

# Local Spanners Revisited

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## Abstract

For a set of points  $P \subseteq \mathbb{R}^2$ , and a family of regions  $\mathcal{F}$ , a *local  $t$ -spanner* of  $P$ , is a sparse graph  $G$  over  $P$ , that remains a  $t$ -spanner even when restricted to a region  $r \in \mathcal{F}$ . That is, for any region  $r \in \mathcal{F}$ , the subgraph restricted to  $r$ , denoted by  $G \cap r$ , is still a  $t$ -spanner for all the points of  $r \cap P$ . Here  $G \cap r$  is the subgraph of  $G$  induced on  $P \cap r$ .

We present algorithms for the construction of local spanners with respect to several families of convex regions. In particular, we construct a near linear size local spanner for homothets of a convex region. Unfortunately, the number of edges in the resulting graph depends logarithmically on the spread of the input point set. We prove that this dependency can not be removed, thus settling an open problem raised by Abam and Borouny. We also show improved constructions of local spanners for fat triangles, and regular  $k$ -gons. In particular, this improves over the known construction for axis parallel squares.

Finally, we study a somewhat weaker notion of local spanner where one allows to shrink the region a “bit”. We show that any spanner is a weak local spanner if one is allowed to shrink the region by some fraction of its diameter. Surprisingly, we show a near linear size construction of a weak spanner for axis-parallel rectangles, where the shrinkage is *multiplicative*.

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## 1 Introduction

For a set  $P$  of points in  $\mathbb{R}^d$ , the *Euclidean graph*  $G_P = (P, \binom{P}{2})$  of  $P$  is an undirected graph. Here, an edge  $pq \in E$  is associated with the segment  $pq$ , and its weight is the (Euclidean) length of the segment. Let  $G = (P, E)$  and  $I = (P, E')$  be two graphs over the same set of vertices (usually  $I$  is a subgraph of  $G$ ). Consider two vertices  $p, q \in P$ , and parameter  $t \geq 1$ . A path  $\pi$  between  $p$  and  $q$  in  $I$ , is a  *$t$ -path*, if the length of  $\pi$  in  $I$  is at most  $t \cdot d_G(p, q)$ , where  $d_G(p, q)$  is the length of the shortest path between  $p$  and  $q$  in  $G$ . The graph  $I$  is a  *$t$ -spanner* of  $G$  if there is a  $t$ -path in  $I$ , for any  $p, q \in P$ . Thus, for a set of points  $P \subseteq \mathbb{R}^d$ , a graph  $G$  over  $P$  is a  *$t$ -spanner* if it is a  $t$ -spanner of the euclidean graph  $G_P$ . There is a lot of work on building geometric spanners, see [NS07] and references there in.

## Fault-tolerant spanners

An  *$\mathcal{F}$ -fault-tolerant spanner* for  $P \subseteq \mathbb{R}^d$ , is a graph  $G = (P, E)$ , such that for any region  $r$  (i.e., the “attack”), the graph  $G - r$  is a  $t$ -spanner of  $G_P - r$  (See Definition 1 for a formal definition of this notation). Surprisingly, as shown by Abam *et al.* [AdBFG09], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

## 2 Local Spanners Revisited

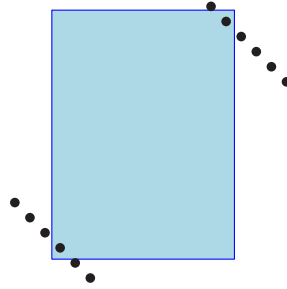
Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size  $k$  of vertices and edges has failed. Levcopoulos *et al.* [LNS02] showed the existence of  $k$ -vertex/edges fault tolerant spanners for a set of points  $P$  in some metric space. Their spanner had  $\mathcal{O}(kn \log n)$  edges, and weight, i.e. sum of edge weights, bounded by  $f(k) \cdot wt(MST(P))$  for some function  $f$ . Lukovszki [Luk99] later achieved a similar construction, improving the number of edges to  $\mathcal{O}(kn)$ , and was able to prove that the result is asymptotically tight.

### Local spanners

Recently, Abam and Borouny [AB21] introduced the notion of local spanners. For a family of regions  $\mathcal{F}$ , a graph  $G = (P, E)$  is a *local  $t$ -spanner* for  $\mathcal{F}$ , if for any  $r \in \mathcal{F}$ , the subgraph of  $G$  induced on  $P \cap r$  is a  $t$ -spanner. Specifically, this induced subgraph  $G \cap r$  contains a  $t$ -path between any  $p, q \in P \cap r$  (note that we keep an edge in the subgraph only if both its endpoints are in  $r$ , see Definition 1).

Abam and Borouny [AB21] showed how to construct such spanners for axis-parallel squares and vertical slabs. In this work, we are further extending their results. They also showed how to construct such spanners for disks if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

To appreciate the difficulty in constructing local spanners, observe that unlike regular spanners, the construction has to take into account many different scenarios as far as which points are available to be used in the spanner. As a concrete example, a local spanner for axis-parallel rectangle requires quadratic number of edges, see Figure 1.1.



**Figure 1.1** For any point in the top diagonal and bottom diagonal, there is a fat axis parallel rectangle that contains only these two points. Thus, a local spanner requires quadratic size in this case.

Namely, regular spanners can rely on using midpoints in their path under the assurance that they are always there. For local spanners this is significantly harder as natural midpoints might “disappear”. Intuitively, a local spanner construction needs to use midpoints that are guaranteed to be present judging only from the source and destination points of the path.

### A good jump is hard to find

Most constructions for spanners can be viewed as searching for a way to build a path from the source to the destination by finding a “good” jump, either by finding a way to move locally from the source to a nearby point in the right direction, as done in the  $\theta$ -graph construction, or alternatively, by finding an edge in the spanner from the neighborhood of the source to the neighborhood of the destination, as done in the spanner constructions using

Region	# edges	Paper	New # edges	Location in paper
Local $(1 + \varepsilon)$ -spanners				
Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AdBFG09]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[AB21]	$\mathcal{O}(\varepsilon^{-3} n \log n)$	Remark 32
Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AB21]		
Disks+Steiner points	$\mathcal{O}_\varepsilon(n)$	[AB21]		
Disks			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 18
			$\Omega(n \log \Phi)$	Lemma 22
Homothets convex shape			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 18
Homothets $\alpha$ -fat triangles			$\mathcal{O}((\alpha\varepsilon)^{-1} n)$	Theorem 28
Homothets triangles			$\Omega(n \log \Phi)$	Lemma 23
$\delta$ -weak local $(1 + \varepsilon)$ -spanners				
Bounded convex shape			$\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$	Lemma 12
$(1 - \delta)$ -local $(1 + \varepsilon)$ -spanners				
Axis-parallel rectangles			$\mathcal{O}((\varepsilon^{-2} + \delta^{-2})n \log^2 n)$	Theorem 38

**Table 1.1** Known and new results. The notation  $\mathcal{O}_\varepsilon$  hides polynomial dependency on  $\varepsilon$  which is not specified in the original work.

well-separated pairs decomposition (WSPD). Usually, one argues inductively that the spanner must have (sufficiently short) paths from the source to the start of the jump, and from the end of the jump to the destination, and then, combining these implies that the resulting new path is short. These ideas guide our constructions as well. However, the availability of specific edges depends on the query region, making the search for a good jump significantly more challenging. The constructions have to guarantee that there are many edges available, and that at least one of them is useful as a jump regardless of the chosen region.

## Our results

### Almost local spanners

We start by showing that regular geometric spanners are local spanners if one is required provide the spanner guarantee only to shrunken regions. Namely, if  $G$  is a  $(1 + \varepsilon)$ -spanner of  $P$ , then for any convex region  $\mathcal{C}$ , the graph  $G \cap \mathcal{C}$  is a spanner for  $\mathcal{C}' \cap P$ , where  $\mathcal{C}'$  is the set of all points in  $\mathcal{C}$  that are in distance at least  $\varepsilon \cdot \text{diam}(\mathcal{C})$  from its boundary.

### Homothets

A *homothet* of a convex region  $\mathcal{C}$ , is a translated and scaled copy of  $\mathcal{C}$ . In Section 3 we present a construction of spanners, which surprisingly, is not only fault-tolerant for all convex regions, but is also a local spanner for homothets of a prespecified convex region. This in particular works for disks, and resolves the aforementioned open problem of Abam and Borouny [AB21]. Our construction is somewhat similar to the original construction of Abam *et al.* [AdBFG09]. For a parameter  $\varepsilon > 0$  the construction of a  $(1 + \varepsilon)$ -local spanner for homothets takes  $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$  time, and the resulted spanner is of size  $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ ,

103 where  $\Phi$  is the spread of the input point set  $P$ , and  $n = |P|$ . We also provide a lower bound  
 104 showing that this logarithmic dependency on  $\Phi$  cannot be avoided.

105 The dependency on the spread  $\Phi$  in the above construction is somewhat disappointing.  
 106 However, the lower bound constructions, provided in [Section 3.3](#), show that this is unavoidable  
 107 for disks or homothets of triangles.

108 Thus, the natural question is what are the cases where one can avoid the “curse of the  
 109 spread” – that is, cases where one can construct local spanners of size independent of the  
 110 spread of the input point set.

### 111 The basic building block: $\mathcal{C}$ -Delaunay triangulation

112 A key ingredient in the above construction is the concept of Delaunay triangulations induced  
 113 by homothets of a convex body. Intuitively, one replaces the unit disk (of the standard  
 114  $L_2$ -norm) by the provided convex region. It is well known ([\[CDI85\]](#)) that such diagrams  
 115 exist, have linear complexity in the plane, and can be computed quickly. In [Section 3.1](#) we  
 116 review these results, and restate the well-known property that the  $\mathcal{C}$ -Delaunay triangulation  
 117 is connected when restricted to a homothet of  $\mathcal{C}$ . By computing these triangulations for  
 118 carefully chosen subsets of the input point set, we get the results stated above.

119 Specifically, we use well-separated and semi-separated decompositions to compute these  
 120 subsets.

### 121 Fat triangles

122 In [Section 3.4](#) we give a construction of local spanners for the family  $\mathcal{F}$  of homothets of a  
 123 given triangle  $\triangle$ , and get a spanner of size  $\mathcal{O}((\alpha\varepsilon)^{-1}n)$  in  $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$  time, where  $\alpha$   
 124 is the smallest angle in  $\triangle$ . This construction is a careful adaptation of the  $\theta$ -graph spanner  
 125 construction to the given triangle, and it is significantly more technically challenging than  
 126 the original construction.

### 127 $k$ -regular polygons

128 It seems natural that if one can handle fat triangles, then homothets of  $k$ -regular polygons  
 129 should readily follow by a simple decomposition of the polygon into fat triangles. Maybe  
 130 surprisingly, this is not the case – a critical configuration might involve two points that are on  
 131 the interior of two non-adjacent edges of a homothet of the input polygon. We overcome this  
 132 by first showing that sufficiently narrow trapezoids, provide us with a good jump somewhere  
 133 inside the trapezoid, assuming one compute the Delaunay triangulation induced by the  
 134 trapezoid, and that the source and destination lie on the two legs of the trapezoid. Next, we  
 135 show that such a polygon can be covered by a small number of narrow trapezoids and fat  
 136 triangles. By building appropriate graphs for each trapezoid/triangle in the collection, we get  
 137 a spanner for homothets of the given  $k$ -regular polygon, with size that has no dependency on  
 138 the spread. Of course, the size does depend on  $k$ . See [Section 3.5](#) for details, and [Theorem 31](#)  
 139 for the precise result.

### 140 Quadrant separated pair decomposition (QSPD)

141 In [Appendix A.1](#), we describe a novel pair-decomposition. Specifically, the QSPD breaks  
 142 the input point set  $P$  into pairs, such that for any pair  $\{X, Y\}$  we have the property that  
 143 there is a translated axis system such that  $X$  and  $Y$  belong to two antipodal quadrants.  
 144 In  $d$  dimensions there is such a decomposition with  $\mathcal{O}(n \log^{d-1} n)$  pairs, and total weight

$\mathcal{O}(n \log^d n)$ . A somewhat similar idea was used by Abam and Borouny [AB21] for the  $d = 1$  case. We believe this decomposition might be useful and is of independent interest.

## Multiplicative weak local spanner for rectangles

In Appendix A.2, we use QSPDs to construct a weak local spanner for axis parallel rectangles. Here, the constructed graph  $G$  over  $P$ , has the property that for any axis-parallel rectangle  $R$ , the graph  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for all the points of  $((1 - \varepsilon)R) \cap P$ , where  $(1 - \varepsilon)R$  is the scaling of the rectangle by a factor of  $1 - \varepsilon$  around its center. Importantly, this works for narrow rectangles where this form of multiplicative shrinking is still meaningful (unlike the diameter based shrinking mentioned above). Contrast this with the lower bound (illustrated in Figure 1.1) of  $\Omega(n^2)$  on the size of local spanner if one does not shrink the rectangles. See Theorem 38 for details of the precise result.

See Table 1.1 for a summary of known results and comparisons to the results of this paper.

## 2 Preliminaries

### Residual graphs

► **Definition 1.** Let  $\mathcal{F}$  be a family of regions in the plane. For a fault region  $r \in \mathcal{F}$  and a geometric graph  $G$  on a point set  $P$ , let  $G - r$  be the residual graph after removing from it all the points of  $P$  in  $r$ . Similarly, let  $G \cap r$  denote the graph restricted to  $r$ . Formally, let

$$G - r = (P \setminus r, \{uv \in E \mid uv \cap \text{int}(r) = \emptyset\}) \quad \text{and} \quad G \cap r = (P \cap r, \{uv \in E \mid uv \subseteq r\}).$$

where  $\text{int}(r)$  denotes the interior of  $r$ .

### 2.1 On various pair decompositions

For sets  $X, Y$ , let  $X \otimes Y = \{\{x, y\} \mid x \in X, y \in Y, x \neq y\}$  be the set of all the (unordered) pairs of points formed by the sets  $X$  and  $Y$ .

► **Definition 2** (Pair decomposition). For a point set  $P$ , a **pair decomposition** of  $P$  is a set of pairs

$$\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\},$$

such that (I)  $X_i, Y_i \subseteq P$  for every  $i$ , (II)  $X_i \cap Y_i = \emptyset$  for every  $i$ , and (III)  $\bigcup_{i=1}^s X_i \otimes Y_i = P \otimes P$ . Its **weight** is  $\omega(\mathcal{W}) = \sum_{i=1}^s (|X_i| + |Y_i|)$ .

The **closest pair** distance of a set of points  $P \subseteq \mathbb{R}^d$ , is  $\text{cp}(P) = \min_{p, q \in P, p \neq q} \|pq\|$ . The **diameter** of  $P$  is  $\text{diam}(P) = \max_{p, q \in P} \|pq\|$ . The **spread** of  $P$  is  $\Phi(P) = \text{diam}(P) / \text{cp}(P)$ , which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD (defined below) can be quadratic, if the spread is bounded, the weight is near linear. For  $X, Y \subseteq \mathbb{R}^d$ , let  $\text{d}(X, Y) = \min_{p \in X, q \in Y} \|pq\|$  be the **distance** between the two sets.

► **Definition 3.** Two sets  $X, Y \subseteq \mathbb{R}^d$  are

$$\begin{aligned} & \text{1/}\varepsilon\text{-well-separated} && \text{if} && \max(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot \text{d}(X, Y), \\ \text{and} & \text{1/}\varepsilon\text{-semi-separated} && \text{if} && \min(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot \text{d}(X, Y). \end{aligned}$$

For a point set  $P$ , a **well-separated pair decomposition (WSPD)** of  $P$  with parameter  $1/\varepsilon$  is a pair decomposition of  $P$  with a set of pairs  $\mathcal{W} = \{\{B_1, C_1\}, \dots, \{B_s, C_s\}\}$ , such that for all  $i$ , the sets  $B_i$  and  $C_i$  are  $(1/\varepsilon)$ -separated. The notion of  $1/\varepsilon$ -SSPD (a.k.a **semi-separated pairs decomposition**) is defined analogously.

► **Lemma 4** ([AH12]). Let  $P$  be a set of  $n$  points in  $\mathbb{R}^d$ , with spread  $\Phi = \Phi(P)$ , and let  $\varepsilon > 0$  be a parameter. Then, one can compute a  $(1/\varepsilon)$ -WSPD  $\mathcal{W}$  for  $P$  of total weight  $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$ . Furthermore, any point of  $P$  participates in at most  $\mathcal{O}(\varepsilon^{-d} \log \Phi)$  pairs.

► **Theorem 5** ([AH12, Har11]). Let  $P$  be a set of  $n$  points in  $\mathbb{R}^d$ , and let  $\varepsilon > 0$  be a parameter. Then, one can compute a  $(1/\varepsilon)$ -SSPD for  $P$  of total weight  $\mathcal{O}(n\varepsilon^{-d} \log n)$ . The number of pairs in the SSPD is  $\mathcal{O}(n\varepsilon^{-d})$ , and the computation time is  $\mathcal{O}(n\varepsilon^{-d} \log n)$ .

► **Lemma 6** (Proof in Appendix B.1). Given an  $\alpha$ -SSPD  $\mathcal{W}$  of a set  $P$  of  $n$  points in  $\mathbb{R}^d$  and a parameter  $\beta \geq 2$ , one can refine  $\mathcal{W}$  into an  $\alpha\beta$ -SSPD  $\mathcal{W}'$ , such that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$ .

► **Definition 7.** An  $\varepsilon$ -double-wedge is a region between two lines, where the angle between the two lines is at most  $\varepsilon$ .

Two point sets  $X$  and  $Y$  that each lie in their own cone of a shared  $\varepsilon$ -double-wedge are  $\varepsilon$ -angularly separated.

► **Lemma 8** (Proof in Appendix B.2). Given a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}$  of  $n$  points in the plane, one can refine  $\mathcal{W}$  into a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}'$ , such that each pair  $\Xi = \{X, Y\} \in \mathcal{W}'$  is contained in a  $\varepsilon$ -double-wedge  $\times_\Xi$ , such that  $X$  and  $Y$  are contained in the two different faces of the double wedge  $\times_\Xi$ . We have that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$ . The construction time is proportional to the weight of  $\mathcal{W}'$ .

► **Corollary 9.** Let  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon > 0$  be a parameter. Then, one can compute a  $(1/\varepsilon)$ -SSPD for  $P$  such that every pair is  $\varepsilon$ -angularly separated. The total weight of the SSPD is  $\mathcal{O}(n\varepsilon^{-3} \log n)$ , the number of pairs in the SSPD is  $\mathcal{O}(n\varepsilon^{-3})$ , and the computation time is  $\mathcal{O}(n\varepsilon^{-3} \log n)$ .

## 2.2 Weak local spanners for fat convex regions

► **Definition 10.** Given a convex region  $C$ , let

$$C_{\Box\delta} = \{p \in C \mid d(p, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)\}.$$

Formally,  $C_{\Box\delta}$  is the Minkowski difference of  $C$  with a disk of radius  $\delta \cdot \text{diam}(C)$ .

► **Definition 11.** Consider a (bounded) set  $C$  in the plane. Let  $r_{\text{in}}(C)$  be the radius of the largest disk contained inside  $C$ . Similarly,  $R_{\text{out}}(C)$  is the smallest radius of a disk containing  $C$ .

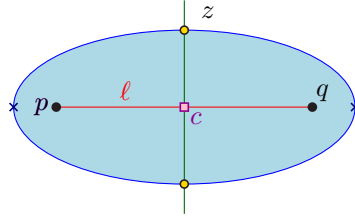
The **aspect ratio** of a region  $C$  in the plane is  $\text{ar}(C) = R_{\text{out}}(C)/r_{\text{in}}(C)$ . Given a family  $\mathcal{F}$  of regions in the plane, its aspect ratio is  $\text{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \text{ar}(C)$ .

Note, that if a convex region  $C$  has bounded aspect ratio, then  $C_{\Box\delta}$  is similar to the result of scaling  $C$  by a factor of  $1 - \mathcal{O}(\delta)$ . On the other hand, if  $C$  is long and skinny then this region is much smaller. Specifically, if  $C$  has width smaller than  $2\delta \cdot \text{diam}(C)$ , then  $C_{\Box\delta}$  is empty.

221 ► **Lemma 12.** *Given a set  $P$  of  $n$  points in the plane, and parameters  $\delta, \varepsilon \in (0, 1)$ . One*  
 222 *can construct a graph  $G$  over  $P$ , in  $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$  time, and with  $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$*   
 223 *edges, such that for any (bounded) convex  $C$  in the plane, we have that for any two points*  
 224  *$p, q \in P \cap C_{\square\delta}$  the graph  $C \cap P$  has  $(1 + \varepsilon)$ -path between  $p$  and  $q$ .*

225 **Proof.** Let  $\vartheta = \min(\varepsilon, \delta^2)$ . Construct, in  $\mathcal{O}(\vartheta^{-1}n \log n)$  time, a standard  $(1 + \vartheta)$ -spanner  $G$   
 226 for  $P$  using  $\mathcal{O}(\vartheta^{-1}n)$  edges [AMS99].

227 So, consider any body  $C \in \mathcal{F}$ , and any two vertices  $p, q \in P \cap C'$ , where  $C' = C_{\square\delta}$ . Let  
 228  $\ell = \|pq\|$ , let  $\pi$  be the shortest path between  $p$  and  $q$  in  $G$ , and let  $\mathcal{E}$  be the locus of all  
 229 points  $u$ , such that  $\|pu\| + \|uq\| \leq (1 + \vartheta)\ell$ . The region  $\mathcal{E}$  is an ellipse that contains  $\pi$ . The  
 230 furthest point from the segment  $pq$  in this ellipse is realized by the co-vertex of the ellipse.  
 231 Formally, it is one of the two intersection points of the boundary of the ellipse with the line  
 232 orthogonal to  $pq$  that passes through the middle point  $c$  of this segment, see Figure 2.1. Let  
 233  $z$  be one of these points.



234 ■ **Figure 2.1** An illustration of the settings in the proof of Lemma 12 with  $\mathcal{E}$  shown in blue.

235 We have that  $\|pz\| = (1 + \vartheta)\ell/2$ . Setting  $h = \|zc\|$ , we have that

$$236 \quad h = \sqrt{\|pz\|^2 - \|pc\|^2} = \frac{\ell}{2} \sqrt{(1 + \vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2 + \vartheta)}}{2} \ell \leq \sqrt{\vartheta} \ell \leq \sqrt{\vartheta} \cdot \text{diam}(C).$$

237 as  $\ell \leq \text{diam}(C') \leq \text{diam}(C)$ .

238 For any point  $x \in C'$ , we have that  $d(x, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)$ . As such, to ensure that  
 239  $\pi \subseteq \mathcal{E} \subseteq C$ , we need that  $\delta \cdot \text{diam}(C) \geq h$ , which holds if  $\delta \cdot \text{diam}(C) \geq \sqrt{\vartheta} \cdot \text{diam}(C)$ . This  
 240 in turn holds if  $\vartheta \leq \delta^2$ . Namely, we have the desired properties if  $\vartheta = \min(\varepsilon, \delta^2)$ . ◀

### 241 3 Local spanners of homothets of convex region

242 Let  $\mathcal{C}$  be a bounded convex and closed region in the plane (e.g., a disk). A *homothet* of  $\mathcal{C}$  is  
 243 a scaled and translated copy of  $\mathcal{C}$ . A point set  $P$  is in *general position* with respect to  $\mathcal{C}$ , if  
 244 no four points of  $P$  lie on the boundary of a homothet of  $\mathcal{C}$ , and no three points are colinear.

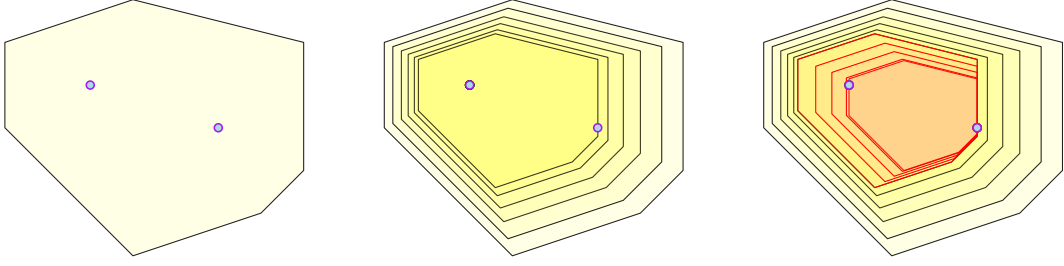
245 A graph  $G = (P, E)$  is a  $\mathcal{C}$ -local  $t$ -spanner for  $P$  if for any homothet  $\mathcal{r}$  of  $\mathcal{C}$  we have that  
 246  $G \cap \mathcal{r}$  is a  $t$ -spanner of  $G_P \cap \mathcal{r}$ .

#### 247 3.1 Delaunay triangulation for homothets

248 ► **Definition 13** ([CDI85]). *Given  $\mathcal{C}$  as above, and a point set  $P$  in general position with*  
 249 *respect to  $\mathcal{C}$ , the  $\mathcal{C}$ -Delaunay triangulation of  $P$ , denoted  $\mathcal{D}_{\mathcal{C}}(P)$ , is the graph formed by*  
 250 *edges between any two points  $p, q \in P$  such that there exist a homothet of  $\mathcal{C}$  that contains*  
 251 *only  $p$  and  $q$  and no other point of  $P$ .*

252 ► **Theorem 14** ([CDI85]). *For any convex shape  $\mathcal{C}$  and a set of points  $P$ ,  $\mathcal{D}_{\mathcal{C}}(P)$  can be*  
 253 *computed in  $\mathcal{O}(n \log n)$  time. Furthermore, the triangulation  $\mathcal{D}_{\mathcal{C}}(P)$  has  $\mathcal{O}(n)$  edges, vertices,*  
 254 *and faces.*





255 ■ **Figure 3.1** Shrinking of homothets so two points becomes on the boundary of the homothet.

256 ► **Lemma 15.** *Let  $\mathcal{C}$  be a convex bounded body, and let  $P$  be a set of points in general position with respect to  $\mathcal{C}$ . Then, if  $C$  is a homothet of  $\mathcal{C}$  that contains two points  $p, q \in C \cap P$ , then there exists a homothet  $C' \subseteq C$  of  $\mathcal{C}$  such that  $p, q \in \partial C'$ .*

259 **Proof.** The idea is to apply a shrinking process of  $C$ , as illustrated in Figure 3.1. Consider the mapping  $f_{\beta, v} : x \mapsto \beta(x - v) + v$ . It is a scaling of the plane around  $v$  by a factor of  $\beta$ . Let  $\beta'$  be the minimum value of  $\beta$  such that  $C_1 = f_{\beta, p}(C)$  contains  $q$  (i.e., we shrink  $C$  around  $p$  till  $q$  becomes a boundary point). Next, shrink  $C'$  around  $q$ , till  $p$  becomes a boundary point – formally, let  $\beta''$  be the minimum value of  $\beta$  such that  $C' = f_{\beta, q}(C_1)$  contains  $p$ . Since  $C' \subseteq C_1 \subseteq C$ , and  $p, q \in \partial C'$ , the claim follows. ◀

265 The following standard claim, usually stated about the standard Delaunay triangulations, also holds for homothets.

267 ► **Claim 16 (Proof in Appendix B.3).** Let  $\mathcal{C}$  be a bounded close convex shape. Given a set of points  $P \subseteq \mathbb{R}^2$  in general position with respect to  $\mathcal{C}$ , let  $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$  be the  $\mathcal{C}$ -Delaunay triangulation of  $P$ . For any homothet  $C$  of  $\mathcal{C}$ , we have that  $\mathcal{D} \cap C$  is connected.

## 270 3.2 The generic construction

271 The input is a set  $P$  of  $n$  points in the plane (in general position) with spread  $\Phi = \Phi(P)$ , and a parameter  $\varepsilon \in (0, 1)$ . We have a convex body  $\mathcal{C}$  that defines the “unit” ball. The task is to construct a local spanner for any homothet of  $\mathcal{C}$ .

274 The algorithm computes a  $(1/\vartheta)$ -WSPD  $\mathcal{W}$  of  $P$  using the algorithm of Lemma 4, where  $\vartheta = \varepsilon/6$ . For each pair  $\Xi = \{X, Y\} \in \mathcal{W}$ , the algorithm computes the  $\mathcal{C}$ -Delaunay triangulation  $\mathcal{D}_{\Xi} = \mathcal{D}_{\mathcal{C}}(X \cup Y)$ . The algorithm adds all the edges in  $\mathcal{D}_{\Xi} \cap (X \otimes Y)$  to the computed graph  $G$ .

### 278 3.2.1 Analysis

#### 279 Size

280 For each pair  $\Xi = \{X, Y\}$  in the WSPD, its  $\mathcal{C}$ -Delaunay triangulation contains at most  $\mathcal{O}(|X| + |Y|)$  edges. As such, the number of edges in the resulting graph is bounded by  $\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}(|X| + |Y|) = \mathcal{O}(\omega(\mathcal{W})) = \mathcal{O}\left(\frac{n \log \Phi}{\vartheta^2}\right)$ , by Lemma 4.

#### 283 Construction time

284 The construction time is bounded by  $\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}((|X| + |Y|) \log(|X| + |Y|)) = \mathcal{O}(\omega(\mathcal{W}) \log n) = \mathcal{O}\left(\frac{n \log \Phi \log n}{\vartheta^2}\right)$ .



286 ► **Lemma 17** (Local spanner property). *For  $P, \mathcal{C}, \varepsilon$  as above, let  $G$  be the graph constructed*  
 287 *above for the point set  $P$ . Then, for any homothet  $C$  of  $\mathcal{C}$  and any two points  $x, y \in P \cap C$ , we*  
 288 *have that  $G \cap C$  has a  $(1 + \varepsilon)$ -path between  $x$  and  $y$ . That is,  $G$  is a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner.*

289 **Proof.** Fix a homothet  $C$  of  $\mathcal{C}$ , and consider two points  $p, q \in P \cap C$ . The proof is by  
 290 induction on the distance between  $p$  and  $q$  (or more precisely, the rank of their distance  
 291 among the  $\binom{n}{2}$  pairwise distances). Consider the pair  $\Xi = \{X, Y\}$  such that  $x \in X$  and  
 292  $y \in Y$ .

293 If  $xy \in \mathcal{D}_\Xi$  then the claim holds, so assume this is not the case. By the connectivity of  
 294  $\mathcal{D}_\Xi \cap C$ , see [Claim 16](#), there must be points  $x' \in X \cap C$ ,  $y' \in Y \cap C$ , such that  $x'y' \in E(\mathcal{D}_\Xi)$ .  
 295 As such, by construction, we have that  $x'y' \in E(G)$ . Furthermore, by the separation property,  
 296 we have that

$$297 \quad \max(\text{diam}(X), \text{diam}(Y)) \leq \vartheta \, d(X, Y) \leq \vartheta \ell,$$

298 where  $\ell = \|xy\|$ . In particular,  $\|x'x\| \leq \vartheta \ell$  and  $\|y'y\| \leq \vartheta \ell$ . As such, by induction, we have  
 299  $d_G(x, x') \leq (1 + \varepsilon) \|xx'\| \leq (1 + \varepsilon) \vartheta \ell$  and  $d_G(y, y') \leq (1 + \varepsilon) \|yy'\| \leq (1 + \varepsilon) \vartheta \ell$ . Furthermore,  
 300  $\|x'y'\| \leq (1 + 2\vartheta) \ell$ . As  $x'y' \in E(G)$ , we have

$$301 \quad d_G(x, y) \leq d_G(x, x') + \|x'y'\| + d_G(y', y) \leq (1 + \varepsilon) \vartheta \ell + (1 + 2\vartheta) \ell + (1 + \varepsilon) \vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta) \ell$$

$$302 \quad = (1 + 6\vartheta) \ell \leq (1 + \varepsilon) \|xy\|,$$

303 if  $\vartheta \leq \varepsilon/6$ . ◀

### 304 The result

305 ► **Theorem 18.** *Let  $\mathcal{C}$  be a bounded convex shape in the plane, let  $P$  be a given set of  $n$*   
 306 *points in the plane (in general position), and let  $\varepsilon \in (0, 1/2)$  be a parameter. The above*  
 307 *algorithm constructs a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner  $G$ . The spanner has  $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$  edges, and*  
 308 *the construction time is  $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ . Formally, for any homothet  $C$  of  $\mathcal{C}$ , and any*  
 309 *two points  $p, q \in P \cap C$ , we have a  $(1 + \varepsilon)$ -path in  $G \cap C$ .*

### 310 3.2.2 Applications and comments

311 The following defines a “visibility” graph when we are restricted to a region  $R$ , where two  
 312 points are visible if there is a witness homothet contained in  $R$  having both points on its  
 313 boundary.

314 ► **Definition 19.** *Let  $\mathcal{C}$  be a bounded convex shape in the plane. Given a region  $R$  in the*  
 315 *plane and a point set  $P$ , consider two points  $p, q \in P$ . The edge  $pq$  is safe in  $R$  if there is a*  
 316 *homothet  $C$  of  $\mathcal{C}$ , such that  $p, q \in C \subseteq R$ . The **safe graph** for  $P$  and  $R$ , denoted by  $\mathcal{G}(P, R)$ ,*  
 317 *is the graph formed by all the safe edges in  $P$  for  $R$ . Note, that this graph might have a*  
 318 *quadratic number of edges in the worst case.*

319 Observe that  $\mathcal{G}(P, \mathbb{R}^2)$  is a clique. Surprisingly, the spanner graph described above, when  
 320 restricted to region  $R$ , is a spanner for  $\mathcal{G}(P, R)$ .

321 ► **Corollary 20** (Proof in [Appendix B.4](#)). *Let  $\mathcal{C}$  be a bounded convex body,  $P$  be a set of  $n$*   
 322 *points in the plane,  $\varepsilon \in (0, 1)$  be a parameter, and let  $G$  be a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner of  $P$ .*

323 Consider a region  $R$  in the plane, and the associated graph  $H = \mathcal{G}(P, R)$ , we have that  
 324  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for  $H$ . Formally, for any two points  $p, q \in P \cap R$ , we have that  
 325  $d_{G \cap R}(p, q) \leq (1 + \varepsilon) d_H(p, q)$ .

326 In particular, for any convex region  $D$ , the graph  $G - D$  is a  $(1 + \varepsilon)$ -spanner for  
 327  $\mathcal{G}(P, \mathbb{R}^2) - D$ .

► Remark 21. The above implies that local spanners for homothets are also robust to convex region faults.

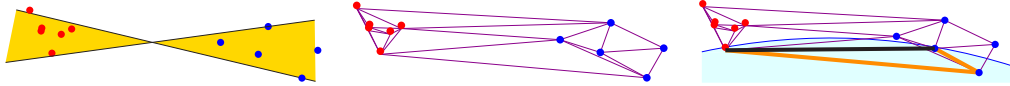
### 3.3 Lower bounds

#### 3.3.1 A lower bound for local spanner for disks

The result of Theorem 18 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). Next, we show a lower bound proving that this dependency is unavoidable, even in the case of disks.

##### Some intuition

A natural way to attempt a spread-independent construction is to try and emulate the construction of Abam *et al.* [AdBFG09] and use a SSPD instead of a WSPD, as the total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair  $\Xi = \{X, Y\}$  is angularly  $\varepsilon$ -separated – that is, there is a double wedge with angle  $\leq \varepsilon$ , such that  $X$  and  $Y$  are of different sides of the double wedge. The problem is that for the local disk  $\circ$ , it might be that the bridge edge between  $X$  and  $Y$  that is in  $\mathcal{D}_\Xi \cap \circ$  is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 3.2.



■ **Figure 3.2** A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

► **Lemma 22.** [Proof in Appendix B.5] For  $\varepsilon = 1/4$ , and parameters  $n$  and  $\Phi \geq 1$ , there is a point set  $P$  of  $n + \lceil \log \Phi \rceil$  points in the plane, with spread  $\mathcal{O}(n\Phi)$ , such that any local  $(1 + \varepsilon)$ -spanner of  $P$  for disks, must have  $\Omega(n \log \Phi)$  edges.

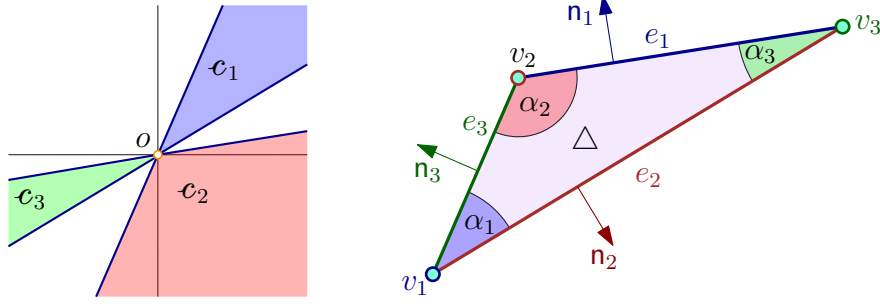
#### 3.3.2 A lower bound for triangles

► **Lemma 23.** [Proof in Appendix B.6] For any  $n > 0$ , and  $\Phi = \Omega(n)$ , one can compute a set  $P$  of  $n + \mathcal{O}(\log \Phi)$  points, with spread  $\mathcal{O}(\Phi n)$ , and a triangle  $\triangle$ , such that any  $\triangle$ -local  $(3/2)$ -spanner of  $P$  requires  $\Omega(n \log \Phi)$  edges.

### 3.4 Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex shape are costly, if we are given a triangle  $\triangle$  with the single constraint that  $\triangle$  is not too “thin”, then one can construct a  $\triangle$ -local  $t$ -spanner with a number of edges that does not depend on the spread of the points. See Figure B.5 for an illustration of a construction showing that dependency if “thin” triangles are allowed.

► **Definition 24.** A triangle  $\triangle$  is  $\alpha$ -fat if the smallest angle in  $\triangle$  is at least  $\alpha$ .



360 **Figure 3.3** For the triangle  $\Delta$  with angles  $\alpha_1, \alpha_2$ , and  $\alpha_3$  we create the cones  $c_1, c_2$ , and  $c_3$ .

### 361 3.4.1 Construction

362 The input is a set  $P$  of  $n$  points in the plane, an  $\alpha$ -fat triangle  $\Delta$ , and an approximation  
 363 parameter  $\varepsilon \in (0, 1)$ . Let  $v_i$  denote the  $i$ th vertex of  $\Delta$ ,  $\alpha_i$  be the adjacent angle, and let  
 364  $e_i$  denote the opposing edge, for  $i \in [3]$ . Let  $c_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$  denote the  
 365 cone with an apex at the origin induced by the  $i$ th vertex of  $\Delta$ . Let  $n_i$  be the outer normal of  
 366  $\Delta$  orthogonal to  $e_i$ . See Figure 3.3 for an illustration. Let  $\mathcal{C}_i$  be a minimum size partition of  
 367  $c_i$  into cones each with angle in the range  $[\beta/2, \beta]$ , where  $\beta = \varepsilon\alpha/\gamma$ , and  $\gamma > 1$  is a constant  
 368 to be determined shortly. For each point  $p \in P$ , and a cone  $c \in \mathcal{C}_i$ , let  $\text{nn}_i(p, c)$  be the first  
 369 point in  $(P - p) \cap (p + c)$  ordered by the direction  $n_i$  (it is the “nearest-neighbor” to  $p$  in  
 370  $p + c$  with respect to the direction  $n_i$ ).

### 371 The construction

372 Let  $G$  be the graph over  $P$  formed by connecting every point  $p \in P$  to  $\text{nn}_i(p, c)$ , for all  
 373  $i \in [3]$  and  $c \in \mathcal{C}_i$ .

### 374 3.4.2 Analysis

375 **► Lemma 25** (Proof in Appendix B.7). *Let  $p \in P$ ,  $c \in \mathcal{C}_i$ , and  $u = \text{nn}_i(p, c)$ , and let  $q$*   
 376 *be a point in  $(P \cap (p + c)) \setminus \{p, u\}$ . We have that  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$  and*  
 377  *$\|qu\| \leq \|pq\|$ .*

378 **► Lemma 26** (Proof in Appendix B.8). *Let  $\Delta$  be a triangle that contains two points  $p, q$ .*  
 379 *Then, there is a homothet  $\Delta' \subseteq \Delta$  of  $\Delta$ , such that one of these points is a vertex of  $\Delta'$ , and*  
 380 *the other point lies on a facing edge of  $\Delta'$ .*

### 381 Local spanner property

382 **► Lemma 27.** *Let  $\Delta'$  be a homothet of  $\Delta$ . For any two points  $p, q \in P \cap \Delta'$ , we have a*  
 383  *$(1 + \varepsilon)$ -path in  $G' = G \cap \Delta'$ .*

384 **Proof.** Consider the closest pair  $p, q \in P \cap \Delta$ . They must be connected directly in  $G'$ , as  
 385 otherwise there is a point  $u \in P' = P \cap \Delta'$  in the cone containing the segment  $pq$ , such that  
 386  $pu \in E(G')$ . But then, by Lemma 25, we have  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ , which  
 387 implies that either  $pu$  or  $qu$  are the closest pair, which is a contradiction.

388 For any other pair  $p, q \in P'$  we have from Lemma 26 that there exists a homothet  
 389  $\Delta'' \subseteq \Delta'$  with one of the two points, say  $p$ , at a vertex, and the other on the opposite edge.  
 390 We therefore have a cone  $c$  with apex at  $p$  such that  $q \in c \cap \Delta''$ . If  $pq$  is an edge in  $G$

then we are done. Otherwise, we have a vertex  $u \in \mathcal{C}$  such that  $pu$  is an edge in  $G$ , and by Lemma 25 we have  $\|qu\| \leq \|pq\|$ , which, by induction, means that there exists a  $(1 + \varepsilon)$  path between  $u$  and  $q$  in  $G$ . Lemma 25 now implies that  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ . Thus, there is a  $(1 + \varepsilon)$  path between  $p$  and  $q$  in  $G'$ , as stated. ◀

### Size and running time

► **Theorem 28.** *Let  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon \in (0, 1)$  be an approximation parameter. The above algorithm computes a  $\triangle$ -local  $(1 + \varepsilon)$ -spanner  $G$  for an  $\alpha$ -fat triangle  $\triangle$ . The construction time is  $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ , and the spanner  $G$  has  $\mathcal{O}((\alpha\varepsilon)^{-1}n)$  edges.*

**Proof.** The local-spanning property is proven in Lemma 27, and we are only left with bounding the size and the running time of the algorithm. The bound on the size is immediate from the construction, as every point  $p$  is the apex of  $\mathcal{O}(\frac{2\pi}{\varepsilon\alpha})$  cones, each giving rise to a single edge incident to  $p$ . The construction time is bounded by the construction time for a  $\theta$ -graph with cone size  $\alpha\varepsilon$ , which is  $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$  ([Cla87]). ◀

## 3.5 A local spanner for nice polygons

### 3.5.1 A good jump for narrow trapezoids

As a reminder, a trapezoid is a quadrilateral with two parallel edges, known as its *bases*. The other two edges are its *legs*. For  $\varepsilon \in (0, 1/4)$ , a trapezoid  $T$  is  $\varepsilon$ -*narrow* if the length of each of its legs is at most  $\varepsilon \cdot \text{diam}(T)$ .

► **Lemma 29** (Proof in Appendix B.9). *Let  $\varepsilon \in (0, 1)$  be some parameter, and  $\vartheta = \varepsilon/16$ . Let  $X, Y$  be two points sets that are  $\vartheta$ -semi separated and  $\vartheta$ -angularly separated (see Definition 7), and let  $T$  be a  $\vartheta$ -narrow trapezoid, with two points  $p \in X$  and  $q \in Y$  lying on the two legs of  $T$ . Then, one can compute a homothet  $T' \subseteq T$  of  $T$ , such that:*

- (I) *There are two points  $p' \in X$  and  $q' \in Y$ , such that  $p'q'$  is an edge of the  $T$ -Delaunay triangulation of  $X \cup Y$ .*
- (II) *We have that  $(1 + \varepsilon)\|pp'\| + \|p'q'\| + (1 + \varepsilon)\|q'q\| \leq (1 + \varepsilon)\|pq\|$ .*

### 3.5.2 Breaking a nice polygon into narrow trapezoids

For a convex polygon  $\mathcal{C}$ , its *sensitivity*, denoted by  $\text{sen}(\mathcal{C})$ , is the minimum distance between any two non-adjacent edges (this quantity is no bigger than the length of the shortest edge in the polygon). A convex polygon  $\mathcal{C}$  is  $t$ -*nice*, if the outer angle at any vertex of the polygon is at least  $2\pi/t$ , and the length of the longest edge of  $\mathcal{C}$  is  $\mathcal{O}(\text{sen}(\mathcal{C}))$ . As an example, a  $k$ -regular polygon is  $k$ -nice.

► **Lemma 30** (Proof in Appendix B.10). *Let  $t$  be a positive integer. Given a  $t$ -nice polygon  $\mathcal{C}$ , and a parameter  $\vartheta$ , one can cover it by a set  $\mathcal{T}$  of  $\mathcal{O}(t^4/\vartheta^3)$   $\vartheta$ -narrow trapezoids, such that for any two points  $p, q \in \partial\mathcal{C}$  that belong to two edges of  $\mathcal{C}$  that are not adjacent, there exists a narrow trapezoid  $T \in \mathcal{T}$ , such that  $p$  and  $q$  are located on two different short legs of  $T$ .*

### 3.5.3 Constructing the local spanner for nice polygons

► **Theorem 31.** *Let  $\mathcal{C}$  be a  $k$ -nice convex polygon,  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon \in (0, 1)$  be a parameter. Then, one can construct a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner of  $P$ . The construction time is  $\mathcal{O}((k^4/\varepsilon^6)n \log^2 n)$ , and the resulting graph has  $\mathcal{O}((k^4/\varepsilon^6)n \log n)$  edges. In particular these bounds hold if  $\mathcal{C}$  is a  $k$ -regular gon.*

**Proof.** Let  $\vartheta = \varepsilon/c_4$ , for  $c_4$  sufficiently large constant. We construct  $\Delta$ , a family of triangles induced by a vertex of  $\mathcal{C}$ , and an non-adjacent edge of  $\mathcal{C}$ . This family has  $\mathcal{O}(k^2)$  triangles. Each such triangle is  $\Omega(1/k)$ -fat, and for each such triangle we construct the  $(1 + \vartheta)$ -spanner of Theorem 28 for  $P$ . Next, we cover  $\mathcal{C}$  by a set  $\mathcal{T}$  of  $k' = \mathcal{O}(k^4/\vartheta^3)$   $\vartheta$ -narrow trapezoids using Lemma 30.

We compute an  $\vartheta$ -angular  $(1/\vartheta)$ -SSPD  $\mathcal{W}$  decomposition of  $P$  using Corollary 9 – the total weight of the decomposition is  $w = \mathcal{O}(n\vartheta^{-3} \log n)$ . For each pair  $\{X, Y\} \in \mathcal{W}$ , and each trapezoid  $T \in \mathcal{T}$ , we compute the  $T$ -Delaunay triangulation of  $X \cup Y$ .

Let  $G$  denote the union of all these graphs. We claim that it is the desired spanner. The construction time is

$$\mathcal{O}((k^3/\vartheta)n \log n + k'w \log n) = \mathcal{O}\left(\frac{k^3}{\vartheta}n \log n + \frac{k^4}{\vartheta^3} \cdot \frac{n}{\vartheta^3} \log n \cdot \log n\right) = \mathcal{O}\left(\frac{k^4}{\vartheta^6}n \log^2 n\right),$$

and the resulting graph has  $\mathcal{O}((k^4/\vartheta^6)n \log n)$  edges.

As for correctness, consider a homothet  $\mathcal{C}'$  of  $\mathcal{C}$  that contains two points  $p, q \in P$ . By Lemma 15, there is a homothet  $\mathcal{C}'' \subseteq \mathcal{C}'$  of  $\mathcal{C}$  such that  $p, q \in \partial\mathcal{C}''$ . There are two possibilities:

- The point  $p$  is on a vertex of  $\mathcal{C}''$  and  $q$  is on an edge. In this case, the vertex and the edge induce a fat triangle, that is a homothet of a triangle  $\Delta \in \Delta$ . Since the graph  $G$  contains a  $\Delta$ -local  $(1 + \varepsilon)$ -spanner for  $P$ , it follows readily that  $G$  is a  $(1 + \varepsilon)$ -spanner for these points, and the path is strictly inside  $\mathcal{C}''$ .
- The points  $p$  and  $q$  are on two non-adjacent edges of  $\mathcal{C}''$ . Then, there is an  $\vartheta$ -narrow trapezoid  $T'$  that has  $p$  and  $q$  on its two legs, and a homothet of  $T'$ , denoted by  $T$ , is in  $\mathcal{T}$ . There is a pair  $\{X, Y\} \in \mathcal{W}$  that is  $(1/\vartheta)$ -semi separated (and  $\vartheta$ -angularly separated), such that  $p \in X$  and  $q \in Y$ . By Lemma 29, there are two points  $p' \in X$  and  $q' \in Y$ , such that  $p'q'$  is an edge of the  $T$ -Delaunay triangulation of  $X \cup Y$ , and by construction this edge is in  $G$ . We now use induction on the shortest paths from  $p$  to  $p'$  and from  $q$  to  $q'$  in  $G$ . By induction, and Lemma 29, we have that

$$d(p, q) \leq d(p, p') + \|p'q'\| + d(q', q) \leq (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|,$$

which implies that there is  $(1 + \varepsilon)$ -path from  $p$  to  $q$  inside  $\mathcal{C}'$ .

► **Remark 32.** For axis-parallel squares Theorem 31 implies a local spanner with  $\mathcal{O}(\varepsilon^{-6}n \log n)$  edges. However, for this special case, the decomposition into narrow trapezoid can be skipped. In particular, in this case, the resulting spanner has  $\mathcal{O}(\varepsilon^{-3}n \log n)$  edges. We do not provide the details here, as it is only a minor improvement over the above, and requires quite a bit of additional work – essentially, one has to prove a version of Lemma 29 for squares.

## References

- AB21** Mohammad Ali Abam and Mohammad Sadegh Borouny. Local geometric spanners. To appear in *Algorithmica*, 2021.
- AdBFG09** Mohammad Ali Abam, Mark de Berg, Mohammad Farshi, and Joachim Gudmundsson. Region-fault tolerant geometric spanners. *Discret. Comput. Geom.*, 41(4):556–582, 2009.
- AH12** M. A. Abam and S. Har-Peled. New constructions of SSPDs and their applications. *Comput. Geom. Theory Appl.*, 45(5–6):200–214, 2012.
- AMS99** Arya, D. M. Mount, and M. Smid. Dynamic algorithms for geometric spanners of small diameter: Randomized solutions. *Comput. Geom. Theory Appl.*, 13(2):91–107, 1999.
- CDI85** Paul Chew and Robert L Dyrsdale III. Voronoi diagrams based on convex distance functions. In *Proc. 1st Annu. Sympos. Comput. Geom. (SoCG)*, pages 235–244, 1985.

- 475 **Cl87** Kenneth L. Clarkson. Approximation algorithms for shortest path motion planning (extended  
 476 abstract). In Alfred V. Aho, editor, *Proceedings of the 19th Annual ACM Symposium on*  
 477 *Theory of Computing, 1987, New York, New York, USA*, pages 56–65. ACM, 1987.
- 478 **Har11S** Har-Peled. *Geometric Approximation Algorithms*, volume 173 of *Math. Surveys & Mono-*  
 479 *graphs*. Amer. Math. Soc., Boston, MA, USA, 2011.
- 480 **LNS02** Christos Levcopoulos, Giri Narasimhan, and Michiel H. M. Smid. Improved algorithms for  
 481 constructing fault-tolerant spanners. *Algorithmica*, 32(1):144–156, 2002.
- 482 **Luk99** Tamás Lukovszki. New results of fault tolerant geometric spanners. In Frank K. H. A. Dehne,  
 483 Arvind Gupta, Jörg-Rüdiger Sack, and Roberto Tamassia, editors, *Algorithms and Data*  
 484 *Structures, 6th International Workshop, WADS '99, Vancouver, British Columbia, Canada,*  
 485 *August 11-14, 1999, Proceedings*, volume 1663 of *Lecture Notes in Computer Science*, pages  
 486 193–204. Springer, 1999.
- 487 **NS07** Giri Narasimhan and Michiel H. M. Smid. *Geometric spanner networks*. Cambridge University  
 488 Press, 2007.

## 489 **A** Weak local spanners for axis-parallel rectangles

### 490 **A.1** Quadrant separated pair decomposition

491 For two points  $p = (p_1, \dots, p_d)$  and  $q = (q_1, \dots, q_d)$  in  $\mathbb{R}^d$ , let  $p \prec q$  denotes that  $q$  *dominates*  
 492  $p$  coordinate-wise. That is  $p_i < q_i$ , for all  $i$ . More generally, let  $p <_i q$  denote that  $p_i < q_i$ .  
 493 For two point sets  $X, Y \subseteq \mathbb{R}^d$ , we use  $X <_i Y$  to denote that  $\forall x \in X, y \in Y \quad x <_i y$ . In  
 494 particular  $X$  and  $Y$  are *i-coordinate separated* if  $X <_i Y$  or  $Y <_i X$ . A pair  $\{X, Y\}$  is  
 495 *quadrant-separated*, if  $X$  and  $Y$  are *i-coordinate separated*, for  $i = 1, \dots, d$ .

496 A *quadrant-separated pair decomposition* of a point set  $P \subseteq \mathbb{R}^d$ , is a pair de-  
 497 composition (see [Definition 2](#))  $\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\}$  of  $P$ , such that  $\{X_i, Y_i\}$  are  
 498 quadrant-separated for all  $i$ .

499 ► **Lemma 33.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}$ , one can compute, in  $\mathcal{O}(n \log n)$  time, a QSPD*  
 500 *of  $P$  with  $\mathcal{O}(n)$  pairs, and of total weight  $\mathcal{O}(n \log n)$ .*

501 **Proof.** If  $P$  is a singleton then there is nothing to do. If  $P = \{p, q\}$ , then the decomposition  
 502 is the pair formed by the two singleton points.

503 Otherwise, let  $x$  be the median of  $P$ , such that  $P_{\leq x} = \{p \in P \mid p \leq x\}$  contains exactly  
 504  $\lfloor n/2 \rfloor$  points, and  $P_{> x} = P \setminus P_{\leq x}$  contains  $\lfloor n/2 \rfloor$  points. Construct the pair  $\Xi = \{P_{\leq x}, P_{> x}\}$ ,  
 505 and compute recursively a QSPDs  $\mathcal{Q}_{\leq x}$  and  $\mathcal{Q}_{> x}$  for  $P_{\leq x}$  and  $P_{> x}$ , respectively. The desired  
 506 QSPD is  $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{> x} \cup \{\Xi\}$ . The bounds on the size and weight of the desired QSPD are  
 507 immediate. ◀

508 ► **Lemma 34.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^d$ , one can compute, in  $\mathcal{O}(n \log^d n)$  time, a*  
 509 *QSPD of  $P$  with  $\mathcal{O}(n \log^{d-1} n)$  pairs, and of total weight  $\mathcal{O}(n \log^d n)$ .*

510 **Proof.** The construction algorithm is recursive on the dimensions, using the algorithm of  
 511 [Lemma 33](#) in one dimension.

512 The algorithm computes a value  $\alpha_d$  that partitions the values of the points'  $d$ th coordinates  
 513 roughly equally (and is distinct from all of them), and let  $h$  be a hyperplane parallel to the  
 514 first  $d - 1$  coordinate axes, and having value  $\alpha_d$  in the  $d$ th coordinate.

515 Let  $P_{\uparrow}$  and  $P_{\downarrow}$  be the subset of points of  $P$  that are above and below  $h$ , respectively. The  
 516 algorithm recursively computes QSPDs  $\mathcal{Q}_{\uparrow}$  and  $\mathcal{Q}_{\downarrow}$  for  $P_{\uparrow}$  and  $P_{\downarrow}$ , respectively. Next, the  
 517 algorithm projects the points of  $P$  on  $h$ , let  $P'$  be the resulting  $d - 1$  dimensional point set  
 518 (after we ignore the  $d$ th coordinate), and recursively computes a QSPD  $\mathcal{Q}'$  for  $P'$ .



For a point set  $X' \subseteq P'$ , let  $\text{lift}(X')$  be the subset of points of  $P$  whose projection on  $h$  is  $X'$ . The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\text{lift}(X') \cap P_{\uparrow}, \text{lift}(Y') \cap P_{\downarrow}\}, \{\text{lift}(X') \cap P_{\downarrow}, \text{lift}(Y') \cap P_{\uparrow}\} \mid \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is  $\widehat{\mathcal{Q}} \cup \mathcal{Q}_{\uparrow} \cup \mathcal{Q}_{\downarrow}$ .

To observe that this is indeed a QSPD, observe that all the pairs in  $\mathcal{Q}_{\uparrow}, \mathcal{Q}_{\downarrow}$  are quadrant separated by induction. As for pairs in  $\widehat{\mathcal{Q}}$ , they are quadrant separated in the first  $d-1$  coordinates by induction on the dimension, and separated in the  $d$  coordinate since one side of the pair comes from  $P_{\uparrow}$ , and the other side from  $P_{\downarrow}$ .

As for coverage, consider any pair of points  $p, q \in P$ , and observe that the claim holds by induction if they are both in  $P_{\uparrow}$  or  $P_{\downarrow}$ . As such, assume that  $p \in P_{\uparrow}$  and  $q \in P_{\downarrow}$ . But then there is a pair  $\{X', Y'\} \in \mathcal{Q}'$  that separates the two projected points in  $h$ , and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates  $p$  and  $q$  as desired.

The number pairs in the decomposition is  $N(n, d) = 2N(n, d-1) + 2N(n/2, d)$  with  $N(n, 1) = \mathcal{O}(n)$ . The solution to this recurrence is  $N(n, d) = \mathcal{O}(n \log^{d-1} n)$ . The total weight of the decomposition is  $W(n, d) = 2W(n, d-1) + 2W(n/2, d)$  with  $W(n, 1) = \mathcal{O}(n \log n)$ . The solution to this recurrence is  $W(n, d) = \mathcal{O}(n \log^d n)$ . Clearly, this also bounds the construction time.  $\blacktriangleleft$

## A.2 Weak local spanner for axis parallel rectangles

For a parameter  $\delta \in (0, 1)$ , and an interval  $I = [b, c]$ , let  $(1-\delta)I = [t - (1-\delta)r, t + (1-\delta)r]$ , where  $t = (b+c)/2$ , and  $r = (c-b)/2$ , be the shrinking of  $I$  by a factor of  $1-\delta$ .

Let  $\mathcal{R}$  be the set of all axis parallel rectangles in the plane. For a rectangle  $R \in \mathcal{R}$ , with  $R = I \times J$ , let  $(1-\delta)R = (1-\delta)I \times (1-\delta)J$  denote the rectangle resulting from shrinking  $R$  by a factor of  $1-\delta$ .

**Definition 35.** Given a set  $P$  of  $n$  points in the plane, and parameters  $\varepsilon, \delta \in (0, 1)$ , a graph  $G$  is a  $(1-\delta)$ -local  $(1+\varepsilon)$ -spanner for rectangles, if for any axis-parallel rectangle  $R$ , we have that  $G \cap R$  is a  $(1+\varepsilon)$ -spanner for all the points in  $((1-\delta)R) \cap P$ .

Observe that rectangles in  $\mathcal{R}$  might be quite “skinny”, so the previous notion of shrinkage used before is not useful in this case.

### A.2.1 Construction for a single quadrant separated pair

Consider a pair  $\Xi = \{X, Y\}$  in a QSPD of  $P$ . The set  $X$  is quadrant-separated from  $Y$ . That is, there is a point  $c_{\Xi}$ , such that  $X$  and  $Y$  are contained in two opposing quadrants in the partition of the plane formed by the vertical and horizontal line through  $c_{\Xi}$ .

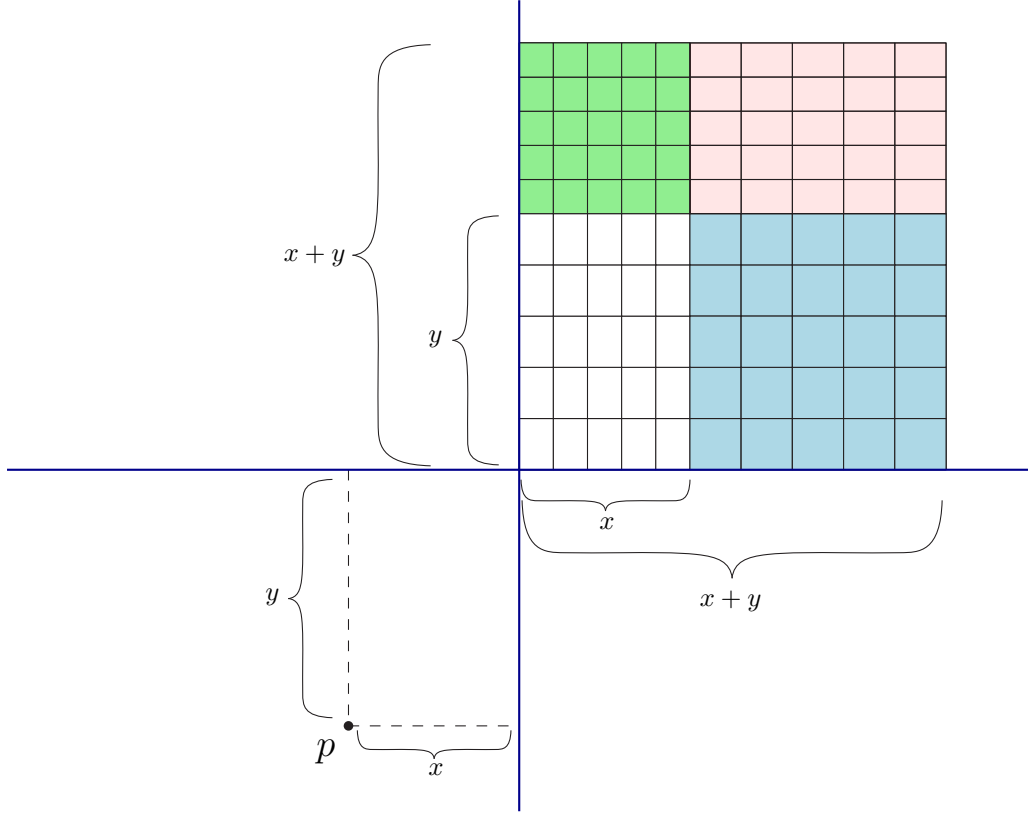
For simplicity of exposition, assume that  $c_{\Xi} = (0, 0)$ , and  $X \prec (0, 0) \prec Y$ . That is, the points of  $X$  are in the negative quadrant, and the points of  $Y$  are in the positive quadrant.

We construct a non-uniform grid  $K(p, \Xi)$  in the square  $[0, x+y]^2$ . To this end, we first partition it into four subrectangles

$$\begin{array}{|l|l|} \hline B_{\swarrow} = [0, x] \times [y, x+y] & B_{\nearrow} = [x, x+y] \times [y, x+y] \\ \hline B_{\swarrow} = [0, x] \times [0, y] & B_{\nwarrow} = [x, x+y] \times [0, y]. \\ \hline \end{array}$$

Let  $\tau \geq 4/\varepsilon + 4/\delta$  be an integer number. We partition each of these rectangles into a  $\tau \times \tau$  grid, where each cell is a copy of the rectangle scaled by a factor of  $1/\tau$ . See Figure A.1.





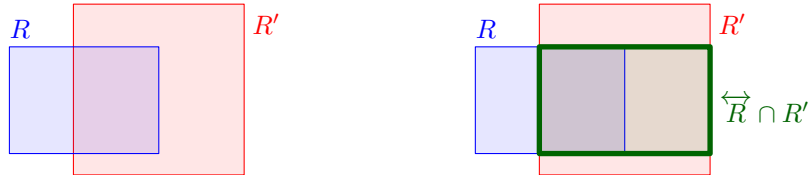
548 ■ **Figure A.1** The construction of the grid  $K(p, \Xi)$  for a point  $p = (-x, -y)$  and a pair  $\Xi$ .

559 This grid has  $\mathcal{O}(\tau^2)$  cells. For a cell  $C$  in this grid, let  $Y \cap C$  be the points of  $Y$  contained in  
 560 it. We connect  $p$  to the left-most and bottom-most points in  $Y \cap C$ . This process generates  
 561 two edges in the constructed graph for each grid cell (that contains at least two points), and  
 562  $\mathcal{O}(\tau^2)$  edges overall.

563 The algorithm repeats this construction for all the points  $p \in X$ , and does the symmetric  
 564 construction for all the points of  $Y$ .

### 565 A.2.2 The construction algorithm

566 The algorithm computes a QSPD  $\mathcal{W}$  of  $P$ . For each pair  $\Xi \in \mathcal{W}$ , the algorithm generates  
 567 edges for  $\Xi$  using the algorithm of Section A.2.1 and adds them to the generated spanner  $G$ .



568 ■ **Figure A.2** Left: The two rectangles  $R, R'$ . Right: In green  $\overleftrightarrow{R} \cap R'$ , the restriction of the slab  
 569  $\overleftrightarrow{R}$  to the rectangle  $R'$ .

### A.2.3 Correctness

For a rectangle  $R$ , let  $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists(x', y) \in R\}$  be its expansion into a horizontal slab. Restricted to a rectangle  $R'$ , the resulting set is  $\overleftrightarrow{R} \cap R'$ , depicted in Figure A.2. Similarly, we denote  $\uparrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists(x, y') \in R\}$ .

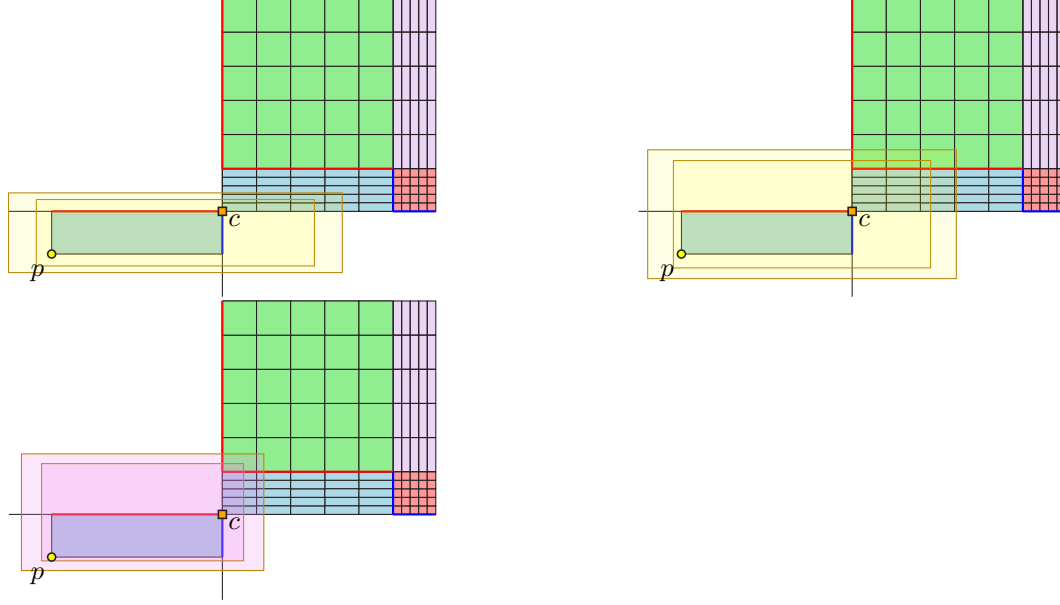


Figure A.3 An illustration of  $K(p, \Xi)$  with three rectangles and their shrunken version.

**Lemma 36.** Assume that  $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$ . Consider a pair  $\Xi = \{X, Y\}$  in the above construction, and a point  $p = (-x, -y) \in X$  with its associated grid  $K = K(p, \Xi)$ . Consider any axis parallel rectangle  $R$ , such that  $p \in (1 - \delta)R = I \times J$ , and  $(1 - \delta)R$  intersects a cell  $C \in K$ . We have that:

- (I) If  $C \subseteq (1 - \delta)R$  then  $(1 - \delta)^{-1}C \subseteq R$ .
- (II)  $\text{diam}(C) \leq (\varepsilon/4)d(p, C)$ .
- (III) If  $x \geq y$  and  $C \subseteq R_{\swarrow} \cup R_{\searrow}$  then  $(1 - \delta)^{-1}C \subseteq R$ .
- (IV) If  $x \leq y$  and  $C \subseteq R_{\swarrow} \cup R_{\searrow}$  then  $(1 - \delta)^{-1}C \subseteq R$ .
- (V) If  $x \geq y$  and  $C \subseteq R_{\nwarrow}$ , then  $(1 - \delta)^{-1}(\overleftrightarrow{(1 - \delta)R} \cap C) \subseteq R$ .
- (VI) If  $x \leq y$  and  $C \subseteq R_{\nearrow}$ , then  $(1 - \delta)^{-1}(\uparrow((1 - \delta)R) \cap C) \subseteq R$ .

**Proof.** (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

(II) We have that  $\text{diam}(C) \leq (x + y)/\tau = \|p\|_1/\tau \leq (\varepsilon/4)d(p, C)$ .

(III) The width, denoted  $\text{wd}(\cdot)$ , of  $(1 - \delta)R$  is at least  $x$ , as it contains both  $p$  and the origin. As such,

$$(\text{wd}(R) - \text{wd}((1 - \delta)R))/2 \geq 2(x/\tau) \geq 2\text{wd}(C).$$

That is, the width of the “expanded” rectangle  $R$  is enough to cover  $C$ , and a grid cell adjacent to it to the right.

A similar argument about the height shows that  $R$  covers the region immediately above  $C$  – in particular, the vertical distance from  $C$  to the top boundary of  $R$  is at least the height of  $C$ . This implies that the expanded cell  $(1 - \delta)^{-1}C$  is contained in  $R$ , as claimed, as  $\delta < 1/2$ .

(V) We decompose the claim to the two dimensions of the region. Let  $B = \overrightarrow{((1-\delta)R \cap C)}$ . Observe that containment in the  $x$ -axis follows by arguing as in (III). As for the  $y$ -interval of  $B$ , observe that it is contained in the  $y$ -interval of  $(1-\delta)R$ , which implies that when expanded by  $(1-\delta)^{-1}$ , it would be contained in the  $y$ -interval of  $R$ . Combining the two implies the result.  $\blacktriangleleft$

► **Lemma 37.** *For any axis-parallel rectangle  $R$ , and any two points  $p, q \in (1-\delta)R \cap P$ , there exists a  $(1+\varepsilon)$ -path between  $p$  and  $q$  in  $G$ .*

**Proof.** The proof is by induction over the size of  $R$  (i.e. area, width, or height). Let  $\Xi = \{X, Y\} \in \mathcal{W}$  be the pair in the QSPD that separates  $p$  and  $q$ , let  $c$  be the separation point of the pair, and assume for the simplicity of exposition that  $p \in X$ ,  $X \prec c \prec Y$ , and  $c = (0, 0)$ . Furthermore, assume that  $\|p\|_1 \geq \|q\|_1$ .

Let  $p = (-x, -y)$ , and let  $C$  be the grid cell of  $K(p, \Xi)$  that contains  $q$ . If  $C \subseteq (1-\delta)R$ , then  $(1-\delta)^{-1}C \subseteq R$  by Lemma 36 (I). As such, let  $u$  be the leftmost point in  $C \cap P$ . Both  $q, u \in (1-\delta)^{-1}C$ , and by induction, there is an  $(1+\varepsilon)$ -path  $\pi$  between them in  $G$  (note that the induction applies to the two points, and the “expanded” rectangle  $(1-\delta)^{-1}C$ ). Since  $pu$  is an edge of  $G$ , prefixing  $\pi$  by this edge results in an  $(1+\varepsilon)$ -path, as  $\|qu\| \leq (\varepsilon/4)\|pq\|$ , by Lemma 36 (II) (verifying this requires some standard calculations which we omit).

Otherwise, one need to apply the same argument using the appropriate case of Lemma 36. So assume that  $x \geq y$  (the case that  $y \geq x$  is handled symmetrically). If  $C \subseteq R_{\swarrow} \cup R_{\searrow}$ , then (III) implies that  $(1-\delta)^{-1}C \subseteq R$ . Which implies that induction applies, and the claim holds.

The remaining case is that  $x \geq y$  and  $C \subseteq R_{\nwarrow}$ . Let  $D = \overrightarrow{(1-\delta)R \cap C}$ . By (V), we have  $(1-\delta)^{-1}D \subseteq R$ . Namely,  $q \in (1-\delta)R \cap C \subseteq D$ , and let  $u$  be the lowest point in  $C \cap P$ . By construction  $pu \in E(G)$ ,  $q, u \in D$ ,  $(1-\delta)^{-1}D \subseteq R$ . As such, we can apply induction to  $q, u$ , and  $(1-\delta)^{-1}D$ , and conclude that  $d_G(q, u) \leq (1+\varepsilon)\|qu\|$ . Plugging this into the regular machinery implies the claim.  $\blacktriangleleft$

► **Theorem 38.** *Let  $P$  be a set of  $n$  points in the plane, and let  $\varepsilon, \delta \in (0, 1)$  be parameters. The above algorithm constructs, in  $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$  time, a graph  $G$  with  $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$  edges. The graph  $G$  is a  $(1-\delta)$ -local  $(1+\varepsilon)$ -spanner for axis parallel rectangles. Formally, for any axis-parallel rectangle  $R$ , we have that  $R \cap P$  is an  $(1+\varepsilon)$ -spanner for all the points of  $((1-\delta)R) \cap P$ .*

**Proof.** Computing the QSPD  $\mathcal{W}$  takes  $\mathcal{O}(n \log^2 n)$  time. For each pair  $\{X, Y\}$  in the decomposition with  $m = |X| + |Y|$  points, we need to compute the lowest and leftmost points in  $(X \cup Y) \cap C$ , for each cell in the constructed grid. This can readily be done using orthogonal range trees in  $\mathcal{O}(\log^2 n)$  time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is  $\mathcal{O}(\omega(\mathcal{W})\tau^2) = \mathcal{O}((1/\delta^2 + 1/\varepsilon^2)n \log^2 n)$ .

The desired local spanner property is provided by Lemma 37.  $\blacktriangleleft$

## B Some missing proofs

### B.1 Proof of Lemma 6

**Restatement of Lemma 6.** *Given an  $\alpha$ -SSPD  $\mathcal{W}$  of a set  $P$  of  $n$  points in  $\mathbb{R}^d$  and a parameter  $\beta \geq 2$ , one can refine  $\mathcal{W}$  into an  $\alpha\beta$ -SSPD  $\mathcal{W}'$ , such that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$ .*

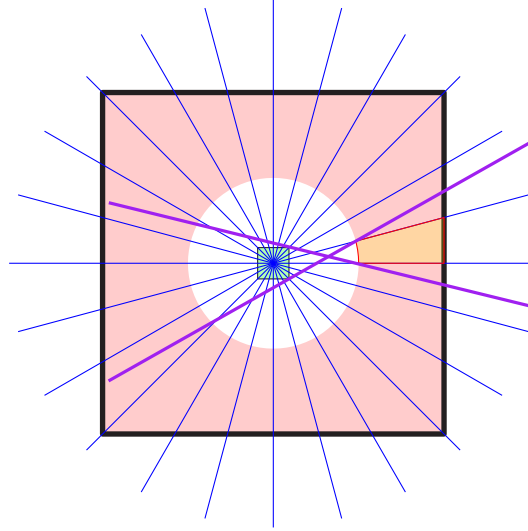
637 **Proof.** The algorithm scans the pairs of  $\mathcal{W}$ . For each pair  $\Xi = \{X, Y\} \in \mathcal{W}$ , assume that  
 638  $\text{diam}(X) < \text{diam}(Y)$ . Let  $\mathfrak{s}$  be the smallest axis-parallel cube containing  $X$ , and denote its  
 639 sidelength by  $r$ . Let  $r' = r / \lceil \sqrt{d}\beta \rceil$ . Partition  $\mathfrak{s}$  into a grid of cubes of sidelength  $r'$ , and let  
 640  $T_\Xi$  be the resulting set of squares. The algorithm now add the set pairs

$$641 \quad \{\{X \cap t, Y\} \mid t \in T_\Xi\}$$

642 to the output SSPD. Clearly, the resulting set is now  $\alpha\beta$ -semi separated, as we chopped the  
 643 smaller part of each pair into  $\beta$  smaller portions.  $\blacktriangleleft$

## 644 B.2 Proof of Lemma 8

645 **Restatement of Lemma 8.** *Given a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}$  of  $n$  points in the plane, one can*  
 646 *refine  $\mathcal{W}$  into a  $(1/\varepsilon)$ -SSPD  $\mathcal{W}'$ , such that each pair  $\Xi = \{X, Y\} \in \mathcal{W}'$  is contained in a*  
 647  *$\varepsilon$ -double-wedge  $\times_\Xi$ , such that  $X$  and  $Y$  are contained in the two different faces of the double*  
 648 *wedge  $\times_\Xi$ . We have that  $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$  and  $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$ . The construction time*  
 649 *is proportional to the weight of  $\mathcal{W}'$ .*



650 **Figure B.1** An illustration of refining the pairs in a SSPD into pairs contained in opposite parts  
 651 of an  $\varepsilon$ -double-wedge.  $X$  is contained in the green square  $\square$ , while  $Y$  is contained in the red square,  
 652 and the white gap between them is a result of the separation property. The set of cones with the  
 653 apex at the center of  $\square$  gives us the desired partition as demonstrated by the purple double-wedge.

654 **Proof.** By using Lemma 6, we can assume that  $\mathcal{W}$  is (say)  $(10/\varepsilon)$ -separated. Now, the  
 655 algorithm scans the pairs of  $\mathcal{W}$ . For each pair  $\Xi = \{X, Y\} \in \mathcal{W}$ , assume that  $\text{diam}(X) <$   
 656  $\text{diam}(Y)$ . Let  $\square$  be the smallest axis-parallel square containing  $X$ , centered at point  $o$ .  
 657 Partition the plane around  $o$ , by drawing  $\mathcal{O}(1/\varepsilon)$  lines intersecting  $o$  with the angle between  
 658 any two consecutive lines being at most (say)  $\varepsilon/4$ , see Figure B.1. This partitions the plane  
 659 into a set of cones  $\mathcal{C}$ . For a cone  $c \in \mathcal{C}$ , we show that there exists an  $\varepsilon$ -double-wedge that  
 660 contains  $X$  in one side, and  $Y \cap c$  in the other.

661 To see that, take the double-wedge formed by the cross tangents between  $\text{ch}(X)$  and  
 662  $\text{ch}(Y \cap c)$ , where  $\text{ch}(X)$  denotes the convex-hull of  $X$ . Assume w.l.o.g that  $\square$  has side length

1, and let  $c$  be a cone of angle  $\varepsilon/4$  with apex  $o$ , whose angular bisector is a horizontal ray in the positive direction of the  $x$  axis. See figure Figure B.2 for an illustration.

We would like to find a vertical segment  $s$  such that all points of  $Y$  lie to its right, with one endpoint on the upper line of  $c$ , and the other on the lower line of  $c$ . Using the segments' height and distance from the right side of  $\square$  we will be able to get a bound on the angle of the cross tangents. We first find a segment  $s$  with all points of  $Y$  to its right. A trivial bound on that distance is given by the segment from, say, the lower left corner of  $\square$ , denoted  $p$ , of length  $10/\varepsilon$  with its right endpoint on the upper line of  $c$ , denote this point by  $q$ . We know that all points of  $Y$  lie to the right of  $q$  due to the  $10/\varepsilon$  separation property of the SSPD. The segment  $pq$  creates an angle  $\leq \pi/4$  with the  $x$ -axis (by the choice of the angle of  $c$ ). We therefore get that the  $x$ -coordinate difference between  $\square$  and  $q$  is at most  $10/\varepsilon \cdot \cos \frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$ . So let  $s'$  be a vertical segment between the upper and lower rays of  $c$ , with  $x$ -coordinate distance of  $6/\varepsilon - \frac{1}{2}$  from  $\square$  (in order to make calculations easier). We get that  $s'$  is of length  $2 \cdot \frac{6}{\varepsilon} \tan \frac{\varepsilon}{8}$ . Finally, we take  $s$  to be a vertical segment of length  $\frac{12}{\varepsilon} \tan \frac{\varepsilon}{8}$ , with its center on the  $x$ -axis at a distance of  $5/\varepsilon + \frac{1}{2}$  away from  $o$ . The angle of the  $x$ -axis and the segment between the lower end of the right side of  $\square$  and the upper end of  $s$  is now given by:

$$\arctan\left(\frac{\frac{6}{\varepsilon} \tan \frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5} \tan \frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) \leq \varepsilon$$

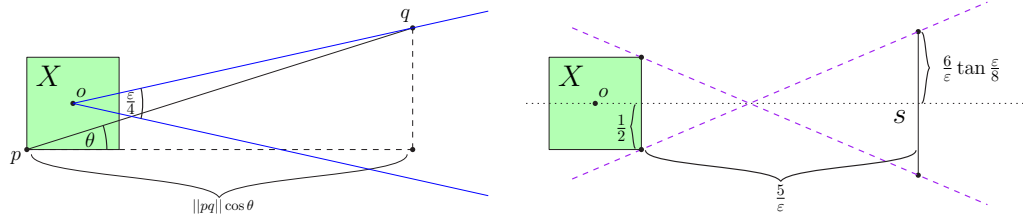


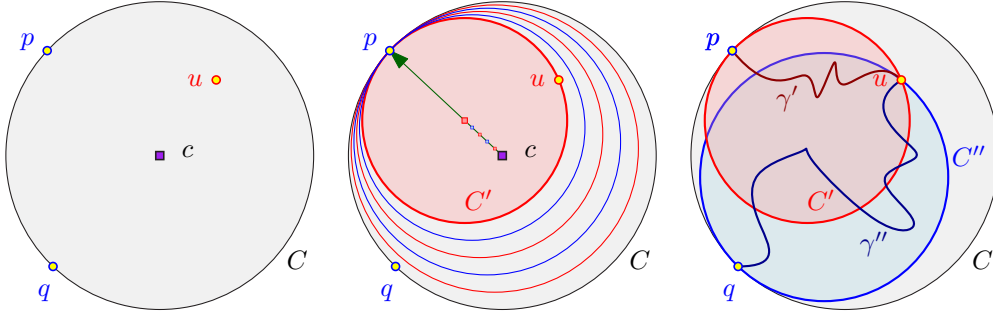
Figure B.2 An illustration of the proof for Lemma 8

### B.3 Proof of Claim 16

**Restatement of Claim 16.** Let  $C$  be a bounded close convex shape. Given a set of points  $P \subseteq \mathbb{R}^2$  in general position with respect to  $C$ , let  $\mathcal{D} = \mathcal{D}_C(P)$  be the  $C$ -Delaunay triangulation of  $P$ . For any homothet  $C'$  of  $C$ , we have that  $\mathcal{D} \cap C'$  is connected.

**Proof.** We prove that for any homothet  $C$  with two points  $p, q \in P$  on its boundary, there is a path between  $p$  and  $q$  in  $\mathcal{D} \cap C$ , and Lemma 15 will immediately imply the general statement. The proof is by induction over the number  $m$  of points of  $P$  in the interior of  $C$ . If  $m = 0$  then  $C$  contains no points of  $P$  in its interior, and thus  $pq$  is an edge of the Delaunay triangulation, as  $C$  testifies.

Otherwise, let  $u \in P$  be a point in the interior of  $C$ . From Lemma 15 we get that there exists a homothet  $C'$  of  $C$  with  $C' \subseteq C$ , such that  $p$  and  $u$  lie on the boundary of  $C'$ . Thus, by induction, there is a path  $\gamma'$  between  $p$  and  $u$  in  $\mathcal{D} \cap C' \subseteq \mathcal{D} \cap C$ . Similarly, there must be a homothet  $C''$ , that gives rise to a path  $\gamma''$  between  $u$  and  $q$ , and concatenating the two paths results in a path between  $p$  and  $q$  in  $\mathcal{D} \cap C$ .



692 **Figure B.3** An illustration of the proof of Claim 16 in the case that  $C$  is a disk.

## 699 B.4 Proof of Corollary 20

700 **Restatement of Corollary 20.** Let  $\mathcal{C}$  be a bounded convex body,  $P$  be a set of  $n$  points in  
 701 the plane,  $\varepsilon \in (0, 1)$  be a parameter, and let  $G$  be a  $\mathcal{C}$ -local  $(1 + \varepsilon)$ -spanner of  $P$ .

702 Consider a region  $R$  in the plane, and the associated graph  $H = \mathcal{G}(P, R)$ , we have that  
 703  $G \cap R$  is a  $(1 + \varepsilon)$ -spanner for  $H$ . Formally, for any two points  $p, q \in P \cap R$ , we have that  
 704  $d_{G \cap R}(p, q) \leq (1 + \varepsilon)d_H(p, q)$ .

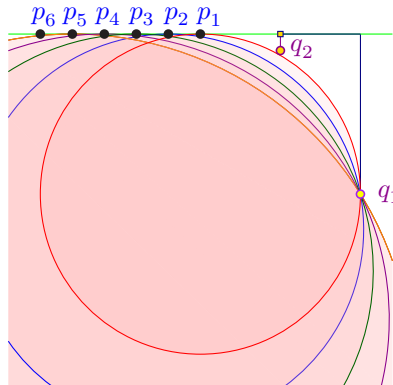
705 In particular, for any convex region  $D$ , the graph  $G - D$  is a  $(1 + \varepsilon)$ -spanner for  
 706  $\mathcal{G}(P, \mathbb{R}^2) - D$ .

707 **Proof.** Consider the shortest path  $\pi = u_1 u_2 \dots u_k$  between  $p$  and  $q$  realizing  $d_H(p, q)$ . Every  
 708 edge  $e_i = u_i u_{i+1}$  has a homothet  $C_i$  such that  $u_i, u_{i+1} \in C_i \subseteq R$ . As such, there is a  
 709  $(1 + \varepsilon)$ -path between  $u_i$  and  $u_{i+1}$  in  $G \cap C_i \subseteq G \cap R$ . Concatenating these paths directly  
 710 yields the desired result.

711 The second claim follows by observing that the complement of  $D$  is the union of halfspaces,  
 712 and halfspaces can be considered to be “infinite” homothets of  $\mathcal{C}$ . As such, the above argument  
 713 applies verbatim. ◀

## 714 B.5 Proof of Lemma 22

715 **Restatement of Lemma 22.** For  $\varepsilon = 1/4$ , and parameters  $n$  and  $\Phi \geq 1$ , there is a point  
 716 set  $P$  of  $n + \lceil \log \Phi \rceil$  points in the plane, with spread  $\mathcal{O}(n\Phi)$ , such that any local  $(1 + \varepsilon)$ -spanner  
 717 of  $P$  for disks, must have  $\Omega(n \log \Phi)$  edges.



718 **Figure B.4** The set of disks  $D_1$ , and the construction of  $q_2$ .

**Proof.** Let  $p_i = (-i, 0)$ , for  $i = 1, \dots, n$ . Let  $M = 1 + \lceil \log_2 \Phi \rceil$  and  $q_1 = (n2^M, -1)$ . For a point  $p$  on the  $x$ -axis, and a point  $q$  below the  $x$ -axis and to the right of  $p$ , let  $\odot_{\downarrow}^p(q)$  be the disk whose boundary passes through  $p$  and  $q$ , and its center has the same  $x$ -coordinate as  $p$ .

In the  $j$ th iteration, for  $j = 2, \dots, M-1$ , Let  $x_j = n2^{M-j+1} = x(q_{j-1})/2$ , and let  $y_j < 0$  be the maximum  $y$ -coordinate of a point that lies on the intersection of the vertical line  $x = x_j$  and the disks of  $D_1 \cup \dots \cup D_j$  where

$$D_j = \left\{ \odot_{\downarrow}^{p_i}(q_{j-1}) \mid i = 1, \dots, n \right\},$$

see Figure B.4 for an illustration of  $D_1$ .

Let  $q_j = (x_j, 0.99y_j)$ .

Clearly, the point  $q_j$  lies outside all the disks of  $D_1 \cup \dots \cup D_j$ . The construction now continues to the next value of  $j$ . Let  $P = \{p_1, \dots, p_n, q_2, \dots, q_M\}$ . We have that  $|P| = n + M - 1$ .

The minimum distance between any points in the construction is 1 (i.e.,  $\|p_1 p_2\|$ ). Indeed  $x(q_{M-1}) = 4n$  and thus  $\|q_{M-1} p_1\| \geq 2n$ . The diameter of  $P$  is  $\|p_1 q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$ . As such, the spread of  $P$  is bounded by  $\leq n2^{M+1} = \mathcal{O}(n\Phi)$ .

For any  $i$  and  $j$ , consider the disk  $\odot_{\downarrow}^{p_i}(q_j)$ . This disk does not contain any point of  $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$  since its interior lies below the  $x$ -axis. By construction it does not contain any point  $q_{j+1}, \dots, q_{M-1}$ . This disk potentially contains the points  $q_{j-1}, \dots, q_1$ , but observe that for any index  $k \in \llbracket j-1 \rrbracket$ , we have that

$$\|p_i q_k\| = \sqrt{(i + n2^{M-k+1})^2 + (y(q_j))^2},$$

which implies that  $n2^{M-k+1} \leq \|p_i q_k\| < n(2^{M-k+1} + 2)$ . We thus have that

$$\frac{\|p_i q_k\|}{\|p_i q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since  $j \in \llbracket M-1 \rrbracket$ . Namely, the shortest path in  $G$  between  $p_i$  and  $q_j$ , can not use any of the points  $q_1, \dots, q_{j-1}$ . As such, the graph  $G$  must contain the edge  $p_i q_j$ . This implies that  $|E(G)| \geq n(M-1)$ , which implies the claim.  $\blacktriangleleft$

## B.6 Proof of Lemma 23

**Restatement of Lemma 23.** For any  $n > 0$ , and  $\Phi = \Omega(n)$ , one can compute a set  $P$  of  $n + \mathcal{O}(\log \Phi)$  points, with spread  $\mathcal{O}(\Phi n)$ , and a triangle  $\triangle$ , such that any  $\triangle$ -local  $(3/2)$ -spanner of  $P$  requires  $\Omega(n \log \Phi)$  edges.

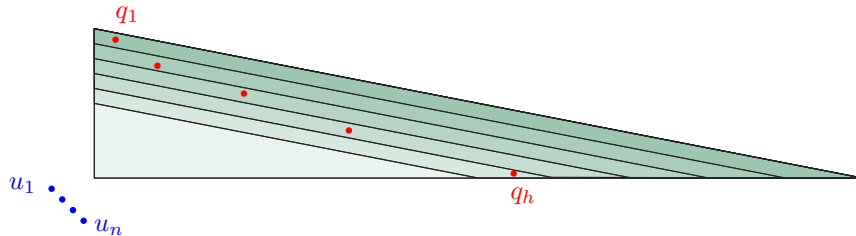


Figure B.5 An Illustration of the construction of Lemma 23.



**Proof.** Let  $h = \lceil \log_2 \Phi \rceil$ . Let  $\triangle$  be the triangle formed by the points  $(0, 0)$ ,  $(0, 1)$  and  $(8\Phi h, 0)$ . The hypotenuse of this triangle lies on the line  $\ell \equiv \frac{1}{8\Phi h}x + y = 1$ , and let  $v = (\frac{1}{8\Phi h}, 1)$  be the vector orthogonal to this line.

For  $i \in \llbracket h \rrbracket$  and  $j \in \llbracket n \rrbracket$ , let

$$q_i = (2^{i+1}, 1 - i/h) \quad \text{and} \quad u_j = (\frac{j}{n} - 1, -\frac{j}{n}),$$

and let  $P = \{q_1, \dots, q_h, u_1, \dots, u_n\}$ , see Figure B.5. Observe that  $\text{cp}(P) = \|u_1 u_2\| = \sqrt{2}/n$ , and as such we have that  $\Phi(P) = n \cdot \text{diam}(P)/\sqrt{2} \leq n(4\Phi + 2n) \leq 8\Phi n$ , as  $\Phi \geq n$ . Observe that

$$\langle q_{i+1} - q_i, v \rangle = \langle (2^{i+1}, -\frac{1}{h}), (\frac{1}{8\Phi h}, 1) \rangle \leq \frac{4\Phi}{8\Phi h} - \frac{1}{h} < 0.$$

That is, the points  $q_1, \dots, q_i$  are increasing in distance from  $\ell$ .

Let  $\triangle_{i,j}$  be the homothet of  $\triangle$ , that has its bottom left corner at  $u_j$ , and its hypotenuse passes through  $q_i$ . By the above,  $P(i, j) = \triangle_{i,j} \cap P = \{u_j, q_i, q_{i+1}, \dots, q_h\}$ . Any  $(1 + \varepsilon)$ -spanner for  $P(i, j)$  must contain the edge  $u_j q_i$ . Indeed, we have, for any  $k$ , that  $2^{k+1} \leq \|u_j q_k\| \leq 2^{k+1} + 3$ . As such, any path on a graph induced on  $P(i, j)$  from  $u_j$  to  $q_i$  that uses (say) a midpoint  $q_k$ , for  $k > i$ , must have dilation at least

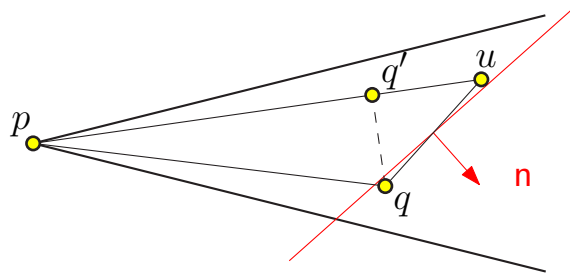
$$\frac{\|u_j q_k\| + \|q_k q_i\|}{\|u_j q_i\|} \geq \frac{2^{k+1} + 2^k}{2^{i+1} + 3} \geq \frac{3 \cdot 2^{i+1}}{(1 + 3/4)2^{i+1}} = \frac{12}{7} > \frac{3}{2}.$$

Thus, any  $\triangle$ -local  $3/2$ -spanner for homothets of  $\triangle$ , must contain the edge  $q_i u_j$ , for any  $i \in \llbracket h \rrbracket$  and  $j \in \llbracket n \rrbracket$ . Thus, such a spanner must have  $\Omega(n \log \Phi)$  edges, as claimed.  $\blacktriangleleft$

## B.7 Proof of Lemma 25

**Restatement of Lemma 25.** Let  $p \in P$ ,  $c \in \mathcal{C}_i$ , and  $u = \text{nn}_i(p, c)$ , and let  $q$  be a point in  $(P \cap (p + c)) \setminus \{p, u\}$ . We have that  $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$  and  $\|qu\| \leq \|pq\|$ .

**Proof.** Consider the triangle  $\triangle pqu$  and denote the angles at  $p, q$ , and  $u$  by  $\angle p, \angle q$ , and  $\angle u$  respectively. Since the angle of  $c$  is smaller than 60 degrees (for an appropriate choice of  $\gamma$ ), we have that  $\|qu\| \leq \max\{\|pu\|, \|pq\|\}$ .



**Figure B.6** The case that  $\|pq\| \leq \|pu\|$  in Lemma 25. The vector used to determine  $\text{nn}_i(p, c)$  is shown in red, and denoted  $n$

Consider the case that  $\|pq\| \leq \|pu\|$ , illustrated in Figure B.6. Observe that  $\angle u \leq \angle q$ . As such  $\angle u \leq \pi/2$ . Furthermore,  $\angle u \geq \alpha \gg \varepsilon\alpha/\gamma = \beta \geq \angle p$ . Similarly,  $\angle q \in [\alpha, \pi - \alpha]$ . By the 1-Lipshitz of  $\sin$ , and as  $\sin x \approx x$ , for small  $x$ , and for  $\gamma$  sufficiently large, we have that

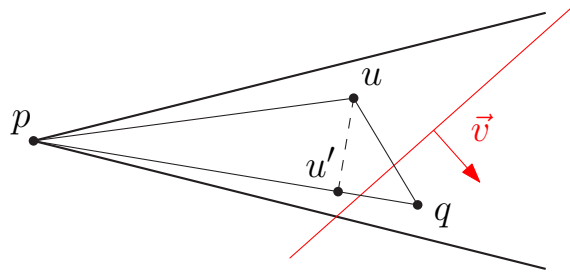
$$\sin(\angle q + \angle p) \in [1 - \varepsilon/4, 1 + \varepsilon/4] \sin \angle q \quad \text{and} \quad \sin \angle p \leq (\varepsilon/4) \sin \angle u.$$

As such, by the law of sines, we have that  $\frac{\|qu\|}{\sin \angle p} = \frac{\|pq\|}{\sin \angle u} = \frac{\|pu\|}{\sin \angle q}$ . This implies that

$$\|pu\| + (1 + \varepsilon) \|qu\| = \left( \frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \right) \|pq\|.$$

Observe, by the above that

$$\frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \leq \frac{\sin \angle q}{\sin (\angle p + \angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq \frac{\sin \angle q}{(1 - \varepsilon/4) \sin (\angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$



■ **Figure B.7** The case that  $\|pq\| > \|pu\|$  in Lemma 25.

The other possibility is that  $\|pq\| > \|pu\|$ , illustrated in Figure B.7. Let  $u'$  be the projection of  $u$  to  $pq$ . Observe that

$$\|uu'\| = \|pu'\| \tan \angle p \leq 2\beta \|pu'\| \leq (\varepsilon/8) \|pu'\|.$$

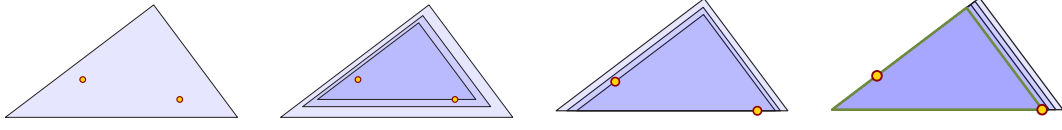
Observe that  $\cos \angle p \geq 1 - (\angle p)^2/2 \geq 1 - \varepsilon^2/8$  as  $\angle p$  is an angle smaller than (say)  $\varepsilon/16$ . As such  $1/\cos \angle p \leq 1 + \varepsilon^2/4$ . This implies that  $\|pu\| \leq \|pu'\|/\cos \angle p \leq (1 + \varepsilon^2/4) \|pu'\|$ . We thus have that

$$\begin{aligned} \tau &= \|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon^2/4) \|pu'\| + (1 + \varepsilon) (\|uu'\| + \|u'q\|) \\ &\leq (1 + \varepsilon^2/4 + (1 + \varepsilon)\varepsilon/8) \|pu'\| + (1 + \varepsilon) \|u'q\| \leq (1 + \varepsilon) \|pq\|. \end{aligned}$$

## B.8 Proof of Lemma 26

**Restatement of Lemma 26.** Let  $\triangle$  be a triangle that contains two points  $p, q$ . Then, there is a homothet  $\triangle' \subseteq \triangle$  of  $\triangle$ , such that one of these points is a vertex of  $\triangle'$ , and the other point lies on a facing edge of  $\triangle'$ .

**Proof.** This follows by the same shrinking argument as Lemma 15, with the addition of a single step. When a homothet  $\triangle'$  with  $p, q \in \partial \triangle'$  is found, if neither point is on a vertex, we “push” the only edge that does not contain one of the points towards the vertex  $v$  opposite of it (this the same mapping described in Lemma 15 with center  $v$ ), until one of the points, say  $p$  lies on the edge.  $p$  now lies on two edges, meaning, at a vertex, while  $q$  lies on the only remaining edge which must be opposite of that vertex. See Figure B.8. ◀

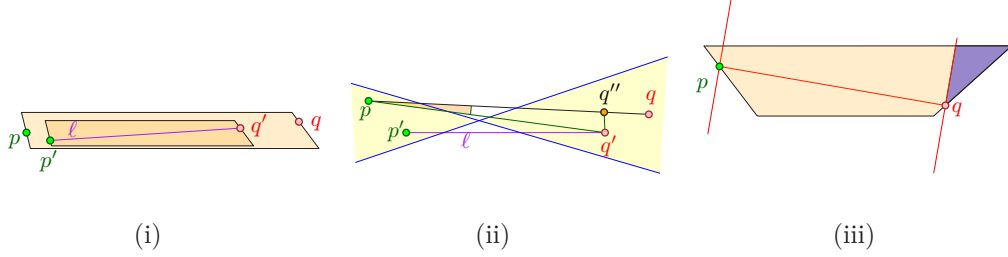


803 **Figure B.8** An illustration of the shrinking process of Lemma 26. The three left subfigures are  
 804 an activation of Lemma 15 for the case that the convex region  $\mathcal{C}$  is a triangle, and the rightmost  
 805 subfigure is the additional final step.

## 806 B.9 Proof of Lemma 29

807 **Restatement of Lemma 29.** Let  $\varepsilon \in (0, 1)$  be some parameter, and  $\vartheta = \varepsilon/16$ . Let  $X, Y$   
 808 be two points sets that are  $\vartheta$ -semi separated and  $\vartheta$ -angularly separated (see Definition 7), and  
 809 let  $T$  be a  $\vartheta$ -narrow trapezoid, with two points  $p \in X$  and  $q \in Y$  lying on the two legs of  $T$ .  
 810 Then, one can compute a homothet  $T' \subseteq T$  of  $T$ , such that:

- 811 (I) There are two points  $p' \in X$  and  $q' \in Y$ , such that  $p'q'$  is an edge of the  $T$ -Delaunay  
 812 triangulation of  $X \cup Y$ .  
 813 (II) We have that  $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$ .



814 **Figure B.9** Illustration of the settings in the proof of Lemma 29. Left: A  $\vartheta$ -narrow trapezoid  
 815 with  $p$  and  $q$  on its legs. Center:  $p$  and  $q$  are  $\vartheta$ -semi separated and  $\vartheta$ -angularly separated. Right:  
 816 The triangle of all the points of the trapezoids that their nearest point on  $pq$  is  $q$ .

817 **Proof.** Let  $\mathcal{D} = \mathcal{D}_T(X \cup Y)$ . Claim 16 implies that  $\mathcal{D} \cap T$  is connected. Thus, there is a  
 818 path in  $\mathcal{D} \cap T$  between  $p$  and  $q$ , and thus, there must be an edge  $p'q'$  along this path with  
 819  $p' \in X$  and  $q' \in Y$ . This implies part (I).

820 Let  $\ell = \|p'q'\|$ . Assume for concreteness that  $\|pp'\| \leq \text{diam}(X) \leq \vartheta d(X, Y) \leq \vartheta \ell \leq \vartheta d$ ,  
 821 where  $d = \text{diam}(T)$ . Let  $q''$  be the closest point on  $pq$  to  $q'$ .

822 We first consider the case that  $q'' \in \text{int}(pq)$ . We have that

$$823 \quad \|pq''\| = \|pq'\| \cos \angle q'pq \geq (\|p'q'\| - \|pp'\|) \cos \angle q'pq \geq (1 - \vartheta)\ell \cdot (1 - \vartheta^2/2) \geq (1 - 2\vartheta)\ell,$$

824 since  $\cos \vartheta \geq 1 - \vartheta^2/2$ , for  $\vartheta < 1/2$ . Similar argumentation implies that  $\|pq''\| \leq (1 + \vartheta)\ell$ .

825 As such, we have

$$826 \quad \|q'q''\| \leq (1 + \vartheta)\ell \sin \angle p'pq' \leq 2\vartheta\ell.$$

827 Thus, we have that

$$828 \quad \|qq'\| \leq \|qq''\| + \|q''q'\| \leq \|pq\| - \|pq''\| + 2\vartheta\ell \leq \|pq\| - (1 - 2\vartheta)\ell + 2\vartheta\ell \leq \|pq\| - \ell.$$

829 Thus, we have that

$$830 \quad (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon)\vartheta\ell + \ell + (1 + \varepsilon)(\|pq\| - \ell) \\
831 \quad = (1 + \varepsilon) \|pq\| + (1 + \varepsilon)\vartheta\ell + \ell - (1 + \varepsilon)\ell \leq (1 + \varepsilon) \|pq\|,$$

832 for  $\vartheta \leq \varepsilon/2$ . Which establish the claim in this case.

833 The case that  $q'' = p$  is impossible, because of the angular separation property. Thus,  
 834 the only remaining possibility is that  $q'' = q$ . This however implies that  $q'$  must be in the  
 835 triangle of all the points of the trapezoids that their nearest point on  $pq$  is  $q$ . The diameter  
 836 of this triangle is bounded by the length of the leg of the trapezoid, which is bounded by  
 837  $\vartheta d$ . Namely, we have  $\|qq'\| \leq \vartheta d$ . Similarly, we have  $(1 - 2\vartheta)d \leq \|pq\| \leq (1 + 2\vartheta)d$ . Since  
 838  $\|pp'\|, \|qq'\| \leq \vartheta d$ , it follows that

$$839 \quad (1 - 4\vartheta)d \leq \ell \leq (1 + 4\vartheta)d.$$

840 As such, for  $\vartheta \leq \varepsilon/8$  and  $\varepsilon \leq 1$ , we have

$$841 \quad (1 + \varepsilon)\|pp'\| + \ell + (1 + \varepsilon)\|q'q\| \leq 4\vartheta d + (1 + 4\vartheta)d = (1 + 8\vartheta)d \leq (1 + \varepsilon)\|pq\|.$$

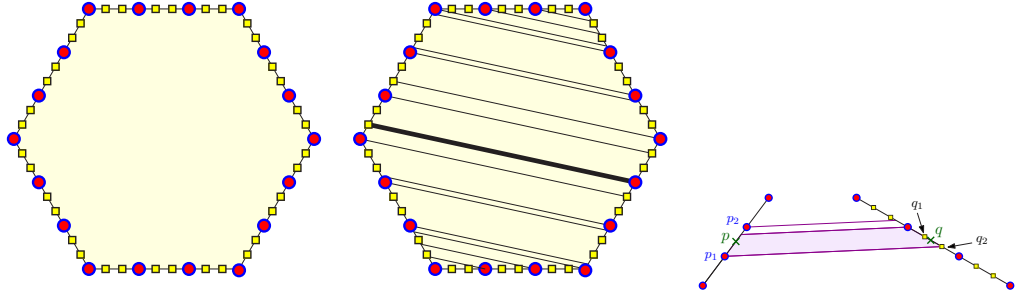
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## 843 B.10 Proof of Lemma 30

844 **Restatement of Lemma 30.** Let  $t$  be a positive integer. Given a  $t$ -nice polygon  $\mathcal{C}$ , and a  
 845 parameter  $\vartheta$ , one can cover it by a set  $\mathcal{T}$  of  $\mathcal{O}(t^4/\vartheta^3)$   $\vartheta$ -narrow trapezoids, such that for any  
 846 two points  $p, q \in \partial\mathcal{C}$  that belong to two edges of  $\mathcal{C}$  that are not adjacent, there exists a narrow  
 847 trapezoid  $T \in \mathcal{T}$ , such that  $p$  and  $q$  are located on two different short legs of  $T$ .

848 **Proof.** We show a somewhat suboptimal but simple construction. A  $t$ -nice polygon has at  
 849 most  $t$  edges. Let  $\psi$  be the sensitivity of  $\mathcal{C}$ , and place a minimum set of points  $P$  on the  
 850 boundary of  $\mathcal{C}$ , which includes all the vertices of  $\mathcal{C}$ , and such that the distance between any  
 851 consecutive pair of points is in the range  $[c_1, 2c_1]$ , where  $c_1 = \vartheta\psi/c_2$ , for some sufficiently  
 852 large constant  $c_2$ . In particular, let  $M = \max_{e \in E(\mathcal{C})} \lceil \|e\|/c_1 \rceil = \mathcal{O}(1/\vartheta)$ .

853 In addition, place  $c_3 t$  equally spaced points between any two consecutive points of  $P$ ,  
 854 where  $c_3$  is a constant to be determined shortly. Let  $Q$  be the set resulting from  $P$  after  
 855 adding all these points.

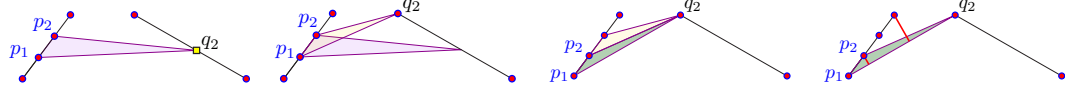


856 ■ **Figure B.10** The points of  $P$  (round), and all the points added to  $P$  in order to create  $Q$  (square).  
 857 On the right, a “vertical” decomposition induced by one of the directions of  $P \times Q$ .

858 We have that  $|P| = \mathcal{O}(t/\vartheta)$  and  $|Q| = \mathcal{O}(t^2/\vartheta)$ . For a direction  $v$ , let  $\mathcal{T}_v$  be the  
 859 decomposition into trapezoids formed by shooting rays from inside  $\mathcal{C}$  in the direction of  $v$  (or  
 860  $-v$ ) from all the points of  $P$ , see Figure B.10. Let  $\mathcal{T}'_v$  be the set resulting from throwing away  
 861 trapezoids with legs that lie on adjacent edges. It is easy to verify that all the trapezoids of  
 862  $\mathcal{T}'_v$  are  $\vartheta$ -narrow. Let  $U$  be the set of all directions induced by pairs of points of  $P \times Q$ , and  
 863 let  $\mathcal{T} = \cup_{u \in U} \mathcal{T}'_u$ . We have that  $|\mathcal{T}| = \mathcal{O}(|P| \cdot |U|) = \mathcal{O}(|P|^2 |Q|) = \mathcal{O}(t^4/\vartheta^3)$ .

Consider any two points  $p, q$  on non-adjacent edges of  $\mathcal{C}$ , and let  $p_1, p_2$  be the two adjacent points of  $P$  such that  $p \in p_1p_2$ . Now, let  $q_1, q_2$  be the adjacent points of  $Q$  such that  $q \in q_1q_2$ . We assume that  $p_1, p_2, q_1, q_2$  are in this clockwise order along the boundary of  $\mathcal{C}$ .

Observe that when we project the interval  $p_1p_2$ , to the line induced by  $q_1q_2$ , in the direction  $\overrightarrow{p_1q_2}$ , the projected interval contains  $q_1q_2$ . The last claim is intuitively obvious, but requires some work to see formally. The minimum height of a triangle involving three vertices of  $\mathcal{C}$  is formed by three consecutive vertices. In the worst case, this is an isosceles triangle with sidelength  $\psi$  and base angle  $\pi/t$ . As such, the height of such a triangle is  $h = \psi \sin(\pi/t) \geq \psi/t$ .



**Figure B.11** The height of the triangle  $\triangle p_1p_2q_2$  is minimized as  $q_2$  and  $p_1$  are moved to vertices of  $\mathcal{C}$ .

The height of the triangle  $\triangle p_1p_2q_2$  is minimized when  $p_1$  or  $p_2$  is a vertex of  $\mathcal{C}$ , and  $q_2$  is at a vertex of  $\mathcal{C}$ , see Figure B.11. Assume, for concreteness, that  $p_1$  is a vertex of  $\mathcal{C}$ , and observe that  $\|p_1p_2\| \geq \|e\|/M$ , where  $e$  is the edge of  $\mathcal{C}$  containing this segment. Using similar triangles, it is straightforward to show that the height of this triangle is at least  $h' = h/M = \Omega(\varepsilon\psi/t)$ . The quantity  $h'$  is a lower bound on the length of the projection of  $p_1p_2$  on the line spanned by  $q_1q_2$ . However,  $\|q_1q_2\| \leq 2c_1/c_3t = \mathcal{O}(\vartheta\psi/c_3t) < h'$ , by picking  $c_3$  to be sufficiently large constant.

This readily implies that the trapezoid induced by the direction  $u = \overrightarrow{p_1q_2}$  in  $\mathcal{T}'_u$  that contains  $p$  on its leg, contains  $q$  on its other leg. ◀