Fault tolerant spanners

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1 Introduction

Let $\mathcal{F}$ be a family of regions in the plane, which we call the fault regions. For a fault region $F \in \mathcal{F}$ and a geometric graph G on a point set P , we define $G \ominus F$ to be the part of G that remains after the points of P that are contained in F, and all the edges of G that intersect F have been removed from the graph. For simplicity, we assume that a region fault F does not contain its boundary, i.e., only vertices and edges intersecting the interior of F will be affected.

Let $\mathcal{L}$ be a family of regions in the plane, which we call the local regions. For a local region $L \in \mathcal{L}$ and a geometric graph G on a point set P , we define $G\mid_F$ to be the part of G contained in the interior of F, meaning only the vertices and edges that are fully containd in the interior of F.

Formally, given $G = (P, E)$:

$$G \ominus F = (p \setminus F, \{e \in E \mid e \cap F = \emptyset\})$$
$$G\mid_F = (p \cap F, \{e \in E \mid e \subseteq F\})$$

The problems:

1. Given a set P of points in $\mathbb{R}^2$, and a family $\mathcal{F}$ of regions, compute a graph G such that $G \ominus F$ is a t-spanner of P for any fault $F \in \mathcal{F}$

2. Given a set P of points in $\mathbb{R}^2$, and a family $\mathcal{L}$ of regions, compute a graph G such that $G\mid_F$ is a t-spanner for any local region $L \in \mathcal{L}$

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2 Complement of disk faults / disk local spanners

3 Convex faults / disk local spanners

Let $\mathcal{F}$ be the set of convex regions, and $\varepsilon > 0$. We use the construction of Abam et al. [**?**] in order to create a $(1 + \varepsilon)$-$\mathcal{F}$ tolerant spanner. In their paper, Abam et al. build a *semi separated pair decomposition* (SSPD), and add a set of carefully chosen edges between every two sets $A, B \subseteq P$ that compose a pair in the SSPD. Given a pair (A, B), the algorithm partitions the larger set, w.l.o.g it is B, by shooting rays at fixed angular intervals from a disk that contains A, and then adds a planar set of edges E_i between the convex hulls of A and every part B of B, that has the following property:

For any half-plane H such that $A \cap H \neq \emptyset$ and $B \cap H \neq \emptyset$, there exists an edge $e \in E_i$ such that $e \subseteq H$. This property, together with the properties of the SSPD, makes the resulted graph an $\mathcal{F}$-fault tolerant spanner.

We notice, that a similar construction that chooses E_i to be the edges of the Delaunay triangulation with one end in A and the other in B has the same property. We now prove that for any disk $d \subseteq \mathbb{R}^2$, and for any set P' of points we have that $\mathcal{DT}(P') \mid_d$ is connected. This is enough as half-planes can be simulated by complements of disks.

Claim 1. *For a set of points $P \subseteq \mathbb{R}^2$ and for any disk d, $\mathcal{DT}(P) \mid_d$ is connected.*

Proof. We prove a different claim that immediately implies the desired one. Let d be a disk with two points $p, q \in P$ on its boundary. Then there is a path between p and q in $\mathcal{DT}(P) \mid_d$. This is enough as for every two points p, q and a disk d containing them, we can get a disk d' that contains p and q on it's boundary by moving the center of d in an arbitrary direction until either of them, say p is on the boundary, and them moving the center of the disk towards p while shrinking the size of the disk to maintain p on the boundary, until q as well is on the boundary.

We prove by induction over the number points in the interior of d.

$|d \cap (P \setminus \partial d)| = 0$: Then by construction of the Delaunay triangulation the edge $\{p, q\}$ is in $\mathcal{DT}(P)$ and is contained in the interior of D.

$|d \cap (P \setminus \partial d)| > 0$: Let $x \in P$ be a point in the interior of d. We move the center of d in the direction of p, shrinking d in the process, until we get a disk $d' \subseteq d$ such that x is on the boundary of d'. By induction there is a path between p and x in $\mathcal{DT} \mid_{d'}$, and since $\mathcal{DT} \mid_{d'} \subseteq \mathcal{DT} \mid_d$ we have that the same path exists in $\mathcal{DT} \mid_d$. The same proof gives us a path between x and q and thus we are done.

□

Since the triangulation is planar, all of the arguments from the original paper regarding the size of the spanner hold, and we get a $(1+\varepsilon) - \mathcal{F}$ fault tolerant spanner of size $O(\varepsilon^{-3} n \log n)$ in $O(\varepsilon^{-2} n \log n)$ time.

4 Complement of convex faults / convex local spanners

Using the same argument, we can extend the result for the case where $\mathcal{L}$ is the set of all scaled and translated copies, homothets, of a convex shape $\mathcal{C}$.

While the Delaunay triangulation is not well defined for all convex shapes, the operation of creating edges between two points $p, q \in P$ such that there exist a homothet of $\mathcal{C}$ that contains only p and q and no other point of P is always well defined, and gives us a graph known as the $\mathcal{C}$-Delaunay graph of P, and denoted $\mathcal{DG}_{\mathcal{C}}(P)$. The above proof applies almost verbatim for any convex $\mathcal{C}$, and proves the connectivity of $\mathcal{DG}_{\mathcal{C}}(P)$ for any $L \in \mathcal{L}$.

We need only to define a suitable shrinking operation for convex region towards a point, which is possible, for example, by parameterizing the curve defining the region and leaving the desired point in the same coordinate of the smaller curve. So, we we get a $(1+\varepsilon) - \mathcal{L}$ local spanner of size $O(\varepsilon^{-3} n \log n)$ in $O(\varepsilon^{-2} n \log n)$ time.(

5 ε-shadow

In this section we consider a weaker form of fault tolerance. Given a family $\mathcal{L}$ of shapes, we say that G is a $(\mathcal{L}, \varepsilon)$-local spanner if for any $L \in \mathcal{L}$ we have that $G \mid_{L_\varepsilon}$, where L_ε is L rescaled by $(1-\varepsilon)$, is a t-spanner. We call $L \setminus L_\varepsilon$ the *shadow* of L, and we say that a point p is *truly contained* in L w.r.t. ε, and denote $p \in_\varepsilon L$, if $p \in L_\varepsilon$.

5.1 Bounded aspect ratio rectangles

Let $\mathcal{L}$ be the set of axis parallel rectangles with aspect ratio at most $1 < \alpha$. We repeatedly preform the algorithm for convex local spanners with rectangles of different aspect-ratio, where in the i-th iteration we use a rectangle with aspect ratio $(1+\varepsilon)^i$, where $i \in \{0, ..., \log_{1+\varepsilon}(\alpha)\}$.

Let r be a rectangle with aspect ratio α, and let (A, B) be a pair in an SSPD such that $A \cap r \neq \emptyset$, and $B \cap r \neq \emptyset$. We assume w.l.o.g that the height of r is 1, and its width is $\alpha' \in [1, \alpha]$.

Let $i \in \{0, ..., \log_{1+\varepsilon}(\alpha)\}$ be an index for which $\alpha' \leq (1+\varepsilon)^i \leq \frac{\alpha}{1-\varepsilon}$ if such an index exists, then let r' be the rectangle with width α' and aspect ratio $(1+\varepsilon)^i$, whose horizontal bisector coincides with that of r. Since $(1+\varepsilon)^i \leq \frac{\alpha}{1-\varepsilon}$, we have that $r \setminus r'$ is contained within the shadow of r, and therefore r' contains points of both A and B, from the correctness of the convex local spanner, we will have an edge between a point in $A \cap r'$ and a point in $B \cap r'$. As before, this, together with the properties of the SSPD, is enough to guarantee that the constructed graph is indeed a $(\mathcal{L}, \varepsilon)$-$t$-spanner (for the appropriate choice of the parameter s of the SSPD).

We are left with proving that there exists an index $i \in \{0, ..., \log_{1+\varepsilon}(\alpha)\}$ for which $\alpha' \leq (1+\varepsilon)^i \leq \frac{\alpha}{1-\varepsilon}$.

$$\alpha' \leq (1+\varepsilon)^i \leq \frac{\alpha}{1-\varepsilon}$$

$$\log_{1+\varepsilon}(\alpha') \leq i \leq \log_{1+\varepsilon}\left(\frac{\alpha}{1-\varepsilon}\right)$$

$$\log_{1+\varepsilon}(\alpha') \le i \le \log_{1+\varepsilon}(\alpha) - \log_{1+\varepsilon}(1-\varepsilon)$$

If $\log_{1+\varepsilon}(1-\varepsilon) < -1$, then there must be an integer i with the required properties. We now notice that $(1+\varepsilon)^{-1} = \frac{1}{1+\varepsilon} > (1-\varepsilon)$ [since $1 > (1-\varepsilon)(1+\varepsilon) = (1-\varepsilon^2)$], and so i exists.

The size of the spanner is $\log_{1+\varepsilon}(\alpha)$ times the number of edges in a convex local spanner, and since $\log_{1+\varepsilon}(\alpha) = O\left(\frac{\log(\alpha)}{\varepsilon}\right)$, we have a spanner of size $O\left(\frac{\log(\alpha)}{\varepsilon(t-1)^{-3}} n \log n\right)$

5.2 Arbitrary rectangles

In order to construct local spanners for the family $\mathcal{L}$ of axis parallel rectangles with ε-shadow, we describe a decomposition of the the point set $P \subseteq \mathbb{R}^2$ in to pairs of sets, a decomposition which we name a Quadrant Separated Pair Decomposition (QSPD). This decomposition gives us $O(n \log^2 n)$ pairs (A_i, B_i) of subsets of P, such that the sets can be separated by a vertical line and also by a horizontal line, and for every two points $p, q \in P$, there exists a single pair (A_i, B_i) such that (w.l.o.g) $p \in A_i$, $q \in B_i$. This separation can be viewed as if on of the sets lies in the first quadrant of the plane (i.e. every point has positive x and y values), and the other is in the third quadrant (i.e. every point has negative x and y values), hence the name.

The construction of the decomposition can be described as the repeated recursive invocation of two fairly simple subroutines denoted S_1 and S_2. The first subroutine S_1 goes as follows. Given a set of points P, and a horizontal line l_y, find the median of P w.r.t. the x-coordinates of the points, and create the vertical line l_x passing through it. l_x and l_y now divide the plane into 4 quadrants, add both pairs of diagonally opposing quadrants to the decomposition, and recurse twice, once on the points to the left of l_x, and once on the points to its right.

The second operation is now even easier to describe. Find the median of P w.r.t. the y-coordinates of the points, create the horizontal line l_y passing through that point, call $S_1(P, l_y)$, and and recurse twice, once on the points to below of l_y, and once on the points above it.

Claim 2. *The subroutine $S_2(P)$ creates a QSPD with size $O(n \log^2 n)$.*

Proof. By construction, each reported pair is separated w.r.t. to both dimensions, and any two point appear in diagonally opposing quadrants exactly once, as every recursive calls to both S_1 and S_2 will include only one of the points.

Every call to S_1 creates two pairs, and generates two recursive calls, each with exactly half of the points. The formula for the size of the pairs created by S_1 is therefore $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$, which solves to $O(n)$. Very similarly, each call to S_2 calls S_1 once, and generates two recursive calls, each with exactly half of the points. The total number of pairs is therefore $S(n) = 2S\left(\frac{n}{2}\right) + O(n \log n)$, which solves to $O(n \log^2 n)$.

$\square$

We first describe a subroutine for connecting two sets of points, A and B, where A is contained in Q^-, the negative quadrant of the plane (i.e., have a negative value x-coordinate and a negative value y-coordinate), and B is contained in Q^+, the positive quadrant of the plane.

Our algorithm will connect every point in A to $O\left(\frac{1}{\varepsilon^2}\right)$ points in the positive quadrant, and after performing the same process for the points of the symmetrically defined B', we will have that every rectangle that truly contains points from A and B will have an edge (a, b) with $a \in A$ and $b \in B$.

For every point $a = (x', y') \in A$ we define partition the positive quadrant into $O\left(\frac{1}{\varepsilon^2}\right)$ sets. We consider the following $\frac{1}{\varepsilon}$ horizontal stripes - $\forall j \in \{1, ..., \frac{1}{\varepsilon}\}$:

$$H_j := \{(x, y) \mid 0 \leq x \leq x' + y',\ (j-1) \cdot \varepsilon y' < y \leq j \cdot \varepsilon y'\}$$

On top of these we add similarly built vertical stripes:

$$V_i := \{(x, y) \mid (j-1) \cdot \varepsilon x' < x \leq j \cdot \varepsilon x',\ 0 \leq y \leq x' + y'\}$$

These stripes create a grid which partitions the rectangle r whose opposite corners are $(0, 0)$ and $(|x'|, |y'|)$ into $\frac{1}{\varepsilon^2}$ cells of width εx and height εy. Formally:

$$C_{i,j} := \{(x', y') \mid (i-1) \cdot \varepsilon x < x' \leq i \cdot \varepsilon x,\ (j-1) \cdot \varepsilon y < y' \leq j \cdot \varepsilon y\}$$

We now divide the parts of the stripes that lie outside of the rectangle r. The horizontal stripes are divided into cells of width $\varepsilon(x+y)$ and height εy, and the vertical stripes are divided into cells of width $\varepsilon y$and height $\varepsilon(x+y)$. The extremal cell in each stripe may be smaller if x or y are not divisible by $\varepsilon(x+y)$. Formally:

$$C_{H_{i,j}} := \{(x', y') \mid x' + (i-1) \cdot \varepsilon(x+y) < x' \leq x' + i \cdot \varepsilon(x+y),\ (j-1) \cdot \varepsilon y < y' \leq j \cdot \varepsilon y\}$$

$$C_{V_{i,j}} := \{(x', y') \mid (i-1) \cdot \varepsilon x < x' \leq i \cdot \varepsilon x,\ y + (j-1) \cdot \varepsilon(x+y) < y' \leq y + j \cdot \varepsilon(x+y)\}$$

The entire construction can be seen in Figure 5.2.

Claim 3. *For every rectangle $r \in \mathcal{L}$ and a pair (A, B) of the SSPD s.t. $r_{1-\varepsilon} \cap A \neq \emptyset$ and $r_{1-\varepsilon} \cap B \neq \emptyset$, there are two points $a \in A, b \in B$ connected by an edge.*

Proof. Let $A' = A \cap r_{1-\varepsilon}, B' = B \cap r_{1-\varepsilon}$, and let $p = \underset{p'}{argmax}\{||p'||_\infty \ :\ p' \in A \cup B\}$, and assume w.l.o.g that $p \in A'$ and prove that there exist a point $q \in B'$ connected to p by an edge.

We take a point $q' \in B'$. Due to the choice of p we have that one of the coordinates of q' has a smaller absolute value than the same respective

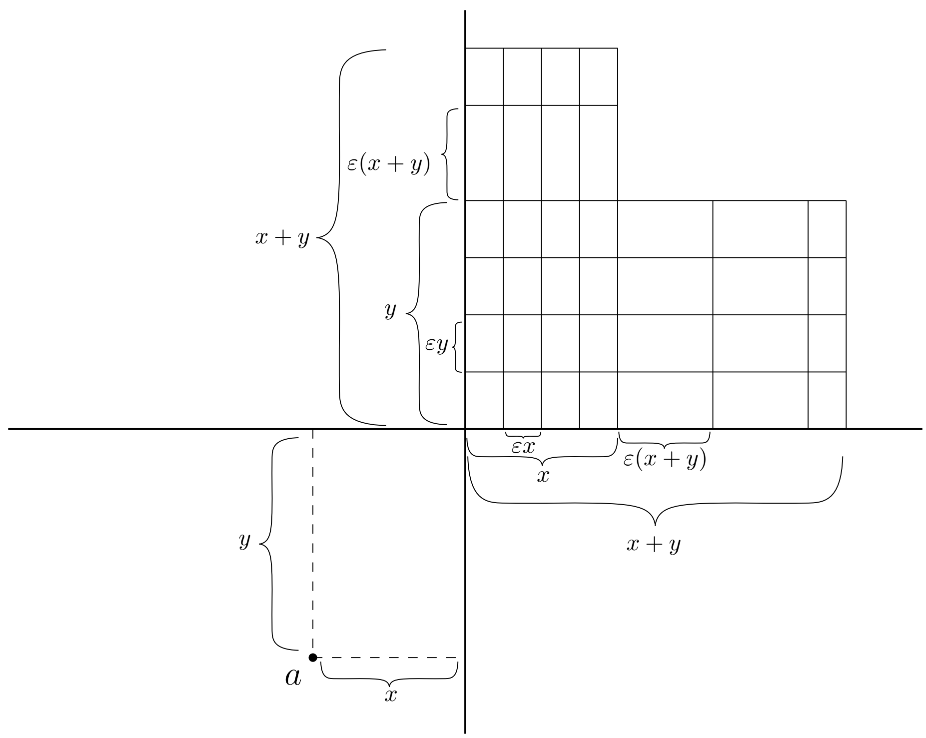
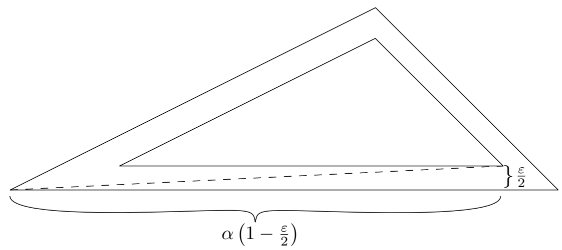


Figure 1: The construction of the grid for the arbitrary axis parallel rectangle local spanner.



coordinate of p, and assume w.l.o.g that it is the x-coordinate. Now, since $\bigcup C_{i,j} \bigcup V_i$ cover the entire part of Q^+ with an absolute x value lower that that of p, we have that either there is an edge $\{p, q\}$ in the graph, or there is another point q in the same cell as q'. Regardless, since the cells are of width $\varepsilon \cdot p.x$ and height $\varepsilon \cdot p.y$,and r is of width at least $p.x$ and height at least $p.y$, we get that the entire cell is inside r, and therefore there exists an edge as described in the claim.

□

5.3 Bounded aspect ratio triangles

The aspect ratio of a triangle is defined as the length of its longest edge divided by its height as it is measured from that edge. Let $\mathcal{L}$ be the set of all triangles with aspect ratio at most α for some $1 < \alpha$. We define a set of slopes, and for each subset of 3 slopes we run the convex region algorithm with $\mathcal{L}$ as homothets of a triangle with edges of the 3 chosen slopes. As long as the fixed angular interval is smaller than $\varepsilon' = \arctan\left(\frac{\varepsilon/2}{\alpha(1-\varepsilon/2)}\right)$ (see figure 5.3).

This construction creates $\frac{1}{\varepsilon'}$ different convex local spanners, and so we get a $(1+\varepsilon)$-local spanner for triangles with bounded aspect ratio in $O\left(\frac{1}{\varepsilon'^3\varepsilon^3} n \log n\right)$.