\mathcal{C}$  be a bounded convex shape in the plane, let  $P$  be a given set of  $n$  points in the plane (in general position), and let  $\varepsilon \in (0, 1/2)$  be a parameter. The above algorithm constructs a local  $(1 + \varepsilon)$ -spanner  $G$  for homothets of  $\mathcal{C}$ . The spanner has  $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$  edges, and the construction time is  $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ . Formally, for any homothet  $C$  of  $\mathcal{C}$ , and any two points  $p, q \in P \cap C$ , we have a  $(1 + \varepsilon)$ -path in  $G \cap C$ .*

### 3.2.2 Applications and comments

The following defines a “visibility” graph when we are restricted to a region  $R$ , where two points are visible if there is a witness homothet contained in  $R$  having both points on its boundary.

► **Definition 18.** *Let  $\mathcal{C}$  be a bounded convex shape in the plane. Given a region  $R$  in the plane and a point set  $P$ , consider two points  $p, q \in P$ . The edge  $pq$  is **safe** in  $R$ , if there is a homothet  $C$  of  $\mathcal{C}$ , such that  $p, q \in C \subseteq R$ . The **safe graph** for  $P$  and  $R$ , denoted by  $\mathcal{G}(P, R)$ , is the graph formed by all the safe edges in  $P$  for  $R$ . Note, that this graph might have a quadratic number of edges in the worst case.*

Observe that  $\mathcal{G}(P, \mathbb{R}^2)$  is a clique. Surprisingly, the spanner graph described above, when restricted to region  $R$ , is a spanner for  $\mathcal{G}(P, R)$ .

► **Corollary 19 (Proof in [Appendix B.4](#)).** *Let  $\mathcal{C}$  be a bounded convex body,  $P$  be a set of  $n$  points in the plane,  $\varepsilon \in (0, 1)$  be a parameter, and let  $G$  be a local  $(1 + \varepsilon)$ -spanner of  $P$  for homothets of  $\mathcal{C}$ . Let  $G$  be the above spanner constructed for  $P$  and  $\mathcal{C}$ .*

*Consider a region  $R$  in the plane, and the associated graph  $H = \mathcal{G}(P, R)$ , we have that  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for  $H$ . Formally, for any two points  $p, q \in P \cap R$ , we have that  $\text{d}_{G \cap R}(p, q) \leq (1 + \varepsilon) \text{d}_H(p, q)$ .*

*In particular, for any convex region  $D$ , the graph  $G - D$  is a  $(1 + \varepsilon)$ -spanner for  $\mathcal{G}(P, \mathbb{R}^2) - D$ .*

► Remark 20. The above implies that local spanners for homothets are also robust to convex region faults.

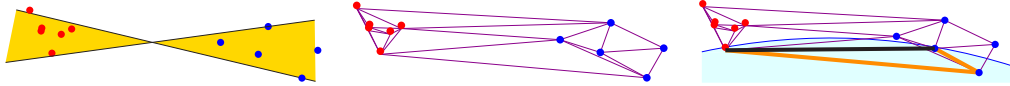
### 3.3 Lower bounds

#### 3.3.1 A lower bound for local spanner for disks

The result of Theorem 17 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). We next show a lower bound proving that this dependency is unavoidable, even in the case of disks.

##### Some intuition

A natural way to attempt a spread-independent construction is to try and emulate the construction of Abam *et al.*, [AdBFG09] and use a SSPD instead of a WSPD. The total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair  $\Xi = \{X, Y\}$  is angularly  $\varepsilon$ -separated – that is, there is a double wedge with angle  $\leq \varepsilon$ , such that  $X$  and  $Y$  are of different sides of the double wedge. The problem is that for the local disk  $\circ$ , it might be that the bridge edge between  $X$  and  $Y$  that is in  $\mathcal{D}_\Xi \cap \circ$  is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 3.2.



■ **Figure 3.2** A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

► **Lemma 21.** [Proof in Appendix B.5] For  $\varepsilon = 1/4$ , and parameters  $n$  and  $\Phi \geq 1$ , there is a point set  $P$  of  $n + \lceil \log \Phi \rceil$  points in the plane, with spread  $\mathcal{O}(n\Phi)$ , such that any local  $(1 + \varepsilon)$ -spanner of  $P$  for disks, must have  $\Omega(n \log \Phi)$  edges.

#### 3.3.2 A lower bound for triangles

► **Lemma 22.** [Proof in Appendix B.6] For any  $n > 0$ , and  $\Phi = \Omega(n)$ , one can compute a set  $P$  of  $n + \mathcal{O}(\log \Phi)$  points, with spread  $\mathcal{O}(\Phi n)$ , and a triangle  $\triangle$ , such that any  $\triangle$ -local  $3/2$ -spanner of  $P$  requires  $\Omega(n \log \Phi)$  edges.

### 3.4 Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex shape are costly, if we are given a triangle  $\triangle$  with the single constraint that  $\triangle$  is not too “thin”, then one can construct a  $\triangle$ -local spanner with a number of edges that does not depend on the spread of the points. See Figure B.5 for an illustration of a construction showing that dependency if “thin” triangles are allowed.

► **Definition 23.** A triangle  $\triangle$  is  $\alpha$ -fat if the smallest angle in  $\triangle$  is at least  $\alpha$ .

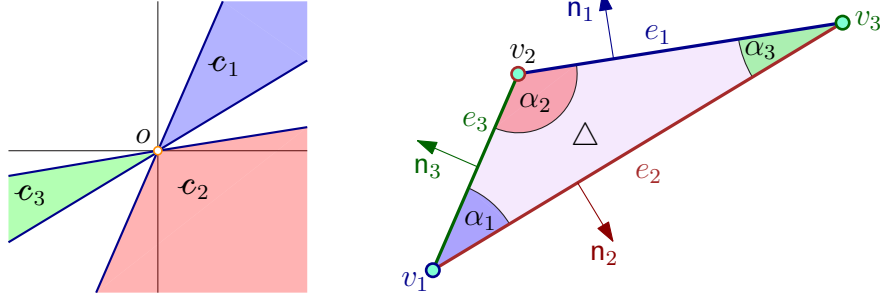


Figure 3.3 For the triangle  $\Delta$  with angles  $\alpha_1, \alpha_2$ , and  $\alpha_3$  we create the cones  $c_1, c_2$ , and  $c_3$ .

### 3.4.1 Construction

The input is a set  $P$  of  $n$  points in the plane, an  $\alpha$ -fat triangle  $\Delta$ , and an approximation parameter  $\varepsilon \in (0, 1)$ . Let  $v_i$  denote the  $i$ th vertex of  $\Delta$ ,  $\alpha_i$  be the adjacent angle, and let  $e_i$  denote the facing edges, for  $i \in [3]$ . Let  $c_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$  denote the cone with an apex at the origin induced by the  $i$ th vertex of  $\Delta$ . Let  $n_i$  be the outer normal of  $\Delta$  orthogonal to  $e_i$ . See Figure 3.3 for an illustration. Let  $\mathcal{C}_i$  be a minimum partition of  $c_i$  into cones each with angle in the range  $[\beta/2, \beta]$ , where  $\beta = \varepsilon\alpha/\gamma$ , and  $\gamma > 1$  is a constant to be determined shortly. For each point  $p \in P$ , and a cone  $c \in \mathcal{C}_i$ , let  $\text{nn}_i(p, c)$  be the first point in  $(P - p) \cap (p + c)$  ordered by the direction  $n_i$  (it is the “nearest-neighbor” to  $p$  in  $p + c$ ).

### The construction

Let  $G$  be the graph over  $P$  formed by connecting every point  $p \in P$  to  $\text{nn}_i(p, c)$ , for all  $i \in [3]$  and  $c \in \mathcal{C}_i$ .

### 3.4.2 Analysis

► **Lemma 24.** [Proof in Appendix B.7] Let  $p \in P$ ,  $c \in \mathcal{C}_i$ , and  $u = \text{nn}_i(p, c)$ , and let  $q$  be a point in  $(P \cap (p + c)) \setminus \{p, u\}$ . We have that  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$  and  $\|qu\| \leq \|pq\|$ .

► **Lemma 25.** Let  $\Delta$  be a triangle that contains two points  $p, q$ . Then, there is a homothet  $\Delta' \subseteq \Delta$  of  $\Delta$ , such that one of these points is a vertex of  $\Delta'$ , and the other point lies on a facing edge of  $\Delta'$ .

**Proof.** This follows by the same shrinking argument as Lemma 14, and is illustrated in Figure 3.4. ◀

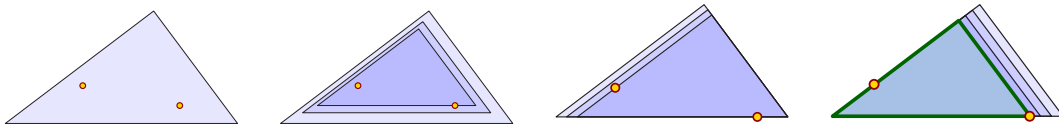


Figure 3.4 An illustration of the shrinking process mentioned in Lemma 25.

386 **Local spanner property**

387 ► **Lemma 26.** *Let  $\Delta'$  be a homothet of  $\Delta$ , for any two points  $p, q \in P \cap \Delta'$ , we have a*  
 388  *$(1 + \varepsilon)$ -path in  $G' = G \cap \Delta'$ .*

389 **Proof.** Consider the closest pair  $p, q \in P \cap \Delta$ . They must be connected directly in  $G'$ , as  
 390 otherwise there is a point  $u \in P' = P \cap \Delta'$  in the cone containing the segment  $pq$ , such that  
 391  $pu \in E(G')$ . But then, by Lemma 24, we have  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ , which  
 392 implies that either  $pu$  or  $qu$  are the closest pairs, which is a contradiction.

393 For any other pair  $p, q \in P'$  we have from Lemma 25 that there exists a homothet  
 394  $\Delta'' \subseteq \Delta'$  with one of the two points, say  $p$ , at a vertex, and the other on the opposite edge.  
 395 We therefore have a cone  $c$  with apex at  $p$  such that  $q \in c \cap \Delta''$ . If  $pq$  is an edge in  $G$   
 396 then we are done. Otherwise, we have a vertex  $u \in c$  such that  $pu$  is an edge in  $G$ , and by  
 397 Lemma 24 we have  $\|qu\| \leq \|pq\|$ , which, by induction, means that there exists a  $(1 + \varepsilon)$  path  
 398 between  $u$  and  $q$  in  $G$ . Lemma 24 now implies that  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ . Thus,  
 399 there is a  $(1 + \varepsilon)$  path between  $p$  and  $q$  in  $G'$ , as stated. ◀

400 **Size and running time**

401 ► **Theorem 27.** *Let  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon \in (0, 1)$  be an approximation*  
 402 *parameter. The above algorithm computes a local  $(1 + \varepsilon)$ -spanner  $G$  for homothets of an  $\alpha$ -fat*  
 403 *triangle  $\Delta$ . The construction time is  $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ , and the spanner  $G$  has  $\mathcal{O}((\alpha\varepsilon)^{-1}n)$*   
 404 *edges.*

405 **Proof.** The local-spanning property is proven in Lemma 26, and we are only left with  
 406 bounding the size and the running time of the algorithm. The bound on the size is immediate  
 407 from the construction, as every point  $p$  is the apex of  $\mathcal{O}(\frac{2\pi}{\varepsilon\alpha})$  cones, each giving rise to a  
 408 single edge incident to  $p$ . The construction time is bounded by the construction time for a  
 409  $\theta$ -graph with cone size  $\alpha\varepsilon$ , which is  $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$  ([Cla87]). ◀

410 **3.5 A local spanner for nice polygons**411 **3.5.1 A good jump for narrow trapezoids**

412 As a reminder, a trapezoid is a quadrilateral with two parallel edges, known as its *bases*.  
 413 The other two edges are its legs. For  $\varepsilon \in (0, 1/4)$ , a trapezoid  $T$  is  $\varepsilon$ -*narrow* if the length of  
 414 each of its legs is at most  $\varepsilon \cdot \text{diam}(T)$ .

415 ► **Lemma 28** (Proof in Appendix B.8). *Let  $\varepsilon \in (0, 1)$  be some parameter, and  $\vartheta = \varepsilon/16$ . Let*  
 416  *$X, Y$  be two points sets that are  $\vartheta$ -semi separated and  $\vartheta$ -angularly separated (see Definition 6),*  
 417 *and let  $T$  be a  $\vartheta$ -narrow trapezoid, with two points  $p \in X$  and  $q \in Y$  lying on the two legs of*  
 418  *$T$ . Then, one can compute a homothet  $T' \subseteq T$  of  $T$ , such that:*

419 (I) *There are two points  $p' \in X$  and  $q' \in Y$ , such that  $p'q'$  is an edge of the  $T$ -Delaunay*  
 420 *triangulation of  $X \cup Y$ .*

421 (II) *We have that  $(1 + \varepsilon)\|pp'\| + \|p'q'\| + (1 + \varepsilon)\|q'q\| \leq (1 + \varepsilon)\|pq\|$ .*

422 **3.5.2 Breaking a nice polygon into narrow trapezoids**

423 For a convex polygon  $\mathcal{C}$ , its *sensitivity*, denoted by  $\text{sen}(\mathcal{C})$ , is the minimum distance between  
 424 any two non-adjacent edges (this quantity is no bigger than the length of the shortest edge in  
 425 the polygon). A convex polygon  $\mathcal{C}$  is *t-nice*, if the outer angle at any vertex of the polygon

is at least  $2\pi/t$ , and the length of the longest edge of  $\mathcal{C}$  is  $\mathcal{O}(\text{sen}(\mathcal{C}))$ . As an example, a  $k$ -regular polygon is  $k$ -nice.

► **Lemma 29** (Proof in [Appendix B.9](#)). *Let  $t$  be a positive integer. Given a  $t$ -nice polygon  $\mathcal{C}$ , and a parameter  $\vartheta$ , one can cover it by a set  $\mathcal{T}$  of  $\mathcal{O}(t^4/\vartheta^3)$   $\vartheta$ -narrow trapezoids, such that for any two points  $p, q \in \partial\mathcal{C}$  that belong to two edges of  $\mathcal{C}$  that are not adjacent, there exists a narrow trapezoid  $T \in \mathcal{T}$ , such that  $p$  and  $q$  are located on two different short legs of  $T$ .*

### 3.5.3 Constructing the local spanner for nice polygons

► **Theorem 30.** *Let  $\mathcal{C}$  be a  $k$ -nice convex polygon,  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon \in (0, 1)$  be a parameter. Then one can construct a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner of  $P$ . The construction time is  $\mathcal{O}((k^4/\varepsilon^6)n \log^2 n)$ , and the resulting graph has  $\mathcal{O}((k^4/\varepsilon^6)n \log n)$  edges. In particular these bounds hold if  $\mathcal{C}$  is a  $k$ -regular gon.*

**Proof.** Let  $\vartheta = \varepsilon/c_4$ , for  $c_4$  sufficiently large constant. We construct  $\Delta$ , a family of triangles induced by a vertex of  $\mathcal{C}$ , and an non-adjacent edge of  $\mathcal{C}$ . This family has  $\mathcal{O}(k^2)$  triangles. Each such triangle is  $\Omega(1/k)$ -fat, and for each such triangle we construct the  $(1 + \vartheta)$ -spanner of [Theorem 27](#) for  $P$ . Next, we cover  $\mathcal{C}$  by a set  $\mathcal{T}$  of  $k' = \mathcal{O}(k^4/\vartheta^3)$   $\vartheta$ -narrow trapezoids using [Lemma 29](#).

We compute an  $\vartheta$ -angular  $(1/\vartheta)$ -SSPD  $\mathcal{W}$  decomposition of  $P$  using [Corollary 8](#) – the total weight of the decomposition is  $w = \mathcal{O}(n\vartheta^{-3} \log n)$ . For each pair  $\{X, Y\} \in \mathcal{W}$ , and each trapezoid  $T \in \mathcal{T}$ , we compute the  $T$ -Delaunay triangulation of  $X \cup Y$ .

Let  $G$  denote the union of all these graphs. We claim that it is the desired spanner. The construction time is

$$\mathcal{O}((k^3/\vartheta)n \log n + k'w \log n) = \mathcal{O}\left(\frac{k^3}{\vartheta}n \log n + \frac{k^4}{\vartheta^3} \cdot \frac{n}{\vartheta^3} \log n \cdot \log n\right) = \mathcal{O}\left(\frac{k^4}{\vartheta^6}n \log^2 n\right).$$

And the resulting graph has  $\mathcal{O}((k^4/\vartheta^6)n \log n)$  edges.

As for correctness, consider a homothet  $\mathcal{C}'$  of  $\mathcal{C}$  that contains two points  $p, q \in P$ . By [Lemma 14](#), there is a homothet  $\mathcal{C}'' \subseteq \mathcal{C}'$  of  $\mathcal{C}$  such that  $p, q \in \partial\mathcal{C}''$ . There are two possibilities:

- The point  $p$  is on a vertex of  $\mathcal{C}''$  and  $q$  is on an edge. In this case, the vertex and the edge induce a fat triangle, that is a homothet of a triangle  $\Delta \in \Delta$ . Since the graph  $G$  contains a  $\Delta$ -local  $(1 + \varepsilon)$ -spanner for  $P$ , it follows readily that  $G$  is a  $(1 + \varepsilon)$ -spanner for these points, and the path is strictly inside  $\mathcal{C}''$ .
- The points  $p$  and  $q$  are on two non-adjacent edges of  $\mathcal{C}''$ . Then, there is an  $\vartheta$ -narrow trapezoid  $T'$  that has  $p$  and  $q$  on its two legs, and a homothet of  $T'$ , denoted by  $T$ , is in  $\mathcal{T}$ . There is a pair  $\{X, Y\} \in \mathcal{W}$  that is  $(1/\vartheta)$ -semi separated (and  $\vartheta$ -angularly separated), such that  $p \in X$  and  $q \in Y$ . By [Lemma 28](#), there are two points  $p' \in X$  and  $q' \in Y$ , such that  $p'q'$  is an edge of the  $T$ -Delaunay triangulation of  $X \cup Y$ , and by construction this edge is in  $G$ . We now use induction on the shortest paths from  $p$  to  $p'$  and from  $q$  to  $q'$  in  $G$ . By induction, and [Lemma 28](#), we have that

$$d(p, q) \leq d(p, p') + \|p'q'\| + d(q', q) \leq (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|,$$

which implies that there is  $(1 + \varepsilon)$ -path from  $p$  to  $q$  inside  $\mathcal{C}'$ . ◀

► **Remark 31.** For axis-parallel squares [Theorem 30](#) implies a local spanner with  $\mathcal{O}(\varepsilon^{-6}n \log n)$  edges. However, for this special case, the decomposition into narrow trapezoid can be skipped. In particular, in this case, the resulting spanner has  $\mathcal{O}(\varepsilon^{-3}n \log n)$  edges. We do not provide

the details here, as it is only a minor improvement over the above, and requires quite a bit of additional work – essentially, one has to prove a version of [Lemma 28](#) for squares.

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## A Weak local spanners for axis-parallel rectangles

### A.1 Quadrant separated pair decomposition

For two points  $p = (p_1, \dots, p_d)$  and  $q = (q_1, \dots, q_d)$  in  $\mathbb{R}^d$ , let  $p \prec q$  denotes that  $q$  *dominates*  $p$  coordinate-wise. That is  $p_i < q_i$ , for all  $i$ . More generally, let  $p <_i q$  denote that  $p_i < q_i$ . For two point sets  $X, Y \subseteq \mathbb{R}^d$ , we use  $X <_i Y$  to denote that  $\forall x \in X, y \in Y \quad x <_i y$ . In particular  $X$  and  $Y$  are *i-coordinate separated* if  $X <_i Y$  or  $Y <_i X$ . A pair  $\{X, Y\}$  is *quadrant-separated*, if  $X$  and  $Y$  are *i-coordinate separated*, for  $i = 1, \dots, d$ .

A *quadrant-separated pair decomposition* of a point set  $P \subseteq \mathbb{R}^d$ , is a pair decomposition (see [Definition 1](#))  $\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\}$  of  $P$ , such that  $\{X_i, Y_i\}$  are quadrant-separated for all  $i$ .

► **Lemma 32.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}$ , one can compute, in  $\mathcal{O}(n \log n)$  time, a QSPD of  $P$  with  $\mathcal{O}(n)$  pairs, and of total weight  $\mathcal{O}(n \log n)$ .*

**Proof.** If  $P$  is a singleton then there is nothing to do. If  $P = \{p, q\}$ , then the decomposition is the pair formed by the two singleton points.

Otherwise, let  $x$  be the median of  $P$ , such that  $P_{\leq x} = \{p \in P \mid p \leq x\}$  contains exactly  $\lceil n/2 \rceil$  points, and  $P_{> x} = P \setminus P_{\leq x}$  contains  $\lfloor n/2 \rfloor$  points. Construct the pair  $\Xi = \{P_{\leq x}, P_{> x}\}$ , and compute recursively a QSPDs  $\mathcal{Q}_{\leq x}$  and  $\mathcal{Q}_{> x}$  for  $P_{\leq x}$  and  $P_{> x}$ , respectively. The desired

QSPD is  $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{> x} \cup \{\Xi\}$ . The bounds on the size and weight of the desired QSPD are immediate.  $\blacktriangleleft$

► **Lemma 33.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^d$ , one can compute, in  $\mathcal{O}(n \log^d n)$  time, a QSPD of  $P$  with  $\mathcal{O}(n \log^{d-1} n)$  pairs, and of total weight  $\mathcal{O}(n \log^d n)$ .*

**Proof.** The construction algorithm is recursive on the dimensions, using the algorithm of Lemma 32 in one dimension.

The algorithm computes a value  $\alpha_d$  that partitions the values of the points'  $d$ th coordinates roughly equally (and is distinct from all of them), and let  $h$  be a hyperplane parallel to the first  $d - 1$  coordinate axes, and having value  $\alpha_d$  in the  $d$ th coordinate.

Let  $P_\uparrow$  and  $P_\downarrow$  be the subset of points of  $P$  that are above and below  $h$ , respectively. The algorithm recursively computes QSPDs  $\mathcal{Q}_\uparrow$  and  $\mathcal{Q}_\downarrow$  for  $P_\uparrow$  and  $P_\downarrow$ , respectively. Next, the algorithm projects the points of  $P$  on  $h$ , let  $P'$  be the resulting  $d - 1$  dimensional point set (after we ignore the  $d$ th coordinate), and recursively computes a QSPD  $\mathcal{Q}'$  for  $P'$ .

For a point set  $X' \subseteq P'$ , let  $\text{lift}(X')$  be the subset of points of  $P$  whose projection on  $h$  is  $X'$ . The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\text{lift}(X') \cap P_\uparrow, \text{lift}(Y') \cap P_\downarrow\}, \{\text{lift}(X') \cap P_\downarrow, \text{lift}(Y') \cap P_\uparrow\} \mid \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is  $\widehat{\mathcal{Q}} \cup \mathcal{Q}_\uparrow \cup \mathcal{Q}_\downarrow$ .

To observe that this is indeed a QSPD, observe that all the pairs in  $\mathcal{Q}_\uparrow, \mathcal{Q}_\downarrow$  are quadrant separated by induction. As for pairs in  $\widehat{\mathcal{Q}}$ , they are quadrant separated in the first  $d - 1$  coordinates by induction on the dimension, and separated in the  $d$  coordinate since one side of the pair comes from  $P_\uparrow$ , and the other side from  $P_\downarrow$ .

As for coverage, consider any pair of points  $p, q \in P$ , and observe that the claim holds by induction if they are both in  $P_\uparrow$  or  $P_\downarrow$ . As such, assume that  $p \in P_\uparrow$  and  $q \in P_\downarrow$ . But then there is a pair  $\{X', Y'\} \in \mathcal{Q}'$  that separates the two projected points in  $h$ , and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates  $p$  and  $q$  as desired.

The number pairs in the decomposition is  $N(n, d) = 2N(n, d - 1) + 2N(n/2, d)$  with  $N(n, 1) = \mathcal{O}(n)$ . The solution to this recurrence is  $N(n, d) = \mathcal{O}(n \log^{d-1} n)$ . The total weight of the decomposition is  $W(n, d) = 2W(n, d - 1) + 2W(n/2, d)$  with  $W(n, 1) = \mathcal{O}(n \log n)$ . The solution to this recurrence is  $W(n, d) = \mathcal{O}(n \log^d n)$ . Clearly, this also bounds the construction time.  $\blacktriangleleft$

## A.2 Weak local spanner for axis parallel rectangles

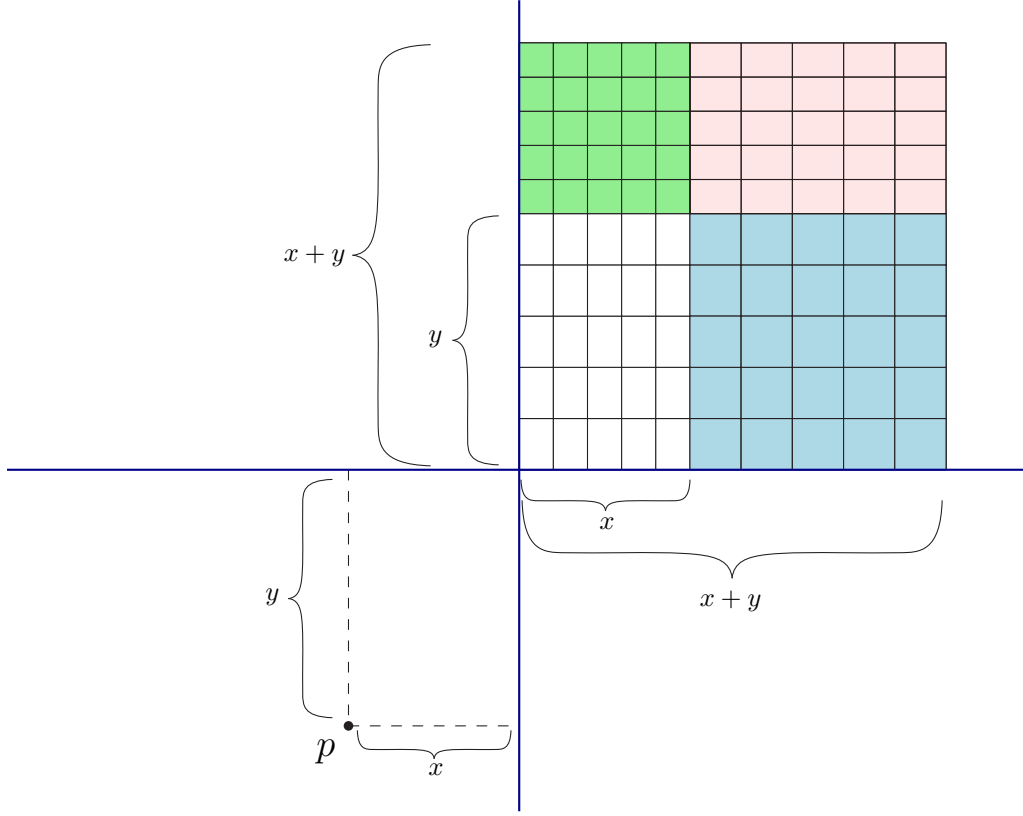
For a parameter  $\delta \in (0, 1)$ , and an interval  $I = [b, c]$ , let  $(1 - \delta)I = [t - (1 - \delta)r, t + (1 - \delta)r]$ , where  $t = (b + c)/2$ , and  $r = (c - b)/2$ , be the shrinking of  $I$  by a factor of  $1 - \delta$ .

Let  $\mathcal{R}$  be the set of all axis parallel rectangles in the plane. For a rectangle  $R \in \mathcal{R}$ , with  $R = I \times J$ , let  $(1 - \delta)R = (1 - \delta)I \times (1 - \delta)J$  denote the rectangle resulting from shrinking  $R$  by a factor of  $1 - \delta$ .

► **Definition 34.** *Given a set  $P$  of  $n$  points in the plane, and parameters  $\varepsilon, \delta \in (0, 1)$ , a graph  $G$  is a  $(1 - \delta)$ -local  $(1 + \varepsilon)$ -spanner for rectangles, if for any axis-parallel rectangle  $R$ , we have that  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for all the points in  $((1 - \delta)R) \cap P$ .*

Observe that rectangles in  $\mathcal{R}$  might be quite “skinny”, so the previous notion of shrinkage used before is not useful in this case.





554 ■ **Figure A.1** The construction of the grid  $K(p, \Xi)$  for a point  $p = (-x, -y)$  and a pair  $\Xi$ .

### 553 A.2.1 Construction for a single quadrant separated pair

555 Consider a pair  $\Xi = \{X, Y\}$  in a QSPD of  $P$ . The set  $X$  is quadrant-separated from  $Y$ . That  
 556 is, there is a point  $c_\Xi$ , such that  $X$  and  $Y$  are contained in two opposing quadrants in the  
 557 partition of the plane formed by the vertical and horizontal line through  $c_\Xi$ .

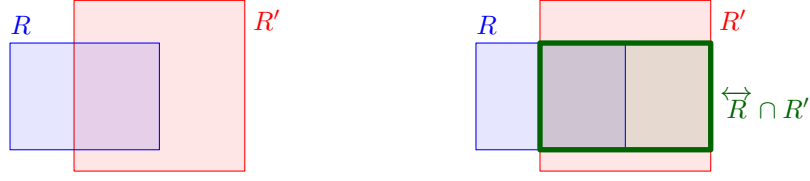
558 For simplicity of exposition, assume that  $c_\Xi = (0, 0)$ , and  $X \prec (0, 0) \prec Y$ . That is, the  
 559 points of  $X$  are in the negative quadrant, and the points of  $Y$  are in the positive quadrant.

560 We construct a non-uniform grid  $K(p, \Xi)$  in the square  $[0, x+y]^2$ . To this end, we first  
 561 partition it into four subrectangles

$$\begin{array}{c|c}
 B_{\swarrow} = [0, x] \times [y, x+y] & B_{\nearrow} = [x, x+y] \times [y, x+y] \\
 \hline
 B_{\swarrow} = [0, x] \times [0, y] & B_{\searrow} = [x, x+y] \times [0, y].
 \end{array}$$

563 Let  $\tau \geq 4/\varepsilon + 4/\delta$  be an integer number. We partition each of these rectangles into a  
 564  $\tau \times \tau$  grid, where each cell is a copy of the rectangle scaled by a factor of  $1/\tau$ . See Figure A.1.  
 565 This grid has  $\mathcal{O}(\tau^2)$  cells. For a cell  $C$  in this grid, let  $Y \cap C$  be the points of  $Y$  contained in  
 566 it. We connect  $p$  to the left-most and bottom-most points in  $Y \cap C$ . This process generates  
 567 two edges in the constructed graph for each grid cell (that contains at least two points), and  
 568  $\mathcal{O}(\tau^2)$  edges overall.

569 The algorithm repeats this construction for all the points  $p \in X$ , and does the symmetric  
 570 construction for all the points of  $Y$ .



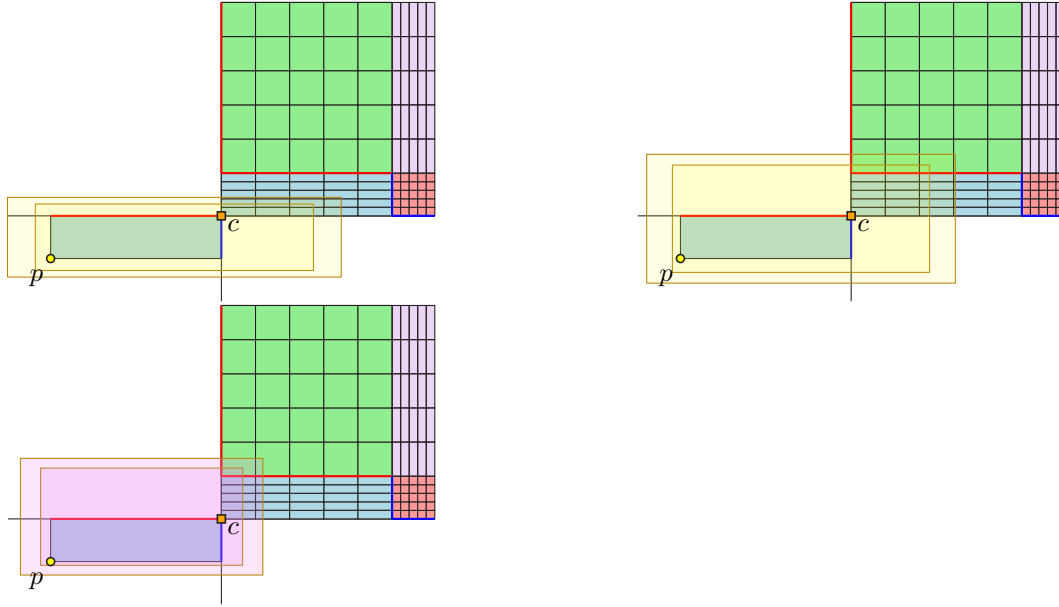
574 **Figure A.2** Left: The two rectangles  $R, R'$ . Right: In green  $\overleftrightarrow{R} \cap R'$ , the restriction of the slab  
 575  $\overleftrightarrow{R}$  to the rectangle  $R'$ .

### 571 A.2.2 The construction algorithm

572 The algorithm computes a QSPD  $\mathcal{W}$  of  $P$ . For each pair  $\Xi \in \mathcal{W}$ , the algorithm generates  
 573 edges for  $\Xi$  using the algorithm of Section A.2.1 and adds them to the generated spanner  $G$ .

### 576 A.2.3 Correctness

577 For a rectangle  $R$ , let  $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists(x', y) \in R\}$  be its expansion into a horizontal  
 578 slab. Restricted to a rectangle  $R'$ , the resulting set is  $\overleftrightarrow{R} \cap R'$ , depicted in Figure A.2.  
 579 Similarly, we denote  $\uparrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists(x, y') \in R\}$ .



580 **Figure A.3** An illustration of  $K(p, \Xi)$  with three rectangles and their shrunken version.

581 **► Lemma 35.** Assume that  $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$ . Consider a pair  $\Xi = \{X, Y\}$  in the above  
 582 construction, and a point  $p = (-x, -y) \in X$  with its associated grid  $K = K(p, \Xi)$ . Consider  
 583 any axis parallel rectangle  $R$ , such that  $p \in (1 - \delta)R = I \times J$ , and  $(1 - \delta)R$  intersects a cell  
 584  $C \in K$ . We have that:

- 585 (I) If  $C \subseteq (1 - \delta)R$  then  $(1 - \delta)^{-1}C \subseteq R$ .
- 586 (II)  $\text{diam}(C) \leq (\varepsilon/4)d(p, C)$ .
- 587 (III) If  $x \geq y$  and  $C \subseteq R_{\swarrow} \cup R_{\searrow}$  then  $(1 - \delta)^{-1}C \subseteq R$ .
- 588 (IV) If  $x \leq y$  and  $C \subseteq R_{\swarrow} \cup R_{\searrow}$  then  $(1 - \delta)^{-1}C \subseteq R$ .

589 (V) If  $x \geq y$  and  $C \subseteq R_{\swarrow}$ , then  $(1 - \delta)^{-1}(\overrightarrow{(1 - \delta)R} \cap C) \subseteq R$ .

590 (VI) If  $x \leq y$  and  $C \subseteq R_{\searrow}$ , then  $(1 - \delta)^{-1}(\overleftarrow{(1 - \delta)R} \cap C) \subseteq R$ .

591 **Proof.** (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

592 (II) We have that  $\text{diam}(C) \leq (x + y)/\tau = \|p\|_1/\tau \leq (\varepsilon/4)d(p, C)$ .

593 (III) The width, denoted  $\text{wd}(\cdot)$ , of  $(1 - \delta)R$  is at least  $x$ , as it contains both  $p$  and the origin.

594 As such,

$$595 \quad (\text{wd}(R) - \text{wd}((1 - \delta)R))/2 \geq 2(x/\tau) \geq 2\text{wd}(C).$$

596 That is, the width of the “expanded” rectangle  $R$  is enough to cover  $C$ , and a grid cell  
597 adjacent to it to the right.

598 A similar argument about the height shows that  $R$  covers the region immediately above  $C$   
599 – in particular, the vertical distance from  $C$  to the top boundary of  $R$  is at least the height of  
600  $C$ . This implies that the expanded cell  $(1 - \delta)^{-1}C$  is contained in  $R$ , as claimed, as  $\delta < 1/2$ .

601 (V) We decompose the claim to the two dimensions of the region. Let  $B = \overrightarrow{(1 - \delta)R} \cap C$ .  
602 Observe that containment in the  $x$ -axis follows by arguing as in (III). As for the  $y$ -interval  
603 of  $B$ , observe that it is contained in the  $y$ -interval of  $(1 - \delta)R$ , which implies that when  
604 expanded by  $(1 - \delta)^{-1}$ , it would be contained in the  $y$ -interval of  $R$ . Combining the two  
605 implies the result. ◀

606 ▶ **Lemma 36.** *For any axis-parallel rectangle  $R$ , and any two points  $p, q \in (1 - \delta)R \cap P$ ,  
607 there exists a  $(1 + \varepsilon)$ -path between  $p$  and  $q$  in  $G$ .*

608 **Proof.** The proof is by induction over the size of  $R$  (i.e. area, width, or height). Let  
609  $\Xi = \{X, Y\} \in \mathcal{W}$  be the pair in the QSPD that separates  $p$  and  $q$ , let  $c$  be the separation  
610 point of the pair, and assume for the simplicity of exposition that  $p \in X$ ,  $X \prec c \prec Y$ , and  
611  $c = (0, 0)$ . Furthermore, assume that  $\|p\|_1 \geq \|q\|_1$ .

612 Let  $p = (-x, -y)$ , and let  $C$  be the grid cell of  $K(p, \Xi)$  that contains  $q$ . If  $C \subseteq (1 - \delta)R$ ,  
613 then  $(1 - \delta)^{-1}C \subseteq R$  by Lemma 35 (I). As such, let  $u$  be the leftmost point in  $C \cap P$ . Both  
614  $q, u \in (1 - \delta)^{-1}C$ , and by induction, there is an  $(1 + \varepsilon)$ -path  $\pi$  between them in  $G$  (note that  
615 the induction applies to the two points, and the “expanded” rectangle  $(1 - \delta)^{-1}C$ ). Since  $pu$   
616 is an edge of  $G$ , prefixing  $\pi$  by this edge results in an  $(1 + \varepsilon)$ -path, as  $\|qu\| \leq (\varepsilon/4)\|pq\|$ , by  
617 Lemma 35 (II) (verifying this requires some standard calculations which we omit).

618 Otherwise, one need to apply the same argument using the appropriate case of Lemma 35.  
619 So assume that  $x \geq y$  (the case that  $y \geq x$  is handled symmetrically). If  $C \subseteq R_{\swarrow} \cup R_{\searrow}$ , then  
620 (III) implies that  $(1 - \delta)^{-1}C \subseteq R$ . Which implies that induction applies, and the claim holds.

621 The remaining case is that  $x \geq y$  and  $C \subseteq R_{\swarrow}$ . Let  $D = \overrightarrow{(1 - \delta)R} \cap C$ . By (V), we have  
622  $(1 - \delta)^{-1}D \subseteq R$ . Namely,  $q \in (1 - \delta)R \cap C \subseteq D$ , and let  $u$  be the lowest point in  $C \cap P$ . By  
623 construction  $pu \in E(G)$ ,  $q, u \in D$ ,  $(1 - \delta)^{-1}D \subseteq R$ . As such, we can apply induction to  $q, u$ ,  
624 and  $(1 - \delta)^{-1}D$ , and conclude that  $d_G(q, u) \leq (1 + \varepsilon)\|qu\|$ . Plugging this into the regular  
625 machinery implies the claim. ◀

626 ▶ **Theorem 37.** *Let  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon, \delta \in (0, 1)$  be parameters.  
627 The above algorithm constructs, in  $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$  time, a graph  $G$  with  $\mathcal{O}((1/\varepsilon^2 +$   
628  $1/\delta^2)n \log^2 n)$  edges. The graph  $G$  is a  $(1 - \delta)$ -local  $(1 + \varepsilon)$ -spanner for axis parallel rectangles.  
629 Formally, for any axis-parallel rectangle  $R$ , we have that  $R \cap P$  is an  $(1 + \varepsilon)$ -spanner for all  
630 the points of  $((1 - \delta)R) \cap P$ .*

**Proof.** Computing the QSPD  $\mathcal{W}$  takes  $\mathcal{O}(n \log^2 n)$  time. For each pair  $\{X, Y\}$  in the decomposition with  $m = |X| + |Y|$  points, we need to compute the lowest and leftmost points in  $(X \cup Y) \cap \mathcal{C}$ , for each cell in the constructed grid. This can readily be done using orthogonal range trees in  $\mathcal{O}(\log^2 n)$  time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is  $\mathcal{O}(\omega(\mathcal{W})\tau^2) = \mathcal{O}((1/\delta^2 + 1/\varepsilon^2)n \log^2 n)$ .

The desired local spanner property is provided by [Lemma 36](#). ◀

## B Some missing proofs

### B.1 Proof of [Lemma 5](#)

**Restatement of [Lemma 5](#).** *Given an  $\alpha$ -SSPD  $\mathcal{W}$  of a set  $P$  of  $n$  points in  $\mathbb{R}^d$  and a parameter  $\beta \geq 2$ , one can refine  $\mathcal{W}$  into an  $\alpha\beta$ -SSPD  $\mathcal{W}'$ , such that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$ .*

**Proof.** The algorithm scans the pairs of  $\mathcal{W}$ . For each pair  $\Xi = \{X, Y\} \in \mathcal{W}$ , assume that  $\text{diam}(X) < \text{diam}(Y)$ . Let  $\mathfrak{z}$  be the smallest axis-parallel cube containing  $X$ , and denote its sidelength by  $r$ . Let  $r' = r / \lceil \sqrt{d}\beta \rceil$ . Partition  $\mathfrak{z}$  into a grid of cubes of sidelength  $r'$ , and let  $T_\Xi$  be the resulting set of squares. The algorithm now add the set pairs

$$\{\{X \cap t, Y\} \mid t \in T_\Xi\}$$

to the output SSPD. Clearly, the resulting set is now  $\alpha\beta$ -semi separated, as we chopped the smaller part of each pair into  $\beta$  smaller portions. ◀

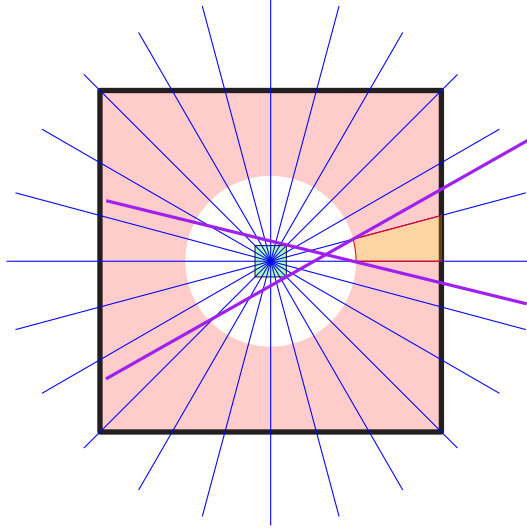
### B.2 Proof of [Lemma 7](#)

**Restatement of [Lemma 7](#).** *Given a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}$  of  $n$  points in the plane, one can refine it, into a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}'$ , such that each pair  $\Xi = \{X, Y\} \in \mathcal{W}'$  is contained in a  $\varepsilon$ -double-wedge  $\times_\Xi$ , such that  $X$  and  $Y$  are contained in the two different faces of the double wedge  $\times_\Xi$ . We have that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$ . The construction time is proportional to the weight of  $\mathcal{W}'$ .*

**Proof.** By using [Lemma 5](#), we can assume that  $\mathcal{W}$  is (say)  $(10/\varepsilon)$ -separated. Now, the algorithm scans the pairs of  $\mathcal{W}$ . For each pair  $\Xi = \{X, Y\} \in \mathcal{W}$ , assume that  $\text{diam}(X) < \text{diam}(Y)$ . Let  $\square$  be the smallest axis-parallel square containing  $X$ , centered at point  $o$ . Partition the plane around  $o$ , by drawing  $\mathcal{O}(1/\varepsilon)$  lines intersecting  $o$  with the angle between any two consecutive lines being at most (say)  $\varepsilon/4$ , see [Figure B.1](#). This partitions the plane into a set of cones  $\mathcal{C}$ . For a cone  $c \in \mathcal{C}$ , we show that there exists an  $\varepsilon$ -double-wedge that contains  $X$  in one side, and  $Y \cap c$  in the other.

To see that, take the double-wedge formed by the cross tangents between  $\text{ch}(X)$  and  $\text{ch}(Y \cap c)$ , where  $\text{ch}(X)$  denotes the convex-hull of  $X$ . Assume w.l.o.g that  $\square$  has side length 1, and let  $c$  be a cone of angle  $\varepsilon/4$  with apex  $o$ , whose angular bisector is a horizontal ray in the positive direction of the  $x$  axis. See [Figure B.2](#) for an illustration.

We would like to find a vertical segment  $s$  such that all points of  $Y$  lie to its right, with one endpoint on the upper line of  $c$ , and the other on the lower line of  $c$ . Using the segments' height and distance from the right side of  $\square$  we will be able to get a bound on the angle of the cross tangents. We first find a segment  $s$  with all points of  $Y$  to its right. A trivial bound on that distance is given by the segment from, say, the lower left corner of  $\square$ , denoted

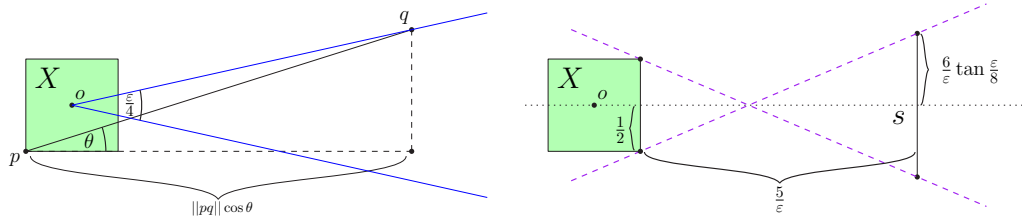


656 **Figure B.1** An illustration of refining the pairs in a SSPD into pairs contained in opposite parts  
 657 of an  $\varepsilon$ -double-wedge.  $X$  is contained in the green square  $\square$ , while  $Y$  is contained in the red square,  
 658 and the white gap between them is a result of the separation property. The set of cones with the  
 659 apex at the center of  $\square$  gives us the desired partition as demonstrated by the purple double-wedge.

676  $p$ , of length  $10/\varepsilon$  with its right endpoint on the upper line of  $c$ , denote this point by  $q$ .  
 677 We know that all points of  $Y$  lie to the right of  $q$  due to the  $10/\varepsilon$  separation property of  
 678 the SSPD. The segment  $pq$  creates an angle  $\leq \pi/4$  with the  $x$ -axis (by the choice of the  
 679 angle of  $c$ ). We therefore get that the  $x$ -coordinate difference between  $\square$  and  $q$  is at most  
 680  $10/\varepsilon \cdot \cos \frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$ . So let  $s'$  be a vertical segment between the upper and lower  
 681 rays of  $c$ , with  $x$ -coordinate distance of  $6/\varepsilon - \frac{1}{2}$  from  $\square$  (in order to make calculations easier).  
 682 We get that  $s'$  is of length  $2 \cdot \frac{6}{\varepsilon} \tan \frac{\varepsilon}{8}$ . Finally, we take  $s$  to be a vertical segment of length  
 683  $\frac{12}{\varepsilon} \tan \frac{\varepsilon}{8}$ , with its center on the  $x$ -axis at a distance of  $5/\varepsilon + \frac{1}{2}$  away from  $o$ . The angle of the  
 684  $x$ -axis and the segment between the lower end of the right side of  $\square$  and the upper end of  $s$   
 685 is now given by:

$$686 \quad \arctan\left(\frac{\frac{6}{\varepsilon} \tan \frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5} \tan \frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) \leq \varepsilon$$

687

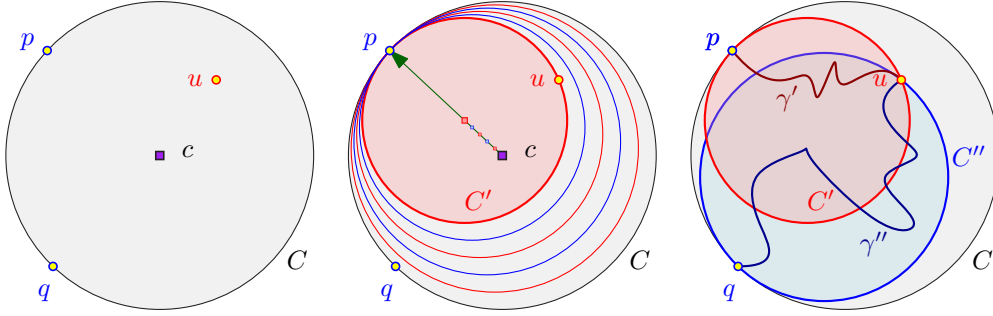


688 **Figure B.2** An illustration of the proof for Lemma 7

### B.3 Proof of Claim 15

**Restatement of Claim 15.** Let  $\mathcal{C}$  be a bounded close convex shape. Given a set of points  $P \subseteq \mathbb{R}^2$  in general position for  $\mathcal{C}$ , let  $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$  be the  $\mathcal{C}$ -Delaunay triangulation of  $P$ . For any homothet  $C$  of  $\mathcal{C}$ , we have that  $\mathcal{D} \cap C$  is connected.

**Proof.** We prove that for any homothet  $C$  with two points  $p, q \in P$  on its boundary, there is a path between  $p$  and  $q$  in  $\mathcal{D} \cap C$ , and Lemma 14 will immediately imply the general statement. The proof is by induction over the number  $m$  of points of  $P$  in the interior of  $C$ . If  $m = 0$  then  $C$  contains no points of  $P$  in its interior, and thus  $pq$  is an edge of the Delaunay triangulation, as  $C$  testifies.



**Figure B.3** An illustration of the proof of Claim 15 in the case that  $C$  is a disk.

Otherwise, let  $u \in P$  be a point in the interior of  $C$ . From Lemma 14 we get that there exists a homothet  $C'$  of  $C$  with  $C' \subseteq C$ , such that  $p$  and  $u$  lie on the boundary of  $C'$ . Thus, by induction, there is a path  $\gamma'$  between  $p$  and  $u$  in  $\mathcal{D} \cap C' \subseteq \mathcal{D} \cap C$ . Similarly, there must be a homothet  $C''$ , that gives rise to a path  $\gamma''$  between  $u$  and  $q$ , and concatenating the two paths results in a path between  $p$  and  $q$  in  $\mathcal{D} \cap C$ .

### B.4 Proof of Corollary 19

**Restatement of Corollary 19.** Let  $\mathcal{C}$  be a bounded convex body,  $P$  be a set of  $n$  points in the plane,  $\varepsilon \in (0, 1)$  be a parameter, and let  $G$  be a local  $(1 + \varepsilon)$ -spanner of  $P$  for homothets of  $\mathcal{C}$ . Let  $G$  be the above spanner constructed for  $P$  and  $\mathcal{C}$ .

Consider a region  $R$  in the plane, and the associated graph  $H = \mathcal{G}(P, R)$ , we have that  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for  $H$ . Formally, for any two points  $p, q \in P \cap R$ , we have that  $d_{G \cap R}(p, q) \leq (1 + \varepsilon)d_H(p, q)$ .

In particular, for any convex region  $D$ , the graph  $G - D$  is a  $(1 + \varepsilon)$ -spanner for  $\mathcal{G}(P, \mathbb{R}^2) - D$ .

**Proof.** Consider the shortest path  $\pi = u_1 u_2 \dots u_k$  between  $p$  and  $q$  realizing  $d_H(p, q)$ . Every edge  $e_i = u_i u_{i+1}$  has a homothet  $C_i$  such that  $u_i, u_{i+1} \in C_i \subseteq R$ . As such, there is a  $(1 + \varepsilon)$ -path between  $u_i$  and  $u_{i+1}$  in  $G \cap C_i \subseteq G \cap R$ . Concatenating these paths directly yields the desired result.

The second claim follows by observing that the complement of  $D$  is the union of halfspaces, and halfspaces can be considered to be “infinite” homothets of  $\mathcal{C}$ . As such, the above argument applies verbatim.

## B.5 Proof of Lemma 21

**Restatement of Lemma 21.** For  $\varepsilon = 1/4$ , and parameters  $n$  and  $\Phi \geq 1$ , there is a point set  $P$  of  $n + \lceil \log \Phi \rceil$  points in the plane, with spread  $\mathcal{O}(n\Phi)$ , such that any local  $(1+\varepsilon)$ -spanner of  $P$  for disks, must have  $\Omega(n \log \Phi)$  edges.

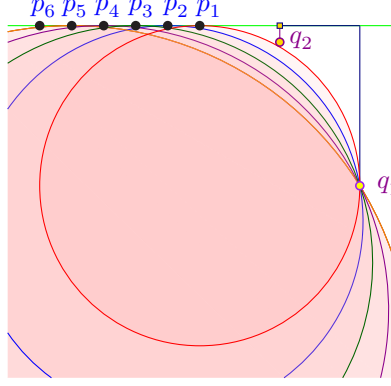


Figure B.4 The set of disks  $D_1$ , and the construction of  $q_2$ .

**Proof.** Let  $p_i = (-i, 0)$ , for  $i = 1, \dots, n$ . Let  $M = 1 + \lceil \log_2 \Phi \rceil$  and  $q_1 = (n2^M, -1)$ . For a point  $p$  on the  $x$ -axis, and a point  $q$  below the  $x$ -axis and to the right of  $p$ , let  $\odot_{\downarrow}^p(q)$  be the disk whose boundary passes through  $p$  and  $q$ , and its center has the same  $x$ -coordinate as  $p$ .

In the  $j$ th iteration, for  $j = 2, \dots, M-1$ , Let  $x_j = n2^{M-j+1} = x(q_{j-1})/2$ , and let  $y_j < 0$  be the maximum  $y$ -coordinate of a point that lies on the intersection of the vertical line  $x = x_j$  and the disks of  $D_1 \cup \dots \cup D_j$  where

$$D_j = \left\{ \odot_{\downarrow}^{p_i}(q_{j-1}) \mid i = 1, \dots, n \right\},$$

see Figure B.4 for an illustration of  $D_1$ .

Let  $q_j = (x_j, 0.99y_j)$ .

Clearly, the point  $q_j$  lies outside all the disks of  $D_1 \cup \dots \cup D_j$ . The construction now continues to the next value of  $j$ . Let  $P = \{p_1, \dots, p_n, q_2, \dots, q_M\}$ . We have that  $|P| = n + M - 1$ .

The minimum distance between any points in the construction is 1 (i.e.,  $\|p_1 p_2\|$ ). Indeed  $x(q_{M-1}) = 4n$  and thus  $\|q_{M-1} p_1\| \geq 2n$ . The diameter of  $P$  is  $\|p_1 q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$ . As such, the spread of  $P$  is bounded by  $\leq n2^{M+1} = \mathcal{O}(n\Phi)$ .

For any  $i$  and  $j$ , consider the disk  $\odot_{\downarrow}^{p_i}(q_j)$ . This disk does not contain any point of  $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$  since its interior lies below the  $x$ -axis. By construction it does not contain any point  $q_{j+1}, \dots, q_{M-1}$ . This disk potentially contains the points  $q_{j-1}, \dots, q_1$ , but observe that for any index  $k \in \llbracket j-1 \rrbracket$ , we have that

$$\|p_i q_k\| = \sqrt{(i + n2^{M-k+1})^2 + (y(q_j))^2},$$

which implies that  $n2^{M-k+1} \leq \|p_i q_k\| < n(2^{M-k+1} + 2)$ . We thus have that

$$\frac{\|p_i q_k\|}{\|p_i q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$



since  $j \in \llbracket M-1 \rrbracket$ . Namely, the shortest path in  $G$  between  $p_i$  and  $q_j$ , can not use any of the points  $q_1, \dots, q_{j-1}$ . As such, the graph  $G$  must contain the edge  $p_i q_j$ . This implies that  $|E(G)| \geq n(M-1)$ , which implies the claim.  $\blacktriangleleft$

## B.6 Proof of Lemma 22

**Restatement of Lemma 22.** For any  $n > 0$ , and  $\Phi = \Omega(n)$ , one can compute a set  $P$  of  $n + \mathcal{O}(\log \Phi)$  points, with spread  $\mathcal{O}(\Phi n)$ , and a triangle  $\Delta$ , such that any  $\Delta$ -local  $3/2$ -spanner of  $P$  requires  $\Omega(n \log \Phi)$  edges.

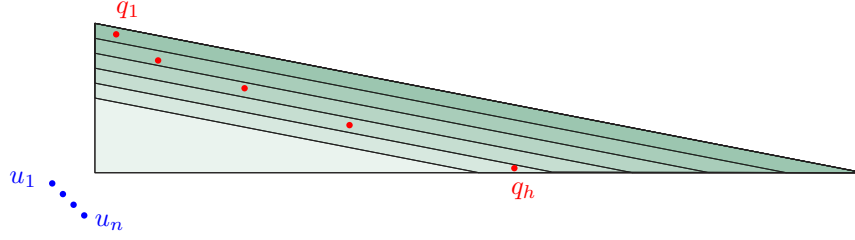


Figure B.5 An Illustration of the construction of Lemma 22.

**Proof.** Let  $h = \lceil \log_2 \Phi \rceil$ . Let  $\Delta$  be the triangle formed by the points  $(0, 0)$ ,  $(0, 1)$  and  $(8\Phi h, 0)$ . The hypotenuse of this triangle lies on the line  $\ell \equiv \frac{1}{8\Phi h}x + y = 1$ , and let  $v = (\frac{1}{8\Phi h}, 1)$  be the vector orthogonal to this line.

For  $i \in \llbracket h \rrbracket$  and  $j \in \llbracket n \rrbracket$ , let

$$q_i = (2^{i+1}, 1 - i/h) \quad \text{and} \quad u_j = (\frac{j}{n} - 1, -\frac{j}{n}),$$

and let  $P = \{q_1, \dots, q_h, u_1, \dots, u_n\}$ , see Figure B.5. Observe that  $\text{cp}(P) = \|u_1 u_2\| = \sqrt{2}/n$ , and as such we have that  $\Phi(P) = n \cdot \text{diam}(P)/\sqrt{2} \leq n(4\Phi + 2n) \leq 8\Phi n$ , as  $\Phi \geq n$ . Observe that

$$\langle q_{i+1} - q_i, v \rangle = \langle (2^{i+1}, -\frac{1}{h}), (\frac{1}{8\Phi h}, 1) \rangle \leq \frac{4\Phi}{8\Phi h} - \frac{1}{h} < 0.$$

That is, the points  $q_1, \dots, q_i$  are increasing in distance from  $\ell$ .

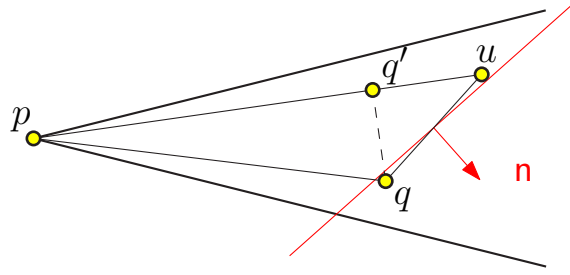
Let  $\Delta_{i,j}$  be the homothet of  $\Delta$ , that has its bottom left corner at  $u_j$ , and its hypotenuse passes through  $q_i$ . By the above,  $P(i, j) = \Delta_{i,j} \cap P = \{u_j, q_i, q_{i+1}, \dots, q_h\}$ . Any  $(1 + \varepsilon)$ -spanner for  $P(i, j)$  must contain the edge  $u_j q_i$ . Indeed, we have, for any  $k$ , that  $2^{k+1} \leq \|u_j q_k\| \leq 2^{k+1} + 3$ . As such, any path on a graph induced on  $P(i, j)$  from  $u_j$  to  $q_i$  that uses (say) a midpoint  $q_k$ , for  $k > i$ , must have dilation at least

$$\frac{\|u_j q_k\| + \|q_k q_i\|}{\|u_j q_i\|} \geq \frac{2^{k+1} + 2^k}{2^{i+1} + 3} \geq \frac{3 \cdot 2^{i+1}}{(1 + 3/4)2^{i+1}} = \frac{12}{7} > \frac{3}{2}.$$

Thus, any  $\Delta$ -local  $3/2$ -spanner for homothets of  $\Delta$ , must contain the edge  $q_i u_j$ , for any  $i \in \llbracket h \rrbracket$  and  $j \in \llbracket n \rrbracket$ . Thus, such a spanner must have  $\Omega(n \log \Phi)$  edges, as claimed.  $\blacktriangleleft$

## B.7 Proof of Lemma 24

**Restatement of Lemma 24.** Let  $p \in P$ ,  $c \in \mathcal{C}_i$ , and  $u = \text{nn}_i(p, c)$ , and let  $q$  be a point in  $(P \cap (p + c)) \setminus \{p, u\}$ . We have that  $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$  and  $\|qu\| \leq \|pq\|$ .



780 **Figure B.6** The case that  $\|pq\| \leq \|pu\|$  in Lemma 24. The vector used to determine  $\text{nn}_i(p, c)$  is  
 781 shown in red, and denoted  $n$

777 **Proof.** Consider the triangle  $\Delta pqu$  and denote the angles at  $p, q$ , and  $u$  by  $\angle p, \angle q$ , and  $\angle u$   
 778 respectively. Since the angle of  $c$  is smaller than 60 degrees (for an appropriate choice of  $\gamma$ ),  
 779 we have that  $\|qu\| \leq \max\{\|pu\|, \|pq\|\}$ .

782 Consider the case that  $\|pq\| \leq \|pu\|$ , illustrated in Figure B.6. Observe that  $\angle u \leq \angle q$ .  
 783 As such  $\angle u \leq \pi/2$ . Furthermore,  $\angle u \geq \alpha \gg \varepsilon\alpha/\gamma = \beta \geq \angle p$ . Similarly,  $\angle q \in [\alpha, \pi - \alpha]$ . By  
 784 the 1-Lipshitz of  $\sin$ , and as  $\sin x \approx x$ , for small  $x$ , and for  $\gamma$  sufficiently large, we have that

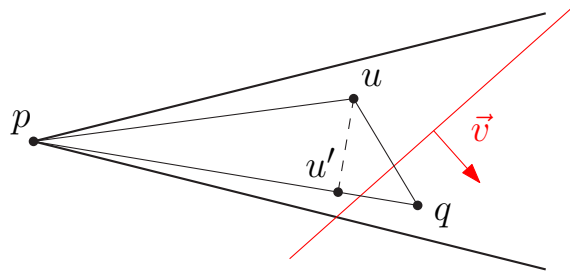
$$785 \quad \sin(\angle q + \angle p) \in [1 - \varepsilon/4, 1 + \varepsilon/4] \sin \angle q \quad \text{and} \quad \sin \angle p \leq (\varepsilon/4) \sin \angle u.$$

786 As such, by the law of sines, we have that  $\frac{\|qu\|}{\sin \angle p} = \frac{\|pq\|}{\sin \angle u} = \frac{\|pu\|}{\sin \angle q}$ . This implies that

$$787 \quad \|pu\| + (1 + \varepsilon) \|qu\| = \left( \frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \right) \|pq\|.$$

788 Observe, by the above that

$$789 \quad \frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \leq \frac{\sin \angle q}{\sin(\angle p + \angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq \frac{\sin \angle q}{(1 - \varepsilon/4) \sin(\angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$



790 **Figure B.7** The case that  $\|pq\| > \|pu\|$  in Lemma 24.

791 The other possibility is that  $\|pq\| > \|pu\|$ , illustrated in Figure B.7. Let  $u'$  be the  
 792 projection of  $u$  to  $pq$ . Observe that

$$793 \quad \|uu'\| = \|pu'\| \tan \angle p \leq 2\beta \|pu'\| \leq (\varepsilon/8) \|pu'\|.$$

794 Observe that  $\cos \angle p \geq 1 - (\angle p)^2/2 \geq 1 - \varepsilon^2/8$  as  $\angle p$  is an angle smaller than (say)  $\varepsilon/16$ . As  
 795 such  $1/\cos \angle p \leq 1 + \varepsilon^2/4$ . This implies that  $\|pu\| \leq \|pu'\|/\cos \angle p \leq (1 + \varepsilon^2/4) \|pu'\|$ . We  
 796 thus have that

$$797 \quad \tau = \|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon^2/4) \|pu'\| + (1 + \varepsilon) (\|uu'\| + \|u'q\|) \\
 798 \quad \leq (1 + \varepsilon^2/4 + (1 + \varepsilon)\varepsilon/8) \|pu'\| + (1 + \varepsilon) \|u'q\| \leq (1 + \varepsilon) \|pq\|.$$

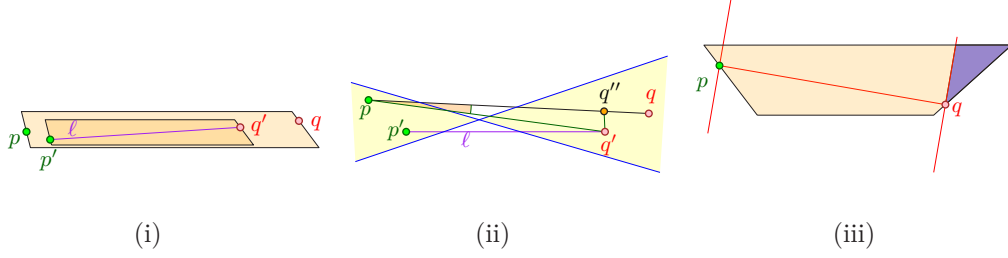
799

## B.8 Proof of Lemma 28

**Restatement of Lemma 28.** Let  $\varepsilon \in (0, 1)$  be some parameter, and  $\vartheta = \varepsilon/16$ . Let  $X, Y$  be two points sets that are  $\vartheta$ -semi separated and  $\vartheta$ -angularly separated (see Definition 6), and let  $T$  be a  $\vartheta$ -narrow trapezoid, with two points  $p \in X$  and  $q \in Y$  lying on the two legs of  $T$ . Then, one can compute a homothet  $T' \subseteq T$  of  $T$ , such that:

(I) There are two points  $p' \in X$  and  $q' \in Y$ , such that  $p'q'$  is an edge of the  $T$ -Delaunay triangulation of  $X \cup Y$ .

(II) We have that  $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$ .



**Figure B.8** Illustration of the settings in the proof of Lemma 28. Left: A  $\vartheta$ -narrow trapezoid with  $p$  and  $q$  on its legs. Center:  $p$  and  $q$  are  $\vartheta$ -semi separated and  $\vartheta$ -angularly separated. Right: The triangle of all the points of the trapezoids that their nearest point on  $pq$  is  $q$ .

**Proof.** Let  $\mathcal{D} = \mathcal{D}_T(X \cup Y)$ . Claim 15 implies that  $\mathcal{D} \cap T$  is connected. Thus, there is a path in  $\mathcal{D} \cap T$  between  $p$  and  $q$ , and thus, there must be an edge  $p'q'$  along this path with  $p' \in X$  and  $q' \in Y$ . This implies part (I).

Let  $\ell = \|p'q'\|$ . Assume for concreteness that  $\|pp'\| \leq \text{diam}(X) \leq \vartheta d(X, Y) \leq \vartheta \ell \leq \vartheta d$ , where  $d = \text{diam}(T)$ . Let  $q''$  be the closest point on  $pq$  to  $q'$ .

We first consider the case that  $q'' \in \text{int}(pq)$ . We have that

$$\|pq''\| = \|pq'\| \cos \angle q'pq \geq (\|p'q'\| - \|pp'\|) \cos \angle q'pq \geq (1 - \vartheta)\ell \cdot (1 - \vartheta^2/2) \geq (1 - 2\vartheta)\ell,$$

since  $\cos \vartheta \geq 1 - \vartheta^2/2$ , for  $\vartheta < 1/2$ . Similar argumentation implies that  $\|pq''\| \leq (1 + \vartheta)\ell$ . As such, we have

$$\|q'q''\| \leq (1 + \vartheta)\ell \sin \angle p'pq' \leq 2\vartheta\ell.$$

Thus, we have that

$$\|qq'\| \leq \|qq''\| + \|q''q'\| \leq \|pq\| - \|pq''\| + 2\vartheta\ell \leq \|pq\| - (1 - 2\vartheta)\ell + 2\vartheta\ell \leq \|pq\| - \ell.$$

Thus, we have that

$$\begin{aligned} (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| &\leq (1 + \varepsilon)\vartheta\ell + \ell + (1 + \varepsilon)(\|pq\| - \ell) \\ &= (1 + \varepsilon) \|pq\| + (1 + \varepsilon)\vartheta\ell + \ell - (1 + \varepsilon)\ell \leq (1 + \varepsilon) \|pq\|, \end{aligned}$$

for  $\vartheta \leq \varepsilon/2$ . Which establish the claim in this case.

The case that  $q'' = p$  is impossible, because of the angular separation property. Thus, the only remaining possibility is that  $q'' = q$ . This however implies that  $q'$  must be in the triangle of all the points of the trapezoids that their nearest point on  $pq$  is  $q$ . The diameter of this triangle is bounded by the length of the leg of the trapezoid, which is bounded by

$\vartheta d$ . Namely, we have  $\|qq'\| \leq \vartheta d$ . Similarly, we have  $(1 - 2\vartheta)d \leq \|pq\| \leq (1 + 2\vartheta)d$ . Since  $\|pp'\|, \|qq'\| \leq \vartheta d$ , it follows that

$$(1 - 4\vartheta)d \leq \ell \leq (1 + 4\vartheta)d.$$

As such, for  $\vartheta \leq \varepsilon/8$  and  $\varepsilon \leq 1$ , we have

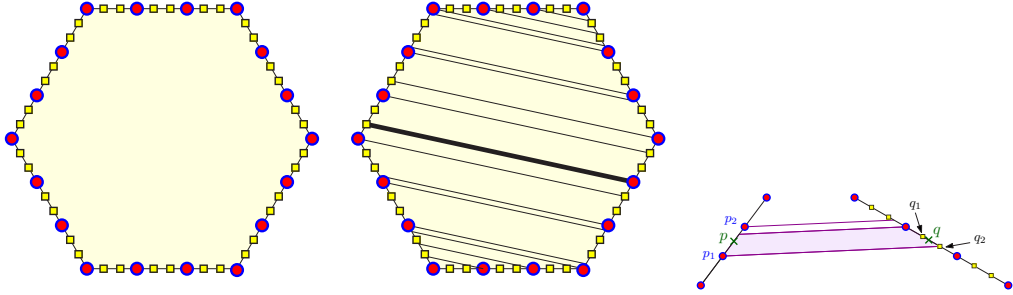
$$(1 + \varepsilon)\|pp'\| + \ell + (1 + \varepsilon)\|q'q\| \leq 4\vartheta d + (1 + 4\vartheta)d = (1 + 8\vartheta)d \leq (1 + \varepsilon)\|pq\|.$$

## B.9 Proof of Lemma 29

**Restatement of Lemma 29.** *Let  $t$  be a positive integer. Given a  $t$ -nice polygon  $\mathcal{C}$ , and a parameter  $\vartheta$ , one can cover it by a set  $\mathcal{T}$  of  $\mathcal{O}(t^4/\vartheta^3)$   $\vartheta$ -narrow trapezoids, such that for any two points  $p, q \in \partial\mathcal{C}$  that belong to two edges of  $\mathcal{C}$  that are not adjacent, there exists a narrow trapezoid  $T \in \mathcal{T}$ , such that  $p$  and  $q$  are located on two different short legs of  $T$ .*

**Proof.** We show a somewhat suboptimal but simple construction. A  $t$ -nice polygon has at most  $t$  edges. Let  $\psi$  be the sensitivity of  $\mathcal{C}$ , and place a minimum set of points  $P$  on the boundary of  $\mathcal{C}$ , which includes all the vertices of  $\mathcal{C}$ , and such that the distance between any consecutive pair of points is in the range  $[c_1, 2c_1]$ , where  $c_1 = \vartheta\psi/c_2$ , for some sufficiently large constant  $c_2$ . In particular, let  $M = \max_{e \in E(\mathcal{C})} \lceil \|e\|/c_1 \rceil = \mathcal{O}(1/\vartheta)$ .

In addition, place  $c_3 t$  equally spaced points between any two consecutive points of  $P$ , where  $c_3$  is a constant to be determined shortly. Let  $Q$  be the set resulting from  $P$  after adding all these points.



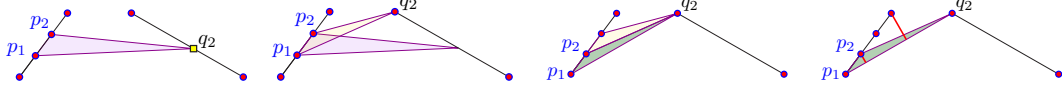
**Figure B.9** The points of  $P$  (round), and all the points added to  $P$  in order to create  $Q$  (square). On the right, a “vertical” decomposition induced by one of the directions of  $P \times Q$ .

We have that  $|P| = \mathcal{O}(t/\vartheta)$  and  $|Q| = \mathcal{O}(t^2/\vartheta)$ . For a direction  $v$ , let  $\mathcal{T}_v$  be the decomposition into trapezoids formed by shooting rays from inside  $\mathcal{C}$  in the direction of  $v$  (or  $-v$ ) from all the points of  $P$ , see Figure B.9. Let  $\mathcal{T}'_v$  be the set resulting from throwing away trapezoids with legs that lie on adjacent edges. It is easy to verify that all the trapezoids of  $\mathcal{T}'_v$  are  $\vartheta$ -narrow. Let  $U$  be the set of all directions induced by pairs of points of  $P \times Q$ , and let  $\mathcal{T} = \cup_{u \in U} \mathcal{T}'_u$ . We have that  $|\mathcal{T}| = \mathcal{O}(|P| \cdot |U|) = \mathcal{O}(|P|^2 |Q|) = \mathcal{O}(t^4/\vartheta^3)$ .

Consider any two points  $p, q$  on non-adjacent edges of  $\mathcal{C}$ , and let  $p_1, p_2$  be the two adjacent points of  $P$  such that  $p \in p_1 p_2$ . Now, let  $q_1, q_2$  be the adjacent points of  $Q$  such that  $q \in q_1 q_2$ . We assume that  $p_1, p_2, q_1, q_2$  are in this clockwise order along the boundary of  $\mathcal{C}$ .

Observe that when we project the interval  $p_1 p_2$ , to the line induced by  $q_1 q_2$ , in the direction  $\overrightarrow{p_1 q_2}$ , the projected interval contains  $q_1 q_2$ . The last claim is intuitively obvious,

but requires some work to see formally. The minimum height of a triangle involving three vertices of  $\mathcal{C}$  is formed by three consecutive vertices. In the worst case, this is an isosceles triangle with sidelength  $\psi$  and base angle  $\pi/t$ . As such, the height of such a triangle is  $h = \psi \sin(\pi/t) \geq \psi/t$ .



**Figure B.10** The height of the triangle  $\triangle p_1 p_2 q_2$  is minimized as  $q_2$  and  $p_1$  are moved to vertices of  $\mathcal{C}$ .

The height of the triangle  $\triangle p_1 p_2 q_2$  is minimized when  $p_1$  or  $p_2$  is a vertex of  $\mathcal{C}$ , and  $q_2$  is at a vertex of  $\mathcal{C}$ , see Figure B.10. Assume, for concreteness, that  $p_1$  is a vertex of  $\mathcal{C}$ , and observe that  $\|p_1 p_2\| \geq \|e\|/M$ , where  $e$  is the edge of  $\mathcal{C}$  containing this segment. Using similar triangles, it is straightforward to show that the height of this triangle is at least  $h' = h/M = \Omega(\varepsilon\psi/t)$ . The quantity  $h'$  is a lower bound on the length of the projection of  $p_1 p_2$  on the line spanned by  $q_1 q_2$ . However,  $\|q_1 q_2\| \leq 2c_1/c_3 t = \mathcal{O}(\vartheta\psi/c_3 t) < h'$ , by picking  $c_3$  to be sufficiently large constant.

This readily implies that the trapezoid induced by the direction  $u = \overrightarrow{p_1 q_2}$  in  $\mathcal{T}'_u$  that contains  $p$  on its leg, contains  $q$  on its other leg.  $\blacktriangleleft$