

Local Spanners Revisited

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Abstract

For a set of points $P \subseteq \mathbb{R}^2$, and a family of regions \mathcal{F} , a *local t -spanner* of P , is a sparse graph G over P , that remains a t -spanner even when restricted to a region $r \in \mathcal{F}$. That is, for any region $r \in \mathcal{F}$, the subgraph restricted to r , denoted by $G \cap r$, is still a t -spanner for all the points of $r \cap P$. Here $G \cap r$ is the subgraph of G induced on $P \cap r$.

We present algorithms for the construction of local spanners with respect to several families of convex regions. In particular, we construct a near linear size local spanner for homothets of a convex region. Unfortunately, the number of edges in the resulting graph depends logarithmically on the spread of the input point set. We prove that this dependency can not be removed, thus settling an open problem raised by Abam and Borouny. We also show improved constructions of local spanners for fat triangles, and regular k -gons. In particular, this improves over the known construction for axis parallel squares.

Finally, we study a somewhat weaker notion of local spanner where one allows to shrink the region a “bit”. We show that any spanner is a weak local spanner if one is allowed to shrink the region by some fraction of its diameter. Surprisingly, we show a near linear size construction of a weak spanner for axis-parallel rectangles, where the shrinkage is *multiplicative*.

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1 Introduction

For a set P of points in \mathbb{R}^d , the *Euclidean graph* $G_P = (P, \binom{P}{2})$ of P is an undirected graph. Here, an edge $pq \in E$ is associated with the segment pq , and its weight is the (Euclidean) length of the segment. Let $G = (P, E)$ and $I = (P, E')$ be two graphs over the same set of vertices (usually I is a subgraph of G). Consider two vertices $p, q \in P$, and parameter $t \geq 1$. A path π between p and q in I , is a *t -path*, if the length of π in I is at most $t \cdot d_G(p, q)$, where $d_G(p, q)$ is the length of the shortest path between p and q in G . The graph I is a *t -spanner* of G if there is a t -path in I , for any $p, q \in P$. Thus, for a set of points $P \subseteq \mathbb{R}^d$, a graph G over P is a *t -spanner* if it is a t -spanner of the euclidean graph G_P . There is a lot of work on building geometric spanners, see [NS07] and references there in.

Fault-tolerant spanners

An *\mathcal{F} -fault-tolerant spanner* for $P \subseteq \mathbb{R}^d$, is a graph $G = (P, E)$, such that for any region r (i.e., the “attack”), the graph $G - r$ is a t -spanner of $G_P - r$ (See Definition 1 for a formal definition of this notation). Surprisingly, as shown by Abam *et al.* [AdBFG09], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

2 Local Spanners Revisited

Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size k of vertices and edges has failed. Levcopoulos *et al.* [LNS02] showed the existence of k -vertex/edges fault tolerant spanners for a set of points P in some metric space. Their spanner had $\mathcal{O}(kn \log n)$ edges, and weight, i.e. sum of edge weights, bounded by $f(k) \cdot wt(MST(P))$ for some function f . Lukovszki [Luk99] later achieved a similar construction, improving the number of edges to $\mathcal{O}(kn)$, and was able to prove that the result is asymptotically tight.

Local spanners

Recently, Abam and Borouny [AB21] introduced the notion of local spanners. For a family of regions \mathcal{F} , a graph $G = (P, E)$ is a *local t -spanner* for \mathcal{F} , if for any $r \in \mathcal{F}$, the subgraph of G induced on $P \cap r$ is a t -spanner. Specifically, this induced subgraph $G \cap r$ contains a t -path between any $p, q \in P \cap r$ (note that we keep an edge in the subgraph only if both its endpoints are in r , see Definition 1).

Abam and Borouny [AB21] showed how to construct such spanners for axis-parallel squares and vertical slabs. In this work, we are further extending their results. They also showed how to construct such spanners for disks if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

To appreciate the difficulty in constructing local spanners, observe that unlike regular spanners, the construction has to take into account many different scenarios as far as which points are available to be used in the spanner. As a concrete example, a local spanner for axis-parallel rectangle requires quadratic number of edges, see Figure 1.1.

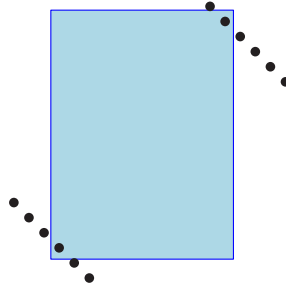


Figure 1.1 For any point in the top diagonal and bottom diagonal, there is a fat axis parallel rectangle that contains only these two points. Thus, a local spanner requires quadratic size in this case.

Namely, regular spanners can rely on using midpoints in their path under the assurance that they are always there. For local spanners this is significantly harder as natural midpoints might “disappear”. Intuitively, a local spanner construction needs to use midpoints that are guaranteed to be present judging only from the source and destination points of the path.

A good jump is hard to find

Most constructions for spanners can be viewed as searching for a way to build a path from the source to the destination by finding a “good” jump, either by finding a way to move locally from the source to a nearby point in the right direction, as done in the θ -graph construction, or alternatively, by finding an edge in the spanner from the neighborhood of the source to the neighborhood of the destination, as done in the spanner constructions using

Region	# edges	Paper	New # edges	Location in paper
Local $(1 + \varepsilon)$ -spanners				
Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AdBFG09]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[AB21]	$\mathcal{O}(\varepsilon^{-3} n \log n)$	Remark 32
Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AB21]		
Disks+Steiner points	$\mathcal{O}_\varepsilon(n)$	[AB21]		
Disks			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 18
			$\Omega(n \log \Phi)$	Lemma 22
Homothets convex shape			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 18
Homothets α -fat triangles			$\mathcal{O}((\alpha\varepsilon)^{-1} n)$	Theorem 28
Homothets triangles			$\Omega(n \log \Phi)$	Lemma 23
δ -weak local $(1 + \varepsilon)$ -spanners				
Bounded convex shape			$\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$	Lemma 12
$(1 - \delta)$ -local $(1 + \varepsilon)$ -spanners				
Axis-parallel rectangles			$\mathcal{O}((\varepsilon^{-2} + \delta^{-2})n \log^2 n)$	Theorem 38

Table 1.1 Known and new results. The notation \mathcal{O}_ε hides polynomial dependency on ε which is not specified in the original work.

well-separated pairs decomposition (WSPD). Usually, one argues inductively that the spanner must have (sufficiently short) paths from the source to the start of the jump, and from the end of the jump to the destination, and then, combining these implies that the resulting new path is short. These ideas guide our constructions as well. However, the availability of specific edges depends on the query region, making the search for a good jump significantly more challenging. The constructions have to guarantee that there are many edges available, and that at least one of them is useful as a jump regardless of the chosen region.

Our results

Almost local spanners

We start by showing that regular geometric spanners are local spanners if one is required provide the spanner guarantee only to shrunken regions. Namely, if G is a $(1 + \varepsilon)$ -spanner of P , then for any convex region \mathcal{C} , the graph $G \cap \mathcal{C}$ is a spanner for $\mathcal{C}' \cap P$, where \mathcal{C}' is the set of all points in \mathcal{C} that are in distance at least $\varepsilon \cdot \text{diam}(\mathcal{C})$ from its boundary.

Homothets

A *homothet* of a convex region \mathcal{C} , is a translated and scaled copy of \mathcal{C} . In Section 3 we present a construction of spanners, which surprisingly, is not only fault-tolerant for all convex regions, but is also a local spanner for homothets of a prespecified convex region. This in particular works for disks, and resolves the aforementioned open problem of Abam and Borouny [AB21]. Our construction is somewhat similar to the original construction of Abam *et al.* [AdBFG09]. For a parameter $\varepsilon > 0$ the construction of a $(1 + \varepsilon)$ -local spanner for homothets takes $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ time, and the resulted spanner is of size $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$,

where Φ is the spread of the input point set P , and $n = |P|$. We also provide a lower bound showing that this logarithmic dependency on Φ cannot be avoided.

The dependency on the spread Φ in the above construction is somewhat disappointing. However, the lower bound constructions, provided in [Section 3.3](#), show that this is unavoidable for disks or homothets of triangles.

Thus, the natural question is what are the cases where one can avoid the “curse of the spread” – that is, cases where one can construct local spanners of size independent of the spread of the input point set.

The basic building block: \mathcal{C} -Delaunay triangulation

A key ingredient in the above construction is the concept of Delaunay triangulations induced by homothets of a convex body. Intuitively, one replaces the unit disk (of the standard L_2 -norm) by the provided convex region. It is well known ([\[CDI85\]](#)) that such diagrams exist, have linear complexity in the plane, and can be computed quickly. In [Section 3.1](#) we review these results, and restate the well-known property that the \mathcal{C} -Delaunay triangulation is connected when restricted to a homothet of \mathcal{C} . By computing these triangulations for carefully chosen subsets of the input point set, we get the results stated above.

Specifically, we use well-separated and semi-separated decompositions to compute these subsets.

Fat triangles

In [Section 3.4](#) we give a construction of local spanners for the family \mathcal{F} of homothets of a given triangle \triangle , and get a spanner of size $\mathcal{O}((\alpha\epsilon)^{-1}n)$ in $\mathcal{O}((\alpha\epsilon)^{-1}n \log n)$ time, where α is the smallest angle in \triangle . This construction is a careful adaptation of the θ -graph spanner construction to the given triangle, and it is significantly more technically challenging than the original construction.

k -regular polygons

It seems natural that if one can handle fat triangles, then homothets of k -regular polygons should readily follow by a simple decomposition of the polygon into fat triangles. Maybe surprisingly, this is not the case – a critical configuration might involve two points that are on the interior of two non-adjacent edges of a homothet of the input polygon. We overcome this by first showing that sufficiently narrow trapezoids, provide us with a good jump somewhere inside the trapezoid, assuming one compute the Delaunay triangulation induced by the trapezoid, and that the source and destination lie on the two legs of the trapezoid. Next, we show that such a polygon can be covered by a small number of narrow trapezoids and fat triangles. By building appropriate graphs for each trapezoid/triangle in the collection, we get a spanner for homothets of the given k -regular polygon, with size that has no dependency on the spread. Of course, the size does depend on k . See [Section 3.5](#) for details, and [Theorem 31](#) for the precise result.

Quadrant separated pair decomposition (QSPD)

In [Appendix A.1](#), we describe a novel pair-decomposition. Specifically, the QSPD breaks the input point set P into pairs, such that for any pair $\{X, Y\}$ we have the property that there is a translated axis system such that X and Y belong to two antipodal quadrants. In d dimensions there is such a decomposition with $\mathcal{O}(n \log^{d-1} n)$ pairs, and total weight

145 $\mathcal{O}(n \log^d n)$. A somewhat similar idea was used by Abam and Borouny [AB21] for the $d = 1$
 146 case. We believe this decomposition might be useful and is of independent interest.

147 Multiplicative weak local spanner for rectangles

148 In [Appendix A.2](#), we use QSPDs to construct a weak local spanner for axis parallel rectangles.
 149 Here, the constructed graph G over P , has the property that for any axis-parallel rectangle
 150 R , the graph $G \cap R$ is a $(1 + \varepsilon)$ -spanner for all the points of $((1 - \varepsilon)R) \cap P$, where $(1 - \varepsilon)R$ is
 151 the scaling of the rectangle by a factor of $1 - \varepsilon$ around its center. Importantly, this works for
 152 narrow rectangles where this form of multiplicative shrinking is still meaningful (unlike the
 153 diameter based shrinking mentioned above). Contrast this with the lower bound (illustrated
 154 in [Figure 1.1](#)) of $\Omega(n^2)$ on the size of local spanner if one does not shrink the rectangles. See
 155 [Theorem 38](#) for details of the precise result.

156 See [Table 1.1](#) for a summary of known results and comparisons to the results of this
 157 paper.

158 2 Preliminaries

159 Residual graphs

160 ► **Definition 1.** Let \mathcal{F} be a family of regions in the plane. For a fault region $r \in \mathcal{F}$ and a
 161 geometric graph G on a point set P , let $G - r$ be the residual graph after removing from it
 162 all the points of P in r . Similarly, let $G \cap r$ denote the graph restricted to r . Formally, let

$$163 \quad G - r = (P \setminus r, \{uv \in E \mid uv \cap \text{int}(r) = \emptyset\}) \quad \text{and} \quad G \cap r = (P \cap r, \{uv \in E \mid uv \subseteq r\}).$$

164 where $\text{int}(r)$ denotes the interior of r .

165 2.1 On various pair decompositions

166 For sets X, Y , let $X \otimes Y = \{\{x, y\} \mid x \in X, y \in Y, x \neq y\}$ be the set of all the (unordered)
 167 pairs of points formed by the sets X and Y .

168 ► **Definition 2** (Pair decomposition). For a point set P , a **pair decomposition** of P is a
 169 set of pairs

$$170 \quad \mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\},$$

171 such that (I) $X_i, Y_i \subseteq P$ for every i , (II) $X_i \cap Y_i = \emptyset$ for every i , and (III) $\bigcup_{i=1}^s X_i \otimes Y_i = P \otimes P$.
 172 Its **weight** is $\omega(\mathcal{W}) = \sum_{i=1}^s (|X_i| + |Y_i|)$.

173 The **closest pair** distance of a set of points $P \subseteq \mathbb{R}^d$, is $\text{cp}(P) = \min_{p, q \in P, p \neq q} \|pq\|$. The
 174 **diameter** of P is $\text{diam}(P) = \max_{p, q \in P} \|pq\|$. The **spread** of P is $\Phi(P) = \text{diam}(P) / \text{cp}(P)$, which
 175 is the ratio between the diameter and closest pair distance. While in general the weight of a
 176 WSPD (defined below) can be quadratic, if the spread is bounded, the weight is near linear.
 177 For $X, Y \subseteq \mathbb{R}^d$, let $\text{d}(X, Y) = \min_{p \in X, q \in Y} \|pq\|$ be the **distance** between the two sets.

178 ► **Definition 3.** Two sets $X, Y \subseteq \mathbb{R}^d$ are

$$179 \quad \begin{array}{ll} 1/\varepsilon\text{-well-separated} & \text{if} \quad \max(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot \text{d}(X, Y), \\ \text{and } 1/\varepsilon\text{-semi-separated} & \text{if} \quad \min(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot \text{d}(X, Y). \end{array}$$

For a point set P , a **well-separated pair decomposition (WSPD)** of P with parameter $1/\varepsilon$ is a pair decomposition of P with a set of pairs $\mathcal{W} = \{\{B_1, C_1\}, \dots, \{B_s, C_s\}\}$, such that for all i , the sets B_i and C_i are $(1/\varepsilon)$ -separated. The notion of $1/\varepsilon$ -SSPD (a.k.a **semi-separated pairs decomposition**) is defined analogously.

► **Lemma 4** ([AH12]). Let P be a set of n points in \mathbb{R}^d , with spread $\Phi = \Phi(P)$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$ -WSPD \mathcal{W} for P of total weight $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}(\varepsilon^{-d} \log \Phi)$ pairs.

► **Theorem 5** ([AH12, Har11]). Let P be a set of n points in \mathbb{R}^d , and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$ -SSPD for P of total weight $\mathcal{O}(n\varepsilon^{-d} \log n)$. The number of pairs in the SSPD is $\mathcal{O}(n\varepsilon^{-d})$, and the computation time is $\mathcal{O}(n\varepsilon^{-d} \log n)$.

► **Lemma 6** (Proof in Appendix B.1). Given an α -SSPD \mathcal{W} of a set P of n points in \mathbb{R}^d and a parameter $\beta \geq 2$, one can refine \mathcal{W} into an $\alpha\beta$ -SSPD \mathcal{W}' , such that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$.

► **Definition 7.** An ε -double-wedge is a region between two lines, where the angle between the two lines is at most ε .

Two point sets X and Y that each lie in their own cone of a shared ε -double-wedge are ε -angularly separated.

► **Lemma 8** (Proof in Appendix B.2). Given a $(1/\varepsilon)$ -SSPD \mathcal{W} of n points in the plane, one can refine \mathcal{W} into a $(1/\varepsilon)$ -SSPD \mathcal{W}' , such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε -double-wedge \times_Ξ , such that X and Y are contained in the two different faces of the double wedge \times_Ξ . We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time is proportional to the weight of \mathcal{W}' .

► **Corollary 9.** Let P be a set of n points in the plane, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$ -SSPD for P such that every pair is ε -angularly separated. The total weight of the SSPD is $\mathcal{O}(n\varepsilon^{-3} \log n)$, the number of pairs in the SSPD is $\mathcal{O}(n\varepsilon^{-3})$, and the computation time is $\mathcal{O}(n\varepsilon^{-3} \log n)$.

2.2 Weak local spanners for fat convex regions

► **Definition 10.** Given a convex region C , let

$$C_{\Box\delta} = \{p \in C \mid d(p, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)\}.$$

Formally, $C_{\Box\delta}$ is the Minkowski difference of C with a disk of radius $\delta \cdot \text{diam}(C)$.

► **Definition 11.** Consider a (bounded) set C in the plane. Let $r_{\text{in}}(C)$ be the radius of the largest disk contained inside C . Similarly, $R_{\text{out}}(C)$ is the smallest radius of a disk containing C .

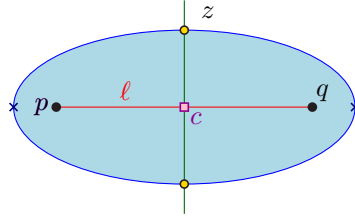
The **aspect ratio** of a region C in the plane is $\text{ar}(C) = R_{\text{out}}(C)/r_{\text{in}}(C)$. Given a family \mathcal{F} of regions in the plane, its aspect ratio is $\text{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \text{ar}(C)$.

Note, that if a convex region C has bounded aspect ratio, then $C_{\Box\delta}$ is similar to the result of scaling C by a factor of $1 - \mathcal{O}(\delta)$. On the other hand, if C is long and skinny then this region is much smaller. Specifically, if C has width smaller than $2\delta \cdot \text{diam}(C)$, then $C_{\Box\delta}$ is empty.

221 ► **Lemma 12.** *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0, 1)$. One*
 222 *can construct a graph G over P , in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$*
 223 *edges, such that for any (bounded) convex C in the plane, we have that for any two points*
 224 *$p, q \in P \cap C_{\square\delta}$ the graph $C \cap P$ has $(1 + \varepsilon)$ -path between p and q .*

225 **Proof.** Let $\vartheta = \min(\varepsilon, \delta^2)$. Construct, in $\mathcal{O}(\vartheta^{-1}n \log n)$ time, a standard $(1 + \vartheta)$ -spanner G
 226 for P using $\mathcal{O}(\vartheta^{-1}n)$ edges [AMS99].

227 So, consider any body $C \in \mathcal{F}$, and any two vertices $p, q \in P \cap C'$, where $C' = C_{\square\delta}$. Let
 228 $\ell = \|pq\|$, let π be the shortest path between p and q in G , and let \mathcal{E} be the locus of all
 229 points u , such that $\|pu\| + \|uq\| \leq (1 + \vartheta)\ell$. The region \mathcal{E} is an ellipse that contains π . The
 230 furthest point from the segment pq in this ellipse is realized by the co-vertex of the ellipse.
 231 Formally, it is one of the two intersection points of the boundary of the ellipse with the line
 232 orthogonal to pq that passes through the middle point c of this segment, see Figure 2.1. Let
 233 z be one of these points.



234 ■ **Figure 2.1** An illustration of the settings in the proof of Lemma 12 with \mathcal{E} shown in blue.

235 We have that $\|pz\| = (1 + \vartheta)\ell/2$. Setting $h = \|zc\|$, we have that

$$236 \quad h = \sqrt{\|pz\|^2 - \|pc\|^2} = \frac{\ell}{2} \sqrt{(1 + \vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2 + \vartheta)}}{2} \ell \leq \sqrt{\vartheta} \ell \leq \sqrt{\vartheta} \cdot \text{diam}(C).$$

237 as $\ell \leq \text{diam}(C') \leq \text{diam}(C)$.

238 For any point $x \in C'$, we have that $d(x, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)$. As such, to ensure that
 239 $\pi \subseteq \mathcal{E} \subseteq C$, we need that $\delta \cdot \text{diam}(C) \geq h$, which holds if $\delta \cdot \text{diam}(C) \geq \sqrt{\vartheta} \cdot \text{diam}(C)$. This
 240 in turn holds if $\vartheta \leq \delta^2$. Namely, we have the desired properties if $\vartheta = \min(\varepsilon, \delta^2)$. ◀

241 3 Local spanners of homothets of convex region

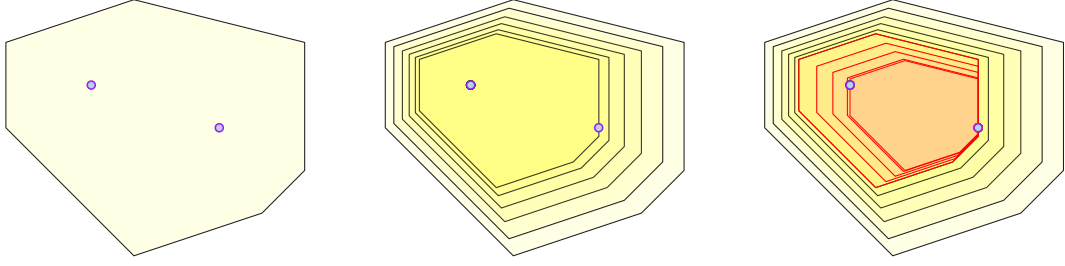
242 Let \mathcal{C} be a bounded convex and closed region in the plane (e.g., a disk). A *homothet* of \mathcal{C} is
 243 a scaled and translated copy of \mathcal{C} . A point set P is in *general position* with respect to \mathcal{C} , if
 244 no four points of P lie on the boundary of a homothet of \mathcal{C} , and no three points are colinear.

245 A graph $G = (P, E)$ is a \mathcal{C} -local t -spanner for P if for any homothet \mathcal{r} of \mathcal{C} we have that
 246 $G \cap \mathcal{r}$ is a t -spanner of $G_P \cap \mathcal{r}$.

247 3.1 Delaunay triangulation for homothets

248 ► **Definition 13** ([CDI85]). *Given \mathcal{C} as above, and a point set P in general position with*
 249 *respect to \mathcal{C} , the \mathcal{C} -Delaunay triangulation of P , denoted $\mathcal{D}_{\mathcal{C}}(P)$, is the graph formed by*
 250 *edges between any two points $p, q \in P$ such that there exist a homothet of \mathcal{C} that contains*
 251 *only p and q and no other point of P .*

252 ► **Theorem 14** ([CDI85]). *For any convex shape \mathcal{C} and a set of points P , $\mathcal{D}_{\mathcal{C}}(P)$ can be*
 253 *computed in $\mathcal{O}(n \log n)$ time. Furthermore, the triangulation $\mathcal{D}_{\mathcal{C}}(P)$ has $\mathcal{O}(n)$ edges, vertices,*
 254 *and faces.*



255 ■ **Figure 3.1** Shrinking of homothets so two points becomes on the boundary of the homothet.

256 ► **Lemma 15.** *Let \mathcal{C} be a convex bounded body, and let P be a set of points in general position with respect to \mathcal{C} . Then, if C is a homothet of \mathcal{C} that contains two points $p, q \in C \cap P$, then there exists a homothet $C' \subseteq C$ of \mathcal{C} such that $p, q \in \partial C'$.*

259 **Proof.** The idea is to apply a shrinking process of C , as illustrated in Figure 3.1. Consider the mapping $f_{\beta, v} : x \mapsto \beta(x - v) + v$. It is a scaling of the plane around v by a factor of β . Let β' be the minimum value of β such that $C_1 = f_{\beta, p}(C)$ contains q (i.e., we shrink C around p till q becomes a boundary point). Next, shrink C' around q , till p becomes a boundary point – formally, let β'' be the minimum value of β such that $C' = f_{\beta, q}(C_1)$ contains p . Since $C' \subseteq C_1 \subseteq C$, and $p, q \in \partial C'$, the claim follows. ◀

265 The following standard claim, usually stated about the standard Delaunay triangulations, also holds for homothets.

267 ► **Claim 16 (Proof in Appendix B.3).** Let \mathcal{C} be a bounded close convex shape. Given a set of points $P \subseteq \mathbb{R}^2$ in general position with respect to \mathcal{C} , let $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$ be the \mathcal{C} -Delaunay triangulation of P . For any homothet C of \mathcal{C} , we have that $\mathcal{D} \cap C$ is connected.

270 3.2 The generic construction

271 The input is a set P of n points in the plane (in general position) with spread $\Phi = \Phi(P)$, and a parameter $\varepsilon \in (0, 1)$. We have a convex body \mathcal{C} that defines the “unit” ball. The task is to construct a local spanner for any homothet of \mathcal{C} .

274 The algorithm computes a $(1/\vartheta)$ -WSPD \mathcal{W} of P using the algorithm of Lemma 4, where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the \mathcal{C} -Delaunay triangulation $\mathcal{D}_{\Xi} = \mathcal{D}_{\mathcal{C}}(X \cup Y)$. The algorithm adds all the edges in $\mathcal{D}_{\Xi} \cap (X \otimes Y)$ to the computed graph G .

278 3.2.1 Analysis

279 Size

280 For each pair $\Xi = \{X, Y\}$ in the WSPD, its \mathcal{C} -Delaunay triangulation contains at most $\mathcal{O}(|X| + |Y|)$ edges. As such, the number of edges in the resulting graph is bounded by $\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}(|X| + |Y|) = \mathcal{O}(\omega(\mathcal{W})) = \mathcal{O}\left(\frac{n \log \Phi}{\vartheta^2}\right)$, by Lemma 4.

283 Construction time

284 The construction time is bounded by $\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}((|X| + |Y|) \log(|X| + |Y|)) = \mathcal{O}(\omega(\mathcal{W}) \log n) = \mathcal{O}\left(\frac{n \log \Phi \log n}{\vartheta^2}\right)$.

286 ► **Lemma 17** (Local spanner property). *For $P, \mathcal{C}, \varepsilon$ as above, let G be the graph constructed*
 287 *above for the point set P . Then, for any homothet C of \mathcal{C} and any two points $x, y \in P \cap C$, we*
 288 *have that $G \cap C$ has a $(1 + \varepsilon)$ -path between x and y . That is, G is a \mathcal{C} -local $(1 + \varepsilon)$ -spanner.*

289 **Proof.** Fix a homothet C of \mathcal{C} , and consider two points $p, q \in P \cap C$. The proof is by
 290 induction on the distance between p and q (or more precisely, the rank of their distance
 291 among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and
 292 $y \in Y$.

293 If $xy \in \mathcal{D}_\Xi$ then the claim holds, so assume this is not the case. By the connectivity of
 294 $\mathcal{D}_\Xi \cap C$, see [Claim 16](#), there must be points $x' \in X \cap C$, $y' \in Y \cap C$, such that $x'y' \in E(\mathcal{D}_\Xi)$.
 295 As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property,
 296 we have that

$$297 \quad \max(\text{diam}(X), \text{diam}(Y)) \leq \vartheta \, d(X, Y) \leq \vartheta \ell,$$

298 where $\ell = \|xy\|$. In particular, $\|x'x\| \leq \vartheta \ell$ and $\|y'y\| \leq \vartheta \ell$. As such, by induction, we have
 299 $d_G(x, x') \leq (1 + \varepsilon) \|xx'\| \leq (1 + \varepsilon) \vartheta \ell$ and $d_G(y, y') \leq (1 + \varepsilon) \|yy'\| \leq (1 + \varepsilon) \vartheta \ell$. Furthermore,
 300 $\|x'y'\| \leq (1 + 2\vartheta) \ell$. As $x'y' \in E(G)$, we have

$$301 \quad d_G(x, y) \leq d_G(x, x') + \|x'y'\| + d_G(y', y) \leq (1 + \varepsilon) \vartheta \ell + (1 + 2\vartheta) \ell + (1 + \varepsilon) \vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta) \ell$$

$$302 \quad = (1 + 6\vartheta) \ell \leq (1 + \varepsilon) \|xy\|,$$

303 if $\vartheta \leq \varepsilon/6$. ◀

304 The result

305 ► **Theorem 18.** *Let \mathcal{C} be a bounded convex shape in the plane, let P be a given set of n*
 306 *points in the plane (in general position), and let $\varepsilon \in (0, 1/2)$ be a parameter. The above*
 307 *algorithm constructs a \mathcal{C} -local $(1 + \varepsilon)$ -spanner G . The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and*
 308 *the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any homothet C of \mathcal{C} , and any*
 309 *two points $p, q \in P \cap C$, we have a $(1 + \varepsilon)$ -path in $G \cap C$.*

310 3.2.2 Applications and comments

311 The following defines a “visibility” graph when we are restricted to a region R , where two
 312 points are visible if there is a witness homothet contained in R having both points on its
 313 boundary.

314 ► **Definition 19.** *Let \mathcal{C} be a bounded convex shape in the plane. Given a region R in the*
 315 *plane and a point set P , consider two points $p, q \in P$. The edge pq is safe in R if there is a*
 316 *homothet C of \mathcal{C} , such that $p, q \in C \subseteq R$. The **safe graph** for P and R , denoted by $\mathcal{G}(P, R)$,*
 317 *is the graph formed by all the safe edges in P for R . Note, that this graph might have a*
 318 *quadratic number of edges in the worst case.*

319 Observe that $\mathcal{G}(P, \mathbb{R}^2)$ is a clique. Surprisingly, the spanner graph described above, when
 320 restricted to region R , is a spanner for $\mathcal{G}(P, R)$.

321 ► **Corollary 20** (Proof in [Appendix B.4](#)). *Let \mathcal{C} be a bounded convex body, P be a set of n*
 322 *points in the plane, $\varepsilon \in (0, 1)$ be a parameter, and let G be a \mathcal{C} -local $(1 + \varepsilon)$ -spanner of P .*

323 Consider a region R in the plane, and the associated graph $H = \mathcal{G}(P, R)$, we have that
 324 $G \cap R$ is a $(1 + \varepsilon)$ -spanner for H . Formally, for any two points $p, q \in P \cap R$, we have that
 325 $d_{G \cap R}(p, q) \leq (1 + \varepsilon) d_H(p, q)$.

326 In particular, for any convex region D , the graph $G - D$ is a $(1 + \varepsilon)$ -spanner for
 327 $\mathcal{G}(P, \mathbb{R}^2) - D$.

► Remark 21. The above implies that local spanners for homothets are also robust to convex region faults.

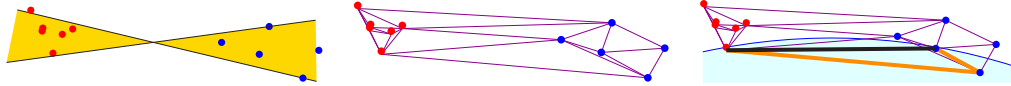
3.3 Lower bounds

3.3.1 A lower bound for local spanner for disks

The result of Theorem 18 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). Next, we show a lower bound proving that this dependency is unavoidable, even in the case of disks.

Some intuition

A natural way to attempt a spread-independent construction is to try and emulate the construction of Abam *et al.* [AdBFG09] and use a SSPD instead of a WSPD, as the total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε -separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are of different sides of the double wedge. The problem is that for the local disk \circ , it might be that the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \circ$ is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 3.2.



■ **Figure 3.2** A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

► **Lemma 22.** [Proof in Appendix B.5] For $\varepsilon = 1/4$, and parameters n and $\Phi \geq 1$, there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $\mathcal{O}(n\Phi)$, such that any local $(1 + \varepsilon)$ -spanner of P for disks, must have $\Omega(n \log \Phi)$ edges.

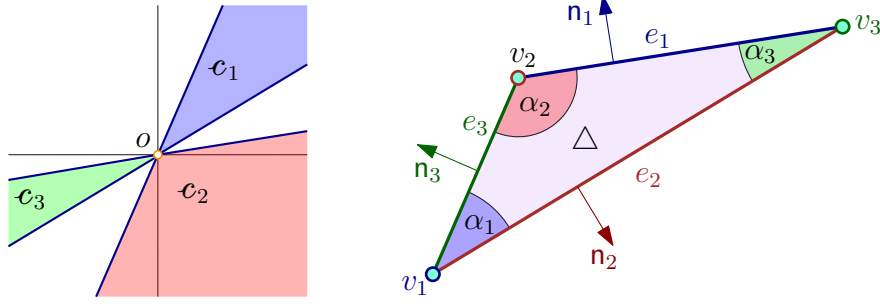
3.3.2 A lower bound for triangles

► **Lemma 23.** [Proof in Appendix B.6] For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a set P of $n + \mathcal{O}(\log \Phi)$ points, with spread $\mathcal{O}(\Phi n)$, and a triangle \triangle , such that any \triangle -local $(3/2)$ -spanner of P requires $\Omega(n \log \Phi)$ edges.

3.4 Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex shape are costly, if we are given a triangle \triangle with the single constraint that \triangle is not too “thin”, then one can construct a \triangle -local t -spanner with a number of edges that does not depend on the spread of the points. See Figure B.5 for an illustration of a construction showing that dependency if “thin” triangles are allowed.

► **Definition 24.** A triangle \triangle is α -fat if the smallest angle in \triangle is at least α .



360 **Figure 3.3** For the triangle Δ with angles α_1, α_2 , and α_3 we create the cones c_1, c_2 , and c_3 .

361 3.4.1 Construction

362 The input is a set P of n points in the plane, an α -fat triangle Δ , and an approximation
 363 parameter $\varepsilon \in (0, 1)$. Let v_i denote the i th vertex of Δ , α_i be the adjacent angle, and let
 364 e_i denote the opposing edge, for $i \in [3]$. Let $c_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$ denote the
 365 cone with an apex at the origin induced by the i th vertex of Δ . Let n_i be the outer normal of
 366 Δ orthogonal to e_i . See Figure 3.3 for an illustration. Let \mathcal{C}_i be a minimum size partition of
 367 c_i into cones each with angle in the range $[\beta/2, \beta]$, where $\beta = \varepsilon\alpha/\gamma$, and $\gamma > 1$ is a constant
 368 to be determined shortly. For each point $p \in P$, and a cone $c \in \mathcal{C}_i$, let $\text{nn}_i(p, c)$ be the first
 369 point in $(P - p) \cap (p + c)$ ordered by the direction n_i (it is the “nearest-neighbor” to p in
 370 $p + c$ with respect to the direction n_i).

371 The construction

372 Let G be the graph over P formed by connecting every point $p \in P$ to $\text{nn}_i(p, c)$, for all
 373 $i \in [3]$ and $c \in \mathcal{C}_i$.

374 3.4.2 Analysis

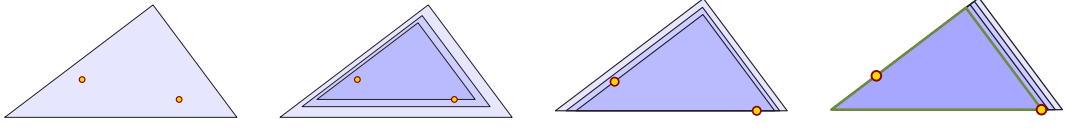
375 **► Lemma 25.** [Proof in [Appendix B.7](#)] Let $p \in P$, $c \in \mathcal{C}_i$, and $u = \text{nn}_i(p, c)$, and let q
 376 be a point in $(P \cap (p + c)) \setminus \{p, u\}$. We have that $\|pu\| + (1 + \varepsilon)\|qu\| \leq (1 + \varepsilon)\|pq\|$ and
 377 $\|qu\| \leq \|pq\|$.

378 **► Lemma 26.** Let Δ be a triangle that contains two points p, q . Then, there is a homothet
 379 $\Delta' \subseteq \Delta$ of Δ , such that one of these points is a vertex of Δ' , and the other point lies on a
 380 facing edge of Δ' .

381 **Proof.** This follows by the same shrinking argument as [Lemma 15](#), with the addition of a
 382 single step. When a homothet Δ' with $p, q \in \partial\Delta'$ is found, if neither point is on a vertex, we
 383 “push” the only edge that does not contain one of the points towards the vertex v opposite of
 384 it (this the same mapping described in [Lemma 15](#) with center v), until one of the points lies
 385 on the edge. That point now lies on two edges. See [Figure 3.4](#). ◀

389 Local spanner property

390 **► Lemma 27.** Let Δ' be a homothet of Δ . For any two points $p, q \in P \cap \Delta'$, we have a
 391 $(1 + \varepsilon)$ -path in $G' = G \cap \Delta'$.



386 **Figure 3.4** An illustration of the shrinking process of Lemma 26. The three left subfigures are
 387 an activation of Lemma 15 for the case that the convex region \mathcal{C} is a triangle, and the rightmost
 388 subfigure is the additional final step.

392 **Proof.** Consider the closest pair $p, q \in P \cap \Delta$. They must be connected directly in G' , as
 393 otherwise there is a point $u \in P' = P \cap \Delta'$ in the cone containing the segment pq , such that
 394 $pu \in E(G')$. But then, by Lemma 25, we have $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$, which
 395 implies that either pu or qu are the closest pair, which is a contradiction.

396 For any other pair $p, q \in P'$ we have from Lemma 26 that there exists a homothet
 397 $\Delta'' \subseteq \Delta'$ with one of the two points, say p , at a vertex, and the other on the opposite edge.
 398 We therefore have a cone c with apex at p such that $q \in c \cap \Delta''$. If pq is an edge in G
 399 then we are done. Otherwise, we have a vertex $u \in c$ such that pu is an edge in G , and by
 400 Lemma 25 we have $\|qu\| \leq \|pq\|$, which, by induction, means that there exists a $(1 + \varepsilon)$ path
 401 between u and q in G . Lemma 25 now implies that $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$. Thus,
 402 there is a $(1 + \varepsilon)$ path between p and q in G' , as stated. \blacktriangleleft

403 Size and running time

404 **► Theorem 28.** Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be an approximation
 405 parameter. The above algorithm computes a local $(1 + \varepsilon)$ -spanner G for homothets of an α -fat
 406 triangle Δ . The construction time is $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$, and the spanner G has $\mathcal{O}((\alpha\varepsilon)^{-1}n)$
 407 edges.

408 **Proof.** The local-spanning property is proven in Lemma 27, and we are only left with
 409 bounding the size and the running time of the algorithm. The bound on the size is immediate
 410 from the construction, as every point p is the apex of $\mathcal{O}(\frac{2\pi}{\varepsilon\alpha})$ cones, each giving rise to a
 411 single edge incident to p . The construction time is bounded by the construction time for a
 412 θ -graph with cone size $\alpha\varepsilon$, which is $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ ([Cla87]). \blacktriangleleft

413 3.5 A local spanner for nice polygons

414 3.5.1 A good jump for narrow trapezoids

415 As a reminder, a trapezoid is a quadrilateral with two parallel edges, known as its *bases*.
 416 The other two edges are its legs. For $\varepsilon \in (0, 1/4)$, a trapezoid T is ε -*narrow* if the length of
 417 each of its legs is at most $\varepsilon \cdot \text{diam}(T)$.

418 **► Lemma 29** (Proof in Appendix B.8). Let $\varepsilon \in (0, 1)$ be some parameter, and $\vartheta = \varepsilon/16$. Let
 419 X, Y be two points sets that are ϑ -semi separated and ϑ -angularly separated (see Definition 7),
 420 and let T be a ϑ -narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on the two legs of
 421 T . Then, one can compute a homothet $T' \subseteq T$ of T , such that:

422 (I) There are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay
 423 triangulation of $X \cup Y$.

424 (II) We have that $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$.

3.5.2 Breaking a nice polygon into narrow trapezoids

For a convex polygon \mathcal{C} , its *sensitivity*, denoted by $\text{sen}(\mathcal{C})$, is the minimum distance between any two non-adjacent edges (this quantity is no bigger than the length of the shortest edge in the polygon). A convex polygon \mathcal{C} is *t -nice*, if the outer angle at any vertex of the polygon is at least $2\pi/t$, and the length of the longest edge of \mathcal{C} is $\mathcal{O}(\text{sen}(\mathcal{C}))$. As an example, a k -regular polygon is k -nice.

► **Lemma 30** (Proof in [Appendix B.9](#)). *Let t be a positive integer. Given a t -nice polygon \mathcal{C} , and a parameter ϑ , one can cover it by a set \mathcal{T} of $\mathcal{O}(t^4/\vartheta^3)$ ϑ -narrow trapezoids, such that for any two points $p, q \in \partial\mathcal{C}$ that belong to two edges of \mathcal{C} that are not adjacent, there exists a narrow trapezoid $T \in \mathcal{T}$, such that p and q are located on two different short legs of T .*

3.5.3 Constructing the local spanner for nice polygons

► **Theorem 31**. *Let \mathcal{C} be a k -nice convex polygon, P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter. Then one can construct a \mathcal{C} -local $(1 + \varepsilon)$ -spanner of P . The construction time is $\mathcal{O}((k^4/\varepsilon^6)n \log^2 n)$, and the resulting graph has $\mathcal{O}((k^4/\varepsilon^6)n \log n)$ edges. In particular these bounds hold if \mathcal{C} is a k -regular gon.*

Proof. Let $\vartheta = \varepsilon/c_4$, for c_4 sufficiently large constant. We construct Δ , a family of triangles induced by a vertex of \mathcal{C} , and an non-adjacent edge of \mathcal{C} . This family has $\mathcal{O}(k^2)$ triangles. Each such triangle is $\Omega(1/k)$ -fat, and for each such triangle we construct the $(1 + \vartheta)$ -spanner of [Theorem 28](#) for P . Next, we cover \mathcal{C} by a set \mathcal{T} of $k' = \mathcal{O}(k^4/\vartheta^3)$ ϑ -narrow trapezoids using [Lemma 30](#).

We compute an ϑ -angular $(1/\vartheta)$ -SSPD \mathcal{W} decomposition of P using [Corollary 9](#) – the total weight of the decomposition is $w = \mathcal{O}(n\vartheta^{-3} \log n)$. For each pair $\{X, Y\} \in \mathcal{W}$, and each trapezoid $T \in \mathcal{T}$, we compute the T -Delaunay triangulation of $X \cup Y$.

Let G denote the union of all these graphs. We claim that it is the desired spanner. The construction time is

$$\mathcal{O}((k^3/\vartheta)n \log n + k'w \log n) = \mathcal{O}\left(\frac{k^3}{\vartheta}n \log n + \frac{k^4}{\vartheta^3} \cdot \frac{n}{\vartheta^3} \log n \cdot \log n\right) = \mathcal{O}\left(\frac{k^4}{\vartheta^6}n \log^2 n\right).$$

And the resulting graph has $\mathcal{O}((k^4/\vartheta^6)n \log n)$ edges.

As for correctness, consider a homothet \mathcal{C}' of \mathcal{C} that contains two points $p, q \in P$. By [Lemma 15](#), there is a homothet $\mathcal{C}'' \subseteq \mathcal{C}'$ of \mathcal{C} such that $p, q \in \partial\mathcal{C}''$. There are two possibilities:

- The point p is on a vertex of \mathcal{C}'' and q is on an edge. In this case, the vertex and the edge induce a fat triangle, that is a homothet of a triangle $\Delta \in \Delta$. Since the graph G contains a Δ -local $(1 + \varepsilon)$ -spanner for P , it follows readily that G is a $(1 + \varepsilon)$ -spanner for these points, and the path is strictly inside \mathcal{C}'' .
- The points p and q are on two non-adjacent edges of \mathcal{C}'' . Then, there is an ϑ -narrow trapezoid T' that has p and q on its two legs, and a homothet of T' , denoted by T , is in \mathcal{T} . There is a pair $\{X, Y\} \in \mathcal{W}$ that is $(1/\vartheta)$ -semi separated (and ϑ -angularly separated), such that $p \in X$ and $q \in Y$. By [Lemma 29](#), there are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay triangulation of $X \cup Y$, and by construction this edge is in G . We now use induction on the shortest paths from p to p' and from q to q' in G . By induction, and [Lemma 29](#), we have that

$$d(p, q) \leq d(p, p') + \|p'q'\| + d(q', q) \leq (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|,$$

which implies that there is $(1 + \varepsilon)$ -path from p to q inside \mathcal{C}' .

467

► Remark 32. For axis-parallel squares Theorem 31 implies a local spanner with $\mathcal{O}(\varepsilon^{-6}n \log n)$ edges. However, for this special case, the decomposition into narrow trapezoid can be skipped. In particular, in this case, the resulting spanner has $\mathcal{O}(\varepsilon^{-3}n \log n)$ edges. We do not provide the details here, as it is only a minor improvement over the above, and requires quite a bit of additional work – essentially, one has to prove a version of Lemma 29 for squares.

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A Weak local spanners for axis-parallel rectangles

A.1 Quadrant separated pair decomposition

For two points $p = (p_1, \dots, p_d)$ and $q = (q_1, \dots, q_d)$ in \mathbb{R}^d , let $p \prec q$ denotes that q *dominates* p coordinate-wise. That is $p_i < q_i$, for all i . More generally, let $p <_i q$ denote that $p_i < q_i$. For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $\forall x \in X, y \in Y \quad x <_i y$. In particular X and Y are *i-coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is *quadrant-separated*, if X and Y are *i-coordinate separated*, for $i = 1, \dots, d$.

A *quadrant-separated pair decomposition* of a point set $P \subseteq \mathbb{R}^d$, is a pair decomposition (see Definition 2) $\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\}$ of P , such that $\{X_i, Y_i\}$ are quadrant-separated for all i .

► Lemma 33. Given a set P of n points in \mathbb{R} , one can compute, in $\mathcal{O}(n \log n)$ time, a QSPD of P with $\mathcal{O}(n)$ pairs, and of total weight $\mathcal{O}(n \log n)$.

Proof. If P is a singleton then there is nothing to do. If $P = \{p, q\}$, then the decomposition is the pair formed by the two singleton points.

Otherwise, let x be the median of P , such that $P_{\leq x} = \{p \in P \mid p \leq x\}$ contains exactly $\lceil n/2 \rceil$ points, and $P_{> x} = P \setminus P_{\leq x}$ contains $\lfloor n/2 \rfloor$ points. Construct the pair $\Xi = \{P_{\leq x}, P_{> x}\}$, and compute recursively a QSPDs $\mathcal{Q}_{\leq x}$ and $\mathcal{Q}_{> x}$ for $P_{\leq x}$ and $P_{> x}$, respectively. The desired QSPD is $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{> x} \cup \{\Xi\}$. The bounds on the size and weight of the desired QSPD are immediate. \blacktriangleleft

► **Lemma 34.** *Given a set P of n points in \mathbb{R}^d , one can compute, in $\mathcal{O}(n \log^d n)$ time, a QSPD of P with $\mathcal{O}(n \log^{d-1} n)$ pairs, and of total weight $\mathcal{O}(n \log^d n)$.*

Proof. The construction algorithm is recursive on the dimensions, using the algorithm of Lemma 33 in one dimension.

The algorithm computes a value α_d that partitions the values of the points' d th coordinates roughly equally (and is distinct from all of them), and let h be a hyperplane parallel to the first $d - 1$ coordinate axes, and having value α_d in the d th coordinate.

Let P_{\uparrow} and P_{\downarrow} be the subset of points of P that are above and below h , respectively. The algorithm recursively computes QSPDs \mathcal{Q}_{\uparrow} and \mathcal{Q}_{\downarrow} for P_{\uparrow} and P_{\downarrow} , respectively. Next, the algorithm projects the points of P on h , let P' be the resulting $d - 1$ dimensional point set (after we ignore the d th coordinate), and recursively computes a QSPD \mathcal{Q}' for P' .

For a point set $X' \subseteq P'$, let $\text{lift}(X')$ be the subset of points of P whose projection on h is X' . The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\text{lift}(X') \cap P_{\uparrow}, \text{lift}(Y') \cap P_{\downarrow}\}, \{\text{lift}(X') \cap P_{\downarrow}, \text{lift}(Y') \cap P_{\uparrow}\} \mid \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is $\widehat{\mathcal{Q}} \cup \mathcal{Q}_{\uparrow} \cup \mathcal{Q}_{\downarrow}$.

To observe that this is indeed a QSPD, observe that all the pairs in $\mathcal{Q}_{\uparrow}, \mathcal{Q}_{\downarrow}$ are quadrant separated by induction. As for pairs in $\widehat{\mathcal{Q}}$, they are quadrant separated in the first $d - 1$ coordinates by induction on the dimension, and separated in the d coordinate since one side of the pair comes from P_{\uparrow} , and the other side from P_{\downarrow} .

As for coverage, consider any pair of points $p, q \in P$, and observe that the claim holds by induction if they are both in P_{\uparrow} or P_{\downarrow} . As such, assume that $p \in P_{\uparrow}$ and $q \in P_{\downarrow}$. But then there is a pair $\{X', Y'\} \in \mathcal{Q}'$ that separates the two projected points in h , and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates p and q as desired.

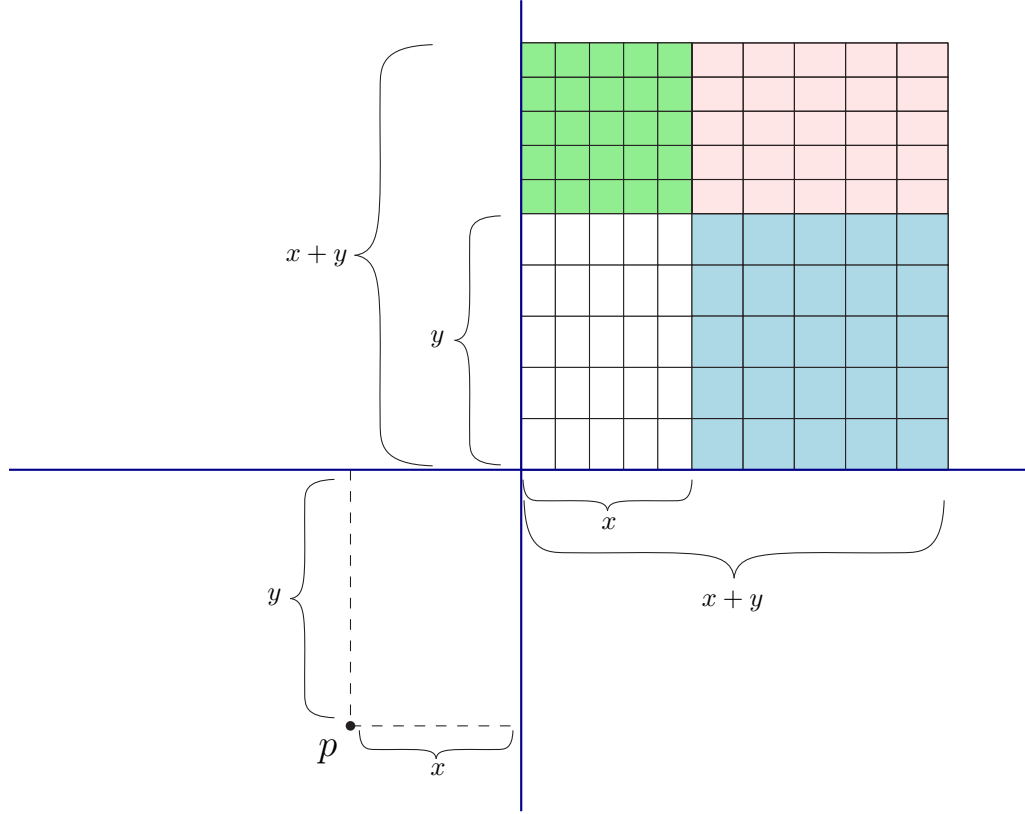
The number pairs in the decomposition is $N(n, d) = 2N(n, d - 1) + 2N(n/2, d)$ with $N(n, 1) = \mathcal{O}(n)$. The solution to this recurrence is $N(n, d) = \mathcal{O}(n \log^{d-1} n)$. The total weight of the decomposition is $W(n, d) = 2W(n, d - 1) + 2W(n/2, d)$ with $W(n, 1) = \mathcal{O}(n \log n)$. The solution to this recurrence is $W(n, d) = \mathcal{O}(n \log^d n)$. Clearly, this also bounds the construction time. \blacktriangleleft

A.2 Weak local spanner for axis parallel rectangles

For a parameter $\delta \in (0, 1)$, and an interval $I = [b, c]$, let $(1 - \delta)I = [t - (1 - \delta)r, t + (1 - \delta)r]$, where $t = (b + c)/2$, and $r = (c - b)/2$, be the shrinking of I by a factor of $1 - \delta$.

Let \mathcal{R} be the set of all axis parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$, with $R = I \times J$, let $(1 - \delta)R = (1 - \delta)I \times (1 - \delta)J$ denote the rectangle resulting from shrinking R by a factor of $1 - \delta$.

► **Definition 35.** *Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0, 1)$, a graph G is a $(1 - \delta)$ -local $(1 + \varepsilon)$ -spanner for rectangles, if for any axis-parallel rectangle R , we have that $G \cap R$ is a $(1 + \varepsilon)$ -spanner for all the points in $((1 - \delta)R) \cap P$.*



557 **Figure A.1** The construction of the grid $K(p, \Xi)$ for a point $p = (-x, -y)$ and a pair Ξ .

554 Observe that rectangles in \mathcal{R} might be quite “skinny”, so the previous notion of shrinkage
 555 used before is not useful in this case.

556 A.2.1 Construction for a single quadrant separated pair

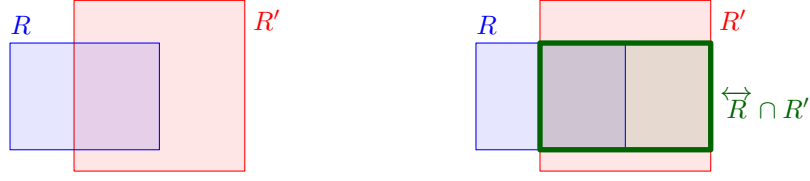
558 Consider a pair $\Xi = \{X, Y\}$ in a QSPD of P . The set X is quadrant-separated from Y . That
 559 is, there is a point c_Ξ , such that X and Y are contained in two opposing quadrants in the
 560 partition of the plane formed by the vertical and horizontal line through c_Ξ .

561 For simplicity of exposition, assume that $c_\Xi = (0, 0)$, and $X \prec (0, 0) \prec Y$. That is, the
 562 points of X are in the negative quadrant, and the points of Y are in the positive quadrant.

563 We construct a non-uniform grid $K(p, \Xi)$ in the square $[0, x + y]^2$. To this end, we first
 564 partition it into four subrectangles

$$\begin{array}{c|c}
 B_{\swarrow} = [0, x] \times [y, x + y] & B_{\nearrow} = [x, x + y] \times [y, x + y] \\
 \hline
 B_{\swarrow} = [0, x] \times [0, y] & B_{\nwarrow} = [x, x + y] \times [0, y].
 \end{array}$$

566 Let $\tau \geq 4/\varepsilon + 4/\delta$ be an integer number. We partition each of these rectangles into a
 567 $\tau \times \tau$ grid, where each cell is a copy of the rectangle scaled by a factor of $1/\tau$. See Figure A.1.
 568 This grid has $\mathcal{O}(\tau^2)$ cells. For a cell C in this grid, let $Y \cap C$ be the points of Y contained in
 569 it. We connect p to the left-most and bottom-most points in $Y \cap C$. This process generates
 570 two edges in the constructed graph for each grid cell (that contains at least two points), and
 571 $\mathcal{O}(\tau^2)$ edges overall.



577 **Figure A.2** Left: The two rectangles R, R' . Right: In green $\overleftrightarrow{R} \cap R'$, the restriction of the slab
 578 \overleftrightarrow{R} to the rectangle R' .

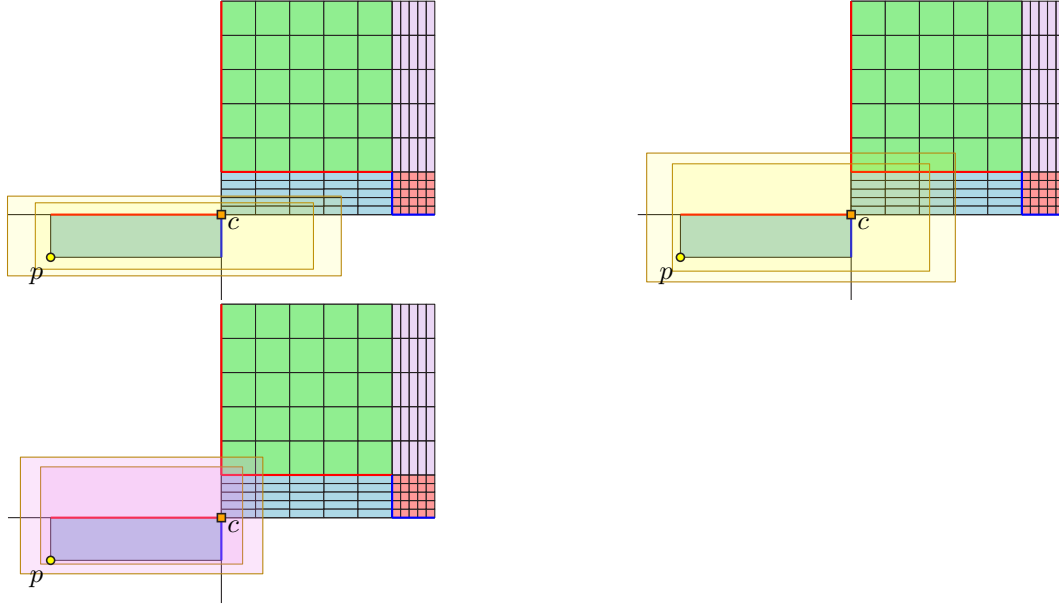
572 The algorithm repeats this construction for all the points $p \in X$, and does the symmetric
 573 construction for all the points of Y .

574 A.2.2 The construction algorithm

575 The algorithm computes a QSPD \mathcal{W} of P . For each pair $\Xi \in \mathcal{W}$, the algorithm generates
 576 edges for Ξ using the algorithm of Section A.2.1 and adds them to the generated spanner G .

579 A.2.3 Correctness

580 For a rectangle R , let $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists (x', y) \in R\}$ be its expansion into a horizontal
 581 slab. Restricted to a rectangle R' , the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure A.2.
 582 Similarly, we denote $\uparrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists (x, y') \in R\}$.



583 **Figure A.3** An illustration of $K(p, \Xi)$ with three rectangles and their shrunk version.

584 **► Lemma 36.** Assume that $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above
 585 construction, and a point $p = (-x, -y) \in X$ with its associated grid $K = K(p, \Xi)$. Consider
 586 any axis parallel rectangle R , such that $p \in (1 - \delta)R = I \times J$, and $(1 - \delta)R$ intersects a cell
 587 $C \in K$. We have that:

588 (I) If $C \subseteq (1 - \delta)R$ then $(1 - \delta)^{-1}C \subseteq R$.

589 (II) $\text{diam}(\mathbf{C}) \leq (\varepsilon/4)d(p, \mathbf{C})$.

590 (III) If $x \geq y$ and $\mathbf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1 - \delta)^{-1}\mathbf{C} \subseteq R$.

591 (IV) If $x \leq y$ and $\mathbf{C} \subseteq R_{\swarrow} \cup R_{\nwarrow}$ then $(1 - \delta)^{-1}\mathbf{C} \subseteq R$.

592 (V) If $x \geq y$ and $\mathbf{C} \subseteq R_{\nwarrow}$, then $(1 - \delta)^{-1}(\overrightarrow{(1 - \delta)R} \cap \mathbf{C}) \subseteq R$.

593 (VI) If $x \leq y$ and $\mathbf{C} \subseteq R_{\searrow}$, then $(1 - \delta)^{-1}(\uparrow((1 - \delta)R) \cap \mathbf{C}) \subseteq R$.

594 **Proof.** (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

595 (II) We have that $\text{diam}(\mathbf{C}) \leq (x + y)/\tau = \|p\|_1/\tau \leq (\varepsilon/4)d(p, \mathbf{C})$.

596 (III) The width, denoted $\text{wd}(\cdot)$, of $(1 - \delta)R$ is at least x , as it contains both p and the origin.

597 As such,

$$598 \quad (\text{wd}(R) - \text{wd}((1 - \delta)R))/2 \geq 2(x/\tau) \geq 2\text{wd}(\mathbf{C}).$$

599 That is, the width of the “expanded” rectangle R is enough to cover \mathbf{C} , and a grid cell
600 adjacent to it to the right.

601 A similar argument about the height shows that R covers the region immediately above \mathbf{C}
602 – in particular, the vertical distance from \mathbf{C} to the top boundary of R is at least the height of
603 \mathbf{C} . This implies that the expanded cell $(1 - \delta)^{-1}\mathbf{C}$ is contained in R , as claimed, as $\delta < 1/2$.

604 (V) We decompose the claim to the two dimensions of the region. Let $B = \overrightarrow{(1 - \delta)R} \cap \mathbf{C}$.
605 Observe that containment in the x -axis follows by arguing as in (III). As for the y -interval
606 of B , observe that it is contained in the y -interval of $(1 - \delta)R$, which implies that when
607 expanded by $(1 - \delta)^{-1}$, it would be contained in the y -interval of R . Combining the two
608 implies the result. \blacktriangleleft

609 **► Lemma 37.** For any axis-parallel rectangle R , and any two points $p, q \in (1 - \delta)R \cap P$,
610 there exists a $(1 + \varepsilon)$ -path between p and q in G .

611 **Proof.** The proof is by induction over the size of R (i.e. area, width, or height). Let
612 $\Xi = \{X, Y\} \in \mathcal{W}$ be the pair in the QSPD that separates p and q , let c be the separation
613 point of the pair, and assume for the simplicity of exposition that $p \in X$, $X \prec c \prec Y$, and
614 $c = (0, 0)$. Furthermore, assume that $\|p\|_1 \geq \|q\|_1$.

615 Let $p = (-x, -y)$, and let \mathbf{C} be the grid cell of $\mathcal{K}(p, \Xi)$ that contains q . If $\mathbf{C} \subseteq (1 - \delta)R$,
616 then $(1 - \delta)^{-1}\mathbf{C} \subseteq R$ by Lemma 36 (I). As such, let u be the leftmost point in $\mathbf{C} \cap P$. Both
617 $q, u \in (1 - \delta)^{-1}\mathbf{C}$, and by induction, there is an $(1 + \varepsilon)$ -path π between them in G (note that
618 the induction applies to the two points, and the “expanded” rectangle $(1 - \delta)^{-1}\mathbf{C}$). Since pu
619 is an edge of G , prefixing π by this edge results in an $(1 + \varepsilon)$ -path, as $\|qu\| \leq (\varepsilon/4)\|pq\|$, by
620 Lemma 36 (II) (verifying this requires some standard calculations which we omit).

621 Otherwise, one need to apply the same argument using the appropriate case of Lemma 36.
622 So assume that $x \geq y$ (the case that $y \geq x$ is handled symmetrically). If $\mathbf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$, then
623 (III) implies that $(1 - \delta)^{-1}\mathbf{C} \subseteq R$. Which implies that induction applies, and the claim holds.

624 The remaining case is that $x \geq y$ and $\mathbf{C} \subseteq R_{\nwarrow}$. Let $D = \overrightarrow{(1 - \delta)R} \cap \mathbf{C}$. By (V), we have
625 $(1 - \delta)^{-1}D \subseteq R$. Namely, $q \in (1 - \delta)R \cap \mathbf{C} \subseteq D$, and let u be the lowest point in $\mathbf{C} \cap P$. By
626 construction $pu \in E(G)$, $q, u \in D$, $(1 - \delta)^{-1}D \subseteq R$. As such, we can apply induction to q, u ,
627 and $(1 - \delta)^{-1}D$, and conclude that $d_G(q, u) \leq (1 + \varepsilon)\|qu\|$. Plugging this into the regular
628 machinery implies the claim. \blacktriangleleft

629 **► Theorem 38.** Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0, 1)$ be parameters.
630 The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 +$
631 $1/\delta^2)n \log^2 n)$ edges. The graph G is a $(1 - \delta)$ -local $(1 + \varepsilon)$ -spanner for axis parallel rectangles.

Formally, for any axis-parallel rectangle R , we have that $R \cap P$ is an $(1 + \varepsilon)$ -spanner for all the points of $((1 - \delta)R) \cap P$.

Proof. Computing the QSPD \mathcal{W} takes $\mathcal{O}(n \log^2 n)$ time. For each pair $\{X, Y\}$ in the decomposition with $m = |X| + |Y|$ points, we need to compute the lowest and leftmost points in $(X \cup Y) \cap \mathcal{C}$, for each cell in the constructed grid. This can readily be done using orthogonal range trees in $\mathcal{O}(\log^2 n)$ time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is $\mathcal{O}(\omega(\mathcal{W})\tau^2) = \mathcal{O}((1/\delta^2 + 1/\varepsilon^2)n \log^2 n)$.

The desired local spanner property is provided by Lemma 37. ◀

B Some missing proofs

B.1 Proof of Lemma 6

Restatement of Lemma 6. Given an α -SSPD \mathcal{W} of a set P of n points in \mathbb{R}^d and a parameter $\beta \geq 2$, one can refine \mathcal{W} into an $\alpha\beta$ -SSPD \mathcal{W}' , such that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$.

Proof. The algorithm scans the pairs of \mathcal{W} . For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\text{diam}(X) < \text{diam}(Y)$. Let \mathfrak{s} be the smallest axis-parallel cube containing X , and denote its sidelength by r . Let $r' = r / \lceil \sqrt{d}\beta \rceil$. Partition \mathfrak{s} into a grid of cubes of sidelength r' , and let T_Ξ be the resulting set of squares. The algorithm now add the set pairs

$$\{\{X \cap t, Y\} \mid t \in T_\Xi\}$$

to the output SSPD. Clearly, the resulting set is now $\alpha\beta$ -semi separated, as we chopped the smaller part of each pair into β smaller portions. ◀

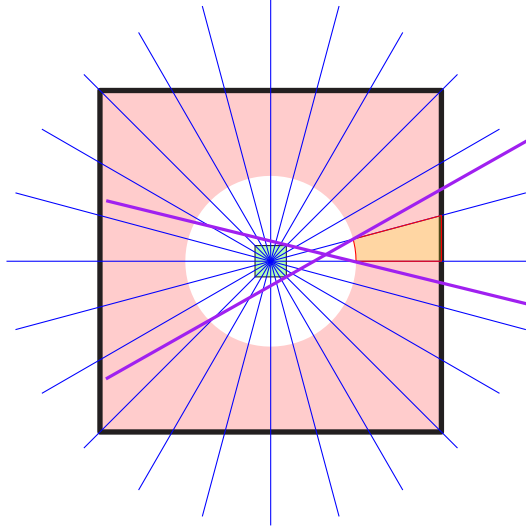
B.2 Proof of Lemma 8

Restatement of Lemma 8. Given a $(1/\varepsilon)$ -SSPD \mathcal{W} of n points in the plane, one can refine \mathcal{W} into a $(1/\varepsilon)$ -SSPD \mathcal{W}' , such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε -double-wedge \times_Ξ , such that X and Y are contained in the two different faces of the double wedge \times_Ξ . We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time is proportional to the weight of \mathcal{W}' .

Proof. By using Lemma 6, we can assume that \mathcal{W} is (say) $(10/\varepsilon)$ -separated. Now, the algorithm scans the pairs of \mathcal{W} . For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\text{diam}(X) < \text{diam}(Y)$. Let \square be the smallest axis-parallel square containing X , centered at point o . Partition the plane around o , by drawing $\mathcal{O}(1/\varepsilon)$ lines intersecting o with the angle between any two consecutive lines being at most (say) $\varepsilon/4$, see Figure B.1. This partitions the plane into a set of cones \mathcal{C} . For a cone $c \in \mathcal{C}$, we show that there exists an ε -double-wedge that contains X in one side, and $Y \cap c$ in the other.

To see that, take the double-wedge formed by the cross tangents between $\text{ch}(X)$ and $\text{ch}(Y \cap c)$, where $\text{ch}(X)$ denotes the convex-hull of X . Assume w.l.o.g that \square has side length 1, and let c be a cone of angle $\varepsilon/4$ with apex o , whose angular bisector is a horizontal ray in the positive direction of the x axis. See figure Figure B.2 for an illustration.

We would like to find a vertical segment s such that all points of Y lie to its right, with one endpoint on the upper line of c , and the other on the lower line of c . Using the segments' height and distance from the right side of \square we will be able to get a bound on the angle

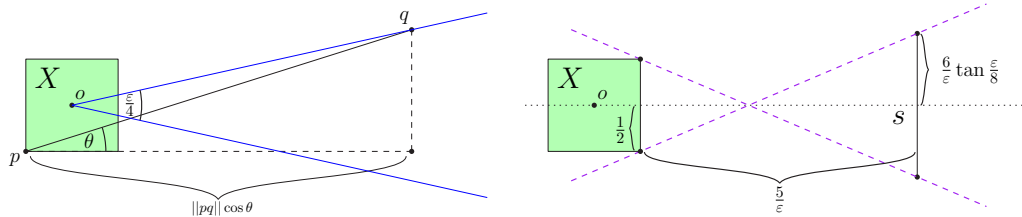


659 **Figure B.1** An illustration of refining the pairs in an SSPD into pairs contained in opposite parts
 660 of an ε -double-wedge. X is contained in the green square \square , while Y is contained in the red square,
 661 and the white gap between them is a result of the separation property. The set of cones with the
 662 apex at the center of \square gives us the desired partition as demonstrated by the purple double-wedge.

677 of the cross tangents. We first find a segment s with all points of Y to its right. A trivial
 678 bound on that distance is given by the segment from, say, the lower left corner of \square , denoted
 679 p , of length $10/\varepsilon$ with its right endpoint on the upper line of c , denote this point by q .
 680 We know that all points of Y lie to the right of q due to the $10/\varepsilon$ separation property of
 681 the SSPD. The segment pq creates an angle $\leq \pi/4$ with the x -axis (by the choice of the
 682 angle of c). We therefore get that the x -coordinate difference between \square and q is at most
 683 $10/\varepsilon \cdot \cos \frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$. So let s' be a vertical segment between the upper and lower
 684 rays of c , with x -coordinate distance of $6/\varepsilon - \frac{1}{2}$ from \square (in order to make calculations easier).
 685 We get that s' is of length $2 \cdot \frac{6}{\varepsilon} \tan \frac{\varepsilon}{8}$. Finally, we take s to be a vertical segment of length
 686 $\frac{12}{\varepsilon} \tan \frac{\varepsilon}{8}$, with its center on the x -axis at a distance of $5/\varepsilon + \frac{1}{2}$ away from o . The angle of the
 687 x -axis and the segment between the lower end of the right side of \square and the upper end of s
 688 is now given by:

$$689 \quad \arctan\left(\frac{\frac{6}{\varepsilon} \tan \frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5} \tan \frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) \leq \varepsilon$$

690



691 **Figure B.2** An illustration of the proof for Lemma 8

B.3 Proof of Claim 16

Restatement of Claim 16. Let \mathcal{C} be a bounded close convex shape. Given a set of points $P \subseteq \mathbb{R}^2$ in general position with respect to \mathcal{C} , let $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$ be the \mathcal{C} -Delaunay triangulation of P . For any homothet C of \mathcal{C} , we have that $\mathcal{D} \cap C$ is connected.

Proof. We prove that for any homothet C with two points $p, q \in P$ on its boundary, there is a path between p and q in $\mathcal{D} \cap C$, and Lemma 15 will immediately imply the general statement. The proof is by induction over the number m of points of P in the interior of C . If $m = 0$ then C contains no points of P in its interior, and thus pq is an edge of the Delaunay triangulation, as C testifies.

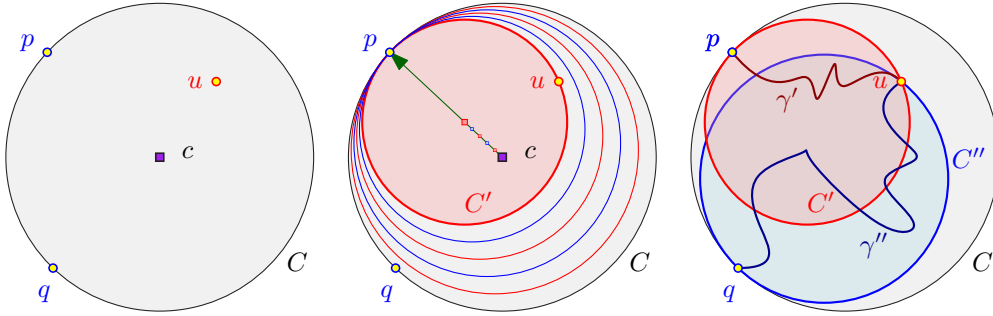


Figure B.3 An illustration of the proof of Claim 16 in the case that C is a disk.

Otherwise, let $u \in P$ be a point in the interior of C . From Lemma 15 we get that there exists a homothet C' of C with $C' \subseteq C$, such that p and u lie on the boundary of C' . Thus, by induction, there is a path γ' between p and u in $\mathcal{D} \cap C' \subseteq \mathcal{D} \cap C$. Similarly, there must be a homothet C'' , that gives rise to a path γ'' between u and q , and concatenating the two paths results in a path between p and q in $\mathcal{D} \cap C$.

B.4 Proof of Corollary 20

Restatement of Corollary 20. Let \mathcal{C} be a bounded convex body, P be a set of n points in the plane, $\varepsilon \in (0, 1)$ be a parameter, and let G be a \mathcal{C} -local $(1 + \varepsilon)$ -spanner of P .

Consider a region R in the plane, and the associated graph $H = \mathcal{G}(P, R)$, we have that $G \cap R$ is a $(1 + \varepsilon)$ -spanner for H . Formally, for any two points $p, q \in P \cap R$, we have that $d_{G \cap R}(p, q) \leq (1 + \varepsilon)d_H(p, q)$.

In particular, for any convex region D , the graph $G - D$ is a $(1 + \varepsilon)$ -spanner for $\mathcal{G}(P, \mathbb{R}^2) - D$.

Proof. Consider the shortest path $\pi = u_1 u_2 \dots u_k$ between p and q realizing $d_H(p, q)$. Every edge $e_i = u_i u_{i+1}$ has a homothet C_i such that $u_i, u_{i+1} \in C_i \subseteq R$. As such, there is a $(1 + \varepsilon)$ -path between u_i and u_{i+1} in $G \cap C_i \subseteq G \cap R$. Concatenating these paths directly yields the desired result.

The second claim follows by observing that the complement of D is the union of halfspaces, and halfspaces can be considered to be “infinite” homothets of \mathcal{C} . As such, the above argument applies verbatim.

B.5 Proof of Lemma 22

Restatement of Lemma 22. For $\varepsilon = 1/4$, and parameters n and $\Phi \geq 1$, there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $\mathcal{O}(n\Phi)$, such that any local $(1+\varepsilon)$ -spanner of P for disks, must have $\Omega(n \log \Phi)$ edges.

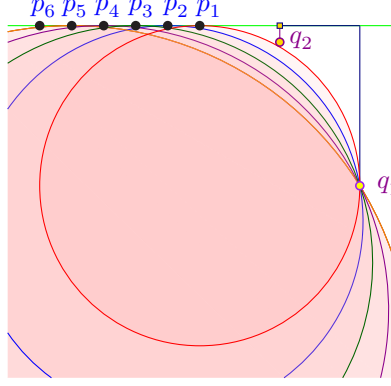


Figure B.4 The set of disks D_1 , and the construction of q_2 .

Proof. Let $p_i = (-i, 0)$, for $i = 1, \dots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$ and $q_1 = (n2^M, -1)$. For a point p on the x -axis, and a point q below the x -axis and to the right of p , let $\odot_{\downarrow}^p(q)$ be the disk whose boundary passes through p and q , and its center has the same x -coordinate as p .

In the j th iteration, for $j = 2, \dots, M-1$, Let $x_j = n2^{M-j+1} = x(q_{j-1})/2$, and let $y_j < 0$ be the maximum y -coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the disks of $D_1 \cup \dots \cup D_j$ where

$$D_j = \left\{ \odot_{\downarrow}^{p_i}(q_{j-1}) \mid i = 1, \dots, n \right\},$$

see Figure B.4 for an illustration of D_1 .

Let $q_j = (x_j, 0.99y_j)$.

Clearly, the point q_j lies outside all the disks of $D_1 \cup \dots \cup D_j$. The construction now continues to the next value of j . Let $P = \{p_1, \dots, p_n, q_2, \dots, q_M\}$. We have that $|P| = n + M - 1$.

The minimum distance between any points in the construction is 1 (i.e., $\|p_1 p_2\|$). Indeed $x(q_{M-1}) = 4n$ and thus $\|q_{M-1} p_1\| \geq 2n$. The diameter of P is $\|p_1 q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = \mathcal{O}(n\Phi)$.

For any i and j , consider the disk $\odot_{\downarrow}^{p_i}(q_j)$. This disk does not contain any point of $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ since its interior lies below the x -axis. By construction it does not contain any point q_{j+1}, \dots, q_{M-1} . This disk potentially contains the points q_{j-1}, \dots, q_1 , but observe that for any index $k \in \llbracket j-1 \rrbracket$, we have that

$$\|p_i q_k\| = \sqrt{(i + n2^{M-k+1})^2 + (y(q_j))^2},$$

which implies that $n2^{M-k+1} \leq \|p_i q_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$\frac{\|p_i q_k\|}{\|p_i q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since $j \in \llbracket M-1 \rrbracket$. Namely, the shortest path in G between p_i and q_j , can not use any of the points q_1, \dots, q_{j-1} . As such, the graph G must contain the edge $p_i q_j$. This implies that $|E(G)| \geq n(M-1)$, which implies the claim. ◀

B.6 Proof of Lemma 23

Restatement of Lemma 23. For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a set P of $n + \mathcal{O}(\log \Phi)$ points, with spread $\mathcal{O}(\Phi n)$, and a triangle Δ , such that any Δ -local $(3/2)$ -spanner of P requires $\Omega(n \log \Phi)$ edges.

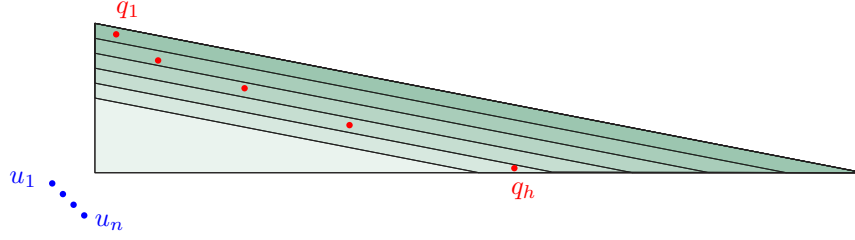


Figure B.5 An Illustration of the construction of Lemma 23.

Proof. Let $h = \lceil \log_2 \Phi \rceil$. Let Δ be the triangle formed by the points $(0, 0)$, $(0, 1)$ and $(8\Phi h, 0)$. The hypotenuse of this triangle lies on the line $\ell \equiv \frac{1}{8\Phi h}x + y = 1$, and let $v = (\frac{1}{8\Phi h}, 1)$ be the vector orthogonal to this line.

For $i \in \llbracket h \rrbracket$ and $j \in \llbracket n \rrbracket$, let

$$q_i = (2^{i+1}, 1 - i/h) \quad \text{and} \quad u_j = (\frac{j}{n} - 1, -\frac{j}{n}),$$

and let $P = \{q_1, \dots, q_h, u_1, \dots, u_n\}$, see Figure B.5. Observe that $\text{cp}(P) = \|u_1 u_2\| = \sqrt{2}/n$, and as such we have that $\Phi(P) = n \cdot \text{diam}(P)/\sqrt{2} \leq n(4\Phi + 2n) \leq 8\Phi n$, as $\Phi \geq n$. Observe that

$$\langle q_{i+1} - q_i, v \rangle = \langle (2^{i+1}, -\frac{1}{h}), (\frac{1}{8\Phi h}, 1) \rangle \leq \frac{4\Phi}{8\Phi h} - \frac{1}{h} < 0.$$

That is, the points q_1, \dots, q_i are increasing in distance from ℓ .

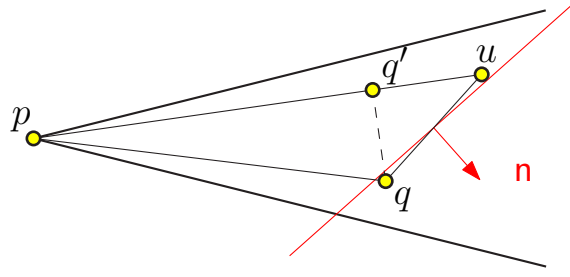
Let $\Delta_{i,j}$ be the homothet of Δ , that has its bottom left corner at u_j , and its hypotenuse passes through q_i . By the above, $P(i, j) = \Delta_{i,j} \cap P = \{u_j, q_i, q_{i+1}, \dots, q_h\}$. Any $(1 + \varepsilon)$ -spanner for $P(i, j)$ must contain the edge $u_j q_i$. Indeed, we have, for any k , that $2^{k+1} \leq \|u_j q_k\| \leq 2^{k+1} + 3$. As such, any path on a graph induced on $P(i, j)$ from u_j to q_i that uses (say) a midpoint q_k , for $k > i$, must have dilation at least

$$\frac{\|u_j q_k\| + \|q_k q_i\|}{\|u_j q_i\|} \geq \frac{2^{k+1} + 2^k}{2^{i+1} + 3} \geq \frac{3 \cdot 2^{i+1}}{(1 + 3/4)2^{i+1}} = \frac{12}{7} > \frac{3}{2}.$$

Thus, any Δ -local $3/2$ -spanner for homothets of Δ , must contain the edge $q_i u_j$, for any $i \in \llbracket h \rrbracket$ and $j \in \llbracket n \rrbracket$. Thus, such a spanner must have $\Omega(n \log \Phi)$ edges, as claimed. ◀

B.7 Proof of Lemma 25

Restatement of Lemma 25. Let $p \in P$, $c \in \mathcal{C}_i$, and $u = \text{nn}_i(p, c)$, and let q be a point in $(P \cap (p + c)) \setminus \{p, u\}$. We have that $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$ and $\|qu\| \leq \|pq\|$.



782 **Figure B.6** The case that $\|pq\| \leq \|pu\|$ in Lemma 25. The vector used to determine $\text{nn}_i(p, c)$ is
 783 shown in red, and denoted n

779 **Proof.** Consider the triangle Δpqu and denote the angles at p, q , and u by $\angle p, \angle q$, and $\angle u$
 780 respectively. Since the angle of c is smaller than 60 degrees (for an appropriate choice of γ),
 781 we have that $\|qu\| \leq \max\{\|pu\|, \|pq\|\}$.

784 Consider the case that $\|pq\| \leq \|pu\|$, illustrated in Figure B.6. Observe that $\angle u \leq \angle q$.
 785 As such $\angle u \leq \pi/2$. Furthermore, $\angle u \geq \alpha \gg \varepsilon\alpha/\gamma = \beta \geq \angle p$. Similarly, $\angle q \in [\alpha, \pi - \alpha]$. By
 786 the 1-Lipshitz of \sin , and as $\sin x \approx x$, for small x , and for γ sufficiently large, we have that

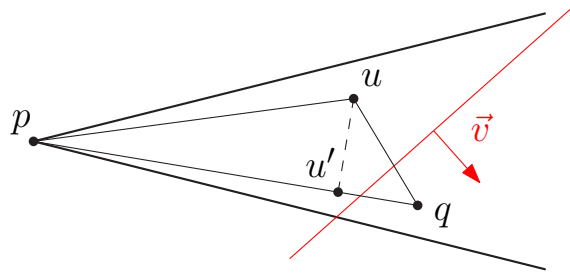
$$787 \quad \sin(\angle q + \angle p) \in [1 - \varepsilon/4, 1 + \varepsilon/4] \sin \angle q \quad \text{and} \quad \sin \angle p \leq (\varepsilon/4) \sin \angle u.$$

788 As such, by the law of sines, we have that $\frac{\|qu\|}{\sin \angle p} = \frac{\|pq\|}{\sin \angle u} = \frac{\|pu\|}{\sin \angle q}$. This implies that

$$789 \quad \|pu\| + (1 + \varepsilon) \|qu\| = \left(\frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \right) \|pq\|.$$

790 Observe, by the above that

$$791 \quad \frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \leq \frac{\sin \angle q}{\sin(\angle p + \angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq \frac{\sin \angle q}{(1 - \varepsilon/4) \sin(\angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$



792 **Figure B.7** The case that $\|pq\| > \|pu\|$ in Lemma 25.

793 The other possibility is that $\|pq\| > \|pu\|$, illustrated in Figure B.7. Let u' be the
 794 projection of u to pq . Observe that

$$795 \quad \|uu'\| = \|pu'\| \tan \angle p \leq 2\beta \|pu'\| \leq (\varepsilon/8) \|pu'\|.$$

796 Observe that $\cos \angle p \geq 1 - (\angle p)^2/2 \geq 1 - \varepsilon^2/8$ as $\angle p$ is an angle smaller than (say) $\varepsilon/16$. As
 797 such $1/\cos \angle p \leq 1 + \varepsilon^2/4$. This implies that $\|pu\| \leq \|pu'\|/\cos \angle p \leq (1 + \varepsilon^2/4) \|pu'\|$. We
 798 thus have that

$$799 \quad \tau = \|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon^2/4) \|pu'\| + (1 + \varepsilon) (\|uu'\| + \|u'q\|) \\
 800 \quad \leq (1 + \varepsilon^2/4 + (1 + \varepsilon)\varepsilon/8) \|pu'\| + (1 + \varepsilon) \|u'q\| \leq (1 + \varepsilon) \|pq\|.$$

B.8 Proof of Lemma 29

Restatement of Lemma 29. Let $\varepsilon \in (0, 1)$ be some parameter, and $\vartheta = \varepsilon/16$. Let X, Y be two points sets that are ϑ -semi separated and ϑ -angularly separated (see Definition 7), and let T be a ϑ -narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on the two legs of T . Then, one can compute a homothet $T' \subseteq T$ of T , such that:

(I) There are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay triangulation of $X \cup Y$.

(II) We have that $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$.

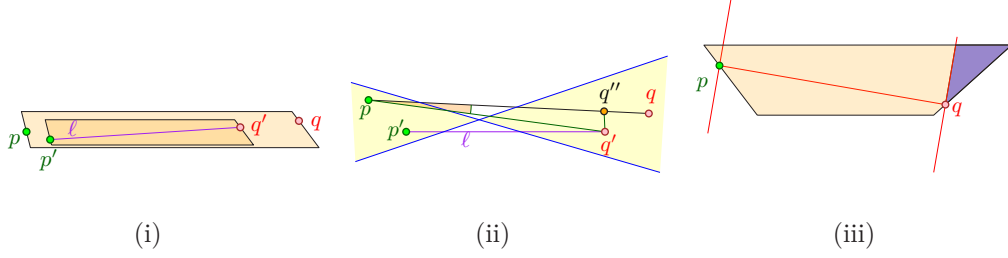


Figure B.8 Illustration of the settings in the proof of Lemma 29. Left: A ϑ -narrow trapezoid with p and q on its legs. Center: p and q are ϑ -semi separated and ϑ -angularly separated. Right: The triangle of all the points of the trapezoids that their nearest point on pq is q .

Proof. Let $\mathcal{D} = \mathcal{D}_T(X \cup Y)$. Claim 16 implies that $\mathcal{D} \cap T$ is connected. Thus, there is a path in $\mathcal{D} \cap T$ between p and q , and thus, there must be an edge $p'q'$ along this path with $p' \in X$ and $q' \in Y$. This implies part (I).

Let $\ell = \|p'q'\|$. Assume for concreteness that $\|pp'\| \leq \text{diam}(X) \leq \vartheta d(X, Y) \leq \vartheta \ell \leq \vartheta d$, where $d = \text{diam}(T)$. Let q'' be the closest point on pq to q' .

We first consider the case that $q'' \in \text{int}(pq)$. We have that

$$\|pq''\| = \|pq'\| \cos \angle q'pq \geq (\|p'q'\| - \|pp'\|) \cos \angle q'pq \geq (1 - \vartheta)\ell \cdot (1 - \vartheta^2/2) \geq (1 - 2\vartheta)\ell,$$

since $\cos \vartheta \geq 1 - \vartheta^2/2$, for $\vartheta < 1/2$. Similar argumentation implies that $\|pq''\| \leq (1 + \vartheta)\ell$. As such, we have

$$\|q'q''\| \leq (1 + \vartheta)\ell \sin \angle p'pq' \leq 2\vartheta\ell.$$

Thus, we have that

$$\|qq'\| \leq \|qq''\| + \|q''q'\| \leq \|pq\| - \|pq''\| + 2\vartheta\ell \leq \|pq\| - (1 - 2\vartheta)\ell + 2\vartheta\ell \leq \|pq\| - \ell.$$

Thus, we have that

$$\begin{aligned} (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| &\leq (1 + \varepsilon)\vartheta\ell + \ell + (1 + \varepsilon)(\|pq\| - \ell) \\ &= (1 + \varepsilon) \|pq\| + (1 + \varepsilon)\vartheta\ell + \ell - (1 + \varepsilon)\ell \leq (1 + \varepsilon) \|pq\|, \end{aligned}$$

for $\vartheta \leq \varepsilon/2$. Which establish the claim in this case.

The case that $q'' = p$ is impossible, because of the angular separation property. Thus, the only remaining possibility is that $q'' = q$. This however implies that q' must be in the triangle of all the points of the trapezoids that their nearest point on pq is q . The diameter of this triangle is bounded by the length of the leg of the trapezoid, which is bounded by

ϑd . Namely, we have $\|qq'\| \leq \vartheta d$. Similarly, we have $(1 - 2\vartheta)d \leq \|pq\| \leq (1 + 2\vartheta)d$. Since $\|pp'\|, \|qq'\| \leq \vartheta d$, it follows that

$$(1 - 4\vartheta)d \leq \ell \leq (1 + 4\vartheta)d.$$

As such, for $\vartheta \leq \varepsilon/8$ and $\varepsilon \leq 1$, we have

$$(1 + \varepsilon)\|pp'\| + \ell + (1 + \varepsilon)\|q'q\| \leq 4\vartheta d + (1 + 4\vartheta)d = (1 + 8\vartheta)d \leq (1 + \varepsilon)\|pq\|.$$

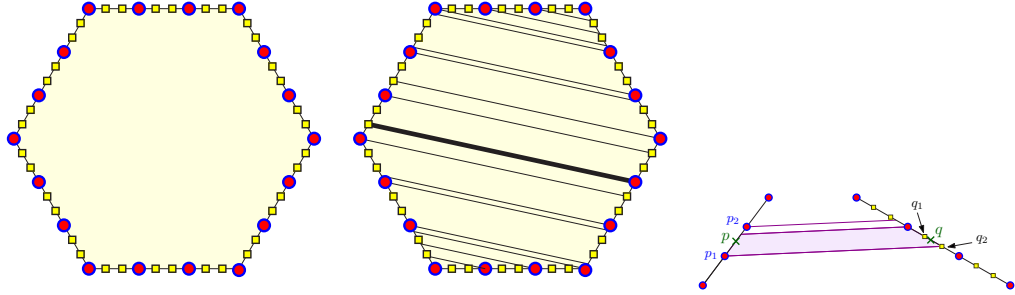
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839 B.9 Proof of Lemma 30

Restatement of Lemma 30. *Let t be a positive integer. Given a t -nice polygon \mathcal{C} , and a parameter ϑ , one can cover it by a set \mathcal{T} of $\mathcal{O}(t^4/\vartheta^3)$ ϑ -narrow trapezoids, such that for any two points $p, q \in \partial\mathcal{C}$ that belong to two edges of \mathcal{C} that are not adjacent, there exists a narrow trapezoid $T \in \mathcal{T}$, such that p and q are located on two different short legs of T .*

Proof. We show a somewhat suboptimal but simple construction. A t -nice polygon has at most t edges. Let ψ be the sensitivity of \mathcal{C} , and place a minimum set of points P on the boundary of \mathcal{C} , which includes all the vertices of \mathcal{C} , and such that the distance between any consecutive pair of points is in the range $[c_1, 2c_1]$, where $c_1 = \vartheta\psi/c_2$, for some sufficiently large constant c_2 . In particular, let $M = \max_{e \in E(\mathcal{C})} \lceil \|e\|/c_1 \rceil = \mathcal{O}(1/\vartheta)$.

In addition, place $c_3 t$ equally spaced points between any two consecutive points of P , where c_3 is a constant to be determined shortly. Let Q be the set resulting from P after adding all these points.



852 ■ **Figure B.9** The points of P (round), and all the points added to P in order to create Q (square).
853 On the right, a “vertical” decomposition induced by one of the directions of $P \times Q$.

We have that $|P| = \mathcal{O}(t/\vartheta)$ and $|Q| = \mathcal{O}(t^2/\vartheta)$. For a direction v , let \mathcal{T}_v be the decomposition into trapezoids formed by shooting rays from inside \mathcal{C} in the direction of v (or $-v$) from all the points of P , see Figure B.9. Let \mathcal{T}'_v be the set resulting from throwing away trapezoids with legs that lie on adjacent edges. It is easy to verify that all the trapezoids of \mathcal{T}'_v are ϑ -narrow. Let U be the set of all directions induced by pairs of points of $P \times Q$, and let $\mathcal{T} = \cup_{u \in U} \mathcal{T}'_u$. We have that $|\mathcal{T}| = \mathcal{O}(|P| \cdot |U|) = \mathcal{O}(|P|^2 |Q|) = \mathcal{O}(t^4/\vartheta^3)$.

Consider any two points p, q on non-adjacent edges of \mathcal{C} , and let p_1, p_2 be the two adjacent points of P such that $p \in p_1 p_2$. Now, let q_1, q_2 be the adjacent points of Q such that $q \in q_1 q_2$. We assume that p_1, p_2, q_1, q_2 are in this clockwise order along the boundary of \mathcal{C} .

Observe that when we project the interval $\overrightarrow{p_1 p_2}$, to the line induced by $q_1 q_2$, in the direction $\overrightarrow{p_1 q_2}$, the projected interval contains $q_1 q_2$. The last claim is intuitively obvious,

but requires some work to see formally. The minimum height of a triangle involving three vertices of \mathcal{C} is formed by three consecutive vertices. In the worst case, this is an isosceles triangle with sidelength ψ and base angle π/t . As such, the height of such a triangle is $h = \psi \sin(\pi/t) \geq \psi/t$.

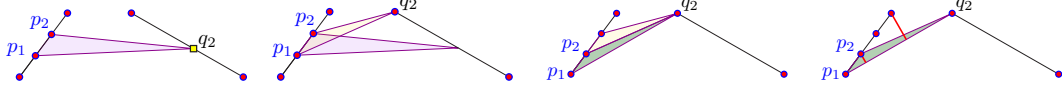


Figure B.10 The height of the triangle $\triangle p_1 p_2 q_2$ is minimized as q_2 and p_1 are moved to vertices of \mathcal{C} .

The height of the triangle $\triangle p_1 p_2 q_2$ is minimized when p_1 or p_2 is a vertex of \mathcal{C} , and q_2 is at a vertex of \mathcal{C} , see Figure B.10. Assume, for concreteness, that p_1 is a vertex of \mathcal{C} , and observe that $\|p_1 p_2\| \geq \|e\|/M$, where e is the edge of \mathcal{C} containing this segment. Using similar triangles, it is straightforward to show that the height of this triangle is at least $h' = h/M = \Omega(\varepsilon\psi/t)$. The quantity h' is a lower bound on the length of the projection of $p_1 p_2$ on the line spanned by $q_1 q_2$. However, $\|q_1 q_2\| \leq 2c_1/c_3 t = \mathcal{O}(\vartheta\psi/c_3 t) < h'$, by picking c_3 to be sufficiently large constant.

This readily implies that the trapezoid induced by the direction $u = \overrightarrow{p_1 q_2}$ in \mathcal{T}'_u that contains p on its leg, contains q on its other leg. \blacktriangleleft