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Local Geometric Spanners

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Abstract We introduce the concept of local spanners for planar point sets with respect to a family of regions, and prove the existence of local spanners of small size for some families. For a geometric graph G on a point set P and a region R belonging to a family $\mathcal{R}$, we define $G \cap R$ to be the part of the graph G that is inside R (or is induced by R). A local t-spanner w.r.t $\mathcal{R}$ is a geometric graph G on P such that for any region $R \in \mathcal{R}$, the graph $G \cap R$ is a t-spanner for $K(P) \cap R$, where $K(P)$ is the complete geometric graph on P. For any set P of n points and any constant $\varepsilon > 0$, we prove that P admits local $(1+\varepsilon)$-spanners of sizes $O(n \log^6 n)$ and $O(n \log n)$ w.r.t axis-parallel squares and vertical slabs, respectively. If adding Steiner points is allowed, then local $(1+\varepsilon)$-spanners with $O(n)$ edges and $O(n \log^2 n)$ edges can be obtained for axis-parallel squares and disks using $O(n)$ Steiner points, respectively.

Keywords Geometric spanner · Local spanner

1 Introduction

Background. A *geometric network* on a point set P in $\mathbb{R}^d$ is an undirected graph $G(P, E)$ with vertex set P whose edges are straight-line segments connecting pairs of points in P. Geometric networks offer a way to model many real-life networks such as water distribution, road networks and telephone services.

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When designing a network for a point set P, distances play a vital role in many applications. It would be ideal to have a direct connection between every pair of points. The network would then be a complete graph, but in most applications, this is unacceptable due to its high cost. This leads to the concept of spanners introduced by Peleg and Schaffer [10] in the context of distributed computing and by Chew [5] in a geometric context.

Let $H = (V, E)$ be a weighted undirected graph and $t \geq 1$ be a real number. The subgraph $G = (V, E')$ of H is a t-spanner if for any two vertices p, q of H, we have $d_G(p, q) \leq t \cdot d_H(p, q)$ where $d_G(p, q)$ and $d_H(p, q)$ denote the length of the shortest path between p and q on G and H, respectively (any path in G from p to q whose length is at most $t \cdot d_H(p, q)$ is called a t-path). The dilation or stretch factor s is the minimum t for which G is a t-spanner of H. In this paper, we are interested in spanners in a geometric context. Here, a geometric network $G(P, E)$ is a geometric t-spanner of $K(P)$, where $K(P)$ is the complete geometric network on P. Precisely, for any two vertices p, q of P we have $d_G(p, q) \leq t \cdot |pq|$ where $|pq|$ denotes the Euclidean distance of p and q. Geometric spanners have received much attention over the past three decades and numerous papers on geometric spanners have been published—see for instance the surveys [8, 11] and the book devoted to geometric spanners [9].

Problem Statement. Suppose one would like to design a road network on a continent that is a $(1+\varepsilon)$-spanner. Locally, each country of the continent would like that the part of the road network inside the country is a $(1 + \varepsilon)$-spanner as exiting and entering a country usually is time and money consuming. This can be simply extended to provinces in a country or divisions in a city. This leads us to define local spanners in an abstract way. Let $\mathcal{R}$ be a family of regions in the plane. For a region $R \in \mathcal{R}$ and a geometric graph $G(P, E)$ on a point set P in the plane, we define $G \cap R$ to be the part of G that is inside R, i.e. the induced subgraph of G over the vertices in $P \cap R$—see Fig. 1. A $(1 + \varepsilon)$-spanner G is local w.r.t. a family $\mathcal{R}$ of regions if and only if for any region $R \in \mathcal{R}$, the graph $G \cap R$ is a $(1 + \varepsilon)$-spanner for $K(P) \cap R$, where $K(P)$ is the complete geometric network on P. Indeed, we want G not only to be a $(1 + \varepsilon)$-spanner globally but also to be a $(1 + \varepsilon)$-spanner locally when G is cut by an arbitrary region R. Ideally, we would like to have a geometric spanner with such a property w.r.t. any convex or non-convex regions while having few number of edges. Unfortunately, this is not possible for any family of regions (we give some examples later). Therefore, we restrict ourselves to some basic families of regions like disks or squares.

For some families of regions, G must have $\Omega(n^2)$ edges to be local. The family of axis-parallel rectangles is such a family. As depicted in Fig. 2, there are $\Theta(n^2)$ different rectangles containing just two points. Therefore, these pairs of points must be connected in G— note that this works also for fat rectangles by taking the two lines containing the points further apart; so fatness does not help. In this paper, we focus on some specific families, namely vertical slabs

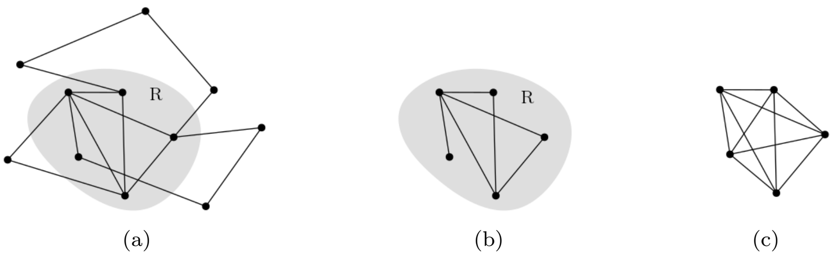

Fig. 1: (a) A geometric network G and a region R, (b) $G \cap R$, (c) $K(P) \cap R$.

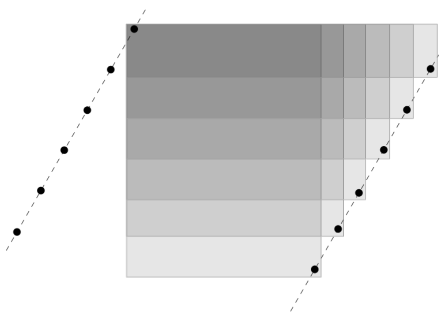

Fig. 2: A configuration of points for which any local $(1+\varepsilon)$-spanner w.r.t. rectangles needs $\Omega(n^2)$ edges.

(i.e. $[a, b] \times [-\infty, \infty]$ for some a and b), axis-parallel squares and disks, and present local $(1+\varepsilon)$-spanners for them with near-linear sizes.

We shall also consider the case where we are allowed to add Steiner points to the graph. In other words, instead of constructing a geometric network for P, we are allowed to construct a network for $P \cup Q$ for some set Q of Steiner points. In this case, we only require short connections between the points in P. Thus, we say that a graph G on $P \cup Q$ is a local Steiner t-spanner for P w.r.t. $\mathcal{R}$ if, for any $R \in \mathcal{R}$ and any two points $u, v \in P \cap R$, the distance between u and v in $G \cap R$ is at most t times their distance in $K(P) \cap R$.

Previous Work. Abam et al. [1] presented a region-fault tolerant $(1+\varepsilon)$-spanner G of size $O(n \log n)$ for a point set P w.r.t. convex regions, i.e. for any convex region R the graph $G - R$ (what remains of G after the vertices and edges of G intersecting R have been removed) is a $(1+\varepsilon)$-spanner for $K(P) - R$. Here, for points p and q the edge $\{p, q\}$ may not exist in $K(P) - R$, and indeed the length of the shortest path from p to q in $G - R$ is guaranteed to be at most $(1+\varepsilon)$ times the length of the shortest path from p to q in $K(P) - R$. When convex regions are half planes, $K(P) - R$ is a complete graph on the remaining points. Therefore, this problem is the inverse of the local spanner problem, and we already have $(1+\varepsilon)$-local spanner of size $O(n \log n)$ for the familly of halfplanes.

Our Results. In Section 2, we present a general method to convert an s-well-separated pairs decomposition (WSPD) [4] for P and s (depending on t) into a local t-spanner for P w.r.t. the family $\mathcal{R}$. In Section 3, we construct a local $(1+\varepsilon)$-spanner of size $O(n \log n)$ for vertical slabs, and moreover, we show any local $(1+\varepsilon)$-spanner w.r.t. vertical slabs needs $\Omega(n \log n)$ edges even if we are allowed to use Steiner points. In Section 4, we consider axis-parallel square regions and show it is possible to construct a local $(1+\varepsilon)$-spanner of size $O(n \log^6 n)$. Moreover, we show that if Steiner points are allowed, we can reduce the number of edges to be $O(n)$. We dedicate Section 5 to disk regions and show that by adding $O(n)$ Steiner points, we can construct a local $(1+\varepsilon)$-spanner of size $O(n \log^2 n)$.

2 Constructing local spanners using WSPD

In this section, we show a general method to obtain a local spanner from a well-separated-pair decomposition of a point set P defined below.

Definition 1 [4] Let P be a set of n points in the plane and let $s > 0$ be a real number. Two point sets A and B in the plane are well-separated w.r.t. s, if there are two disjoint disks of the same radius that each of them covers their respective point set and their distance is at least s times of their radius. A well-separated pair decomposition (WSPD) for P w.r.t. s (called s-WSPD) is a collection $\Psi := \{(A_1, B_1), \ldots, (A_m, B_m)\}$ of pairs of non-empty subsets of P such that

1. A_i and B_i are well-separated w.r.t. s, for all $i = 1, \cdots, m$.
2. for any two distinct points p and q of P, there is exactly one pair $(A_i, B_i) \in \Psi$, such that
   (i) $p \in A_i$ and $q \in B_i$, or (ii) $q \in A_i$ and $p \in B_i$.

The number of pairs, m, is called the size of the WSPD. Callahan and Kosaraju show that any set P admits a WSPD of size $m = O(s^2 n)$.

The spanner construction based on the WSPD [4]. We first construct an s-WSPD for $s = 4 + 8/\varepsilon$, and then for each pair (A_i, B_i) of the WSPD, we add an edge to the spanner G from an arbitrary point of A_i to an arbitrary point of B_i. It was shown that G is a $(1+\varepsilon)$-spanner. The standard proof is based on induction on the distances of the pairs of points. As the basis step, it was shown that the closest pair always exists in G. In the inductive step, to show that there is a $(1+\varepsilon)$-path from p to q in G assuming $|pq|$ is the kth smallest distance, consider the pair (A_i, B_i) of the WSPD where $p \in A_i$ and $q \in B_i$ or vice versa. Assume $p \in A_i$ ans $q \in B_i$. We know that there is an edge connecting one point of A_i to one point of B_i in G; let $\{p', q'\}$ be such an edge ($p' \in A_i$ and $q' \in B_i$). Both $|pp'|$ and $|qq'|$ are known to be smaller than $|pq|$ due to the properties of the WSPD. Therefore, there is a $(1+\varepsilon)$-path from p (q) to p' (q') based on the inductive hypothesis. It was shown that these two $(1+\varepsilon)$-paths together with edge $\{p', q'\}$ provides a $(1+\varepsilon)$-path from p to q in G. Hence, if we guarantee that G contains an edge per pair of an $(4+8/\varepsilon)$-WSPD, G is indeed a $(1+\varepsilon)$-spanner. As we use this fact in all our spanner constructions, we state it in the following corollary.

Corollary 1 *If a graph $G(P, E)$ contains an edge per pair of an $(4+8/\varepsilon)$-WSPD of a point set P, then G is a $(1+\varepsilon)$-spanner for P.*

In the above construction, G might not be local. Indeed, even though for a region $R \in \mathcal{R}$, the sets $R \cap A_i$ and $R \cap B_i$ might not be empty, the selected edge for pair (A_i, B_i) might not exist in R. Therefore, we can not apply the standard proof to show the subgraph of G induced by R is a $(1+\varepsilon)$-spanner.

The general method. We first construct an s-WSPD for $s = 4 + 8/\varepsilon$, and then for each pair (A_i, B_i) of the s-WSPD, we do as follows. For each pair (p, q) where $p \in A_i$ and $q \in B_i$, we add $\{p, q\}$ to the spanner G provided that there is a region $R \in \mathcal{R}$ that interiorly contains no points of $A_i \cup B_i$ and has p and q on its boundary.

The general method may add several edges per pair (A_i, B_i) of the s-WSPD to the spanner G. So, we can not simply say that the size of G is the number of the pairs of the s-WSPD, and indeed there are examples of region families (like rectangles) such that the size of G becomes $\Omega(n^2)$.

For the families of axis-parallel squares and disks, the number of edges added to the spanner for a pair (A_i, B_i) is $O(|A_i| + |B_i|)$ under the general position assumption. Indeed, a subset of the edges of the Delaunay graph w.r.t. disks and axis-parallel squares on point set $A_i \cup B_i$ is added to G, which is known to be linear [3]. So, G has $\sum(|A_i| + |B_i|)$ edges where sum is over all pairs of the s-WSPD. Therefore, we get the following theorem.

Theorem 1 *For a set P of n points in the plane and families of axis-parallel squares and disks, we can construct a local $(1+\varepsilon)$-spanner of size $\sum(|A_i|+|B_i|)$ where (A_i, B_i) are the pairs of a $(4+8/\varepsilon)$-WSPD.*

Proof We explain the proof for disks. The proof for axis-parallel squares is similar. Assume $W = \{(A_i, B_i)\}$ is a $(4 + 8/\epsilon)$-WSPD over point set P, and $R \in \mathcal{R}$ is a disk. We have already proved that the size of G is $\sum(|A_i| + |B_i|)$. It remains to show that G is local. It is easy to see $W \cap R = \{(A_i \cap R, B_i \cap R)\}$ is a $(4+8/\epsilon)$-WSPD for $P \cap R$. Moreover, our construction (explained above) guarantees that for each pair (A_i, B_i), there is at least one edge between two sets $A_i \cap R$ and $B_i \cap R$ in G provided that both sets $A_i \cap R$ and $B_i \cap R$ are not empty. To show this, shrink R (while fixing its center) until one of $A_i \cap R$ and $B_i \cap R$ has only one point and the other has at least one point (assume$A_i \cap R$ has only one point and name this point p). Now shrink R by moving its center toward p until $B_i \cap R$ contains only one point, say q. In the general method, we add $\{p, q\}$ to the spanner G. Now, based on Corollary 1, the subgraph of G induced by R is a $(1+\varepsilon)$-spanner. □

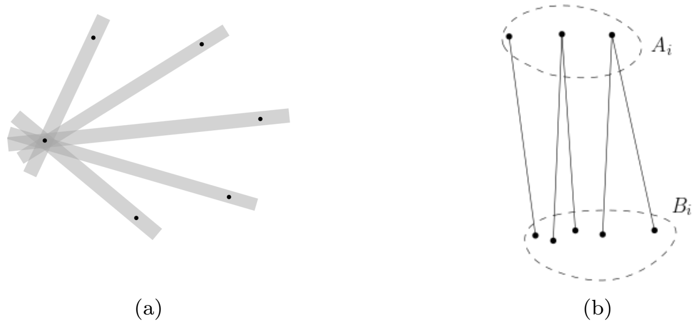

Fig. 3: (a) For points on convex position, any local t-spanner w.r.t. arbitrary oriented slabs must be complete. (b) The first local-spanner construction w.r.t. vertical slabs.

3 Local spanners w.r.t. vertical slabs

For arbitrary oriented slabs (not necessarily vertical), there is an example (Fig. 3(a)) showing that we need $\Omega(n^2)$ edges to have a local spanner—in fact, any set of points in general position could be an example for this claim because any edge of the complete graph can introduce a narrow slab which contains only its two endpoints. Hence, we only focus on vertical slabs (i.e. $[a, b] \times [-\infty, \infty]$ for some a and b). We present two methods similar to the general method presented in the previous section.

First construction. Our first construction uses a $(4 + 8/\epsilon)$-WSPD $W = \{(A_i, B_i)\}$ of size $\sum \min(|A_i|, |B_i|)$ which is bounded by $O(n \log n)$ [9]. We apply the general method given in Section 2 using such a WSPD. Assume (A_i, B_i) is a pair of $(4 + 8/\epsilon)$-WSPD and $|A_i| \leq |B_i|$. Imagine the sorted list of the points in $A_i \cup B_i$ based on their x-coordinate. Each point p of A_i is connected to at most two points of B_i: the leftmost point of B_i right to p and the rightmost of B_i left to p in the sorted list—see Fig. 3(b). Hence, the number of edges added to the spanner G is at most $2 \sum \min(|A_i|, |B_i|) = O(n \log n)$.

Theorem 2 *For a set P of n points in the plane and any $\varepsilon > 0$, there exists a local $(1 + \varepsilon)$-spanner of size $O(n \log n)$ w.r.t. vertical slabs.*

Proof The proof is similar to one given for Theorem 1. Assume $W = \{(A_i, B_i)\}$ is a $(4 + 8/\epsilon)$-WSPD over point set P, and $R \in \mathcal{R}$ is a vertical slab. We have already proved that the size of G is $\sum \min(|A_i|, |B_i|) = O(n \log n)$. It remains to show that G is local. It is easy to see $W \cap R = \{(A_i \cap R, B_i \cap R)\}$ is a $(4 + 8/\epsilon)$-WSPD for $P \cap R$. Moreover, our construction guarantees that for each pair (A_i, B_i), there is at least one edge between two sets $A_i \cap R$ and $B_i \cap R$ in G provided that both sets $A_i \cap R$ and $B_i \cap R$ are not empty. To show this, consider the sorted list of points in $(A_i \cap R) \cup (B_i \cap R)$ based on their x-coordinate. Since both sets $A_i \cap R$ and $B_i \cap R$ are not empty, there must be two consecutive points in the sorted list, say p and q, such that $p \in A_i \cap R$ and $q \in B_i \cap R$. Since there is a vertical slab just containing p and q among point in $A_i \cup B_i$, there is an edge connecting p to q in G. Now, based on Corollary 1, the subgraph of G induced by R is a $(1 + \varepsilon)$-spanner. □

Second construction. Our other construction uses an x-monotone WSPD defined below.

Definition 2 A WSPD $W = \{(A_i, B_i)\}$ is called x-monotone if for all i, $p \in A_i$, and $q \in B_i$, we have $x(p) < x(q)$ where $x(p)$ denotes the x-coordinate of p.

Theorem 3 *For a set P of n points in the plane and any $s > 0$, there exists an x-monotone s-WSPD of size (the number of pairs) $O(s^2 n \log n)$.*

Proof We construct a balanced binary search tree over P based on the x-coordinates of points. At each node w of the tree with children u and v, we construct an s-WSPD of size $O(s^2|P(w)|)$ separating $P(u)$ and $P(v)$ where $P(w)$ denotes the set of points lying at the subtree rooted at w. Indeed at node w we can construct a compressed quad-tree [7] on $P(w)$ such that at the root of the quad-tree, the vertical spiliting line ℓ passes through the x-median of $P(w)$. From all produced pairs, we keep only pairs (A, B) such that A and B are at different sides of ℓ. In total, we construct $O(n)$ compreseed quad-tree (one for each node w of the balanced binary search tree and produce $O(s^2|P(w)|)$ pairs). Therefore, the number of pairs in total becomes $O(s^2 n \log n)$ by a simple recursive formula. □

When an x-monotone $(4 + 8/\epsilon)$-WSPD of size $O(n \log n)$ is available (no matter how it is constructed), we can easily construct a local $(1 + \varepsilon)$-spanner as follows. For any pair (A, B) of the WSPD, we add an edge between the rightmost point (based on x-coordinates) of A and the leftmost point of B. Since the number of pairs is $O(n \log n)$, the number of edges added to the spanner is $O(n \log n)$. Moreover, for each slab R, if $A \cap R$ and $B \cap R$ are not be empty, the edge added to the spanner (coming from the pair (A, B)) exists in R. Therefore, based on Corollary 1 our spanner is a local $(1 + \varepsilon)$-spanner. Also, it is easy to show that there is an x-monotone $(1+\varepsilon)$-path between any two points as we use an x-monotone WSPD in our construction.

Theorem 4 *For a set P of n points in the plane and any $\varepsilon > 0$, there exists a local $(1 + \varepsilon)$-spanner of size $O(n \log n)$ w.r.t. vertical slabs in which there exists an x-monotone $(1 + \varepsilon)$-path between any two points.*

Next, we show that any local $(1 + \varepsilon)$-spanner w.r.t. vertical slabs needs $\Omega(n \log n)$ edges even if we are allowed to use Steiner points. To this end, we define a set P of n points recursively as follows. Consider two disks C_1 and C_2 of radius r (for some positive number r) whose centers have the same x-coordinate and $d(C_1, C_2)$ (their distance) is more than $(2 + 2\varepsilon)r$—see Fig. 4.

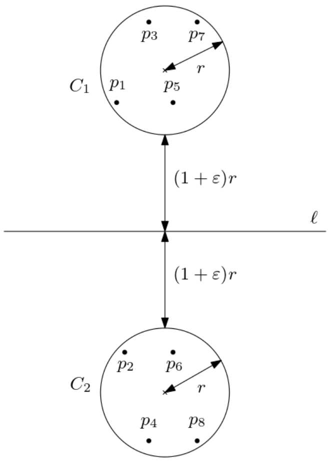

Fig. 4: A lower-bound example for local spanners w.r.t. vertical slabs.

Let points in $P = \{p_1, p_2, \cdots, p_n\}$ be indexed in the sorted order of their x-coordinates. We put all p_{2i+1} into C_1 and all p_{2i} into C_2. Consider a horizontal line ℓ of distance $(1+\varepsilon)r$ from C_1 and C_2. We will show there should be at least $\lfloor (n-1)/2 \rfloor$ edges in the spanner that intersect ℓ and they can not be a part of a $(1+\varepsilon)$-path for any two points inside C_1 or any two points inside C_2. Consider a slab S_i whose left and right sides are passing through p_i and p_{i+1}, respectively $(i = 1, \cdots, n-1)$. S_i contains just points p_i and p_{i+1} from P, but it may contain some Steiner points. There should be a path from p_i to p_{i+1} inside S_i. This path definitely intersects ℓ at some point inside S_i. Let the edge of the path intersecting ℓ be called e_i. As S_is are disjoint for even i (even they do not have a common boundary), e_is are distinct. Hence, we have $\lfloor (n-1)/2 \rfloor$ distinct edges that are only used in $(1+\varepsilon)$-paths whose endpoints are at different sides of ℓ. These $\lfloor (n-1)/2 \rfloor$ edges can not be used on $(1+\varepsilon)$-paths whose endpoints are on the same side of ℓ. Otherwise, the length of the path is more than $(2+2\varepsilon)r$ (note that the distance of ℓ to C_1 and C_2 is $(1+\varepsilon)r$), which is more than $(1+\varepsilon)$ times their Euclidean distance, which is at most $2r$. If we recursively organize points in C_1 and C_2, by a simple recursive function, we can conclude that the number of edges is $\Omega(n \log n)$.

Theorem 5 *For any $\varepsilon > 0$, there is a set P of n points in the plane such that any local $(1+\varepsilon)$-spanner w.r.t. vertical slabs needs $\Omega(n \log n)$ edges even if Steiner points are allowed to be used.*

We recall that if an x-monotone WSPD of size m is available (no matter how it is constructed), we can construct a local $(1+\varepsilon)$-spanner of size m for vertical slabs. This together Theorem 5 gives us the following corollary.

Corollary 2 *In the worst case, any x-monotone WSPD has $\Omega(n \log n)$ pairs.*

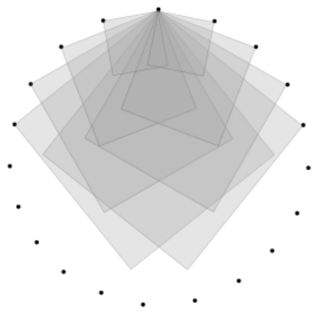

Fig. 5: If points are the vertices of a regular polygon, any local $(1+\varepsilon)$-spanner w.r.t. arbitrary square must contain $\Omega(n^2)$ edges.

4 Local spanners w.r.t. squares

For arbitrary squares (not necessarily axis-parallel), there is an example showing that we may need $\Omega(n^2)$ edges to have a local spanner—see Fig. 5. Hence, we only focus on axis-parallel squares.

We start with the case where we are allowed to use Steiner points. We use the general method given in Section 2, and add a set Q of Steiner points to P such that $|E(A,B)| = O(1)$ for any pair (A,B) in the WSPD where $E(A,B)$ is the set of edges added to the spanner for pair (A,B).

We exploit the idea used in [1] to design region-fault tolerant spanners with Steiner points. We construct a compressed quadtree $\mathcal{T}(P)$ for P [7]. For each internal node v of $\mathcal{T}(P)$, let $\square_v$ and $P(v)$ be the associated square with v and the subset of points from P inside $\square_v$, respectively. The set Q of Steiner points is the corner points of $\square_v$ for all internal nodes of $\mathcal{T}(P)$. Since the compressed quadtree has at most $n-1$ nodes, the set Q has size at most $4(n-1)$.

Now consider $\mathcal{T}(\overline{P})$ as a compressed quadtree for $\overline{P} = P \cup Q$ and let v be an internal node of $\mathcal{T}(\overline{P})$. It was shown [1] that if the convex hull of $\overline{P}(v)$ (denoted by $CH(\overline{P}(v))$) contains more than one member, it is either a square or a kite, both depicted in Fig. 6(a). We call the vertices of $CH(\overline{P}(v))$, the representative points of v. The representative point of a leaf node v is the single point inside $\square_v$.

Fisher and Har-Peled [7] show that one can obtain an s-WSPD of size $O(s^2 n)$ for $\overline{P}$ that consists of pairs $(P(v_1), P(v_2))$ where v_1 and v_2 are nodes in $\mathcal{T}(\overline{P})$. We set $s = 4 + 8/\varepsilon$ and construct the WSPD. Then, we add all edges $\{p, q\}$ to the spanner G where p and q are the representative point of v_1 and v_2, respectively. We only need the following lemma to show the dilation of the spanner is $1 + \varepsilon$.

Lemma 1 *Suppose R is an axis-parallel square region and (A,B) is a pair of the WSPD for $\overline{P}$. If $R \cap A$ and $R \cap B$ are not empty, there must be an edge of G inside R connecting a point of A to a point of B.*

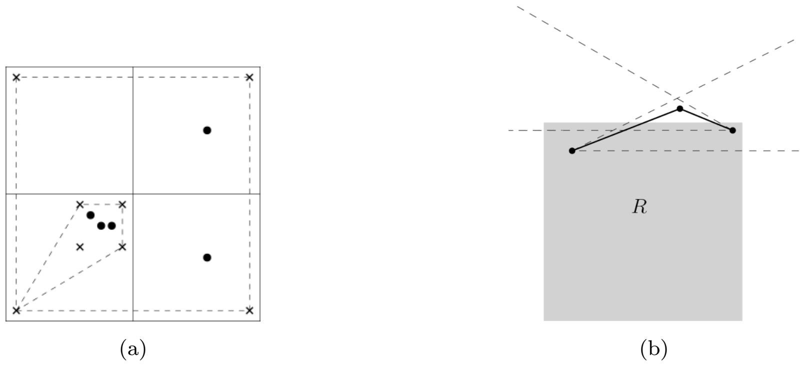

Fig. 6: (a) $\times$ show Steiner points and $\bullet$ show member of P, (b) The closest point to p is outside the square R.

Proof Assume $A = P(v_1)$ and $B = P(v_2)$ for some nodes v_1 and v_2 of the $\mathcal{T}(\overline{P})$. We only need to show that at least one representative point of v_1 (v_2) is in R. If the square $\square_{v_1}$ contains one point from $\overline{P}$, then that member is a representative point. Otherwise, representative points of v_1 are the vertices of a square or a kite, denoted by $xyzw$. Since R intersects $\square_{v_1}$ and $\square_{v_2}$, and $s > 1$, it is clear that R is larger than $\square_{v_1}$ and $\square_{v_2}$. We distinguish two cases:

- $xyzw$ is a square: since $xyzw$ is smaller than R and intersects R, then at least one of x, y, z or w must be inside R.
- $xyzw$ is a kite: let x be the vertex whose incident angle is smaller than $\pi/2$. It was shown in [1] that $CH(P(v_1) - \{x\})$ is a square; denoted by $tyzw$. As R is larger than $\square_{v_1}$ (and therefore is larger than the square $tyzw$) and its sides are parallel to the axes, it is easy to see that it can not happen that t is inside R and x, y, z and w are outside R.

$\square$

Putting all this together, we get the following theorem.

Theorem 6 *For any set P of n points in the plane, and any $\varepsilon > 0$, one can construct a local Steiner $(1 + \varepsilon)$-spanner of size $O(n)$ w.r.t. axis-parallel squares by adding at most $4(n - 1)$ Steiner points.*

Next, we focus our attention to constructing local spanners without Steiner points. Our spanner construction is based on the θ-graph approach [6], sketched below. Let θ be a suitably small (depending on ε) constant. Now let $\mathcal{C}$ be a collection of $O(1/\theta)$ interior-disjoint cones, each with their apex at the origin and angle θ (we can select θ such that $2\pi/\theta$ is integer), that together cover $\mathbb{R}^2$. For a cone $\sigma \in \mathcal{C}$ and a point $p \in \mathbb{R}^2$, let $\sigma(p)$ denote the translated copy of σ whose apex coincides with p. The θ-graph G for P is now constructed by adding at most $|\mathcal{C}|$ edges for each point $p \in P$ to G. Namely, for each cone $\sigma \in \mathcal{C}$, p is connected to the point $q \in P \cap \sigma(p)$ whose projection to the bisector of σ is closest to p. The θ-graph G may not be local. Imagine a square R

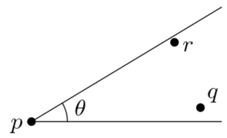

Fig. 7: Illustration for Lemma 2

whose intersection with $\sigma(p)$ contains several points but the closest point to p is outside R—see Fig. 6(b). Therefore, it is not possible to apply the standard proof for θ-graph to show $G \cap R$ is a $(1+\varepsilon)$-spanner. To resolve this problem, we need to add extra edges to G.

We first define the set $\mathcal{C}$ more carefully. Consider the x and y axes and the lines $y = x$ and $y = -x$. They divide the plane into eight cones of angle $\pi/4$. We divide each cone of angle $\pi/4$ above the x-axis into cones of angle θ (we can select θ such that $\pi/(4\theta)$ is integer). We reflect each cone σ of angle θ about the origin to get $\overline{\sigma}$. $\mathcal{C}$ is the set of all these cones σ and $\overline{\sigma}$ of angle θ that altogether cover the plane. We put all cones of $\mathcal{C}$ incident to the lines $y = x$ or $y = -x$ into set $\mathcal{C}_b$. Clearly, $|\mathcal{C}_b| = 8$. We treat the cones in $\mathcal{C}_b$ in a different way to the cones in $\mathcal{C} - \mathcal{C}_b$.

We use the L_∞ distance to determine the closest point in the θ-graph construction. To prove that the θ-graph produced in this manner is a $(1+\varepsilon)$-spanner, we can use the following lemma.

Lemma 2 *Let $\mathcal{C}$ be the collection of θ-cones defined above, where θ is an angle satisfying both $\cot\theta > 1 + \sqrt{2}(2/\varepsilon + 1)$ and $\cos\theta - \sin\theta \geq 1/(1+\varepsilon)$. Let $\sigma \in \mathcal{C}$ be a cone, and let q and r be two points in $\sigma(p)$ such that $L_\infty(p,r) \leq L_\infty(p,q)$ (see Fig. 7). Then*

$$|pr| + (1+\varepsilon)\cdot|rq| \;\leq\; (1+\varepsilon)\cdot|pq|.$$

Proof We distinguish two cases:

(i) $|pq| \geq |pr|$: Because $|pq|$ is the longest edge in the triangle pqr, we have

$$|rq| \leq |pq| - (\cos\angle qpr - \sin\angle qpr)|pr|.$$

Hence,

$$\begin{aligned}|pr| + (1+\varepsilon)\cdot|rq| &\leq |pr| + (1+\varepsilon)\cdot(|pq| - (\cos\theta - \sin\theta)|pr|)\\ &\leq (1+\varepsilon)\cdot|pq|,\end{aligned}$$

because,

$$\cos\theta - \sin\theta \geq 1/(1+\varepsilon).$$

(ii) $|pq| < |pr|$: Since $|pq| < |pr|$, then $\angle prq < \pi/2$. On the other hand, since $L_\infty(p, r) \leq L_\infty(p, q)$, we have $\angle prq > \pi/4 - \angle qpr$. Moreover, since the segments pq and pr are inside $\sigma(p)$, we have $\angle qpr \leq \theta$. Finally, from $0 \leq \angle qpr \leq \theta$ and $\pi/4 - \theta \leq \angle prq \leq \pi/2$ it follows that

$$|rq|/|pq| = \sin\angle qpr / \sin\angle prq \leq \sin\theta/\sin(\pi/4 - \theta) = \sqrt{2}\sin\theta/(\cos\theta - \sin\theta).$$

Therefore,

$$\begin{aligned} |pr| + (1+\varepsilon)\cdot|rq| &\leq |pq| + |rq| + (1+\varepsilon)\cdot|rq| \\ &\leq (1 + \sqrt{2}(2+\varepsilon)\sin\theta/(\cos\theta - \sin\theta))|pq|. \end{aligned}$$

Then to prove the lemma, we just need to show that $1+\sqrt{2}(2+\varepsilon)\sin\theta/(\cos\theta - \sin\theta) \leq 1+\varepsilon$. This inequality is equivalent to $\cot\theta > 1+\sqrt{2}(2/\varepsilon+1)$ which proves our claim.

□

Lemma 2 implies that concatenating the edge pr to a $(1+\varepsilon)$-path from r to q yields a $(1+\varepsilon)$-path from p to q. This can be used to show, by induction, that if we connect every point p to the closest point (in the L_∞ norm) in each of its cones, we get indeed a $(1+\varepsilon)$-spanner.

Let p and q be two arbitrary points of P and let R be a square containing p and q. We can simply shrink R to get a square $R(p, q)$ such that p and q lie in opposite sides of $R(p, q)$. It is sufficient to show that there is a $(1+\varepsilon)$-path from p to q inside $R(p, q)$. From now on, without loss of generality, we assume p and q are on the horizontal sides of $R(p, q)$, and $x(p) \leq x(q)$, and $y(p) < y(q)$.

Let σ be a cone of $\mathcal{C}$ such that $q \in \sigma(p)$. If $R(p, q) \cap \sigma(p)$ is a triangle, we can apply the standard induction proof using Lemma 2 as the closest point inside $\sigma(p)$ to p is definitely inside $R(p, q)$. This is valid as well when $R(p, q) \cap \overline{\sigma}(q)$ is a triangle. The following lemma shows when one of $R(p, q) \cap \sigma(p)$ and $R(p, q) \cap \overline{\sigma}(q)$ is a triangle.

Lemma 3 *If $\sigma \in \mathcal{C} - \mathcal{C}_b$, then one of $R(p, q) \cap \sigma(p)$ and $R(p, q) \cap \overline{\sigma}(q)$ is a triangle.*

Proof If $R(p, q) \cap \sigma(p)$ is a triangle, we are done. Otherwise, $\sigma(p)$ contains a corner of $R(p, q)$. We translate $R(p, q)$ to get a square R' such that one of its corners is p. Let's denote $R' \cap \sigma(p)$ and $R(p, q) \cap \bar{\sigma}(q)$ by Δq_2pq_3 and Δp_4qp_3, respectively—see Fig. 8 for this and the other notations. To prove the lemma, we just need to show that $|p_1p_4| \leq \ell$ where ℓ is the side length of $R(p, q)$ and p_1 is the right-bottom corner of $R(p, q)$. We know $|pp_4| = |qq_2|$ as pq_2 is parallel to qp_4. Let $\angle q_1pq_2 = \theta$. Since $\sigma(p) \in C - C_b$ and p is the corner of R', the triangle Δq_1pq_2 is inside R'. Moreover, it is clear that $|q_2q_3| \leq |q_1q_2|$. We bound $|p_1p_4|$ as follows

$$\begin{aligned} |p_1p_4| &= |p_1p| + |pp_4| = |p_1p| + |qq_2| \leq |p_1p| + |q_2q_3| \\ &\leq |p_1p| + |q_2q_1| \leq |q_2q_4| + |q_2q_1| \leq \ell \end{aligned}$$

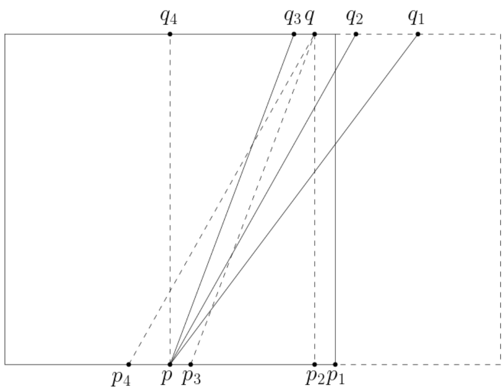

Fig. 8: The illustration for the proof of Lemma 3 (R' is shown by dashed lines).

Hence, p_4 lies in $R(p,q)$ and thus the intersection of $R(p,q)$ and $\overline{\sigma}(q)$ is a triangle.

□

This lemma shows the standard induction proof may not work only when p and q are close to the corners of $R(p,q)$ or precisely $\sigma \in \mathcal{C}_b$—see Fig. 9(a). This forces us to add extra edges to the θ-graph G to keep it local. Let σ_b be a member of $\mathcal{C}_b$ and let r be the closest point to p inside $\sigma_b(p)$. For points inside the gray region depicted in Fig. 9(b) (denoted by $\Delta(p,\sigma_b)$), we can not apply the standard induction proof. Indeed, if $q \in \Delta(p,\sigma_b)$, it is possible r being outside $R(p,q)$. Note that $\Delta(p,\sigma_b)$ is a right triangle whose hypotenuse lies on the side of σ_b not being parallel to the lines $y=x$ or $y=-x$, and whose legs are parallel to the x and y axes and coincide at r. For two points p,q in P with $x(p) \leq x(q)$ and $y(p) < y(q)$, we call (p,q) a bad pair if $q \in \Delta(p,\sigma_b)$ and $p \in \Delta(q,\overline{\sigma}_b)$. Next, we show that we can decompose bad pairs using separated-pair decomposition technique with almost linear total weight.

Definition 3 Let P be a set of points in the plane and $\sigma_b \in \mathcal{C}_b$. A pair decomposition of bad pairs of P with respect to σ_b is a collection $\Psi_{\sigma_b} := \{(A_1,B_1),\ldots,(A_m,B_m)\}$ of pairs of subsets P such that

- For every two points $p,q \in P$ such that $q \in \Delta(p,\sigma_b)$ and $p \in \Delta(q,\overline{\sigma}_b)$, there is a unique pair $(A_i,B_i) \in \Psi_{\sigma_b}$ such that $p \in A_i$ and $q \in B_i$.
- For any pair $(A_i,B_i) \in \Psi_{\sigma_b}$ and every two points $p \in A_i$ and $q \in B_i$, we have $q \in \Delta(p,\sigma_b)$ and $p \in \Delta(q,\overline{\sigma}_b)$

The following lemma shows there is a $\Psi_{\sigma_b} := \{(A_i,B_i)\}$ whose total weight (i.e. $\sum(|A_i|+|B_i|)$) is almost linear.

Lemma 4 *For any set P of n points in the plane, $\sigma_b \in \mathcal{C}_b$, and any constant $s>0$, there is a pair decomposition of bad pairs $\Psi_{\sigma_b} = \{(A_1,B_1),\ldots,(A_m,B_m)\}$ such that $\sum(|A_i|+|B_i|) = O(n\log^6 n)$. Moreover, if we set θ such that* $\cot\theta \geq$

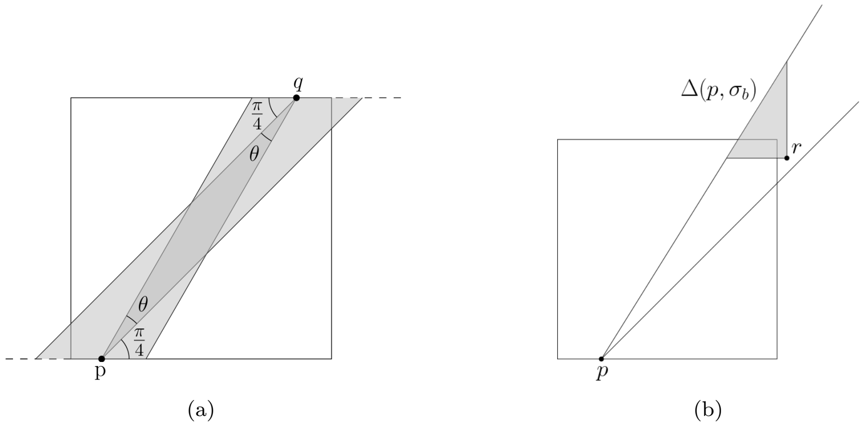

Fig. 9: (a) If p and q are close to the corners of $R(p, q)$, $R(p, q) \cap \sigma(p)$ and $R(p, q) \cap \bar{\sigma}(q)$ may not be a triangle. (b) σ_b and $\Delta(p, \sigma_b)$.

$5 + 8/\varepsilon$, we have $d(A_i, B_i) \geq (4 + 8/\varepsilon) \cdot \max(radius(A_i), radius(B_i))$ where $d(A, B)$ denotes the distance of two sets A and B, and $radius(A)$ is the radius of the smallest enclosing disk of set A.

Proof We can obtain a pair decomposition of bad pairs in a fairly standard manner, using range-searching techniques. Next, we explain this.

We construct a three-level range tree T_{σ_b} over points P where levels correspond to the x-axis, the y-axis and the direction perpendicular to the side of σ_b not being parallel to the lines $y = x$ or $y = -x$. For any $p \in P$, points inside $\Delta(p, \sigma)$ can be reported as the union of $O(\log^3 n)$ disjoint canonical subsets $P(v)$ where v is a node at the third level of T_{σ_b}, and $P(v)$ is the subset of P stored at the subtree rooted at v. Moreover, any point of P appears in $O(\log^3 n)$ canonical subsets $P(v)$. Now, fix a node v at the third level of T_{σ_b} and assume $B = P(v)$. We put a point $p \in P$ into set A provided that B is one of the canonical subsets when reporting $\Delta(p, \sigma)$.

Consider all such a pair (A, B). It is clear that $\sum(|A| + |B|) = O(n \log^3 n)$ as each point appears in $O(\log^3 n)$ canonical subsets and points inside $\Delta(p, \sigma_b)$ can be seen as $O(\log^3 n)$ canonical subsets. Now, fix such a pair (A, B). For any two points $p \in A$ and $q \in B$, we know that $q \in \Delta(p, \sigma_b)$ but p is not necessarily in $\Delta(q, \overline{\sigma}_b)$.

Therefore, we use the same technique, using a three-level range tree $T^A_{\overline{\sigma}_b}$ over set A, and produce pairs $\{(A_i, B_i)\}$ with $\sum(|A_i| + |B_i|) = O((|A| + |B|) \log^3(|A| + |B|)$ such that for any two points $p \in A_i$ and $q \in B_i$, we have $q \in \Delta(p, \sigma_b)$ and $p \in \Delta(q, \overline{\sigma}_b))$. Since $\sum(|A| + |B|) = O(n \log^3 n)$, the total weight becomes $O(n \log^6 n)$.

To prove the second claim, let (p, q) be the closest pair such that $q \in A_i$ and $p \in B_i$ where (A_i, B_i) is a member of a pair decomposition of bad pairs. Based on the definition of the pair decomposition of bad pairs, A_i and B_i are placed on $\Delta(p, \sigma_b)$ and $\Delta(q, \overline{\sigma}_b)$, respectively. So, the maximum

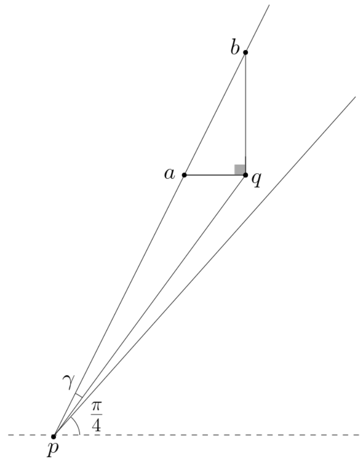

Fig. 10: Illustration for the proof of Lemma 4.

radius of the circumscribed circle of these triangles is a upper bound for $\max(\text{radius}(A_i), \text{radius}(B_i))$. W.l.o.g. we assume $\text{radius}(A_i) \geq \text{radius}(B_i)$. We name the corners of $\Delta(p, \sigma_b)$ by a, q, and b as illustrated in Fig. 10. From $\angle bqa = \pi/2$ we can conclude that $\text{radius}(A_i) < |ab|/2$. It follows that

$$\max(\text{radius}(A_i), \text{radius}(B_i)) \leq \frac{|ab|}{2} = \frac{|aq|}{2\cos(\frac{\pi}{4}+\theta)} = \frac{\sqrt{2}\,|aq|}{2(\cos\theta - \sin\theta)}$$

$$d(A_i, B_i) = |pq| \geq |ap| = \frac{|aq|\ \sin(\frac{\pi}{4}+\theta-\gamma)}{\sin\gamma} \geq \frac{|aq|\ \sin\frac{\pi}{4}}{\sin\theta} = \frac{\sqrt{2}\,|aq|}{2\sin\theta}$$

Then to prove the lemma, we need to show that

$$\frac{\sqrt{2}\,|aq|}{2\sin\theta} \geq \frac{\sqrt{2}\,|aq|}{2(\cos\theta - \sin\theta)} \cdot (4 + \frac{8}{\varepsilon})$$

This inequality is equivalent to $\cot\theta \geq 5 + 8/\varepsilon$ which proves our claim. □

The second claim in Lemma 4 implies for each pair $(A, B) \in \Psi_{\sigma_b}$, $d(A, B)$ is much greater than the radius of the smallest enclosing disks of A and B. So, we can act like the general method. That is, for $p \in A$ and $q \in B$, we add the edge $\{p, q\}$ to our spanner G if it is an edge of the Delaunay graph of $A \cup B$. Since $\sum(|A| + |B|) = O(n \log^6 n)$, the total number of extra edges added to the spanner G is $O(n \log^6 n)$. It remains to show that the resulting spanner G is a $(1+\varepsilon)$-spanner.

Lemma 5 *If θ is set to have all inequalities $\cos\theta - \sin\theta \geq 1/(1+\varepsilon)$, $\cot\theta \geq 5 + 8/\varepsilon$ and $\cot\theta > 1 + \sqrt{2}(2/\varepsilon + 1)$, the graph G is a local $(1+\varepsilon)$-spanner w.r.t. axis-parallel squares.*

Proof We sort all pairs of points (p, q) for all distinct $p, q \in P$ based on their distance on the L_∞ metric. Using induction on distances, we prove for any two distinct points p and q, there is a $(1+\varepsilon)$-path inside any square $R(p, q)$. W.l.o.g. assume $x(p) \leq x(q)$ and $y(p) \leq y(q)$.

Basis step: The closest pair (p, q) is directly connected to each other as q is the closest point to p inside $\sigma(p)$ for some $\sigma \in \mathcal{C}$.

Inductive step: Let (p, q) have rank k among all pair distances and assume for any pair (p', q') of rank $i < k$, we know there is a $(1+\varepsilon)$-path inside any $R(p', q')$ (inductive assumption). We distinguish two cases:

- (p, q) is not a bad pair: let σ be a cone of $\mathcal{C}$ such that $q \in \sigma(p)$. Then one of $\sigma(p) \cap R(p, q)$ or $\overline{\sigma}(q) \cap R(p, q)$ is a triangle. W.l.o.g assume $\sigma(p) \cap R(p, q)$ is a triangle. Let r be the closest point to p inside $\sigma(p)$. Shrink $R(p, q)$ to reach a $R(r, q)$ which is inside $R(p, q)$. Since $L_\infty(q, r) < L_\infty(p, q)$, then there is a $(1+\varepsilon)$-path from q to r inside $R(r, q)$ (and of course inside $R(p, q)$). We can apply Lemma 2 to show there is a $(1+\varepsilon)$-path inside the square $R(p, q)$ from p to q.
- If (p, q) is a bad pair: we know there is a pair (A, B) of the pair decomposition of bad pairs such that $p \in A$ and $q \in B$, and $d(A, B) \geq (4 + 8/\varepsilon)\max(\text{radius}(A), \text{radius}(B))$. By our construction, we know there is a pair (p', q') such that both p' and q' are inside $R(p, q)$, $p' \in A$, $q' \in B$ and p' and q' are connected in our spanner G. Similar to the above case, there are $(1+\varepsilon)$-paths from p to p' and from q' to q both inside $R(p, q)$ as $L_\infty(q, q'), L_\infty(p, p') < L_\infty(p, q)$. Therefore, we can apply the standard proof for $(1+\varepsilon)$-spanner construction based on $(4 + 8/\varepsilon)$-WSPD to show there a $(1+\varepsilon)$-path inside the square $R(p, q)$ from p to q.

□

Putting all this together, we get the main result of this section.

Theorem 7 *Suppose P is a set of n points in the plane, and $\varepsilon > 0$ is a given number. We can construct a local $(1+\varepsilon)$-spanner of size $O(n \log^6 n)$ w.r.t. axis-parallel squares.*

5 Local spanners w.r.t. disks.

Constructing local spanners w.r.t. disks is more challenging. One may guess that the Delaunay triangulation of a point set P is a local t-spanner for some constant t as it is connected inside a disk and it is a constant spanner [12]. Fig. 11 shows a counterexample. Consider a disk just containing p, q and o. The length of the only path from p to q inside the disk can be very larger than $|pq|$. Next, we explain how to construct a local spanner using Steiner points. We start with some definitions and a lemma given in [2].

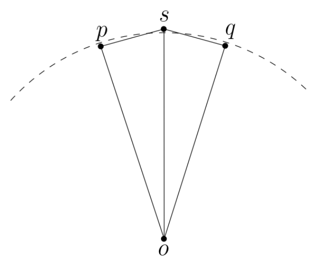

Fig. 11: Delaunay triangulation is not a local spanner w.r.t. disks.

Definition 4 Let $b(p, r)$ be the set of all points of P whose distance to $p \in P$ is at most r. Also, let $\text{ring}(p, r, r')$ be the set of all points of P whose distance to $p \in P$ is at most r' and at least r.

Lemma 6 *([2]) Let P be a set of n points in $\mathbb{R}^2$, $t > 0$ be a parameter, and let c be a sufficiently large constant. Then, one can compute in a linear time a ball $b = b(p, r)$, such that (i) $|b \cap P| \geq n/c$, (ii) $ring(p, r, r(1 + 1/n))$ is empty, and (iii) $|P \backslash b(p, 2r)| \geq n/2$.*

Based on Lemma 6, let $P_{\text{in}} = b(p, r)$, $P_{\text{out}} = b(p, 2r) \backslash b(p, r)$ and $P_{\text{outer}} = P \backslash b(p, 2r)$—indeed, we partition the point set into three sets $P_{\text{in}}, P_{\text{out}}$ and P_{outer} using the ball $b(p, r)$ computed in Lemma 6 and we will use p and r to introduce the Steiner points. We define an angle $\theta = 2\pi/k$ depending on the given ε—note that k is constant depending on ε. We select k new points on the circle centered at p and radius $3r/2$ such that for any two consecutive such points q and q', the angle $\angle qpq'$ is θ. We look at all these points as Steiner points. Then, we connect each Steiner point to all other original points in our spanner; this in total adds $O(n)$ edges to the spanner. These edges guarantee that there is always a $(1 + \varepsilon)$-path from a point in P_{in} to a point in P_{outer} even we cut the spanner by a disk as proved below.

Lemma 7 *Assume an arbitrary disk D contains $u \in P_{in}$ and $v \in P_{outer}$ and $0 < \varepsilon \leq 1$. If we set $\theta \leq \min(2 \arcsin 1/6, \varepsilon/12)$, there is an Steiner point $s \in D$ such that $|us| + |sv| \leq (1 + \varepsilon)|uv|$.*

Proof Let o and r' be the center and radius of D, respectively. Since D contains a point of P_{in} and a point of P_{outer}, then $r' \geq r/2$. Let $C(p, r)$ be a circle centered at p and radius r. We will show that the sub-arc of $C(p, 3r/2)$ which is inside D, is at least $4 \arcsin 1/6$ degrees. This sub-arc gets smaller if we move D in the direction of $\mathbf{po}$ until it is getting tangent to $C(p, r)$. Then, this sub-arc gets smaller if we make D smaller while keeping it tangent to $C(p, r)$ until the radius of D becomes $r/2$. Let M and N be the intersections of the boundary of $C(p, 3r/2)$ and the boundary of D—see Fig. 12(a). The triangle ΔpoM has side lengths $3r/2, 3r/2$ and $r/2$. Therefore, $\angle Mpo = 2 \arcsin 1/6$, then $\angle MpN = 4 \arcsin 1/6$. Our choice of θ (we selected θ to be smaller than $2 \arcsin 1/6$) ensures that there is an Steiner point in this sub-arc.

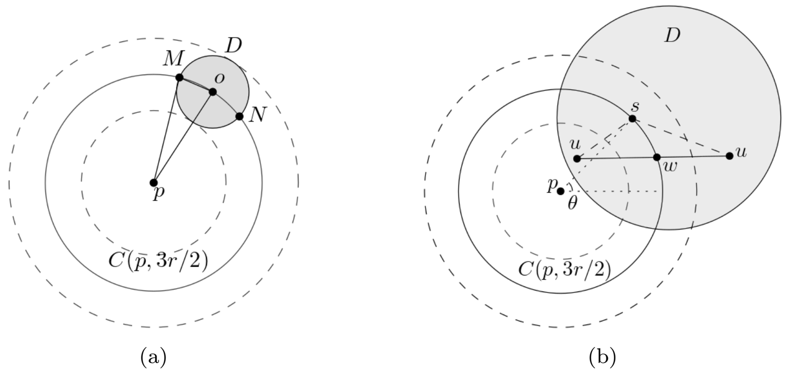

Fig. 12: Proof of Lemma 7.

Let uv intersect $C(p, 3r/2)$ at point w and let s be an Steiner point inside D such that the arc sw is at most θ degrees—see Fig. 12(b). There is such a point s as any arc defined by two consecutive Steiner points on $C(p, 3r/2)$ is θ degrees. We will show $|us| + |sv| \leq (1 + \varepsilon)|uv|$.

We know $|uw|/|ws| = \sin\angle wsu/\sin\angle wus$. Since $|uw| \geq r/2$, $|sw| \leq 3r\sin\theta/2$, and $\sin\angle wsu \leq 1$, then $\sin\angle wus \leq |ws|/|uw| \leq 6\sin\theta/2$. Moreover, $|us| \leq |uw| + |ws| \leq |uw| + 3r\sin\theta/2 \leq |uw| + r/2 \leq |uv|$. Now, as $|us| \leq |uv|$ and $\sin\angle wus \leq 6\sin\theta/2$, we have

$$\begin{aligned}
|us| + |sv| &\leq |us| + (|uv| - (\cos\angle wus - \sin\angle wus)|us|) \\
&= |uv| + (1 + \sin\angle wus - \cos\angle wus)|us| \\
&\leq |uv| + (1 + \sin\angle wus - \cos\angle wus)|uv| \\
&\leq |uv| + (1 + 6\sin\theta/2 - \sqrt{1 - 36\sin^2\theta/2})|uv|
\end{aligned}$$

Then to prove the lemma, we just need to show that $1 + 6\sin\theta/2 - \sqrt{1 - 36\sin^2\theta/2} \leq \varepsilon$. This inequality is equivalent to $\sin\theta/2 \leq (\varepsilon - 1 + \sqrt{1 + 2\varepsilon - \varepsilon^2})/12$. For $\varepsilon \leq 1$ we know $\varepsilon/12 \leq (\varepsilon - 1 + \sqrt{1 + 2\varepsilon - \varepsilon^2})/12$, so our choice of θ guarantees this. □

Next, we explain how to guarantee that there exists a $(1 + \varepsilon)$-path from a point in P_{in} to a point in P_{out} by adding $O(n \log n)$ edges. If we succeed in doing that, then we can recursively continue our construction over P_{in}, and $P_{\text{out}} \cup P_{\text{outer}}$ and get a spanner of size $O(n \log^2 n)$. Note that the number of Steiner points is $O(n)$ as we add a constant number of Steiner points at each node of the recursion tree. Our construction uses the following proposition proved in [2].

Proposition 1 *([2]) Let P be a set of n points in $\mathbb{R}^2$. Let the **spread** of P, denoted by Φ, be $\Phi = (\max_{p,q \in P} |pq|)/(\min_{p,q \in P, p \neq q} |pq|)$, and let $\varepsilon > 0$ be a parameter.*

Then, one can compute an s-WSPD for P of total weight $O(ns^2 \log \Phi)$ (i.e. $\sum(|A| + |B|) = O(ns^2 \log \Phi)$ where sum is over all pair of the s-WSPD).

To construct a local spanner over the Cartesian product $X = P_{\text{in}} \otimes P_{\text{out}}$, we use the general method given in Section 2. Let $\ell = d(P_{\text{in}}, P_{\text{out}})$ and observe that $\ell \geq r/n$. We want to compute the WSPD for all pairs of points in X. It is clear that the length of these pairs is not smaller than ℓ and the diameter of $Q = P_{\text{in}} \cup P_{\text{out}}$ is $2 \cdot \text{radius}(Q) \leq 4\ell n$. We snap the point set Q to a grid of side length $\ell/2\sqrt{2}$—this choice of the side length ensures that there is no cell of the grid containing a point of P_{in} and a point of P_{out} simultaneously as $\text{ring}(p, r, r(1+1/n))$ is empty and separates P_{in} from P_{out}. The resulting point set Q' has spread $O(n)$. So by Proposition 1, we can compute an s-WSPD on Q' of weight $O(n \log n)$. Then we get back to the original points and have an s-WSPD for X of weight $O(n \log n)$. Now by setting $s = 4 + 8/\varepsilon$ and applying Theorem 1 (i.e. for each pair (A, B) of the s-WSPD, we add all edges of the Delaunay graph over $A \cup B$ to our spanner; the total edge added to the spanner in this step is $\sum(|A| + |B|) = O(n \log n)$), we ensure that there is $(1+\varepsilon)$-path from any point in P_{in} to any point in P_{out} even if they are cut by a disk.

Theorem 8 *Suppose ε is a parameter, and P is an arbitrary point set in the plane. One can construct a local $(1 + \varepsilon)$-spanner w.r.t. disks that uses $O(n)$ Steiner point and has $O(n \log^2 n)$ edges.*

6 Conclusion

We introduced the concept of local spanners for planar point sets with respect to a family of regions, and prove the existence of local spanners of small size for the families of squares and vertical slabs. If adding Steiner points is allowed, we obtained a better result for the family of squares, and moreover, we obtained a local spanner of small size for the family of disks. We leave the following interesting problems for the future research:

- constructing a local spanner of small size for the family of disks without using Steiner points.
- constructing a local spanner of small size for the family of rectangles using Steiner points.
- studing local spanners for special point sets like points on convex position

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