


Local Spanners Revisited

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Abstract

For a set of points $P \subseteq \mathbb{R}^2$, and a family of regions \mathcal{F} , a *local t -spanner* of P , is a sparse graph G over P , such that, for any region $r \in \mathcal{F}$, the subgraph restricted to r , denoted by $G \cap r = G_{P \cap r}$, is a t -spanner for all the points of $r \cap P$.

We present algorithms for the construction of local spanners with respect to several families of regions, such as homothets of a convex region. Unfortunately, the number of edges in the resulting graph depends logarithmically on the spread of the input point set. We prove that this dependency can not be removed, thus settling an open problem raised by Abam and Borouny. We also show improved constructions (with no dependency on the spread) of local spanners for fat triangles, and regular k -gons. In particular, this improves over the known construction for axis parallel squares.

We also study a somewhat weaker notion of local spanner where one allows to shrink the region a “bit”. Any spanner is a weak local spanner if the shrinking is proportional to the diameter. Surprisingly, we show a near linear size construction of a weak spanner for axis-parallel rectangles, where the shrinkage is *multiplicative*.

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1 Introduction

For a set P of points in \mathbb{R}^d , the *Euclidean graph* $\mathcal{K}_P = (P, \binom{P}{2})$ of P is an undirected graph. Here, an edge $pq \in E$ is associated with the segment pq , and its weight is the (Euclidean) length of the segment. Let $G = (P, E)$ and $I = (P, E')$ be two graphs over the same set of vertices (usually I is a subgraph of G). Consider two vertices $p, q \in P$, and parameter $t \geq 1$. A path π between p and q in I , is a *t -path*, if the length of π in I is at most $t \cdot d_G(p, q)$, where $d_G(p, q)$ is the length of the shortest path between p and q in G . The graph I is a *t -spanner* of G if there is a t -path in I , for any $p, q \in P$. Thus, for a set of points $P \subseteq \mathbb{R}^d$, a graph G over P is a *t -spanner* if it is a t -spanner of the euclidean graph \mathcal{K}_P . There is a lot of work on building geometric spanners, see [10] and references there in.

Fault-tolerant spanners

An *\mathcal{F} -fault-tolerant spanner* for $P \subseteq \mathbb{R}^d$, is a graph $G = (P, E)$, such that for any region r (i.e., the “attack”), the graph $G - r$ is a t -spanner of $\mathcal{K}_P - r$ (See Definition 1 for a formal definition of this notation). Here $G - r$ denotes the graph after one deletes from G all the vertices in $P \cap r$, and all the edges in G that their corresponding segments intersect r . Surprisingly, as shown by Abam *et al.* [3], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size k of vertices and edges has failed. Levcopoulos *et al.* [8] showed the existence of k -vertex/edges fault tolerant spanners for a set of points P in some metric space. Their spanner had $\mathcal{O}(kn \log n)$ edges, and weight, i.e. sum of edge weights, bounded by $f(k) \cdot wt(MST(P))$ for some function f . Lukovszki [9] later achieved a similar construction, improving the number of edges to $\mathcal{O}(kn)$, and was able to prove that the result is asymptotically tight.

Local spanners

Recently, Abam and Borouny [2] introduced the notion of local spanners. For a family of regions \mathcal{F} , a graph $G = (P, E)$ is a *local t -spanner* for \mathcal{F} , if for any $r \in \mathcal{F}$, the subgraph of G induced on $P \cap r$ is a t -spanner. Specifically, this induced subgraph $G \cap r$ contains a t -path between any $p, q \in P \cap r$ (note that we keep an edge in the subgraph only if both its endpoints are in r , see Definition 1).

Abam and Borouny [2] showed how to construct such spanners for axis-parallel squares and vertical slabs. In this work, we are further extend their results. They also showed how to construct such spanners for disks if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

To appreciate the difficulty in constructing local spanners, observe that unlike regular spanners, the construction has to take into account many different scenarios as far as which points are available to be used in the spanner. As a concrete example, a local spanner for axis-parallel rectangle requires quadratic number of edges, see Figure 1.1.

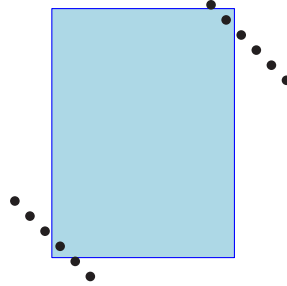


Figure 1.1 For any point in the top diagonal and bottom diagonal, there is a fat axis parallel rectangle that contains only these two points. Thus, a local spanner requires quadratic size in this case.

Namely, regular spanners can rely on using midpoints in their path under the assurance that they are always there. For local spanners this is significantly harder as natural midpoints might “disappear”. Intuitively, a local spanner construction needs to use midpoints that are guaranteed to be present judging only from the source and destination points of the path.

A good jump is hard to find

Most constructions for spanners can be viewed as searching for a way to build a path from the source to the destination by finding a “good” jump, either by finding a way to move locally from the source to a nearby point in the right direction, as done in the θ -graph construction, or alternatively, by finding an edge in the spanner from the neighborhood of the source to the neighborhood of the destination, as done in the spanner constructions using well-separated pairs decomposition (WSPD). Usually, one argues inductively that the spanner

74	Region	# edges	Paper	New # edges	Location in paper
Local $(1 + \varepsilon)$ -spanners					
75	Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[3]		
76	Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[2]	$\mathcal{O}(\varepsilon^{-3} n \log n)$	Remark 32
77	Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[2]		
78	Disks+Steiner points	$\mathcal{O}_\varepsilon(n)$	[2]		
79	Disks			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 18
80				$\Omega(n \log \Phi)$	Lemma 22
81	Homothets convex shape			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 18
82	Homothets α -fat triangles			$\mathcal{O}((\alpha\varepsilon)^{-1} n)$	Theorem 28
83	Homothets triangles			$\Omega(n \log \Phi)$	Lemma 23
δ -weak local $(1 + \varepsilon)$ -spanners					
84	Bounded convex shape			$\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$	Lemma 12
$(1 - \delta)$ -local $(1 + \varepsilon)$ -spanners					
85	Axis-parallel rectangles			$\mathcal{O}((\varepsilon^{-2} + \delta^{-2})n \log^2 n)$	Theorem 38

86 **Table 1.1** Known and new results. The notation \mathcal{O}_ε hides polynomial dependency on ε which is
87 not specified in the original work.

68 must have (sufficiently short) paths from the source to the start of the jump, and from the
69 end of the jump to the destination, and then, combining these implies that the resulting
70 new path is short. These ideas guide our constructions as well. However, the availability of
71 specific edges depends on the query region, making the search for a good jump significantly
72 more challenging. The constructions have to guarantee that there are many edges available,
73 and that at least one of them is useful as a jump regardless of the chosen region.

88 Our results

89 Our results are summarized in Table 1.1.

90 Almost local spanners

91 We start by showing that regular geometric spanners are local spanners if one is required
92 provide the spanner guarantee only to shrunken regions. Namely, if G is a $(1 + \varepsilon)$ -spanner of
93 P , then for any convex region \mathcal{C} , the graph $G \cap \mathcal{C}$ is a spanner for $\mathcal{C}' \cap P$, where \mathcal{C}' is the set
94 of all points in \mathcal{C} that are in distance at least $\varepsilon \cdot \text{diam}(\mathcal{C})$ from its boundary.

95 Homothets

96 A *homothet* of a convex region \mathcal{C} , is a translated and scaled copy of \mathcal{C} . In Section 3 we
97 present a construction of spanners, which surprisingly, is not only fault-tolerant for all convex
98 regions, but is also a local spanner for homothets of a prespecified convex region. This
99 in particular works for disks, and resolves the aforementioned open problem of Abam and
100 Borouny [2]. Our construction is somewhat similar to the original construction of Abam
101 *et al.* [3]. For a parameter $\varepsilon > 0$ the construction of a $(1 + \varepsilon)$ -local spanner for homothets
102 takes $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ time, and the resulted spanner is of size $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$, where Φ

103 is the spread of the input point set P , and $n = |P|$. We also provide a lower bound showing
 104 that this logarithmic dependency on Φ cannot be avoided.

105 The dependency on the spread Φ in the above construction is somewhat disappointing.
 106 However, the lower bound constructions, provided in [Section 3.3](#), show that this is unavoidable
 107 for disks or homothets of triangles.

108 Thus, the natural question is what are the cases where one can avoid the “curse of the
 109 spread” – that is, cases where one can construct local spanners of near-linear size independent
 110 of the spread of the input point set.

111 The basic building block: \mathcal{C} -Delaunay triangulation

112 A key ingredient in the above construction is the concept of Delaunay triangulations induced
 113 by homothets of a convex body. Intuitively, one replaces the unit disk (of the standard
 114 L_2 -norm) by the provided convex region. It is well known [5] that such diagrams exist, have
 115 linear complexity in the plane, and can be computed quickly. In [Section 3.1](#) we review these
 116 results, and restate the well-known property that the \mathcal{C} -Delaunay triangulation is connected
 117 when restricted to a homothet of \mathcal{C} . By computing these triangulations for carefully chosen
 118 subsets of the input point set, we get the results stated above.

119 Specifically, we use well-separated and semi-separated decompositions to compute these
 120 subsets.

121 Fat triangles

122 In [Section 3.4](#) we give a construction of local spanners for the family \mathcal{F} of homothets of a
 123 given triangle \triangle , and get a spanner of size $\mathcal{O}((\alpha\varepsilon)^{-1}n)$ in $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ time, where α
 124 is the smallest angle in \triangle . This construction is a careful adaptation of the θ -graph spanner
 125 construction to the given triangle, and it is significantly more technically challenging than
 126 the original construction.

127 k -regular polygons

128 It seems natural that if one can handle fat triangles, then homothets of k -regular polygons
 129 should readily follow by a simple decomposition of the polygon into fat triangles. Maybe
 130 surprisingly, this is not the case – a critical configuration might involve two points that are on
 131 the interior of two non-adjacent edges of a homothet of the input polygon. We overcome this
 132 by first showing that sufficiently narrow trapezoids, provide us with a good jump somewhere
 133 inside the trapezoid, assuming one computes the Delaunay triangulation induced by the
 134 trapezoid, and that the source and destination lie on the two legs of the trapezoid. Next, we
 135 show that such a polygon can be covered by a small number of narrow trapezoids and fat
 136 triangles. By building appropriate graphs for each trapezoid/triangle in the collection, we get
 137 a spanner for homothets of the given k -regular polygon, with size that has no dependency on
 138 the spread. Of course, the size does depend on k . See [Section 3.5](#) for details, and [Theorem 31](#)
 139 for the precise result.

140 Quadrant separated pair decomposition (QSPD)

141 In [Appendix A.1](#), we describe a novel pair-decomposition. Specifically, the QSPD breaks
 142 the input point set P into pairs, such that for any pair $\{X, Y\}$ we have the property that
 143 there is a translated axis system such that X and Y belong to two antipodal quadrants.
 144 In d dimensions there is such a decomposition with $\mathcal{O}(n \log^{d-1} n)$ pairs, and total weight

145 $\mathcal{O}(n \log^d n)$. A somewhat similar idea was used by Abam and Borouny [2] for the $d = 1$ case.
 146 We believe this decomposition might be useful and is of independent interest.

147 Multiplicative weak local spanner for rectangles

148 In [Appendix A.2](#), we use QSPDs to construct a weak local spanner for axis parallel rectangles.
 149 Here, the constructed graph G over P , has the property that for any axis-parallel rectangle
 150 R , the graph $G \cap R$ is a $(1 + \varepsilon)$ -spanner for all the points of $((1 - \varepsilon)R) \cap P$, where $(1 - \varepsilon)R$ is
 151 the scaling of the rectangle by a factor of $1 - \varepsilon$ around its center. Importantly, this works for
 152 narrow rectangles where this form of multiplicative shrinking is still meaningful (unlike the
 153 diameter based shrinking mentioned above). Contrast this with the lower bound (illustrated
 154 in [Figure 1.1](#)) of $\Omega(n^2)$ on the size of local spanner if one does not shrink the rectangles. See
 155 [Theorem 38](#) for details of the precise result.

156 See [Table 1.1](#) for a summary of known results and comparisons to the results of this
 157 paper.

158 2 Preliminaries

159 Residual graphs

160 ► **Definition 1.** Let \mathcal{F} be a family of regions in the plane. For a fault region $r \in \mathcal{F}$ and a
 161 geometric graph G on a point set P , let $G - r$ be the residual graph after removing from
 162 it all the points of P in r and all the edges that their corresponding segments intersect r .
 163 Similarly, let $G \cap r$ denote the graph restricted to r . Formally, let

$$164 \quad G - r = (P \setminus r, \{uv \in E \mid uv \cap \text{int}(r) = \emptyset\}) \quad \text{and} \quad G \cap r = (P \cap r, \{uv \in E \mid uv \subseteq r\}).$$

165 where $\text{int}(r)$ denotes the interior of r .

166 2.1 On various pair decompositions

167 For sets X, Y , let $X \otimes Y = \{\{x, y\} \mid x \in X, y \in Y, x \neq y\}$ be the set of all the (unordered)
 168 pairs of points formed by the sets X and Y .

169 ► **Definition 2** (Pair decomposition). For a point set P , a **pair decomposition** of P is a
 170 set of pairs

$$171 \quad \mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\},$$

172 such that (I) $X_i, Y_i \subseteq P$ for every i , (II) $X_i \cap Y_i = \emptyset$ for every i , and (III) $\bigcup_{i=1}^s X_i \otimes Y_i = P \otimes P$.
 173 Its **weight** is $\omega(\mathcal{W}) = \sum_{i=1}^s (|X_i| + |Y_i|)$.

174 The **closest pair** distance of a set of points $P \subseteq \mathbb{R}^d$, is $\text{cp}(P) = \min_{p, q \in P, p \neq q} \|pq\|$. The
 175 **diameter** of P is $\text{diam}(P) = \max_{p, q \in P} \|pq\|$. The **spread** of P is $\Phi(P) = \text{diam}(P) / \text{cp}(P)$, which
 176 is the ratio between the diameter and closest pair distance. While in general the weight of a
 177 WSPD (defined below) can be quadratic, if the spread is bounded, the weight is near linear.
 178 For $X, Y \subseteq \mathbb{R}^d$, let $\text{d}(X, Y) = \min_{p \in X, q \in Y} \|pq\|$ be the **distance** between the two sets.

179 ► **Definition 3.** Two sets $X, Y \subseteq \mathbb{R}^d$ are

180 $1/\varepsilon$ -**well-separated** if $\max(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot d(X, Y)$,
 181 and $1/\varepsilon$ -**semi-separated** if $\min(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot d(X, Y)$.

182 For a point set P , a **well-separated pair decomposition (WSPD)** of P with parameter
 183 $1/\varepsilon$ is a pair decomposition of P with a set of pairs $\mathcal{W} = \{\{B_1, C_1\}, \dots, \{B_s, C_s\}\}$, such
 184 that for all i , the sets B_i and C_i are $(1/\varepsilon)$ -separated. The notion of $1/\varepsilon$ -SSPD (a.k.a
 185 **semi-separated pairs decomposition**) is defined analogously.

186 ► **Lemma 4** ([1]). Let P be a set of n points in \mathbb{R}^d , with spread $\Phi = \Phi(P)$, and let
 187 $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$ -WSPD \mathcal{W} for P of total weight
 188 $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}(\varepsilon^{-d} \log \Phi)$
 189 pairs.

190 ► **Theorem 5** ([1, 7]). Let P be a set of n points in \mathbb{R}^d , and let $\varepsilon > 0$ be a parameter. Then,
 191 one can compute a $(1/\varepsilon)$ -SSPD for P of total weight $\mathcal{O}(n\varepsilon^{-d} \log n)$. The number of pairs in
 192 the SSPD is $\mathcal{O}(n\varepsilon^{-d})$, and the computation time is $\mathcal{O}(n\varepsilon^{-d} \log n)$.

193 ► **Lemma 6** (Proof in Appendix B.1). Given an α -SSPD \mathcal{W} of a set P of n points in \mathbb{R}^d and
 194 a parameter $\beta \geq 2$, one can refine \mathcal{W} into an $\alpha\beta$ -SSPD \mathcal{W}' , such that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$
 195 and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$.

196 ► **Definition 7.** An ε -**double-wedge** is a region between two lines, where the angle between
 197 the two lines is at most ε .

198 Two point sets X and Y that each lie in their own cone of a shared ε -double-wedge are
 199 ε -**angularly separated**.

200 ► **Lemma 8** (Proof in Appendix B.2). Given a $(1/\varepsilon)$ -SSPD \mathcal{W} of n points in the plane, one
 201 can refine \mathcal{W} into a $(1/\varepsilon)$ -SSPD \mathcal{W}' , such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a
 202 ε -double-wedge \times_Ξ , such that X and Y are contained in the two different faces of the double
 203 wedge \times_Ξ . We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time
 204 is proportional to the weight of \mathcal{W}' .

205 ► **Corollary 9.** Let P be a set of n points in the plane, and let $\varepsilon > 0$ be a parameter. Then,
 206 one can compute a $(1/\varepsilon)$ -SSPD for P such that every pair is ε -angularly separated. The total
 207 weight of the SSPD is $\mathcal{O}(n\varepsilon^{-3} \log n)$, the number of pairs in the SSPD is $\mathcal{O}(n\varepsilon^{-3})$, and the
 208 computation time is $\mathcal{O}(n\varepsilon^{-3} \log n)$.

209 2.2 Weak local spanners for fat convex regions

210 ► **Definition 10.** Given a convex region C , let

$$211 \quad C_{\square\delta} = \{p \in C \mid d(p, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)\}.$$

212 Formally, $C_{\square\delta}$ is the Minkowski difference of C with a disk of radius $\delta \cdot \text{diam}(C)$.

213 ► **Definition 11.** Consider a (bounded) set C in the plane. Let $r_{\text{in}}(C)$ be the radius of the
 214 largest disk contained inside C . Similarly, $R_{\text{out}}(C)$ is the smallest radius of a disk containing
 215 C .

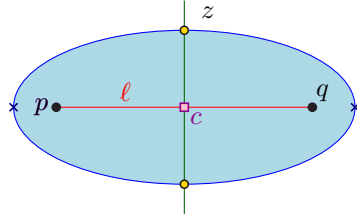
216 The **aspect ratio** of a region C in the plane is $\text{ar}(C) = R_{\text{out}}(C)/r_{\text{in}}(C)$. Given a family
 217 \mathcal{F} of regions in the plane, its aspect ratio is $\text{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \text{ar}(C)$.

Note, that if a convex region C has bounded aspect ratio, then $C_{\Box\delta}$ is similar to the result of scaling C by a factor of $1 - \mathcal{O}(\delta)$. On the other hand, if C is long and skinny then this region is much smaller. Specifically, if C has width smaller than $2\delta \cdot \text{diam}(C)$, then $C_{\Box\delta}$ is empty.

► **Lemma 12.** *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0, 1)$. One can construct a graph G over P , in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$ edges, such that for any (bounded) convex C in the plane, we have that for any two points $p, q \in P \cap C_{\Box\delta}$ the graph $C \cap P$ has $(1 + \varepsilon)$ -path between p and q .*

Proof. Let $\vartheta = \min(\varepsilon, \delta^2)$. Construct, in $\mathcal{O}(\vartheta^{-1}n \log n)$ time, a standard $(1 + \vartheta)$ -spanner G for P using $\mathcal{O}(\vartheta^{-1}n)$ edges [4].

So, consider any body $C \in \mathcal{F}$, and any two vertices $p, q \in P \cap C'$, where $C' = C_{\Box\delta}$. Let $\ell = \|pq\|$, let π be the shortest path between p and q in G , and let \mathcal{E} be the locus of all points u , such that $\|pu\| + \|uq\| \leq (1 + \vartheta)\ell$. The region \mathcal{E} is an ellipse that contains π . The furthest point from the segment pq in this ellipse is realized by the co-vertex of the ellipse. Formally, it is one of the two intersection points of the boundary of the ellipse with the line orthogonal to pq that passes through the middle point c of this segment, see Figure 2.1. Let z be one of these points.



■ **Figure 2.1** An illustration of the settings in the proof of Lemma 12 with \mathcal{E} shown in blue.

We have that $\|pz\| = (1 + \vartheta)\ell/2$. Setting $h = \|zc\|$, we have that

$$h = \sqrt{\|pz\|^2 - \|pc\|^2} = \frac{\ell}{2} \sqrt{(1 + \vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2 + \vartheta)}}{2} \ell \leq \sqrt{\vartheta} \ell \leq \sqrt{\vartheta} \cdot \text{diam}(C).$$

as $\ell \leq \text{diam}(C') \leq \text{diam}(C)$.

For any point $x \in C'$, we have that $d(x, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)$. As such, to ensure that $\pi \subseteq \mathcal{E} \subseteq C$, we need that $\delta \cdot \text{diam}(C) \geq h$, which holds if $\delta \cdot \text{diam}(C) \geq \sqrt{\vartheta} \cdot \text{diam}(C)$. This in turn holds if $\vartheta \leq \delta^2$. Namely, we have the desired properties if $\vartheta = \min(\varepsilon, \delta^2)$. ◀

3 Local spanners of homothets of convex region

Let \mathcal{C} be a bounded convex and closed region in the plane (e.g., a disk). A *homothet* of \mathcal{C} is a scaled and translated copy of \mathcal{C} . A point set P is in *general position* with respect to \mathcal{C} , if no four points of P lie on the boundary of a homothet of \mathcal{C} , and no three points are colinear.

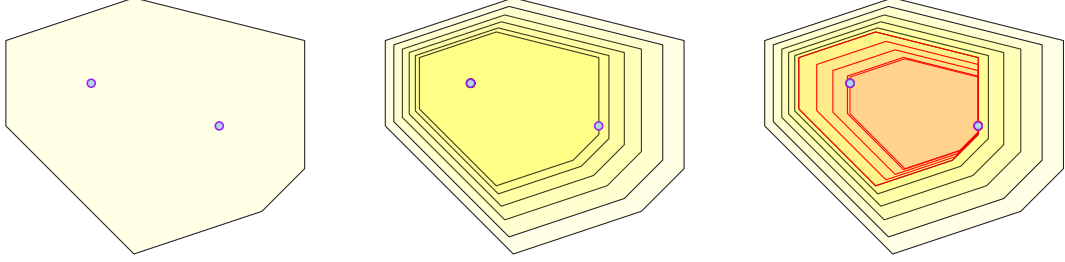
A graph $G = (P, E)$ is a \mathcal{C} -local t -spanner for P if for any homothet \mathcal{r} of \mathcal{C} we have that $G \cap \mathcal{r}$ is a t -spanner of $\mathcal{K}_P \cap \mathcal{r}$.

3.1 Delaunay triangulation for homothets

► **Definition 13** ([5]). *Given \mathcal{C} as above, and a point set P in general position with respect to \mathcal{C} , the \mathcal{C} -Delaunay triangulation of P , denoted by $\mathcal{D}_{\mathcal{C}}(P)$, is the graph formed by edges*

251 between any two points $p, q \in P$ such that there is a homothet of \mathcal{C} that contains only p and
 252 q and no other point of P .

253 ► **Theorem 14** ([5]). For any convex shape \mathcal{C} and a set of points P , $\mathcal{D}_{\mathcal{C}}(P)$ can be computed
 254 in $\mathcal{O}(n \log n)$ time. Furthermore, the triangulation $\mathcal{D}_{\mathcal{C}}(P)$ has $\mathcal{O}(n)$ edges, vertices, and
 255 faces.



256 ■ **Figure 3.1** Shrinking of homothets so two points becomes on the boundary of the homothet.

257 ► **Lemma 15.** Let \mathcal{C} be a convex bounded body, and let P be a set of points in general position
 258 with respect to \mathcal{C} . Then, if C is a homothet of \mathcal{C} that contains two points $p, q \in C \cap P$, then
 259 there exists a homothet $C' \subseteq C$ of \mathcal{C} such that $p, q \in \partial C'$.

260 **Proof.** The idea is to apply a shrinking process of C , as illustrated in [Figure 3.1](#). Consider
 261 the mapping $f_{\beta, v} : x \mapsto \beta(x - v) + v$. It is a scaling of the plane around v by a factor of
 262 β . Let β' be the minimum value of β such that $C_1 = f_{\beta, p}(C)$ contains q (i.e., we shrink
 263 C around p till q becomes a boundary point). Next, shrink C' around q , till p becomes
 264 a boundary point – formally, let β'' be the minimum value of β such that $C' = f_{\beta, q}(C_1)$
 265 contains p . Since $C' \subseteq C_1 \subseteq C$, and $p, q \in \partial C'$, the claim follows. ◀

266 The following standard claim, usually stated for the standard Delaunay triangulations,
 267 also holds for homothets.

268 ► **Claim 16** (Proof in [Appendix B.3](#)). Let \mathcal{C} be a bounded close convex shape. Given a set
 269 of points $P \subseteq \mathbb{R}^2$ in general position with respect to \mathcal{C} , let $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$ be the \mathcal{C} -Delaunay
 270 triangulation of P . For any homothet C of \mathcal{C} , we have that $\mathcal{D} \cap C$ is connected.

271 3.2 The generic construction

272 The input is a set P of n points in the plane (in general position) with spread $\Phi = \Phi(P)$,
 273 and a parameter $\varepsilon \in (0, 1)$. We have a convex body \mathcal{C} that defines the “unit” ball. The task
 274 is to construct a local spanner for any homothet of \mathcal{C} .

275 The algorithm computes a $(1/\vartheta)$ -WSPD \mathcal{W} of P using the algorithm of [Lemma 4](#),
 276 where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the \mathcal{C} -Delaunay
 277 triangulation $\mathcal{D}_{\Xi} = \mathcal{D}_{\mathcal{C}}(X \cup Y)$. The algorithm adds all the edges in $\mathcal{D}_{\Xi} \cap (X \otimes Y)$ to the
 278 computed graph G .

279 3.2.1 Analysis

280 **Size.** For each pair $\Xi = \{X, Y\}$ in the WSPD, its \mathcal{C} -Delaunay triangulation contains at
 281 most $\mathcal{O}(|X| + |Y|)$ edges. As such, the number of edges in the resulting graph is bounded by
 282 $\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}(|X| + |Y|) = \mathcal{O}(\omega(\mathcal{W})) = \mathcal{O}\left(\frac{n \log \Phi}{\vartheta^2}\right)$, by [Lemma 4](#).

Construction time. The construction time is bounded by $\sum_{\{X,Y\} \in \mathcal{W}} O((|X|+|Y|) \log(|X|+|Y|)) = \mathcal{O}(\omega(\mathcal{W}) \log n) = \mathcal{O}\left(\frac{n \log \Phi \log n}{\vartheta^2}\right)$.

► **Lemma 17** (Local spanner property). *For $P, \mathcal{C}, \varepsilon$ as above, let G be the graph constructed above for the point set P . Then, for any homothet C of \mathcal{C} and any two points $x, y \in P \cap C$, we have that $G \cap C$ has a $(1 + \varepsilon)$ -path between x and y . That is, G is a \mathcal{C} -local $(1 + \varepsilon)$ -spanner.*

Proof. Fix a homothet C of \mathcal{C} , and consider two points $p, q \in P \cap C$. The proof is by induction on the distance between p and q (or more precisely, the rank of their distance among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and $y \in Y$.

If $xy \in \mathcal{D}_\Xi$ then the claim holds, so assume this is not the case. By the connectivity of $\mathcal{D}_\Xi \cap C$, see Claim 16, there must be points $x' \in X \cap C$, $y' \in Y \cap C$, such that $x'y' \in E(\mathcal{D}_\Xi)$. As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property, we have that

$$\max(\text{diam}(X), \text{diam}(Y)) \leq \vartheta \mathbf{d}(X, Y) \leq \vartheta \ell,$$

where $\ell = \|xy\|$. In particular, $\|x'x\| \leq \vartheta \ell$ and $\|y'y\| \leq \vartheta \ell$. As such, by induction, we have $\mathbf{d}_G(x, x') \leq (1 + \varepsilon) \|xx'\| \leq (1 + \varepsilon) \vartheta \ell$ and $\mathbf{d}_G(y, y') \leq (1 + \varepsilon) \|yy'\| \leq (1 + \varepsilon) \vartheta \ell$. Furthermore, $\|x'y'\| \leq (1 + 2\vartheta) \ell$. As $x'y' \in E(G)$, we have

$$\begin{aligned} \mathbf{d}_G(x, y) &\leq \mathbf{d}_G(x, x') + \|x'y'\| + \mathbf{d}_G(y', y) \leq (1 + \varepsilon) \vartheta \ell + (1 + 2\vartheta) \ell + (1 + \varepsilon) \vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta) \ell \\ &= (1 + 6\vartheta) \ell \leq (1 + \varepsilon) \|xy\|, \end{aligned}$$

if $\vartheta \leq \varepsilon/6$. ◀

The result. We thus get the following.

► **Theorem 18.** *Let \mathcal{C} be a bounded convex shape in the plane, let P be a given set of n points in the plane (in general position), and let $\varepsilon \in (0, 1/2)$ be a parameter. The above algorithm constructs a \mathcal{C} -local $(1 + \varepsilon)$ -spanner G . The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any homothet C of \mathcal{C} , and any two points $p, q \in P \cap C$, we have a $(1 + \varepsilon)$ -path in $G \cap C$.*

3.2.2 Applications and comments

The following defines a “visibility” graph when we are restricted to a region R , where two points are visible if there is a witness homothet contained in R having both points on its boundary.

► **Definition 19.** *Let \mathcal{C} be a bounded convex shape in the plane. Given a region R in the plane and a point set P , consider two points $p, q \in P$. The edge pq is **safe** in R if there is a homothet C of \mathcal{C} , such that $p, q \in C \subseteq R$. The **safe graph** for P and R , denoted by $\mathcal{S}(P, R)$, is the graph formed by all the safe edges in P for R . Note, that this graph might have a quadratic number of edges in the worst case.*

Observe that $\mathcal{S}(P, \mathbb{R}^2)$ is a clique. Surprisingly, the spanner graph described above, when restricted to region R , is a spanner for $\mathcal{S}(P, R)$.

► **Corollary 20** (Proof in Appendix B.4). *Let \mathcal{C} be a bounded convex body, P be a set of n points in the plane, $\varepsilon \in (0, 1)$ be a parameter, and let G be a \mathcal{C} -local $(1 + \varepsilon)$ -spanner of P .*

Consider a region R in the plane, and the associated graph $H = \mathcal{S}(P, R)$, we have that $G \cap R$ is a $(1 + \varepsilon)$ -spanner for H . Formally, for any two points $p, q \in P \cap R$, we have that $d_{G \cap R}(p, q) \leq (1 + \varepsilon)d_H(p, q)$.

In particular, for any convex region D , the graph $G - D$ is a $(1 + \varepsilon)$ -spanner for $\mathcal{S}(P, \mathbb{R}^2) - D$.

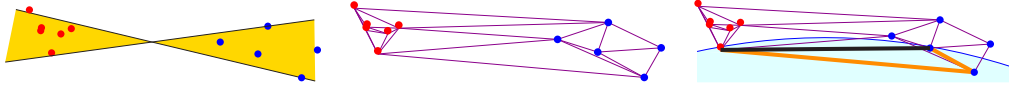
► **Remark 21.** The above implies that local spanners for homothets are also robust to convex region faults. Namely, this construction both provides a local spanner and a fault tolerant spanner, where the locality is for homothets of the given shape, and the faults are for any convex regions.

3.3 Lower bounds

3.3.1 A lower bound for local spanner for disks

The result of [Theorem 18](#) is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). Next, we show a lower bound proving that this dependency is unavoidable, even in the case of disks.

Some intuition. A natural way to attempt a spread-independent construction is to try and emulate the construction of Abam *et al.* [3] and use a SSPD instead of a WSPD, as the total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε -separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are of different sides of the double wedge. The problem is that for the local disk \odot , it might be that the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \odot$ is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in [Figure 3.2](#).



■ **Figure 3.2** A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

► **Lemma 22.** For $\varepsilon = 1/4$, and parameters n and $\Phi \geq 1$, there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $\mathcal{O}(n\Phi)$, such that any local $(1 + \varepsilon)$ -spanner of P for disks, must have $\Omega(n \log \Phi)$ edges.

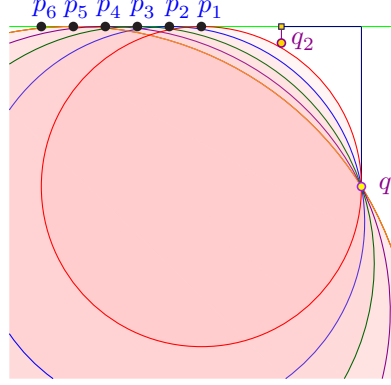
Proof. Let $p_i = (-i, 0)$, for $i = 1, \dots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$ and $q_1 = (n2^M, -1)$. For a point p on the x -axis, and a point q below the x -axis and to the right of p , let $\odot_{\downarrow}^p(q)$ be the disk whose boundary passes through p and q , and its center has the same x -coordinate as p .

In the j th iteration, for $j = 2, \dots, M - 1$, Let $x_j = n2^{M-j+1} = x(q_{j-1})/2$, and let $y_j < 0$ be the maximum y -coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the disks of $D_1 \cup \dots \cup D_j$ where

$$D_j = \left\{ \odot_{\downarrow}^{p_i}(q_{j-1}) \mid i = 1, \dots, n \right\},$$

see [Figure 3.3](#) for an illustration of D_1 .

Let $q_j = (x_j, 0.99y_j)$.



349 ■ **Figure 3.3** The set of disks D_1 , and the construction of q_2 .

350 Clearly, the point q_j lies outside all the disks of $D_1 \cup \dots \cup D_j$. The construction
 360 now continues to the next value of j . Let $P = \{p_1, \dots, p_n, q_2, \dots, q_M\}$. We have that
 361 $|P| = n + M - 1$.

362 The minimum distance between any points in the construction is 1 (i.e., $\|p_1 p_2\|$). Indeed
 363 $x(q_{M-1}) = 4n$ and thus $\|q_{M-1} p_1\| \geq 2n$. The diameter of P is $\|p_1 q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq$
 364 $2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = \mathcal{O}(n\Phi)$.

365 For any i and j , consider the disk $\odot_{\downarrow}^{p_i}(q_j)$. This disk does not contain any point of
 366 $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ since its interior lies below the x -axis. By construction it does not
 367 contain any point q_{j+1}, \dots, q_{M-1} . This disk potentially contains the points q_{j-1}, \dots, q_1 , but
 368 observe that for any index $k \in \llbracket j-1 \rrbracket$, we have that

$$369 \quad \|p_i q_k\| = \sqrt{(i + n2^{M-k+1})^2 + (y(q_j))^2},$$

370 which implies that $n2^{M-k+1} \leq \|p_i q_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$371 \quad \frac{\|p_i q_k\|}{\|p_i q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

372 since $j \in \llbracket M-1 \rrbracket$. Namely, the shortest path in G between p_i and q_j , can not use any of
 373 the points q_1, \dots, q_{j-1} . As such, the graph G must contain the edge $p_i q_j$. This implies that
 374 $|E(G)| \geq n(M-1)$, which implies the claim. ◀

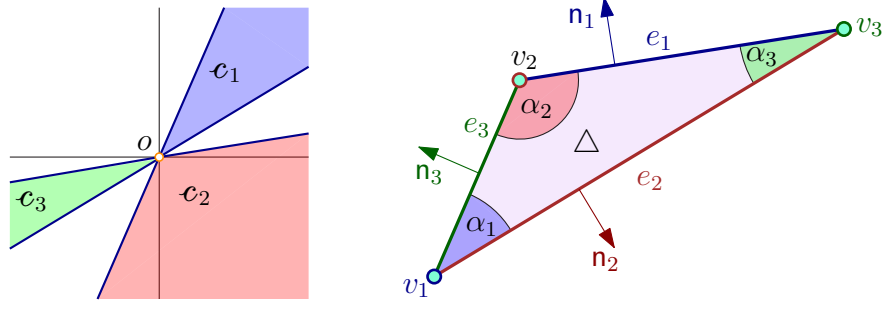
375 3.3.2 A lower bound for triangles

376 ► **Lemma 23.** [Proof in [Appendix B.5](#)] For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a
 377 set P of $n + \mathcal{O}(\log \Phi)$ points, with spread $\mathcal{O}(\Phi n)$, and a triangle \triangle , such that any \triangle -local
 378 $(3/2)$ -spanner of P requires $\Omega(n \log \Phi)$ edges.

379 3.4 Local spanners for fat triangles

380 While local spanners for homothets of an arbitrary convex shape are costly, if we are given
 381 a triangle \triangle with the single constraint that \triangle is not too “thin”, then one can construct
 382 a \triangle -local t -spanner with a number of edges that does not depend on the spread of the
 383 points. See [Figure B.4](#) for an illustration of a construction showing that dependency if “thin”
 384 triangles are allowed.

385 ► **Definition 24.** A triangle \triangle is α -fat if the smallest angle in \triangle is at least α .



386 ■ **Figure 3.4** For the triangle \triangle with angles α_1, α_2 , and α_3 we create the cones c_1, c_2 , and c_3 .

387 3.4.1 Construction

388 The input is a set P of n points in the plane, an α -fat triangle \triangle , and an approximation
 389 parameter $\varepsilon \in (0, 1)$. Let v_i denote the i th vertex of \triangle , α_i be the adjacent angle, and let
 390 e_i denote the opposing edge, for $i \in \llbracket 3 \rrbracket$. Let $c_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$ denote the
 391 cone with an apex at the origin induced by the i th vertex of \triangle . Let n_i be the outer normal of
 392 \triangle orthogonal to e_i . See Figure 3.4 for an illustration. Let \mathcal{C}_i be a minimum size partition of
 393 c_i into cones each with angle in the range $[\beta/2, \beta]$, where $\beta = \varepsilon\alpha/\gamma$, and $\gamma > 1$ is a constant
 394 to be determined shortly. For each point $p \in P$, and a cone $c \in \mathcal{C}_i$, let $\text{nn}_i(p, c)$ be the first
 395 point in $(P - p) \cap (p + c)$ ordered by the direction n_i (it is the “nearest-neighbor” to p in
 396 $p + c$ with respect to the direction n_i).

397 The construction

398 Let G be the graph over P formed by connecting every point $p \in P$ to $\text{nn}_i(p, c)$, for all
 399 $i \in \llbracket 3 \rrbracket$ and $c \in \mathcal{C}_i$.

400 3.4.2 Analysis

401 ► **Lemma 25** (Proof in Appendix B.6). *Let $p \in P$, $c \in \mathcal{C}_i$, and $u = \text{nn}_i(p, c)$, and let q
 402 be a point in $(P \cap (p + c)) \setminus \{p, u\}$. We have that $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ and
 403 $\|qu\| \leq \|pq\|$.*

404 ► **Lemma 26** (Proof in Appendix B.7). *Let \triangle be a triangle that contains two points p, q .
 405 Then, there is a homothet $\triangle' \subseteq \triangle$ of \triangle , such that one of these points is a vertex of \triangle' , and
 406 the other point lies on a facing edge of \triangle' .*

407 Local spanner property

408 ► **Lemma 27.** *Let \triangle' be a homothet of \triangle . For any two points $p, q \in P \cap \triangle'$, we have a
 409 $(1 + \varepsilon)$ -path in $G' = G \cap \triangle'$.*

410 **Proof.** Consider the closest pair $p, q \in P \cap \triangle$. They must be connected directly in G' , as
 411 otherwise there is a point $u \in P' = P \cap \triangle'$ in the cone containing the segment pq , such that
 412 $pu \in E(G')$. But then, by Lemma 25, we have $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$, which
 413 implies that either pu or qu are the closest pair, which is a contradiction.

414 For any other pair $p, q \in P'$ we have from Lemma 26 that there exists a homothet
 415 $\triangle'' \subseteq \triangle'$ with one of the two points, say p , at a vertex, and the other on the opposite edge.
 416 We therefore have a cone c with apex at p such that $q \in c \cap \triangle''$. If pq is an edge in G

then we are done. Otherwise, we have a vertex $u \in \mathcal{C}$ such that pu is an edge in G , and by Lemma 25 we have $\|qu\| \leq \|pq\|$, which, by induction, means that there exists a $(1 + \varepsilon)$ path between u and q in G . Lemma 25 now implies that $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$. Thus, there is a $(1 + \varepsilon)$ path between p and q in G' , as stated. \blacktriangleleft

Size and running time

► **Theorem 28.** *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be an approximation parameter. The above algorithm computes a \triangle -local $(1 + \varepsilon)$ -spanner G for an α -fat triangle \triangle . The construction time is $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$, and the spanner G has $\mathcal{O}((\alpha\varepsilon)^{-1}n)$ edges.*

Proof. The local-spanning property is proven in Lemma 27, and we are only left with bounding the size and the running time of the algorithm. The bound on the size is immediate from the construction, as every point p is the apex of $\mathcal{O}(\frac{2\pi}{\varepsilon\alpha})$ cones, each giving rise to a single edge incident to p . The construction time is bounded by the construction time for a θ -graph with cone size $\alpha\varepsilon$, which is $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ ([6]). \blacktriangleleft

3.5 A local spanner for nice polygons

3.5.1 A good jump for narrow trapezoids

As a reminder, a trapezoid is a quadrilateral with two parallel edges, known as its *bases*. The other two edges are its *legs*. For $\varepsilon \in (0, 1/4)$, a trapezoid T is ε -*narrow* if the length of each of its legs is at most $\varepsilon \cdot \text{diam}(T)$.

► **Lemma 29** (Proof in Appendix B.8). *Let $\varepsilon \in (0, 1)$ be some parameter, and $\vartheta = \varepsilon/16$. Let X, Y be two points sets that are ϑ -semi separated and ϑ -angularly separated (see Definition 7), and let T be a ϑ -narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on the two legs of T . Then, one can compute a homothet $T' \subseteq T$ of T , such that:*

- (I) *There are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay triangulation of $X \cup Y$.*
- (II) *We have that $(1 + \varepsilon)\|pp'\| + \|p'q'\| + (1 + \varepsilon)\|q'q\| \leq (1 + \varepsilon)\|pq\|$.*

3.5.2 Breaking a nice polygon into narrow trapezoids

For a convex polygon \mathcal{C} , its *sensitivity*, denoted by $\text{sen}(\mathcal{C})$, is the minimum distance between any two non-adjacent edges (this quantity is no bigger than the length of the shortest edge in the polygon). A convex polygon \mathcal{C} is t -*nice*, if the outer angle at any vertex of the polygon is at least $2\pi/t$, and the length of the longest edge of \mathcal{C} is $\mathcal{O}(\text{sen}(\mathcal{C}))$. As an example, a k -regular polygon is k -nice.

► **Lemma 30** (Proof in Appendix B.9). *Let t be a positive integer. Given a t -nice polygon \mathcal{C} , and a parameter ϑ , one can cover it by a set \mathcal{T} of $\mathcal{O}(t^4/\vartheta^3)$ ϑ -narrow trapezoids, such that for any two points $p, q \in \partial\mathcal{C}$ that belong to two edges of \mathcal{C} that are not adjacent, there exists a narrow trapezoid $T \in \mathcal{T}$, such that p and q are located on two different short legs of T .*

3.5.3 Constructing the local spanner for nice polygons

► **Theorem 31.** *Let \mathcal{C} be a k -nice convex polygon, P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter. Then, one can construct a \mathcal{C} -local $(1 + \varepsilon)$ -spanner of P . The construction time is $\mathcal{O}((k^4/\varepsilon^6)n \log^2 n)$, and the resulting graph has $\mathcal{O}((k^4/\varepsilon^6)n \log n)$ edges. In particular these bounds hold if \mathcal{C} is a k -regular polygon.*

Proof. Let $\vartheta = \varepsilon/c_4$, for c_4 sufficiently large constant. We construct Δ , a family of triangles induced by a vertex of \mathcal{C} , and an non-adjacent edge of \mathcal{C} . This family has $\mathcal{O}(k^2)$ triangles. Each such triangle is $\Omega(1/k)$ -fat, and for each such triangle we construct the $(1 + \vartheta)$ -spanner of Theorem 28 for P . Next, we cover \mathcal{C} by a set \mathcal{T} of $k' = \mathcal{O}(k^4/\vartheta^3)$ ϑ -narrow trapezoids using Lemma 30.

We compute an ϑ -angular $(1/\vartheta)$ -SSPD \mathcal{W} decomposition of P using Corollary 9 – the total weight of the decomposition is $w = \mathcal{O}(n\vartheta^{-3} \log n)$. For each pair $\{X, Y\} \in \mathcal{W}$, and each trapezoid $T \in \mathcal{T}$, we compute the T -Delaunay triangulation of $X \cup Y$.

Let G denote the union of all these graphs. We claim that it is the desired spanner. The construction time is

$$\mathcal{O}((k^3/\vartheta)n \log n + k'w \log n) = \mathcal{O}\left(\frac{k^3}{\vartheta}n \log n + \frac{k^4}{\vartheta^3} \cdot \frac{n}{\vartheta^3} \log n \cdot \log n\right) = \mathcal{O}\left(\frac{k^4}{\vartheta^6}n \log^2 n\right),$$

and the resulting graph has $\mathcal{O}((k^4/\vartheta^6)n \log n)$ edges.

As for correctness, consider a homothet \mathcal{C}' of \mathcal{C} that contains two points $p, q \in P$. By Lemma 15, there is a homothet $\mathcal{C}'' \subseteq \mathcal{C}'$ of \mathcal{C} such that $p, q \in \partial\mathcal{C}''$. There are two possibilities:

(A) The point p is on a vertex of \mathcal{C}'' and q is on an edge. In this case, the vertex and the edge induce a fat triangle, that is a homothet of a triangle $\Delta \in \Delta$. Since the graph G contains a Δ -local $(1 + \varepsilon)$ -spanner for P , it follows readily that G is a $(1 + \varepsilon)$ -spanner for these points, and the path is strictly inside \mathcal{C}'' .

(B) The points p and q are on two non-adjacent edges of \mathcal{C}'' . Then, there is an ϑ -narrow trapezoid T' that has p and q on its two legs, and a homothet of T' , denoted by T , is in \mathcal{T} . There is a pair $\{X, Y\} \in \mathcal{W}$ that is $(1/\vartheta)$ -semi separated (and ϑ -angularly separated), such that $p \in X$ and $q \in Y$. By Lemma 29, there are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay triangulation of $X \cup Y$, and by construction this edge is in G . We now use induction on the shortest paths from p to p' and from q to q' in G . By induction, and Lemma 29, we have that

$$d(p, q) \leq d(p, p') + \|p'q'\| + d(q', q) \leq (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|,$$

which implies that there is $(1 + \varepsilon)$ -path from p to q inside \mathcal{C}' . \blacktriangleleft

► Remark 32. For axis-parallel squares Theorem 31 implies a local spanner with $\mathcal{O}(\varepsilon^{-6}n \log n)$ edges. However, for this special case, the decomposition into narrow trapezoid can be skipped. In particular, in this case, the resulting spanner has $\mathcal{O}(\varepsilon^{-3}n \log n)$ edges. We do not provide the details here, as it is only a minor improvement over the above, and requires quite a bit of additional work – essentially, one has to prove a version of Lemma 29 for squares.

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A Weak local spanners for axis-parallel rectangles

A.1 Quadrant separated pair decomposition

522 For two points $p = (p_1, \dots, p_d)$ and $q = (q_1, \dots, q_d)$ in \mathbb{R}^d , let $p \prec q$ denotes that q *dominates*
523 p coordinate-wise. That is $p_i < q_i$, for all i . More generally, let $p <_i q$ denote that $p_i < q_i$.
524 For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $\forall x \in X, y \in Y \quad x <_i y$. In
525 particular X and Y are *i-coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is
526 *quadrant-separated*, if X and Y are i -coordinate separated, for $i = 1, \dots, d$.

527 A *quadrant-separated pair decomposition* of a point set $P \subseteq \mathbb{R}^d$, is a pair de-
528 composition (see Definition 2) $\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\}$ of P , such that $\{X_i, Y_i\}$ are
529 quadrant-separated for all i .

530 ► **Lemma 33.** *Given a set P of n points in \mathbb{R} , one can compute, in $\mathcal{O}(n \log n)$ time, a QSPD*
531 *of P with $\mathcal{O}(n)$ pairs, and of total weight $\mathcal{O}(n \log n)$.*

532 **Proof.** If P is a singleton then there is nothing to do. If $P = \{p, q\}$, then the decomposition
533 is the pair formed by the two singleton points.

534 Otherwise, let x be the median of P , such that $P_{\leq x} = \{p \in P \mid p \leq x\}$ contains exactly
535 $\lceil n/2 \rceil$ points, and $P_{> x} = P \setminus P_{\leq x}$ contains $\lfloor n/2 \rfloor$ points. Construct the pair $\Xi = \{P_{\leq x}, P_{> x}\}$,
536 and compute recursively a QSPDs $\mathcal{Q}_{\leq x}$ and $\mathcal{Q}_{> x}$ for $P_{\leq x}$ and $P_{> x}$, respectively. The desired
537 QSPD is $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{> x} \cup \{\Xi\}$. The bounds on the size and weight of the desired QSPD are
538 immediate. ◀

539 ► **Lemma 34.** *Given a set P of n points in \mathbb{R}^d , one can compute, in $\mathcal{O}(n \log^d n)$ time, a*
540 *QSPD of P with $\mathcal{O}(n \log^{d-1} n)$ pairs, and of total weight $\mathcal{O}(n \log^d n)$.*

541 **Proof.** The construction algorithm is recursive on the dimensions, using the algorithm of
542 Lemma 33 in one dimension.

543 The algorithm computes a value α_d that partitions the values of the points' d th coordinates
544 roughly equally (and is distinct from all of them), and let h be a hyperplane parallel to the
545 first $d - 1$ coordinate axes, and having value α_d in the d th coordinate.

Let P_\uparrow and P_\downarrow be the subset of points of P that are above and below h , respectively. The algorithm recursively computes QSPDs \mathcal{Q}_\uparrow and \mathcal{Q}_\downarrow for P_\uparrow and P_\downarrow , respectively. Next, the algorithm projects the points of P on h , let P' be the resulting $d-1$ dimensional point set (after we ignore the d th coordinate), and recursively computes a QSPD \mathcal{Q}' for P' .

For a point set $X' \subseteq P'$, let $\text{lift}(X')$ be the subset of points of P whose projection on h is X' . The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\text{lift}(X') \cap P_\uparrow, \text{lift}(Y') \cap P_\downarrow\}, \{\text{lift}(X') \cap P_\downarrow, \text{lift}(Y') \cap P_\uparrow\} \mid \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is $\widehat{\mathcal{Q}} \cup \mathcal{Q}_\uparrow \cup \mathcal{Q}_\downarrow$.

To observe that this is indeed a QSPD, observe that all the pairs in $\mathcal{Q}_\uparrow, \mathcal{Q}_\downarrow$ are quadrant separated by induction. As for pairs in $\widehat{\mathcal{Q}}$, they are quadrant separated in the first $d-1$ coordinates by induction on the dimension, and separated in the d coordinate since one side of the pair comes from P_\uparrow , and the other side from P_\downarrow .

As for coverage, consider any pair of points $p, q \in P$, and observe that the claim holds by induction if they are both in P_\uparrow or P_\downarrow . As such, assume that $p \in P_\uparrow$ and $q \in P_\downarrow$. But then there is a pair $\{X', Y'\} \in \mathcal{Q}'$ that separates the two projected points in h , and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates p and q as desired.

The number pairs in the decomposition is $N(n, d) = 2N(n, d-1) + 2N(n/2, d)$ with $N(n, 1) = \mathcal{O}(n)$. The solution to this recurrence is $N(n, d) = \mathcal{O}(n \log^{d-1} n)$. The total weight of the decomposition is $W(n, d) = 2W(n, d-1) + 2W(n/2, d)$ with $W(n, 1) = \mathcal{O}(n \log n)$. The solution to this recurrence is $W(n, d) = \mathcal{O}(n \log^d n)$. Clearly, this also bounds the construction time. \blacktriangleleft

A.2 Weak local spanner for axis parallel rectangles

For a parameter $\delta \in (0, 1)$, and an interval $I = [b, c]$, let $(1-\delta)I = [t - (1-\delta)r, t + (1-\delta)r]$, where $t = (b+c)/2$, and $r = (c-b)/2$, be the shrinking of I by a factor of $1-\delta$.

Let \mathcal{R} be the set of all axis parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$, with $R = I \times J$, let $(1-\delta)R = (1-\delta)I \times (1-\delta)J$ denote the rectangle resulting from shrinking R by a factor of $1-\delta$.

Definition 35. Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0, 1)$, a graph G is a $(1-\delta)$ -local $(1+\varepsilon)$ -spanner for rectangles, if for any axis-parallel rectangle R , we have that $G \cap R$ is a $(1+\varepsilon)$ -spanner for all the points in $((1-\delta)R) \cap P$.

Observe that rectangles in \mathcal{R} might be quite “skinny”, so the previous notion of shrinkage used before is not useful in this case.

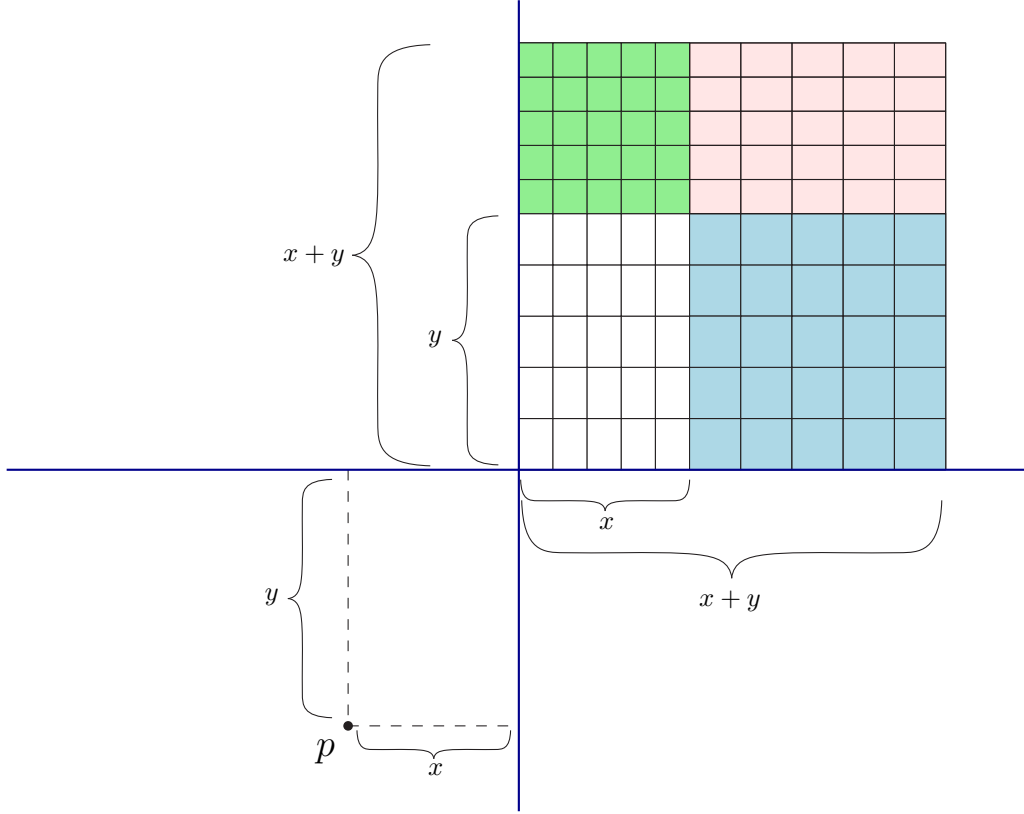
A.2.1 Construction for a single quadrant separated pair

Consider a pair $\Xi = \{X, Y\}$ in a QSPD of P . The set X is quadrant-separated from Y . That is, there is a point c_Ξ , such that X and Y are contained in two opposing quadrants in the partition of the plane formed by the vertical and horizontal line through c_Ξ .

For simplicity of exposition, assume that $c_\Xi = (0, 0)$, and $X \prec (0, 0) \prec Y$. That is, the points of X are in the negative quadrant, and the points of Y are in the positive quadrant.

We construct a non-uniform grid $K(p, \Xi)$ in the square $[0, x+y]^2$. To this end, we first partition it into four subrectangles

$$\begin{array}{c|c} B_{\swarrow} = [0, x] \times [y, x+y] & B_{\nearrow} = [x, x+y] \times [y, x+y] \\ \hline B_{\swarrow} = [0, x] \times [0, y] & B_{\nwarrow} = [x, x+y] \times [0, y]. \end{array}$$



579 **Figure A.1** The construction of the grid $K(p, \Xi)$ for a point $p = (-x, -y)$ and a pair Ξ .

588 Let $\tau \geq 4/\varepsilon + 4/\delta$ be an integer number. We partition each of these rectangles into a
 589 $\tau \times \tau$ grid, where each cell is a copy of the rectangle scaled by a factor of $1/\tau$. See Figure A.1.
 590 This grid has $\mathcal{O}(\tau^2)$ cells. For a cell C in this grid, let $Y \cap C$ be the points of Y contained in
 591 it. We connect p to the left-most and bottom-most points in $Y \cap C$. This process generates
 592 two edges in the constructed graph for each grid cell (that contains at least two points), and
 593 $\mathcal{O}(\tau^2)$ edges overall.

594 The algorithm repeats this construction for all the points $p \in X$, and does the symmetric
 595 construction for all the points of Y .

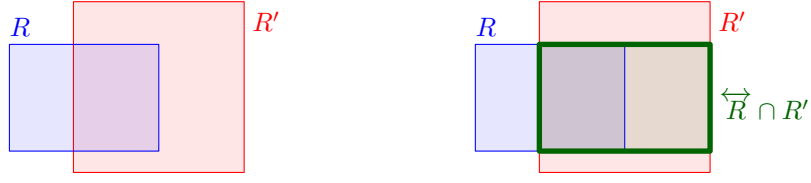
596 A.2.2 The construction algorithm

597 The algorithm computes a QSPD \mathcal{W} of P . For each pair $\Xi \in \mathcal{W}$, the algorithm generates
 598 edges for Ξ using the algorithm of Section A.2.1 and adds them to the generated spanner G .

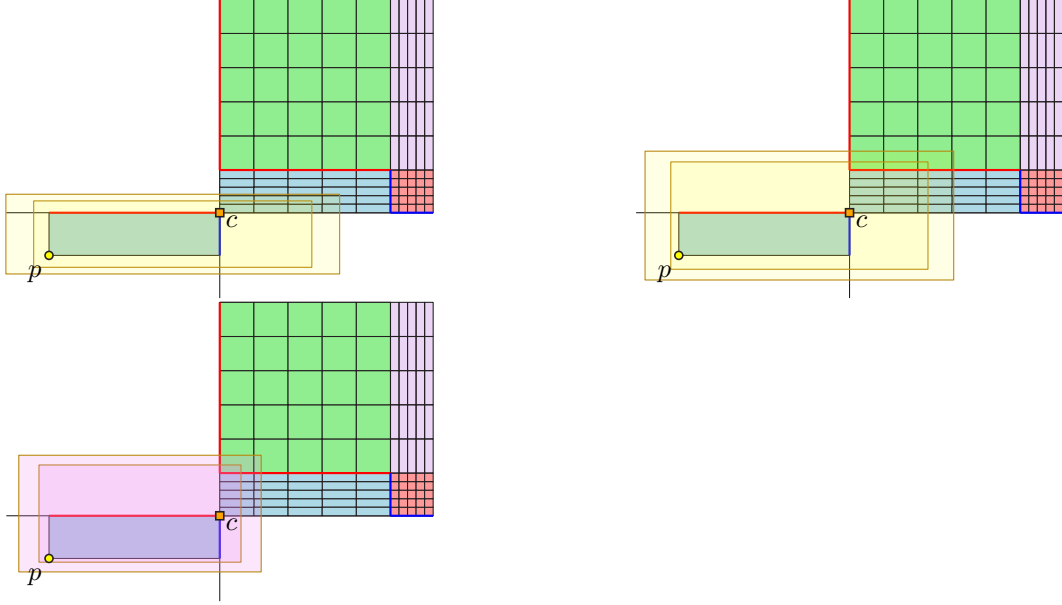
601 A.2.3 Correctness

602 For a rectangle R , let $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists (x', y) \in R\}$ be its expansion into a horizontal
 603 slab. Restricted to a rectangle R' , the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure A.2.
 604 Similarly, we denote $\uparrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists (x, y') \in R\}$.

606 **► Lemma 36.** Assume that $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above
 607 construction, and a point $p = (-x, -y) \in X$ with its associated grid $K = K(p, \Xi)$. Consider



599 **Figure A.2** Left: The two rectangles R, R' . Right: In green $\overleftrightarrow{R} \cap R'$, the restriction of the slab
 600 \overleftrightarrow{R} to the rectangle R' .



605 **Figure A.3** An illustration of $K(p, \Xi)$ with three rectangles and their shrunken version.

608 any axis parallel rectangle R , such that $p \in (1 - \delta)R = I \times J$, and $(1 - \delta)R$ intersects a cell
 609 $C \in K$. We have that:

- 610 (I) If $C \subseteq (1 - \delta)R$ then $(1 - \delta)^{-1}C \subseteq R$.
- 611 (II) $\text{diam}(C) \leq (\varepsilon/4)d(p, C)$.
- 612 (III) If $x \geq y$ and $C \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1 - \delta)^{-1}C \subseteq R$.
- 613 (IV) If $x \leq y$ and $C \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1 - \delta)^{-1}C \subseteq R$.
- 614 (V) If $x \geq y$ and $C \subseteq R_{\nwarrow}$, then $(1 - \delta)^{-1}(\overleftrightarrow{(1 - \delta)R} \cap C) \subseteq R$.
- 615 (VI) If $x \leq y$ and $C \subseteq R_{\nearrow}$, then $(1 - \delta)^{-1}(\overleftrightarrow{(1 - \delta)R} \cap C) \subseteq R$.

616 **Proof.** (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

617 (II) We have that $\text{diam}(C) \leq (x + y)/\tau = \|p\|_1/\tau \leq (\varepsilon/4)d(p, C)$.

618 (III) The width, denoted $\text{wd}(\cdot)$, of $(1 - \delta)R$ is at least x , as it contains both p and the origin.
 619 As such,

$$620 \quad (\text{wd}(R) - \text{wd}((1 - \delta)R))/2 \geq 2(x/\tau) \geq 2\text{wd}(C).$$

621 That is, the width of the “expanded” rectangle R is enough to cover C , and a grid cell
 622 adjacent to it to the right.

A similar argument about the height shows that R covers the region immediately above C – in particular, the vertical distance from C to the top boundary of R is at least the height of C . This implies that the expanded cell $(1 - \delta)^{-1}C$ is contained in R , as claimed, as $\delta < 1/2$.

(V) We decompose the claim to the two dimensions of the region. Let $B = \overrightarrow{((1 - \delta)R \cap C)}$. Observe that containment in the x -axis follows by arguing as in (III). As for the y -interval of B , observe that it is contained in the y -interval of $(1 - \delta)R$, which implies that when expanded by $(1 - \delta)^{-1}$, it would be contained in the y -interval of R . Combining the two implies the result. \blacktriangleleft

► **Lemma 37.** *For any axis-parallel rectangle R , and any two points $p, q \in (1 - \delta)R \cap P$, there exists a $(1 + \varepsilon)$ -path between p and q in G .*

Proof. The proof is by induction over the size of R (i.e. area, width, or height). Let $\Xi = \{X, Y\} \in \mathcal{W}$ be the pair in the QSPD that separates p and q , let c be the separation point of the pair, and assume for the simplicity of exposition that $p \in X$, $X \prec c \prec Y$, and $c = (0, 0)$. Furthermore, assume that $\|p\|_1 \geq \|q\|_1$.

Let $p = (-x, -y)$, and let C be the grid cell of $K(p, \Xi)$ that contains q . If $C \subseteq (1 - \delta)R$, then $(1 - \delta)^{-1}C \subseteq R$ by Lemma 36 (I). As such, let u be the leftmost point in $C \cap P$. Both $q, u \in (1 - \delta)^{-1}C$, and by induction, there is an $(1 + \varepsilon)$ -path π between them in G (note that the induction applies to the two points, and the “expanded” rectangle $(1 - \delta)^{-1}C$). Since pu is an edge of G , prefixing π by this edge results in an $(1 + \varepsilon)$ -path, as $\|qu\| \leq (\varepsilon/4) \|pq\|$, by Lemma 36 (II) (verifying this requires some standard calculations which we omit).

Otherwise, one need to apply the same argument using the appropriate case of Lemma 36. So assume that $x \geq y$ (the case that $y \geq x$ is handled symmetrically). If $C \subseteq R_{\swarrow} \cup R_{\searrow}$, then (III) implies that $(1 - \delta)^{-1}C \subseteq R$. Which implies that induction applies, and the claim holds.

The remaining case is that $x \geq y$ and $C \subseteq R_{\nwarrow}$. Let $D = \overrightarrow{((1 - \delta)R \cap C)}$. By (V), we have $(1 - \delta)^{-1}D \subseteq R$. Namely, $q \in (1 - \delta)R \cap C \subseteq D$, and let u be the lowest point in $C \cap P$. By construction $pu \in E(G)$, $q, u \in D$, $(1 - \delta)^{-1}D \subseteq R$. As such, we can apply induction to q, u , and $(1 - \delta)^{-1}D$, and conclude that $d_G(q, u) \leq (1 + \varepsilon) \|qu\|$. Plugging this into the regular machinery implies the claim. \blacktriangleleft

► **Theorem 38.** *Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0, 1)$ be parameters. The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ edges. The graph G is a $(1 - \delta)$ -local $(1 + \varepsilon)$ -spanner for axis parallel rectangles. Formally, for any axis-parallel rectangle R , we have that $R \cap P$ is an $(1 + \varepsilon)$ -spanner for all the points of $((1 - \delta)R) \cap P$.*

Proof. Computing the QSPD \mathcal{W} takes $\mathcal{O}(n \log^2 n)$ time. For each pair $\{X, Y\}$ in the decomposition with $m = |X| + |Y|$ points, we need to compute the lowest and leftmost points in $(X \cup Y) \cap C$, for each cell in the constructed grid. This can readily be done using orthogonal range trees in $\mathcal{O}(\log^2 n)$ time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is $\mathcal{O}(\omega(\mathcal{W})\tau^2) = \mathcal{O}((1/\delta^2 + 1/\varepsilon^2)n \log^2 n)$.

The desired local spanner property is provided by Lemma 37. \blacktriangleleft

B Some missing proofs

B.1 Proof of Lemma 6

Restatement of Lemma 6. *Given an α -SSPD \mathcal{W} of a set P of n points in \mathbb{R}^d and a parameter $\beta \geq 2$, one can refine \mathcal{W} into an $\alpha\beta$ -SSPD \mathcal{W}' , such that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$ and*

667 $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$.

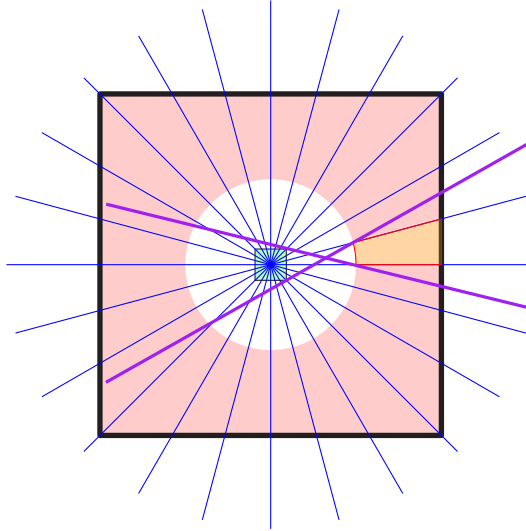
668 **Proof.** The algorithm scans the pairs of \mathcal{W} . For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that
 669 $\text{diam}(X) < \text{diam}(Y)$. Let \mathfrak{s} be the smallest axis-parallel cube containing X , and denote its
 670 sidelength by r . Let $r' = r/\lceil\sqrt{d}\beta\rceil$. Partition \mathfrak{s} into a grid of cubes of sidelength r' , and let
 671 T_Ξ be the resulting set of squares. The algorithm now add the set pairs

$$672 \quad \{\{X \cap t, Y\} \mid t \in T_\Xi\}$$

673 to the output SSPD. Clearly, the resulting set is now $\alpha\beta$ -semi separated, as we chopped the
 674 smaller part of each pair into β smaller portions. ◀

675 B.2 Proof of Lemma 8

676 **Restatement of Lemma 8.** *Given a $(1/\varepsilon)$ -SSPD \mathcal{W} of n points in the plane, one can
 677 refine \mathcal{W} into a $(1/\varepsilon)$ -SSPD \mathcal{W}' , such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a
 678 ε -double-wedge \times_Ξ , such that X and Y are contained in the two different faces of the double
 679 wedge \times_Ξ . We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time
 680 is proportional to the weight of \mathcal{W}' .*



681 ■ **Figure B.1** An illustration of refining the pairs in a SSPD into pairs contained in opposite parts
 682 of an ε -double-wedge. X is contained in the green square \square , while Y is contained in the red square,
 683 and the white gap between them is a result of the separation property. The set of cones with the
 684 apex at the center of \square gives us the desired partition as demonstrated by the purple double-wedge.

685 **Proof.** By using Lemma 6, we can assume that \mathcal{W} is (say) $(10/\varepsilon)$ -separated. Now, the
 686 algorithm scans the pairs of \mathcal{W} . For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\text{diam}(X) <$
 687 $\text{diam}(Y)$. Let \square be the smallest axis-parallel square containing X , centered at point o .
 688 Partition the plane around o , by drawing $\mathcal{O}(1/\varepsilon)$ lines intersecting o with the angle between
 689 any two consecutive lines being at most (say) $\varepsilon/4$, see Figure B.1. This partitions the plane
 690 into a set of cones \mathcal{C} . For a cone $c \in \mathcal{C}$, we show that there exists an ε -double-wedge that
 691 contains X in one side, and $Y \cap c$ in the other.

692 To see that, take the double-wedge formed by the cross tangents between $\text{ch}(X)$ and
 693 $\text{ch}(Y \cap c)$, where $\text{ch}(X)$ denotes the convex-hull of X . Assume w.l.o.g that \square has side length

1, and let c be a cone of angle $\varepsilon/4$ with apex o , whose angular bisector is a horizontal ray in the positive direction of the x axis. See figure Figure B.2 for an illustration.

We would like to find a vertical segment s such that all points of Y lie to its right, with one endpoint on the upper line of c , and the other on the lower line of c . Using the segments' height and distance from the right side of \square we will be able to get a bound on the angle of the cross tangents. We first find a segment s with all points of Y to its right. A trivial bound on that distance is given by the segment from, say, the lower left corner of \square , denoted p , of length $10/\varepsilon$ with its right endpoint on the upper line of c , denote this point by q . We know that all points of Y lie to the right of q due to the $10/\varepsilon$ separation property of the SSPD. The segment pq creates an angle $\leq \pi/4$ with the x -axis (by the choice of the angle of c). We therefore get that the x -coordinate difference between \square and q is at most $10/\varepsilon \cdot \cos \frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$. So let s' be a vertical segment between the upper and lower rays of c , with x -coordinate distance of $6/\varepsilon - \frac{1}{2}$ from \square (in order to make calculations easier). We get that s' is of length $2 \cdot \frac{6}{\varepsilon} \tan \frac{\varepsilon}{8}$. Finally, we take s to be a vertical segment of length $\frac{12}{\varepsilon} \tan \frac{\varepsilon}{8}$, with its center on the x -axis at a distance of $5/\varepsilon + \frac{1}{2}$ away from o . The angle of the x -axis and the segment between the lower end of the right side of \square and the upper end of s is now given by:

$$\arctan\left(\frac{\frac{6}{\varepsilon} \tan \frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5} \tan \frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) \leq \varepsilon$$

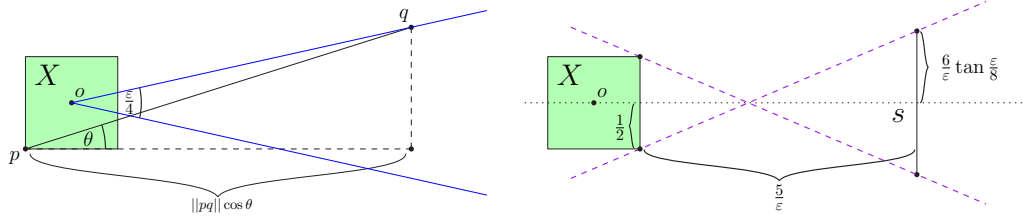


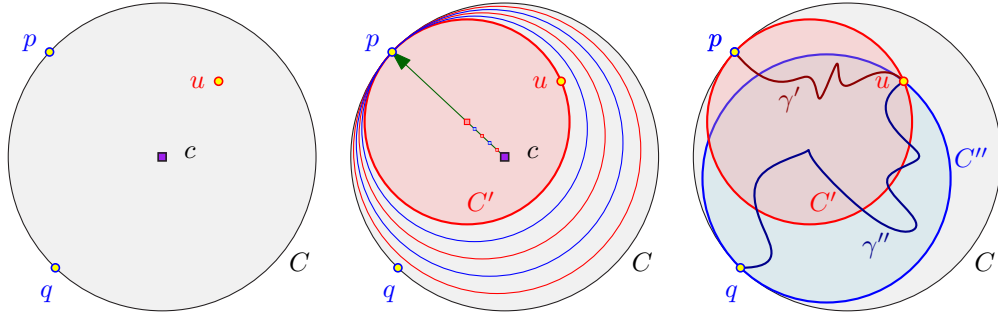
Figure B.2 An illustration of the proof for Lemma 8

B.3 Proof of Claim 16

Restatement of Claim 16. Let C be a bounded close convex shape. Given a set of points $P \subseteq \mathbb{R}^2$ in general position with respect to C , let $\mathcal{D} = \mathcal{D}_C(P)$ be the C -Delaunay triangulation of P . For any homothet C' of C , we have that $\mathcal{D} \cap C'$ is connected.

Proof. We prove that for any homothet C with two points $p, q \in P$ on its boundary, there is a path between p and q in $\mathcal{D} \cap C$, and Lemma 15 will immediately imply the general statement. The proof is by induction over the number m of points of P in the interior of C . If $m = 0$ then C contains no points of P in its interior, and thus pq is an edge of the Delaunay triangulation, as C testifies.

Otherwise, let $u \in P$ be a point in the interior of C . From Lemma 15 we get that there exists a homothet C' of C with $C' \subseteq C$, such that p and u lie on the boundary of C' . Thus, by induction, there is a path γ' between p and u in $\mathcal{D} \cap C' \subseteq \mathcal{D} \cap C$. Similarly, there must be a homothet C'' , that gives rise to a path γ'' between u and q , and concatenating the two paths results in a path between p and q in $\mathcal{D} \cap C$.



723 ■ **Figure B.3** An illustration of the proof of Claim 16 in the case that C is a disk.

730 B.4 Proof of Corollary 20

731 **Restatement of Corollary 20.** Let \mathcal{C} be a bounded convex body, P be a set of n points in
 732 the plane, $\varepsilon \in (0, 1)$ be a parameter, and let G be a \mathcal{C} -local $(1 + \varepsilon)$ -spanner of P .

733 Consider a region R in the plane, and the associated graph $H = \mathcal{S}(P, R)$, we have that
 734 $G \cap R$ is a $(1 + \varepsilon)$ -spanner for H . Formally, for any two points $p, q \in P \cap R$, we have that
 735 $d_{G \cap R}(p, q) \leq (1 + \varepsilon)d_H(p, q)$.

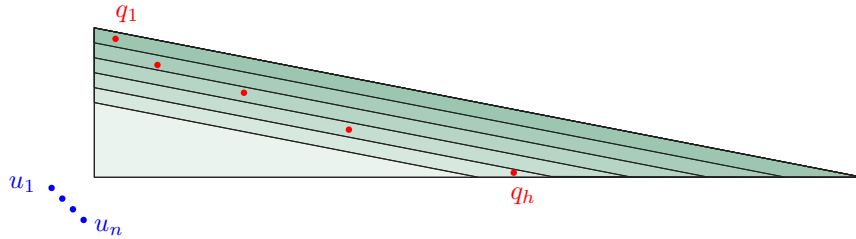
736 In particular, for any convex region D , the graph $G - D$ is a $(1 + \varepsilon)$ -spanner for
 737 $\mathcal{S}(P, \mathbb{R}^2) - D$.

738 **Proof.** Consider the shortest path $\pi = u_1 u_2 \dots u_k$ between p and q realizing $d_H(p, q)$. Every
 739 edge $e_i = u_i u_{i+1}$ has a homothet C_i such that $u_i, u_{i+1} \in C_i \subseteq R$. As such, there is a
 740 $(1 + \varepsilon)$ -path between u_i and u_{i+1} in $G \cap C_i \subseteq G \cap R$. Concatenating these paths directly
 741 yields the desired result.

742 The second claim follows by observing that the complement of D is the union of halfspaces,
 743 and halfspaces can be considered to be “infinite” homothets of \mathcal{C} . As such, the above argument
 744 applies verbatim. ◀

745 B.5 Proof of Lemma 23

746 **Restatement of Lemma 23.** For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a set
 747 P of $n + \mathcal{O}(\log \Phi)$ points, with spread $\mathcal{O}(\Phi n)$, and a triangle Δ , such that any Δ -local
 748 $(3/2)$ -spanner of P requires $\Omega(n \log \Phi)$ edges.



749 ■ **Figure B.4** An Illustration of the construction of Lemma 23.

750 **Proof.** Let $h = \lceil \log_2 \Phi \rceil$. Let Δ be the triangle formed by the points $(0, 0)$, $(0, 1)$ and $(8\Phi h, 0)$.
 751 The hypotenuse of this triangle lies on the line $\ell \equiv \frac{1}{8\Phi h}x + y = 1$, and let $v = (\frac{1}{8\Phi h}, 1)$ be
 752 the vector orthogonal to this line.

For $i \in \llbracket h \rrbracket$ and $j \in \llbracket n \rrbracket$, let

$$q_i = (2^{i+1}, 1 - i/h) \quad \text{and} \quad u_j = \left(\frac{j}{n} - 1, -\frac{j}{n}\right),$$

and let $P = \{q_1, \dots, q_h, u_1, \dots, u_n\}$, see Figure B.4. Observe that $\text{cp}(P) = \|u_1 u_2\| = \sqrt{2}/n$, and as such we have that $\Phi(P) = n \cdot \text{diam}(P)/\sqrt{2} \leq n(4\Phi + 2n) \leq 8\Phi n$, as $\Phi \geq n$. Observe that

$$\langle q_{i+1} - q_i, v \rangle = \langle (2^{i+1}, -\frac{1}{h}), (\frac{1}{8\Phi h}, 1) \rangle \leq \frac{4\Phi}{8\Phi h} - \frac{1}{h} < 0.$$

That is, the points q_1, \dots, q_i are increasing in distance from ℓ .

Let $\triangle_{i,j}$ be the homothet of \triangle , that has its bottom left corner at u_j , and its hypotenuse passes through q_i . By the above, $P(i, j) = \triangle_{i,j} \cap P = \{u_j, q_i, q_{i+1}, \dots, q_h\}$. Any $(1 + \varepsilon)$ -spanner for $P(i, j)$ must contain the edge $u_j q_i$. Indeed, we have, for any k , that $2^{k+1} \leq \|u_j q_k\| \leq 2^{k+1} + 3$. As such, any path on a graph induced on $P(i, j)$ from u_j to q_i that uses (say) a midpoint q_k , for $k > i$, must have dilation at least

$$\frac{\|u_j q_k\| + \|q_k q_i\|}{\|u_j q_i\|} \geq \frac{2^{k+1} + 2^k}{2^{i+1} + 3} \geq \frac{3 \cdot 2^{i+1}}{(1 + 3/4)2^{i+1}} = \frac{12}{7} > \frac{3}{2}.$$

Thus, any \triangle -local $3/2$ -spanner for homothets of \triangle , must contain the edge $q_i u_j$, for any $i \in \llbracket h \rrbracket$ and $j \in \llbracket n \rrbracket$. Thus, such a spanner must have $\Omega(n \log \Phi)$ edges, as claimed. \blacktriangleleft

B.6 Proof of Lemma 25

Restatement of Lemma 25. Let $p \in P$, $c \in \mathcal{C}_i$, and $u = \text{nn}_i(p, c)$, and let q be a point in $(P \cap (p + c)) \setminus \{p, u\}$. We have that $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ and $\|qu\| \leq \|pq\|$.

Proof. Consider the triangle $\triangle pqu$ and denote the angles at p, q , and u by $\angle p, \angle q$, and $\angle u$ respectively. Since the angle of c is smaller than 60 degrees (for an appropriate choice of γ), we have that $\|qu\| \leq \max\{\|pu\|, \|pq\|\}$.

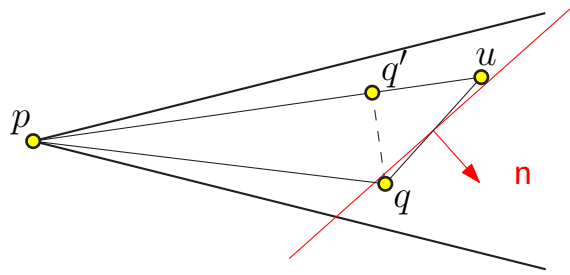


Figure B.5 The case that $\|pq\| \leq \|pu\|$ in Lemma 25. The vector used to determine $\text{nn}_i(p, c)$ is shown in red, and denoted n

Consider the case that $\|pq\| \leq \|pu\|$, illustrated in Figure B.5. Observe that $\angle u \leq \angle q$. As such $\angle u \leq \pi/2$. Furthermore, $\angle u \geq \alpha \gg \varepsilon\alpha/\gamma = \beta \geq \angle p$. Similarly, $\angle q \in [\alpha, \pi - \alpha]$. By the 1-Lipshitz of \sin , and as $\sin x \approx x$, for small x , and for γ sufficiently large, we have that

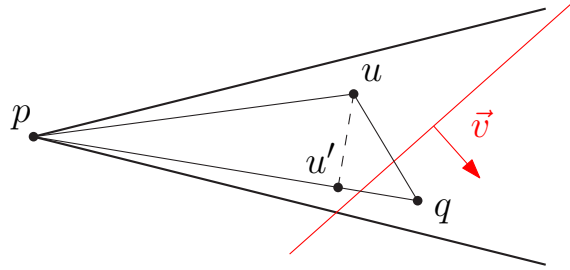
$$\sin(\angle q + \angle p) \in [1 - \varepsilon/4, 1 + \varepsilon/4] \sin \angle q \quad \text{and} \quad \sin \angle p \leq (\varepsilon/4) \sin \angle u.$$

As such, by the law of sines, we have that $\frac{\|qu\|}{\sin \angle p} = \frac{\|pq\|}{\sin \angle u} = \frac{\|pu\|}{\sin \angle q}$. This implies that

$$\|pu\| + (1 + \varepsilon)\|qu\| = \left(\frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \right) \|pq\|.$$

Observe, by the above that

$$\frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \leq \frac{\sin \angle q}{\sin (\angle p + \angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq \frac{\sin \angle q}{(1 - \varepsilon/4) \sin (\angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$



■ **Figure B.6** The case that $\|pq\| > \|pu\|$ in Lemma 25.

The other possibility is that $\|pq\| > \|pu\|$, illustrated in Figure B.6. Let u' be the projection of u to pq . Observe that

$$\|uu'\| = \|pu'\| \tan \angle p \leq 2\beta \|pu'\| \leq (\varepsilon/8) \|pu'\|.$$

Observe that $\cos \angle p \geq 1 - (\angle p)^2/2 \geq 1 - \varepsilon^2/8$ as $\angle p$ is an angle smaller than (say) $\varepsilon/16$. As such $1/\cos \angle p \leq 1 + \varepsilon^2/4$. This implies that $\|pu\| \leq \|pu'\|/\cos \angle p \leq (1 + \varepsilon^2/4) \|pu'\|$. We thus have that

$$\begin{aligned} \tau &= \|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon^2/4) \|pu'\| + (1 + \varepsilon) (\|uu'\| + \|u'q\|) \\ &\leq (1 + \varepsilon^2/4 + (1 + \varepsilon)\varepsilon/8) \|pu'\| + (1 + \varepsilon) \|u'q\| \leq (1 + \varepsilon) \|pq\|. \end{aligned}$$

B.7 Proof of Lemma 26

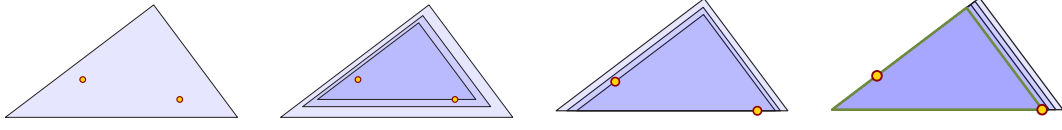
Restatement of Lemma 26. Let \triangle be a triangle that contains two points p, q . Then, there is a homothet $\triangle' \subseteq \triangle$ of \triangle , such that one of these points is a vertex of \triangle' , and the other point lies on a facing edge of \triangle' .

Proof. This follows by the same shrinking argument as Lemma 15, with the addition of a single step. When a homothet \triangle' with $p, q \in \partial \triangle'$ is found, if neither point is on a vertex, we “push” the only edge that does not contain one of the points towards the vertex v opposite of it (this the same mapping described in Lemma 15 with center v), until one of the points, say p lies on the edge. p now lies on two edges, meaning, at a vertex, while q lies on the only remaining edge which must be opposite of that vertex. See Figure B.7. ◀

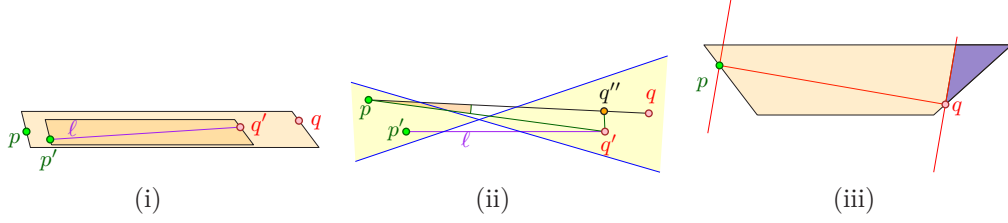
B.8 Proof of Lemma 29

Restatement of Lemma 29. Let $\varepsilon \in (0, 1)$ be some parameter, and $\vartheta = \varepsilon/16$. Let X, Y be two points sets that are ϑ -semi separated and ϑ -angularly separated (see Definition 7), and let T be a ϑ -narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on the two legs of T . Then, one can compute a homothet $T' \subseteq T$ of T , such that:

- (I) There are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay triangulation of $X \cup Y$.



804 **Figure B.7** An illustration of the shrinking process of Lemma 26. The three left figures illustrates
 805 the process of Lemma 15, for the case that the convex region \mathcal{C} is a triangle, and the rightmost
 806 figure is the additional final step.



815 **Figure B.8** Illustration of the settings in the proof of Lemma 29. Left: A ϑ -narrow trapezoid
 816 with p and q on its legs. Center: p and q are ϑ -semi separated and ϑ -angularly separated. Right:
 817 The triangle of all the points of the trapezoids that their nearest point on pq is q .

814 (II) We have that $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$.

818 **Proof.** Let $\mathcal{D} = \mathcal{D}_T(X \cup Y)$. Claim 16 implies that $\mathcal{D} \cap T$ is connected. Thus, there is a
 819 path in $\mathcal{D} \cap T$ between p and q , and thus, there must be an edge $p'q'$ along this path with
 820 $p' \in X$ and $q' \in Y$. This implies part (I).

821 Let $\ell = \|p'q'\|$. Assume for concreteness that $\|pp'\| \leq \text{diam}(X) \leq \vartheta d(X, Y) \leq \vartheta \ell \leq \vartheta d$,
 822 where $d = \text{diam}(T)$. Let q'' be the closest point on pq to q' .

823 We first consider the case that $q'' \in \text{int}(pq)$. We have that

$$824 \quad \|pq''\| = \|p'q'\| \cos \angle q'pq \geq (\|p'q'\| - \|pp'\|) \cos \angle q'pq \geq (1 - \vartheta)\ell \cdot (1 - \vartheta^2/2) \geq (1 - 2\vartheta)\ell,$$

825 since $\cos \vartheta \geq 1 - \vartheta^2/2$, for $\vartheta < 1/2$. Similar argumentation implies that $\|pq''\| \leq (1 + \vartheta)\ell$.
 826 As such, we have

$$827 \quad \|q'q''\| \leq (1 + \vartheta)\ell \sin \angle p'pq' \leq 2\vartheta\ell.$$

828 Thus, we have that

$$829 \quad \|qq'\| \leq \|qq''\| + \|q''q'\| \leq \|pq\| - \|pq''\| + 2\vartheta\ell \leq \|pq\| - (1 - 2\vartheta)\ell + 2\vartheta\ell \leq \|pq\| - \ell.$$

830 Thus, we have that

$$831 \quad (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon)\vartheta\ell + \ell + (1 + \varepsilon)(\|pq\| - \ell) \\
 832 \quad = (1 + \varepsilon) \|pq\| + (1 + \varepsilon)\vartheta\ell + \ell - (1 + \varepsilon)\ell \leq (1 + \varepsilon) \|pq\|,$$

833 for $\vartheta \leq \varepsilon/2$. Which establish the claim in this case.

834 The case that $q'' = p$ is impossible, because of the angular separation property. Thus,
 835 the only remaining possibility is that $q'' = q$. This however implies that q' must be in the
 836 triangle of all the points of the trapezoids that their nearest point on pq is q . The diameter
 837 of this triangle is bounded by the length of the leg of the trapezoid, which is bounded by
 838 ϑd . Namely, we have $\|qq'\| \leq \vartheta d$. Similarly, we have $(1 - 2\vartheta)d \leq \|pq\| \leq (1 + 2\vartheta)d$. Since
 839 $\|pp'\|, \|qq'\| \leq \vartheta d$, it follows that

$$840 \quad (1 - 4\vartheta)d \leq \ell \leq (1 + 4\vartheta)d.$$

As such, for $\vartheta \leq \varepsilon/8$ and $\varepsilon \leq 1$, we have

$$(1 + \varepsilon) \|pp'\| + \ell + (1 + \varepsilon) \|q'q\| \leq 4\vartheta d + (1 + 4\vartheta)d = (1 + 8\vartheta)d \leq (1 + \varepsilon) \|pq\|.$$

B.9 Proof of Lemma 30

Restatement of Lemma 30. *Let t be a positive integer. Given a t -nice polygon \mathcal{C} , and a parameter ϑ , one can cover it by a set \mathcal{T} of $\mathcal{O}(t^4/\vartheta^3)$ ϑ -narrow trapezoids, such that for any two points $p, q \in \partial\mathcal{C}$ that belong to two edges of \mathcal{C} that are not adjacent, there exists a narrow trapezoid $T \in \mathcal{T}$, such that p and q are located on two different short legs of T .*

Proof. We show a somewhat suboptimal but simple construction. A t -nice polygon has at most t edges. Let ψ be the sensitivity of \mathcal{C} , and place a minimum set of points P on the boundary of \mathcal{C} , which includes all the vertices of \mathcal{C} , and such that the distance between any consecutive pair of points is in the range $[c_1, 2c_1]$, where $c_1 = \vartheta\psi/c_2$, for some sufficiently large constant c_2 . In particular, let $M = \max_{e \in E(\mathcal{C})} \lceil \|e\|/c_1 \rceil = \mathcal{O}(1/\vartheta)$.

In addition, place $c_3 t$ equally spaced points between any two consecutive points of P , where c_3 is a constant to be determined shortly. Let Q be the set resulting from P after adding all these points.

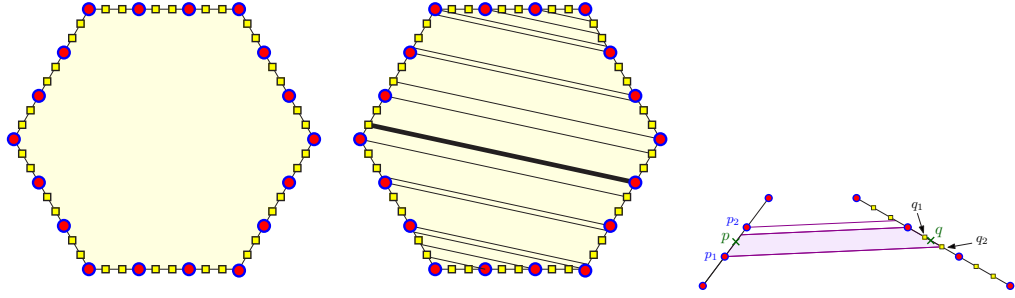
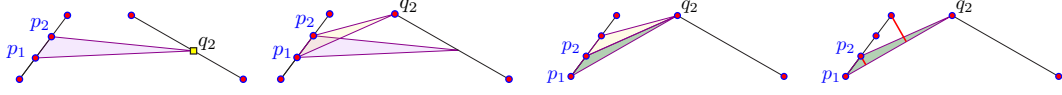


Figure B.9 The points of P (round), and all the points added to P in order to create Q (square). On the right, a “vertical” decomposition induced by one of the directions of $P \times Q$.

We have that $|P| = \mathcal{O}(t/\vartheta)$ and $|Q| = \mathcal{O}(t^2/\vartheta)$. For a direction v , let \mathcal{T}_v be the decomposition into trapezoids formed by shooting rays from inside \mathcal{C} in the direction of v (or $-v$) from all the points of P , see Figure B.9. Let \mathcal{T}'_v be the set resulting from throwing away trapezoids with legs that lie on adjacent edges. It is easy to verify that all the trapezoids of \mathcal{T}'_v are ϑ -narrow. Let U be the set of all directions induced by pairs of points of $P \times Q$, and let $\mathcal{T} = \cup_{u \in U} \mathcal{T}'_u$. We have that $|\mathcal{T}| = \mathcal{O}(|P| \cdot |U|) = \mathcal{O}(|P|^2 |Q|) = \mathcal{O}(t^4/\vartheta^3)$.

Consider any two points p, q on non-adjacent edges of \mathcal{C} , and let p_1, p_2 be the two adjacent points of P such that $p \in p_1 p_2$. Now, let q_1, q_2 be the adjacent points of Q such that $q \in q_1 q_2$. We assume that p_1, p_2, q_1, q_2 are in this clockwise order along the boundary of \mathcal{C} .

Observe that when we project the interval $p_1 p_2$, to the line induced by $q_1 q_2$, in the direction $\overrightarrow{p_1 q_2}$, the projected interval contains $q_1 q_2$. The last claim is intuitively obvious, but requires some work to see formally. The minimum height of a triangle involving three vertices of \mathcal{C} is formed by three consecutive vertices. In the worst case, this is an isosceles triangle with sidelength ψ and base angle π/t . As such, the height of such a triangle is $h = \psi \sin(\pi/t) \geq \psi/t$.



874 **Figure B.10** The height of the triangle $\triangle p_1 p_2 q_2$ is minimized as q_2 and p_1 are moved to vertices
 875 of \mathcal{C} .

876 The height of the triangle $\triangle p_1 p_2 q_2$ is minimized when p_1 or p_2 is a vertex of \mathcal{C} , and
 877 q_2 is at a vertex of \mathcal{C} , see Figure B.10. Assume, for concreteness, that p_1 is a vertex of \mathcal{C} ,
 878 and observe that $\|p_1 p_2\| \geq \|e\|/M$, where e is the edge of \mathcal{C} containing this segment. Using
 879 similar triangles, it is straightforward to show that the height of this triangle is at least
 880 $h' = h/M = \Omega(\varepsilon\psi/t)$. The quantity h' is a lower bound on the length of the projection of
 881 $p_1 p_2$ on the line spanned by $q_1 q_2$. However, $\|q_1 q_2\| \leq 2c_1/c_3 t = \mathcal{O}(\vartheta\psi/c_3 t) < h'$, by picking
 882 c_3 to be sufficiently large constant.

883 This readily implies that the trapezoid induced by the direction $u = \overrightarrow{p_1 q_2}$ in \mathcal{T}'_u that
 884 contains p on its leg, contains q on its other leg. ◀