Fault-Tolerant and Local Spanners Revisited

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Abstract

For a set of points $P \in \mathbb{R}^d$, and a family of regions $\mathcal{F}$, a geometric local t-spanner of P, is a sparse graph G over P, that remains a t-spanner even when restricted to a region $\mathscr{r} \in \mathcal{F}$. That is, for any region $\mathscr{r} \in \mathcal{F}$, the subgraph restricted to $\mathscr{r}$, denoted by $G \cap \mathscr{r}$, is still a t-spanner. Here $G \cap \mathscr{r}$ is the subgraph of G induced on $P \cap \mathscr{r}$. A weak $\mathcal{F}$-local spanner of P provides the same guarantee for some smaller subregion of $\mathscr{r}$.

In this paper, we present algorithms for the construction of local of points in $\mathbb{R}^2$ with respect to several families of convex regions. This includes an improvement of the known construction for axis parallel squares, and a construction for disks that does not require Steiner points, along with a matching lower bound. The last result settles an open problem raised by Abam and Borouny.

1. Introduction

Given a set of points P in $\mathbb{R}^d$ and a parameter $t > 1$, the problem of constructing graph in which shortest paths approximate the Euclidean distance between every pair of points within a factor of t, is a well known and well studied problem in computation geometry. Recently, Abam and Borouny [AB21] introduced the problem of designing such spanners, with the additional requirement that subgraphs of the spanner created by "cropping" it , i.e. only considering points and edges contained in some region taken from a pre-defined set $\mathcal{F}$, maintain the distance approximation property. This property can also be thought of as fault-tolerance, where the faults, or attacks as they are sometimes called, are complements of members of $\mathcal{F}$.

Euclidean graph. For a set P of points in $\mathbb{R}^d$, a *Euclidean graph* $G_P = (P, E)$ is an undirected graph with P as the set of vertices. An edge $pq \in E$ is naturally associated with the segment $\overline{pq}$, and weight of the edge is the (Euclidean) length of the segment.

t-spanners. Let $G = (P, E)$ and $G' = (P, E')$ be two graphs over the same set of vertices. Consider a pair of vertices $p, q \in P$. For a parameter $t \geq 1$, a path between p and q in G' is a *t-path* w.r.t. G, if the length of the path is at most $t \cdot \mathsf{d}(p, q)$, where $\mathsf{d}(p, q)$ is the length of the shortest path between

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$p, q \in P$ in the graph G. The graph G' is a *t-spanner* of G if there is a t-path between any pair of points $p, q \in P$. For $P \subseteq \mathbb{R}^d$ we say that a graph G is a t-spanner of P if it is a t-spanner of G_P. Throughout the paper, n denotes the cardinality of the point set P, unless stated otherwise.

Residual graphs. Let $\mathcal{F}$ be a family of regions in the plane. For a fault region $\boldsymbol{r} \in \mathcal{F}$ and a geometric graph G on a point set P, let $G - \boldsymbol{r}$ be the residual graph after removing from it all the points of P in $\boldsymbol{r}$. and all the edges that intersect $\boldsymbol{r}$. Formally, let

$$G - \boldsymbol{r} = \big(P \setminus \boldsymbol{r}, \{uv \in E \mid uv \cap \mathrm{int}(\boldsymbol{r}) = \emptyset\}\big),$$

where $\mathrm{int}(\boldsymbol{r})$ denotes the interior of $\boldsymbol{r}$. Similarly, let

$$G \cap \boldsymbol{r} = \big(P \cap \boldsymbol{r}, \{uv \in E \mid uv \subseteq \boldsymbol{r}\}\big).$$

be the residual graph after restricting G to the region $\boldsymbol{r}$.

Fault-tolerant and local spanners. An *$\mathcal{F}$-fault-tolerant spanner* for $P \subseteq \mathbb{R}^d$, is a graph $G = (P, E)$, such that for any region $\boldsymbol{r}$ (i.e., the "attack"), the graph $G - \boldsymbol{r}$ is a t-spanner of $G_P - \boldsymbol{r}$. Surprisingly, as shown by Abam *et al.* [AdBFG09], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

In the same spirit, a graph $G = (P, E)$ is an *$\mathcal{F}$-local spanner* for P if for any region $\boldsymbol{r} \in \mathcal{F}$, we have that $G \cap \boldsymbol{r}$ is a t-spanner of $G_P \cap \boldsymbol{r}$. The notion of local-spanners was defined by Abam and Borouny [AB21] who showed how to construct such spanners for axis-parallel squares and vertical slabs. They also showed how to construct such spanners for disks, if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

Relted work

Geometric spanners have been widely studied, see [NS07]. Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size k of vertices and edges has failed. Levcopoulos *et al.* [LNS02] showed the existence of k-vertex/edges fault tolerant spanners for a set of points P in some metric space. Their spanner had $O(kn \log n)$ edges, and weight, i.e. sum of edge weights, bounded by $f(k) \cdot wt(MST(P))$ for some function f. Lukovszki [Luk99] later achieved a similar construction, improving the number of edges to $O(kn)$ and was able to prove that the result is asymptotically tight.

Abam *et al.* [AdBFG09] introduced region-fault tolerant spanners, where the faults were not arbitrary sets of vertices and/or edges, but instead all of the points and geometric edges (segments between points) intersecting some region. They showed several results, including a construction of convex regions fault tolerant t-spanners of size $O(n \log n)$. Later Abam and Borouny [AB21] introduced the concept of local spanners, and showed constructions of local t-spanners of size $O(n \cdot polylog(n))$ for axis-parallel squares and vertical slabs, and also showed that constructions using $O(n)$ Steiner points for the same cases as well as for the case of disk local spanners.

Our results

Disks In Section 2 we present a construction of spanners, which surprisingly, is not only fault-tolerant for convex regions, but it also a local spanner for disks. This resolves the aforementioned open problem from Abam and Borouny [AB21]. Our construction is a variant of the original construction of Abam *et al.* [AdBFG09]. For a parameter $\varepsilon > 0$ the construction of a $(1+\varepsilon)$-local spanner for disks takes $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$ time, and the resulted spanner is of size $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$, where Φ is the spread of the point set. We also provide a lower bound showing that this logarithmic dependency on Φ cannot be avoided.

In Section 2.3 we extend this construction to scaled and translated copies (*homothets*) of a convex shape $\mathcal{C}$.

Squares In Section 3 we show a construction similar to that of Section 2, but prove that for the case of axis parallel square local spanners we are able to produce a spanner of size $\mathcal{O}(\varepsilon^{-3} n \log n)$, that is, independent of the spread of the point set.

Triangles In Section 4 we give a construction of local spanners for the family $\mathcal{F}$ of homothets of a given triangle $\triangle$, and get a spanner of size $O\left((\alpha\varepsilon)^{-1} n\right)$ in $O\left((\alpha\varepsilon)^{-1} n\right) \log n$ time, where α is the smallest angle in $\triangle$. We also show that if we allow α to be arbitrarily small there exists a set of points that requires $O(n^2)$ edges for some triangles.

For the following families of regions, it is possible to show that even after constraining parameter such as fatness/aspect ratio, a local spanner with respect to the set of regions might require a quadratic number of edges. We therefore present constructions of weak spanners for those families, where the mathematical definition of "weak" changes between the two section, even though the two definitions are very closely related. In both cases, the size of the spanner and the construction time both depend on the "weakness" of the requested spanner, parameterized by $\delta \in (0,1)$.

Convex regions In Section 5 we construct weak local spanners for homothets of a given convex region $\mathcal{C}$ with a bounded aspect ratio. We are able to show an algorithm for constructing a δ-weak $\mathcal{C}$-local $(1+\varepsilon)$-spanner of size $\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2})n\big)$ in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time.

Rectangles In Section 6 we describe a new pair decomposition data structure, the *Quadrant Separated Pair Decomposition* (QSPD), and use it to construct a weak local spanner for axis parallel rectangles.We get a δ-weak local $(1+\varepsilon)$-spanner for the axis parallel rectangles, with size $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2) n \log^2 n)$, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2) n \log^2 n)$ time.

See Figure 1.1 for a summary of known results and comparisons to the results of this paper.

2. Local spanner for disks

Our purpose here is to build a local spanner for disks.

2.1. Preliminaries

2.1.1. Well separated pair decomposition

For sets X, Y, let $X \otimes Y = \{\{x, y\} \mid x \in X,\, y \in Y, x \neq y\}$ be the set of all the (unordered) pairs of points formed by the sets X and Y.

Region	Known # edges	Paper	New # edges	Location in paper
Local $(1+\varepsilon)$-spanners				
Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AdBFG09]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[AB21]	$\mathcal{O}(\varepsilon^{-3} n \log n)$	Theorem 3.3
Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AB21]		
Disks with Steiner points	$\mathcal{O}_\varepsilon(n)$	[AB21]		
Disks			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 2.13
			$\Omega(n \log \Phi)$	Lemma 2.16
Homothets of a convex shape			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 2.20
α-fat triangles			$\mathcal{O}((\alpha\varepsilon)^{-1} n)$	Theorem 4.6
δ-weak local $(1+\varepsilon)$-spanners				
Any bounded convex shape			$\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2}) n\big)$	Lemma 5.3
$(1-\delta)$-local $(1+\varepsilon)$-spanners				
Axis-parallel rectangles			$\mathcal{O}\big((\varepsilon^{-2} + \delta^{-2}) n \log^2 n\big)$	Theorem 6.6

Figure 1.1: Known and new results. The notation $\mathcal{O}_\varepsilon$ hides polynomial dependency on ε which is not specified in the original work.

Definition 2.1 (Pair decomposition). For a point set P, a ***pair decomposition*** of P is a set of pairs

$$\mathcal{W} = \big\{ \{X_1, Y_1\}, \ldots, \{X_s, Y_s\} \big\},$$

such that (I) $X_i, Y_i \subseteq P$ for every i, (II) $X_i \cap Y_i = \emptyset$ for every i, and (III) $\bigcup_{i=1}^s X_i \otimes Y_i = P \otimes P$.

Definition 2.2. Given a pair decomposition $\mathcal{W} = \big\{ \{X_1, Y_1\}, \ldots, \{X_s, Y_s\} \big\}$ of a point set P, its ***weight*** is $\omega(\mathcal{W}) = \sum_{i=1}^s (|X_i| + |Y_i|)$.

The ***closest pair*** distance of a set of points $P \subseteq \mathbb{R}^d$, is $\mathrm{cp}(P) = \min_{p,q \in P, p \neq q} \|p - q\|$. The ***diameter*** of P is $\mathrm{diam}(P) = \max_{p,q \in P} \|p - q\|$. The ***spread*** of P is $\Phi(P) = \mathrm{diam}(P)/\mathrm{cp}(P)$, which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD can be quadratic, if the spread is bounded, the weight is near linear.

Definition 2.3. The pair of sets $X, Y \subseteq \mathbb{R}^d$ is $(1/\varepsilon)$***-well-separated*** if

$$\max(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \varepsilon \cdot \mathsf{d}(X, Y),$$

where $\mathsf{d}(X, Y) = \min_{p \in X, q \in Y} \|p - q\|$.

Definition 2.4. For a point set P, a ***well-separated pair decomposition*** (***WSPD***) of P with parameter $1/\varepsilon$ is a pair decomposition of P with a set of pairs $\mathcal{W} = \big\{ \{B_1, C_1\}, \ldots, \{B_s, C_s\} \big\}$, such that, for all i, the sets B_i and C_i are $(1/\varepsilon)$-separated.

Lemma 2.5 ([AH12]). *Let P be a set of n points in $\mathbb{R}^d$, with spread $\Phi = \Phi(P)$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-WSPD $\mathcal{W}$ for P of total weight $\mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}\big(\varepsilon^{-d} \log \Phi\big)$ pairs. Namely, $\omega(\mathcal{W}) = O(\varepsilon^{-d} n \log \Phi)$.*

2.1.2. Semi separated pair decomposition

Definition 2.6. Two sets of points B and C are $(1/\varepsilon)$***-semi-separated*** if

$$\min(\mathrm{diam}(B), \mathrm{diam}(C)) \leq \varepsilon \cdot \mathsf{d}(B, C),$$

For a point set P, a ***semi-separated pair decomposition*** (***SSPD***) of P with parameter $1/\varepsilon$, denoted $(1/\varepsilon)$-SSPD, is a pair decomposition of P formed by a set of pairs $\mathcal{W}$ such that all the pairs are $1/\varepsilon$-semi-separated.

Theorem 2.7 ([AH12, Har11]). *Let P be a set of n points in $\mathbb{R}^d$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-SSPD for P of total weight $\mathcal{O}\left(n\varepsilon^{-d}\log n\right)$. The number of pairs in the SSPD is $\mathcal{O}\left(n\varepsilon^{-d}\right)$, and the computation time is $\mathcal{O}\left(n\varepsilon^{-d}\log n\right)$.*

Definition 2.8. A *ε-double-wedge* is a region between two lines, where the angle between the two lines is at most ε.

Lemma 2.9. *Given an α-SSPD $\mathcal{W}$ of a set P of n points in $\mathbb{R}^d$ and a parameter $\beta \geq 2$, one can refine $\mathcal{W}$ into an $\alpha\beta$-SSPD $\mathcal{W}'$, such that that $|\mathcal{W}'| = O(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\beta^d)$.*

Proof: The algorithm scans the pairs of $\mathcal{W}$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$. Let $\mathfrak{s}$ be the smallest axis-parallel cube containing X, and denote its sidelength by r. Let $r' = r/\left\lceil \sqrt{d}\beta \right\rceil$. Partition $\mathfrak{s}$ into a grid of cubes of sidelength r', and let T_Ξ be the resulting set of squares. The algorithm now add the set pairs

$$\{\{X \cap \mathfrak{t}, Y\} \mid \mathfrak{t} \in T_\Xi\}$$

to the output SSPD. Clearly, the resulting set is now $\alpha\beta$-semi separated, as we chopped the smaller part of each pair into β smaller portions. ■

Lemma 2.10. *Given a ε^{-1}-SSPD $\mathcal{W}$ of n points in the plane, one can refine it, into a ε^{-1}-SSPD $\mathcal{W}'$, such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε-double-wedge $\times_\Xi$, such that X and Y are contained in the two different faces of the double wedge $\times_\Xi$. We have that $|\mathcal{W}'| = O(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\varepsilon)$. The construction time is proportional to the weight of $\mathcal{W}'$.*

Proof: By using Lemma 2.9, we can assume that $\mathcal{W}$ is (say) $(10/\varepsilon)$-separated. Now, the algorithm scans the pairs of $\mathcal{W}$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$. Let $\square$ be the smallest axis-parallel square containing X, centered at point o. Partition the plane around o, by drawing around it $\mathcal{O}(1/\varepsilon)$ lines with the angle between any two consecutive lines being at most (say) $\varepsilon/4$, see Figure 2.1. This partitions the plane into a set of cones $\mathcal{C}$. For a cone $C \in \mathcal{C}$, we show that there exists an ε-double-wedge that contains X in one side, and $Y \cap C$ in the other.

To see that, take the double-wedge formed by the cross tangents between $\mathsf{ch}(X)$ and $\mathsf{ch}(Y \cap C)$, where $\mathsf{ch}(X)$ denotes the convex-hull of X. Assume w.l.o.g that $\square$ has side length 1, and let c be a cone of angle $\varepsilon/4$ with apex o, whose angular bisector is a horizontal ray in the positive direction of the x axis. See figure Figure 2.2 for an illustration.

We would like to find a vertical segment s such that all points of Y lie to its right, with one endpoint on the upper line of c, and the other on the lower line of c. Using the segments' height and distance from the right side of $\square$ we will be able to get a bound on the angle of the cross tangents. We first find

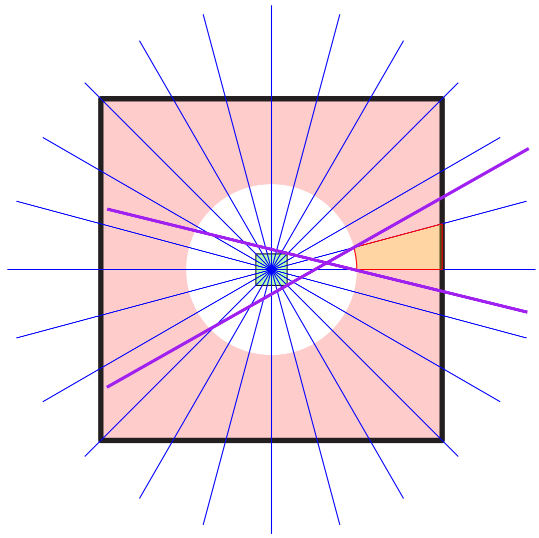

Figure 2.1: An illustration of refining the pairs in a SSPD into pairs in opposite parts of an ε-double-wedge. X is contained in the green square $\square$, while Y is contained in the red square, and the white gap between them is a result of the separation property. The set of cones with the apex at the center of $\square$ give us the desired partition as demonstrated by the purple double-wedge.

a segment s with all points of Y to its right. A trivial bound on that distance is given by the segment from, say, the lower left corner of $\square$, denoted p, of length $10/\varepsilon$ with its right endpoint on the upper line of c, denote this point by q. This is due to the $10/\varepsilon$ separation property of the SSPD. We know that this segment creates an angle of less than $\pi/4$ with the x-axis, since o is the center of $\square$, and lies on the ray with apex p that creates a $\pi/4$ angle with the x-axis. We therefore get that the x-coordinate difference between $\square$ and q is at most $10/\varepsilon \cdot \cos\frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$. So let s' be a vertical segment between the upper and lower rays of c, with x-coordinate distance of $6/\varepsilon - \frac{1}{2}$ from $\square$ (in order to make calculations easier). We get that s' is of length $2 \cdot \frac{6}{\varepsilon}\tan\frac{\varepsilon}{8}$. Finally, we take s to be a vertical segment of length $\frac{12}{\varepsilon}\tan\frac{\varepsilon}{8}$, with its center on the x-axis at a distance of $5/\varepsilon + \frac{1}{2}$ away from o. The angle of the x-axis and the segment between the lower end of the right side of $\square$ and the upper end of s is now given by:

$$\arctan\left(\frac{\frac{6}{\varepsilon}\tan\frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5}\tan\frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) = \Theta(\varepsilon) \qquad \blacksquare$$

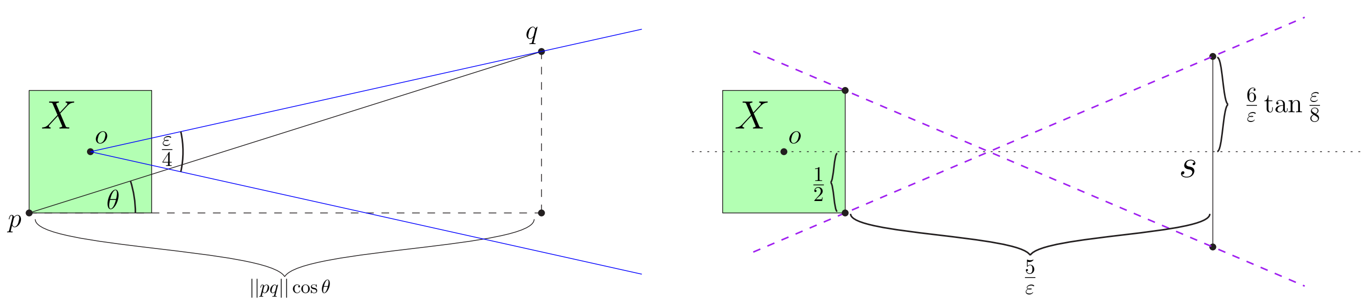

Figure 2.2: An illustration of the proof for Lemma 2.10

2.1.3. Delaunay triangulation

We need the following well known property of Delaunay triangulation, which would play a center role in our construction.

Claim 2.11. *For a set of points $P \subseteq \mathbb{R}^2$ in general position, let $\mathcal{D} = \mathcal{DT}(P)$ denote its Delaunay triangulation. Then, for any closed disk $\bigcirc$, we have $\mathcal{DT}(P) \cap \bigcirc$ is connected.*

Proof: We first prove that for any (closed) disk $\bigcirc$ with two points $p, q \in P$ on its boundary, there is a path between p and q in $\mathcal{D} \cap \bigcirc$. The proof is by induction over the number m of points of P in the interior of $\bigcirc$:

- $m = 0$: The disk $\bigcirc$ contains no points of P in its interior, and thus pq is an edge of the Delaunay triangulation, as $\bigcirc$ testifies.

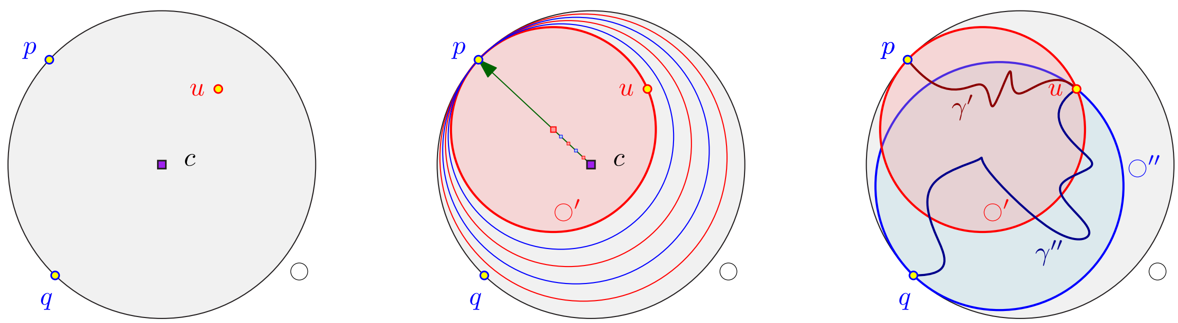

Figure 2.3

- $m > 0$: Let $u \in P$ be a point in the interior of $\bigcirc$. We move the center c of $\bigcirc$ in the direction of p, shrinking $\bigcirc$ in the process, so that the radius the disk is $\|c - p\|$, until we get a disk $\bigcirc' \subseteq \bigcirc$ such that u is on the boundary of $\bigcirc'$, see Figure 2.3. Observe that p and u are on the boundary of the new disk, and $|\mathrm{int}(\bigcirc') \cap P| < |\mathrm{int}(\bigcirc) \cap P|$. Thus, by induction, there is a path γ' between p and u in $\mathcal{D} \cap \bigcirc' \subseteq \mathcal{D} \cap \bigcirc$. Similarly, there must be a path γ'' between u and q, and concatenating the two paths results in a path between p and q in $\mathcal{D} \cap \bigcirc$.

Back to the original claim. For any two points $p, q \in \bigcirc \cap P$ one can get a disk $\bigcirc' \subseteq \bigcirc$ that contains p and q on its boundary. Indeed, shrink the radius of $\bigcirc$ till, say, p is on the boundary, and then move the center of the disk towards p while shrinking the size of the disk to maintain p on the boundary, until q is also on the boundary of the shrunken disk. ■

2.2. The construction of local spanners for disks

2.2.1. The construction

The input is a set P of n points in the plane (in general position) with $\Phi = \Phi(P)$, and a parameter $\varepsilon \in (0, 1)$.

The algorithm computes a $1/\vartheta$-WSPD $\mathcal{W}$ of P using the algorithm of Lemma 2.5, where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the Delaunay triangulation $\mathcal{D}_\Xi = \mathcal{DT}(X \cup Y)$. The algorithm adds all the edges in $\mathcal{D}_\Xi \cap (X \otimes Y)$ to the computed graph G.

2.2.2. Analysis

Size. For each pair $\Xi = \{X, Y\}$ in the WSPD, its Delaunay triangulation contains at most $\mathcal{O}(|X|+|Y|)$ edges. As such, the number of edges in the resulting graph is bounded by

$$\sum_{\{X,Y\}\in\mathcal{W}} O\big(|X| + |Y|\big) = O(\omega(\mathcal{W})) = O\left(\frac{n \log \Phi}{\vartheta^2}\right),$$

by Lemma 2.5.

Construction time. The construction time is bounded by

$$\sum_{\{X,Y\}\in\mathcal{W}} O\big((|X| + |Y|)\log(|X| + |Y|)\big) = O(\omega(\mathcal{W})\log n) = O\left(\frac{n \log \Phi \log n}{\vartheta^2}\right),$$

Local spanner property.

Lemma 2.12. *Let G be the graph constructed above for the point set P. Then, for any (closed) disk $\bigcirc$, and any two points $x, y \in P \cap \bigcirc$, we have that $G \cap \bigcirc$ has a $(1+\varepsilon)$-path between x and y. That is, G is a $(1+\varepsilon)$-local spanner for disks.*

Proof: The proof is by induction on the distance between p and q (or more precisely, the rank of their distance among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and $y \in Y$.

For the base case, consider the case that x is the nearest-neighbor to y in P, and y is the nearest-neighbor to x in P. It must be, because of the separation property of Ξ, that X and Y are singletons. Indeed, if X contains another point, then y would not be the nearest-neighbor to x (this is true for $\vartheta < 0.5$). As such, $xy \in \mathcal{D}_\Xi$, $x, y \in \bigcirc$, and the edge $xy \in E(G)$, implying the claim.

For the inductive step, observe that the claim follows if $xy \in \mathcal{D}_\Xi$, so assume this is not the case. By the connectivity of $\mathcal{D}_\Xi \cap \bigcirc$, see Claim 2.11, there must be points $x' \in X \cap \bigcirc$, $y' \in Y \cap \bigcirc$, such that $x'y' \in E(\mathcal{D}_\Xi)$. As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property, we have that

$$\max(\mathrm{diam}(X), \mathrm{diam}(Y)) \leq \vartheta \cdot \mathsf{d}(X, Y) \leq \vartheta\ell,$$

where $\ell = \|x - y\|$. In particular, $\|x' - x\| \leq \vartheta\ell$ and $\|y' - y\| \leq \vartheta\ell$. As such, by induction, we have $\mathsf{d}_G(x, x') \leq (1+\varepsilon)\|x - x'\| \leq (1+\varepsilon)\vartheta\ell$ and $\mathsf{d}_G(y, y') \leq (1+\varepsilon)\|y - y'\| \leq (1+\varepsilon)\vartheta\ell$. Furthermore, $\|x' - y'\| \leq (1 + 2\vartheta)\ell$. As $x'y' \in E(G)$, we have

$$\begin{aligned}\mathsf{d}_G(x, y) &\leq \mathsf{d}_G(x, x') + \|x' - y'\| + \mathsf{d}_G(y', y) \leq (1+\varepsilon)\vartheta\ell + (1+2\vartheta)\ell + (1+\varepsilon)\vartheta\ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta)\ell \\ &= (1 + 6\vartheta)\ell \leq (1+\varepsilon)\|x - y\|,\end{aligned}$$

if $\vartheta \leq \varepsilon/6$. ∎

The result.

Theorem 2.13. *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter. The above algorithm constructs a local $(1+\varepsilon)$-spanner G for disks. The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any disk $\bigcirc$ in the plane, and any two points $p, q \in P \cap \bigcirc$, we have a $(1+\varepsilon)$-path in $G \cap \bigcirc$.*

2.2.3. Applications and comments

Definition 2.14. Given a region R in the plane and a point set P, consider two points $p, q \in P$. The edge pq is *safe* in R, if there is a disk $\bigcirc$ such that $p, q \in \bigcirc \subseteq R$. Let $\mathcal{G}(P, R)$ be the graph formed by all the safe edges in P for R. Note, that this graph might have a quadratic number of edges in the worst case.

Observe that $\mathcal{G}(\mathbb{R}^2, P)$ is a clique.

Corollary 2.15. *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter, and let G be a local $(1 + \varepsilon)$-spanner of P for disks. Then, for a region R in the plane, if we denote the graph $H = \mathcal{G}(P, R)$, we have that $G \cap R$ is a $(1 + \varepsilon)$-spanner for $H \cap R$. Formally, for any two points $p, q \in P \cap R$, we have that $\mathsf{d}_H(p, q) \leq (1 + \varepsilon)\mathsf{d}_G(p, q)$.*

In particular, for any convex region C, the graph $G - C$ is a $(1 + \varepsilon)$-spanner for $\mathcal{G}(\mathbb{R}^2, P) - C$.

Proof: Consider the shortest path $\pi = u_1 u_2 \dots u_k$ between p and q in $\mathsf{d}_H(p, q)$. Every edge $e_i = u_i u_{i+1}$ has a disk $\bigcirc_i$ such that $u_i, u_{i+1} \in \bigcirc_i \subseteq R$. As such, there is a $(1 + \varepsilon)$-path between u_i and u_{i+1} in $G \cap \bigcirc_i \subseteq G \cap R$. Concatenating these paths directly yields the desired result.

The second claim follows by observing that the complement of C is the union of halfspaces, and halfspaces can be considered to be "infinite" radius disks. As such, the above argument applies verbatim. ■

But why not SSPD? The result of Theorem 2.13 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). A natural way is to try and emulate the construction of Abam *et al.* [AdBFG09] and use a SSPD instead of a WSPD. The total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε-separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are of different sides of the double wedge. The problem is that for the local disk $\bigcirc$, it might be that the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \bigcirc$ is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 2.4.

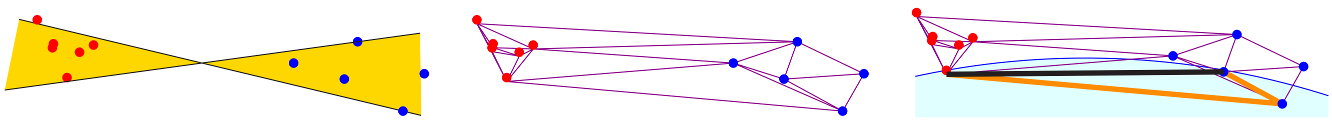

Figure 2.4: A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

2.2.4. A lower bound for local spanner for disks

Lemma 2.16. *For $\varepsilon = 1/4$, and parameters n and $\Phi \geq 1$, there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $O(n\Phi)$, such that any local $(1 + \varepsilon)$-spanner of P for disks, must have $\Omega(n \log \Phi)$ edges.*

Proof: Let $p_i = (-i, 0)$, for $i = 1, \dots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$ and $q_1 = (n2^M, -1)$. For a point p on the x-axis, and a point q below the x-axis, and to the right of p, let $\bigcirc_\downarrow^p(q)$ be the disk whose boundary passes through p and q, and its center has the same x-coordinate as p.

In the jth iteration, for $j = 2, \dots, M - 1$, Let $x_j = n2^{M-j+1} = x(q_{j-1})/2$, and let $y_j < 0$ be the maximum y-coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the disks of $D_1 \cup \dots \cup D_j$. Let $q_j = (x_j, 0.99y_j)$. Consider the set of disks

$$D_j = \left\{ \bigcirc_\downarrow^{p_i}(q_{j-1}) \;\middle|\; i = 1, \dots, n \right\},$$

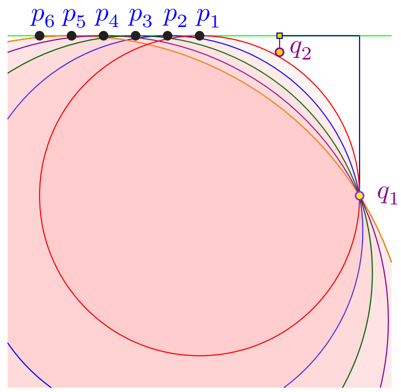

Figure 2.5: The set of disks D_1, and the construction of q_2.

see Figure 2.5.

Clearly, the point q_j lies outside all the disks of $D_1 \cup \ldots \cup D_j$. The construction now continues to the next value of j. Let $P = \{p_1, \ldots, p_n, q_2, \ldots, q_M\}$. We have that $|P| = n + M - 1$.

The minimum distance between any points in the construction is 1 (i.e., $\|p_1 - p_2\|$). Indeed $x(q_M) = 2n$ and thus $\|q_M - p_1\| \geq 2n$. The diameter of P is $\|p_1 - q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = O(n\Phi)$.

For any i and j, consider the disk $\bigcirc_{\downarrow}^{p_i}(q_j)$. This disk does not contain any point of $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ since its interior lies below the x-axis. By construction it does not contain any point $q_{j+1}, \ldots, q_{M-1}$. This disk potentially contains the points $q_{j-1}, \ldots, q_1$, but observe that for any index $k \in [\![j-1]\!]$, we have that

$$\|p_i - q_k\| = \sqrt{\left(i + n2^{M-k+1}\right)^2 + \left(y(q_j)\right)^2},$$

which implies that $n2^{M-k+1} \leq \|p_i - q_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$\frac{\|p_i - q_k\|}{\|p_i - q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since $j \in [\![M-1]\!]$. Namely, the shortest path in G between p_i and q_j, can not use any of the points $q_1, \ldots q_{j-1}$. As such, the graph G must contain the edge $p_i q_j$. This implies that $|E(G)| \geq n(M-1)$, which implies the claim. ■

2.3. Homothets of a convex region

Using arguments similar to those used for disk local spanners, we now extend the results for the case where $\mathcal{F}$ is the set of all scaled and translated copies of a convex shape $\mathcal{C}$. This family of regions is also known as homothets of $\mathcal{C}$. Formally, the homothet of the shape $\mathcal{C}$ with center o and scaling λ is the set $\{o + \lambda op : p \in \mathcal{C}\}$. See Figure 2.6. While the Delaunay triangulation is not well defined for all convex shapes, the operation of creating edges between every two points $p, q \in P$ such that there exist a homothet of $\mathcal{C}$ that contains only p and q and no other point of P, is well defined for a point set in general position, and gives us a graph known as the $\mathcal{C}$-Delaunay graph of P. We denote the $\mathcal{C}$-Delaunay graph of P by $\mathcal{DG}_{\mathcal{C}}(P)$. Note that in these settings, the meaning of general position is that no 3 points of P lie on the boundary of a homothet of $\mathcal{C}$.

Claim 2.17. *Given a set of points $P \subseteq \mathbb{R}^2$ in general position, and a convex shape $\mathcal{C}$, let $\mathcal{DG} = \mathcal{DG}_{\mathcal{C}}(P)$ denote the $\mathcal{C}$-Delaunay graph of P. For any homothet C of $\mathcal{C}$, we have $\mathcal{DG} \cap C$ is connected.*

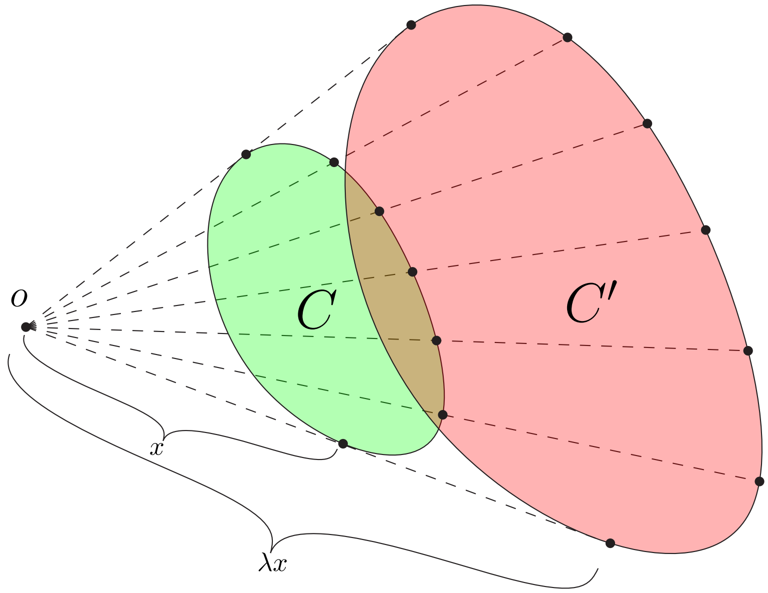

Figure 2.6: C' is a homothet of C of center o and scaling λ.

Proof: This proof very closely resembles that of Claim 2.11. The second part of both claims is actually identical, and we therefore only prove that for any (closed) convex region $\mathcal{C}$ with two points $p, q \in P$ on its boundary, there is a path between p and q in $\mathcal{DG} \cap C$. The proof is again by induction over the number of points of P in the interior of C, and is very similar to its counterpart in Claim 2.11. The only difference here is that shrinking an arbitrary convex region towards a point on the boundary is not well defined. Luckily, it is not hard to prove that such a procedure is possible, and after proving this, the rest of the proof is exactly as that of Claim 2.11. So, let C be a homothet of $\mathcal{C}$ such that $p \in P \cap \partial C$. We define shrinking C by a factor of λ towards p as a homothetical translation of C with center p and scaling factor λ. Since C is convex we have that the resulted region C' is contained in C, and by the definition of a homothet we have that C' is a homothet of C. This finishes the proof since we can now follow the proof of Claim 2.11.

Notice that for any convex region $\mathcal{C} \subseteq \mathbb{R}^2$ and two points $p, q \in C$, there exists a homothet C of $\mathcal{C}$ with p and q on its boundary, and such that $C \subseteq \mathcal{C}$. This can be seen in much the same way that a similar claim is proven for disks, as we can shrink $\mathcal{C}$ towards the center of the segment $\overline{pq}$ until one of the points lie on the boundary of the shape, and then shrink it further toward that point, which now lies on the boundary, until the second point is on the boundary as well. Since the center of the shrinking process are always contained in the shrinking convex shape, we get that at it is always inside the its unshrinked version.

We will need the following from the paper of Chew and Drysdale [CDI85] in order to construct the spanner, and bound its size similarly to the local disk spanner.

Theorem 2.18 ([CDI85]). *For any convex shape $\mathcal{C}$ and a set of points P, $\mathcal{DG}_{\mathcal{C}}(P)$ can be computed in $O(n \log n)$ time.*

Lemma 2.19 ([CDI85]). *For any convex shape $\mathcal{C}$ and a set of points P, $\mathcal{DG}_{\mathcal{C}}(P)$ has $O(n)$ edges, vertices, and faces.*

The result

Theorem 2.20. *Let P be a set of n points in the plane, and let $\varepsilon \in (0,1)$ be a parameter. We can constructs a local $(1+\varepsilon)$-spanner G for homothets of a convex region $\mathcal{C}$. The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any homothet C of the convex region $\mathcal{C} \subseteq \mathbb{R}^2$, and any two points $p, q \in P \cap \bigcirc$, we have a $(1+\varepsilon)$-path in $G \cap C$.*

Proof: Due to Theorem 2.18, Lemma 2.19, and Claim 2.17, we can use the algorithm in Section 2.2.1 verbatim, switching only the conventional Delaunay triangulation with the $\mathcal{C}$-Delaunay graph. Since the $\mathcal{C}$-Delaunay graph has linear complexity for any convex shape $\mathcal{C}$ the size of the spanner can be bounded using the same asymptotic bounds as the disk local spanner, and since the $\mathcal{C}$-Delaunay graph can be constructed in $O(n \log n)$ time, the time complexity bounds hold as well. The correctness proof of Theorem 2.13 uses only the connectivity of the Delaunay triangulation, and thus the same proof suffices.

3. A local spanner for axis parallel squares

One can modify the above construction for axis-parallel squares, and get a local spanner without dependency on the spread.

3.0.1. Construction

The input is a point set P of n points in the plane, and an approximation parameter $\varepsilon \in (0, 1/2)$. We assume that the input point set P is in general position. Specifically, no two points of P share a coordinate value, or appear in opposing corners of an axis-parallel square – this can be ensured by slightly perturbing the points if necessary (or symbolic perturbation).

One can define the Delaunay triangulation when the unit ball is replaced by the unit square. Formally, in this triangulation two points are connected $\iff$ there is a square that contains these two points on its boundary and no points in its interior. Let $\mathcal{D}_\square$ denote the resulting *L_∞-Delaunay triangulation*.

Let $\vartheta = \varepsilon/20$. Instead of constructing a WSPD, the algorithm computes a $1/\vartheta$-SSPD $\mathcal{W}$, using the algorithm of Theorem 2.7. By using the algorithm of Lemma 2.10, and increasing the weight and number of pairs by a factor of $\mathcal{O}(1/\vartheta)$, one can assume that every pair $\{X, Y\} \in \mathcal{W}$ is not only semi-separated, but also that there is an associated double wedge of angle $\leq \vartheta$ containing X and Y in opposing wedges. The algorithm now computes the "square" Delaunay triangulation for each such pair, and adds the edges of the triangulation to the resulting graph G.

3.0.2. Analysis

Size and running time. Computing the SSPD takes $\mathcal{O}(n\vartheta^{-2} \log n)$ time, and the refinement takes $\mathcal{O}(n\vartheta^{-3} \log n)$ time (which is also the weight of the resulting SSPD). The number of edges of each L_∞-Delaunay triangulation for a pair is proportional to its weight, which implies that the total number of edges in the resulting graph G is $\mathcal{O}(\vartheta^{-3} n \log n)$. Computing all these Delaunay triangulations takes $\mathcal{O}\big(\vartheta^{-3} n \log^2 n\big)$ time.

Shrinking squares. We need the following lemma about the shrinking of axis-parallel squares. Observe that this property definitely does not hold for disks, as illustrated in Figure 2.4.

Lemma 3.1. *(A) Let $\mathfrak{s}$ be an axis parallel square in the plane, and let p, q be two arbitrary points in $\mathfrak{s}$. Then, there is a square $\mathfrak{t} \subseteq \mathfrak{s}$ that contains p and q on its boundary.*

(B) Let $\mathfrak{s}$ be as before, and let X, Y be two point sets in the plane, such that $X' = X \cap \mathfrak{s} \neq \emptyset$ and $Y' = Y \cap \mathfrak{s} \neq \emptyset$. Let $x \in X, y \in Y$ be the two points realizing $\mathsf{d}_\infty(X', Y') = \min_{p \in X', q \in Y'} \|p - q\|_\infty$. Then, there is a square $\mathfrak{t} \subseteq \mathfrak{s}$ that contains x and y on its boundary, and $\mathfrak{t}$ does not contain any other point of $X \cup Y$.

Proof: (A) Start shrinking $\mathfrak{s}$ around its center till it contains one of the points (say p is on its boundary. Next, move the center of the square towards p till the boundary of the continuously shrinking square passes through q. If p and q lie on adjacent edges, then continue the shrinking process by moving the center towards the common corner of the shared edges – this process stops when one of the points is on the corner of the square. Clearly, the resulting square $\mathfrak{t}$ is the desired square, see Figure 3.1.

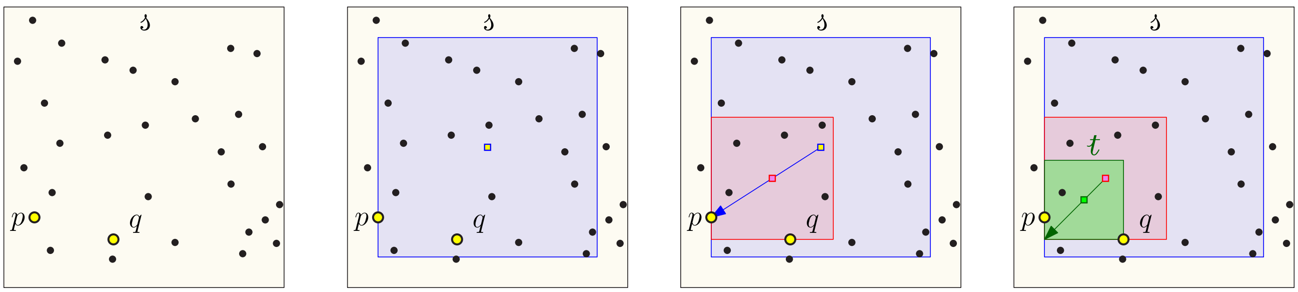

Figure 3.1

(B) Let $r = \mathsf{d}_\infty(X', Y')$. By (A), there is a square $\mathfrak{t} \subseteq \mathfrak{s}$ having x and y on apposing sides. As such, the side length of $\mathfrak{t}$ is r. Assume for contradiction, that there is some other point $x' \in X \cap \mathfrak{t}$. By our general position assumption, x' is in the interior of $\mathfrak{t}$, and in particular, $\|x' - y\|_\infty < r$, which is a contradiction to the choice of x and y. ■

Local spanner property.

Lemma 3.2. *For any axis parallel square $\mathfrak{s}$ in the plane, and any two points $p, q \in P \cap \mathfrak{s}$, we have a $(1 + \varepsilon)$-path in $G \cap \mathfrak{s}$.*

Proof: We prove the existence of a $(1 + \varepsilon)$-path between every pair $x, y \in P \cap \mathfrak{s}$ of points, by induction over the rank of $\|x - y\|$. The base case is simple, as the pair $(\{x\}, \{y\})$ is a pair in $\mathcal{W}$, and xy is thus an edge in G. Now, consider two points $x, y \in P \cap \mathfrak{s}$, where $\mathfrak{s}$ is some arbitrary square. There exists a pair $\Xi = \{X, Y\} \in \mathcal{W}$ such that $x \in X$ and $y \in Y$, and this pair is ϑ^{-1}-semi separated and is also separated by a double wedge of angle $\leq \vartheta$. See Figure 3.2. Furthermore, assume that $\operatorname{diam}(X) < \operatorname{diam}(Y)$.

Let $X' = X \cap \mathfrak{s}$ and $Y' = Y \cap \mathfrak{s}$, and consider the two points $x' \in X'$ and $y' \in Y'$ realizing $r = \mathsf{d}_\infty(X', Y')$. By Lemma 3.1 there exists a square $\mathfrak{t}$ containing x', y' on its boundary (on two apposing edges), such that $\mathfrak{t} \subseteq \mathfrak{s}$, and $\mathfrak{t}$ contains no other points $X \cup Y$. By construction, we have that $x'y'$ is in the L_∞-Delaunay triangulation of Ξ, and thus $x'y' \in G$. Since $\|x - x'\| \ll \|x - y\|$ we have that by induction $\mathsf{d}_G(x, x') \leq (1 + \varepsilon) \|x - x'\|$.

Let $\ell = \|x' - y'\|$. Due to the semi-separation property and since $\operatorname{diam}(X) < \operatorname{diam}(Y)$,

$$\|x - x'\| \leq \operatorname{diam}(X) \leq \vartheta \|X - Y\| \leq \vartheta \sqrt{2} \cdot \mathsf{d}_\infty(X, Y) \leq 2\vartheta\ell.$$

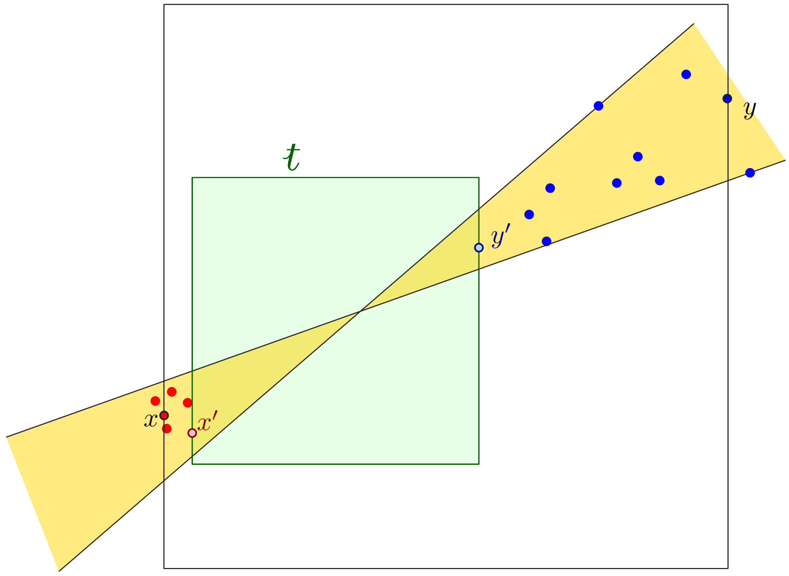

Figure 3.2: A square region t and two double-wedge semi-separated point sets X (red) and Y (blue. Notice that while x' and y' are the closest pair using L_∞, that is not necessarily true for the Euclidean distance.

Thus, we have that

$$\mathsf{d}_G(x, x') \leq (1+\varepsilon)\|x - x'\| \leq (1+\varepsilon)2\vartheta\ell \leq 4\vartheta\ell.$$

By the triangle inequality, we have

$$(1-2\vartheta)\ell \leq \|x' - y'\| - \|x - x'\| \leq \|x - y'\| \leq (1+2\vartheta)\ell.$$

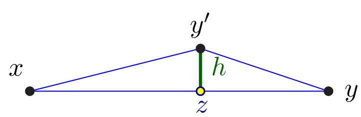

Figure 3.3: An illustration of the relative positions of x, y, y', and z. The angle of the separating double-wedge guarantees that $\angle y'xy$ is small.

Consider the triangle $\triangle xy'y$, and observe that by the double-wedge property $\alpha = \angle y'xy \leq \vartheta$. Let z be the projection of y' to xy, and let

$$h = \|y' - z\| = \|x - y'\| \sin\alpha \leq \|x - y'\| \sin\vartheta \leq \|x - y'\| \vartheta \leq \vartheta(1+2\vartheta)\ell \leq 2\vartheta\ell,$$

as $\vartheta \in (0, 1/10)$, the monotonicity of sin in this range, and as $\sin\vartheta \leq \vartheta$.

We have that $\|x - z\| \leq \|x - y'\| \leq (1+2\vartheta)\ell$. Similarly, we have

$$\|x - z\| = \|x - y'\| \cos\alpha \geq (1 - \alpha^2/2)\|x - y'\| \geq (1 - \vartheta^2/2)(1-2\vartheta)\ell \geq (1-3\vartheta)\ell.$$

By the triangle inequality, we have that

$$\|y' - y\| \geq \|x - y\| - \|y' - x\| \geq \|x - y\| - (1+2\vartheta)\ell.$$

As for an upper bound, we have

$$\begin{aligned}\|y'-y\| &\leq \|z-y\| + h \leq \|x-y\| - \|x-z\| + 2\vartheta\ell \leq \|x-y\| - (1-3\vartheta)\ell + 2\vartheta\ell \\ &= \|x-y\| - (1-5\vartheta)\ell < \|x-y\|.\end{aligned}$$

As such, by induction $\mathsf{d}_G(y',y) \leq (1+\varepsilon)\|y'-y\|$.

We thus have that

$$\begin{aligned}\mathsf{d}_G(x,y) &\leq \mathsf{d}_G(x,x') + \|x'-y'\| + \mathsf{d}_G(y',y) \leq 4\vartheta\ell + \ell + (1+\varepsilon)\|y'-y\| \\ &\leq (1+4\vartheta)\ell + (1+\varepsilon)\big(\|x-y\| - (1-5\vartheta)\ell\big) \\ &= \big[1 + 4\vartheta - (1+\varepsilon)(1-5\vartheta)\big]\ell + (1+\varepsilon)\|x-y\| \\ &\leq (1+\varepsilon)\|x-y\|,\end{aligned}$$

for $\vartheta \leq \varepsilon/20$, as $1 + 4\vartheta - (1+\varepsilon)(1-5\vartheta) \leq 1 + \varepsilon/5 - (1+\varepsilon)(1-\varepsilon/4) = \varepsilon/5 - (3/4)\varepsilon + \varepsilon^2/4 < 0$, as $\varepsilon < 1$. ∎

Theorem 3.3. *Let P be a set of n points in the plane, and let $\varepsilon \in (0,1)$ be an approximation parameter. The above algorithm computes a local $(1+\varepsilon)$-spanner G for axis parallel squares. The construction time is $\mathcal{O}\big(\varepsilon^{-3} n \log^2 n\big)$, and the spanner G has $\mathcal{O}(\varepsilon^{-3} n \log n)$ edges.*

4. Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex shape are costly, if we are given a triangle $\triangle$ with the single constraint that $\triangle$ isn't too "thin", one can construct a $\triangle$-local spanner with a number of edges that does not depend on the spread of the points. See Figure 4.1 for an illustration of a construction showing that dependency if "thin" triangles are allowed.

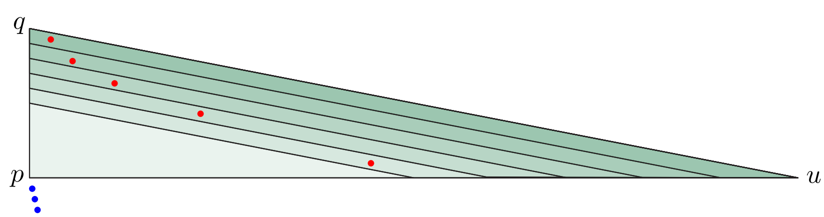

Figure 4.1: For any fixed $t > 1$ we can take a "thin" right angle triangle $\triangle$, and create a point set as shown in the figure, such that every blue point must be connected by an edge to many of the red points (which are of exponentially increasing x-coordinate) in order to create a $\triangle$-local t-spanner.

Definition 4.1. We say that a triangle $\triangle$ is α-fat if the smallest angle in $\triangle$ is of size at least α.

Claim 4.2. *For any fixed $\varepsilon \in (0,1)$, there exists a set P of $n + \log\Phi$ points with spread $O(\Phi) >> n$ and a triangle $\triangle$, such that the $\triangle$-local $(1+\varepsilon)$-spanner of P requires $O(n\log\Phi)$ edges.*

Proof: First, we describe two sets of points A and B of sizes n and $\log\Phi$ respectively, such that every point in A must be connected to $O(\log\Phi)$ points of B in any $\triangle$-local $(1+\varepsilon)$-spanner for a triangle that we will choose accordingly. Let $A = \{a_i := (-i, i/n) \mid 0 \leq i \leq n\}$, and let $B = \{b_i := (n-i, 1+2^i) \mid 1 \leq i \leq \log \Phi\}$. It is not hard to see that the spread of $A \cup B$ is indeed $O(\Phi)$.

We now take $\triangle$ to be the right angle triangle with vertices $p := (1,0), q := (1,1)$, and $u := (x,0)$, where x is sufficiently large so that for every $i \in \{1,...,n\}$ there exist a segment parallel to $\overline{qu}$ with one endpoint on $\overline{pq}$, and the other on $\overline{pu}$, that separates b_i and b_{i-1}. See Figure 4.1 for some intuition. Notice that for every point $a \in A$ and index $i \in \{0,..,\log \phi\}$, there exist a homothet $\triangle'$ of $\triangle$ such that $B \cap \triangle' = \{b_i, ..., b_n\}$.

We now show that $\|a - b_{i+1}\| + \|b_i - b_{i+1}\| \geq (1+\varepsilon)\|a - b_i\|$ for $O(n)$ indices. First we make a couple of simple observations:

1. $1 + \varepsilon \leq 2$,
2. $\|a - b_{i+1}\| \geq 2^{i+1}$,
3. $\|b_i - b_{i+1}\| \geq 2^i$,
4. and $\|a - b_i\| \leq 2n + 2^i$.

So it is enough to show that for $O(n)$ points in B we have that

$$2^{i+1} + 2^i \geq 2(2n + 2^i).$$

Simplifying a bit we get that the above inequality is true when $i \geq \log(4n)$, and we therefore have that for sufficiently many points of B there must be an edge to every point in A. ∎

4.0.1. Construction

The input is a point set P of n points in the plane, an α-fat triangle $\triangle$, and an approximation parameter $\varepsilon \in (0,1)$. We assume that the smallest angle in $\triangle$ is of size α. Let c_1, c_2 and c_3 be the 3 cones with an apex at the origin that coincide each with one of the angles of $\triangle$ when translating its respective vertex to the origin. We also denote the orthogonal direction to the edge opposite of α_i by v_i. See Figure 4.2 for an illustration. Let C_1, C_2, and C_3 be partitions of each of these cones into cones of angle at most $O(\varepsilon) \cdot \alpha$.

For each point $p \in P$ we create the set of cones C_p, which is the set of cones $C_1 \cup C_2 \cup C_3$ translated by p, and for each $c \in C_p$ we connect p to the point $q \in c$ closest to p in an ordering by v_i (where i corresponds to the set C_i that contains the un-translated copy of c).

4.0.2. Analysis

Shrinking triangles. We will need the following lemmas in order to prove the local spanning property of G. The first lemma concerns distances between points in the construction above (Section 4.0.1), and the second deals with triangle shrinking.

Lemma 4.3. *Let $p, q, u \in P$ such that the edge $\{p, u\}$ was added to the graph for the cone c with apex at p, and such that $u \in c$ (as well as p and q).*

We get that

1. $\|p - u\| + (1+\varepsilon)\|q - u\| \leq (1+\varepsilon)\|p - q\|$ *,and*
2. $\|p - u\| \leq \|p - q\|$.

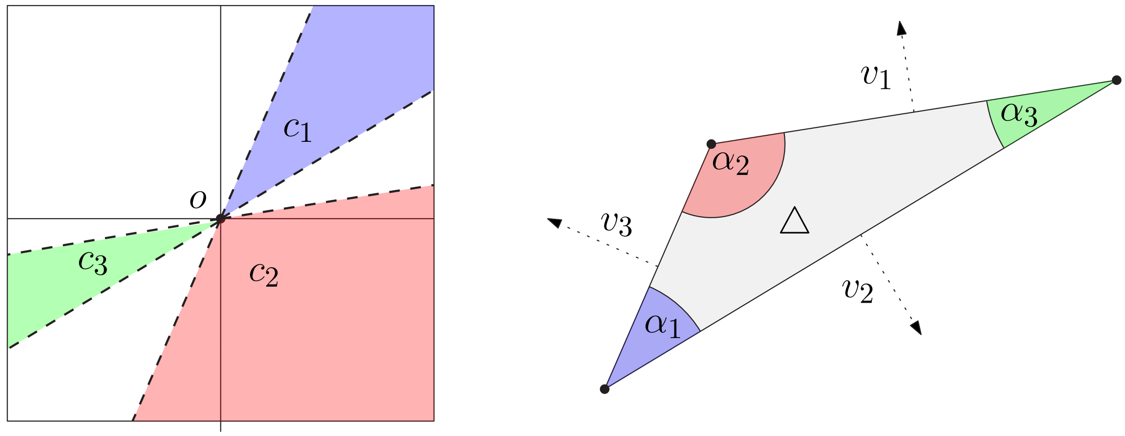

Figure 4.2: For the triangle $\triangle$ with angles α_1, α_2, and α_3 we create the cones c_1, c_2, and c_3.

Proof: We denote the smallest angle of the triangle $\triangle$ by α. Consider the triangle Δpqu and denote the angles at p, q, and u by θ_p, θ_q, and θ_u respectively.

We now consider two possible cases, either $\|p-q\| \leq \|p-u\|$ or $\|p-q\| > \|p-u\|$. Notice that since the angle of the cone c, and therefore θ_p as well, is smaller than the smallest angle of $\triangle$, we have that $\|q-u\| \leq \max\{\|p-u\|, \|p-q\|\}$.

$\|p-q\| \leq \|p-u\|$: Denote by q' the projection of q on $\overline{pu}$. We have that

$$\|q-q'\| = \sin\theta_p \|p-q\| = \sin\theta_u \|q-u\| \implies \|q-u\| = \frac{\sin\theta_p}{\sin\theta_u} \|p-q\|$$

$$\|p-u\| = \|p-q'\| + \|q'-u\| = \cos\theta_p \|p-q\| + \cos\theta_u \|q-u\| = \cos\theta_p \|p-q\| + \frac{\sin\theta_p}{\sin\theta_u} \|p-q\| = (\cos\theta_p + \frac{\sin\theta_p}{\sin\theta_u}) \|$$

We use these equalities and get that:

$$\|p-u\| + (1+\varepsilon)\|q-u\| \leq (1+\varepsilon)\|p-q\|$$

$$\iff \|p-q\| (\cos\theta_p + \frac{\sin\theta_p}{\sin\theta_u}) + (1+\varepsilon)\frac{\sin\theta_p}{\sin\theta_u} \|p-q\| \leq (1+\varepsilon)\|p-q\|$$

$$\iff (\cos\theta_p + \frac{\sin\theta_p}{\sin\theta_u}) + (1+\varepsilon)\frac{\sin\theta_p}{\sin\theta_u} \leq (1+\varepsilon)$$

$$\Longleftarrow (1 + \frac{\sin\theta_p}{\sin\theta_u}) + (1+\varepsilon)\frac{\sin\theta_p}{\sin\theta_u} \leq (1+\varepsilon)$$

$$\iff (2+\varepsilon)\frac{\sin\theta_p}{\sin\theta_u} \leq \varepsilon \iff \frac{\sin\theta_p}{\sin\theta_u} \leq \frac{\varepsilon}{2+\varepsilon} = O(\varepsilon)$$

The last piece of this part is the fact that point u was chosen by being the first point in the cone c encountered by a line parallel to the edge opposite of p. See the red line and direction in Figure 4.3. This implies that the angle θ_u is at least α, and the result then follows.

$\|p-q\| < \|p-u\|$: Denote by u' the projection of u on $\overline{pq}$. We have that

$$\|q-u\| \leq \|u-u'\| + \|q-u'\| = \sin\theta_p \|p-u\| + (\|p-q\| - \|q-u'\|)$$

$$\implies \|q-u\| \leq \sin\theta_p \|p-u\| + (\|p-q\| - \cos\theta_p \|p-u\|)$$

$$\implies \|p-u\| + \frac{1}{\cos\theta_p - \sin\theta_p}\|q-u\| \leq \frac{1}{\cos\theta_p - \sin\theta_p}\|p-q\|$$

Which means that

$$\|p-u\| + (1+\varepsilon)\|q-u\| \leq (1+\varepsilon)\|p-q\| \iff \frac{1}{\cos\theta_p - \sin\theta_p} \leq 1+\varepsilon$$

Since $\theta_p = O(\varepsilon\alpha)$ we get that for a sufficiently large constant the condition $\frac{1}{\cos\theta_p - \sin\theta_p} \leq 1+\varepsilon$ holds. The second part of the lemma is derived from the first:

$$\|p-u\| + (1+\varepsilon)\|q-u\| \leq (1+\varepsilon)\|p-q\| \iff \|q-u\| \leq \|p-q\| - \frac{\|p-u\|}{(1+\varepsilon)} \leq \|p-q\|$$

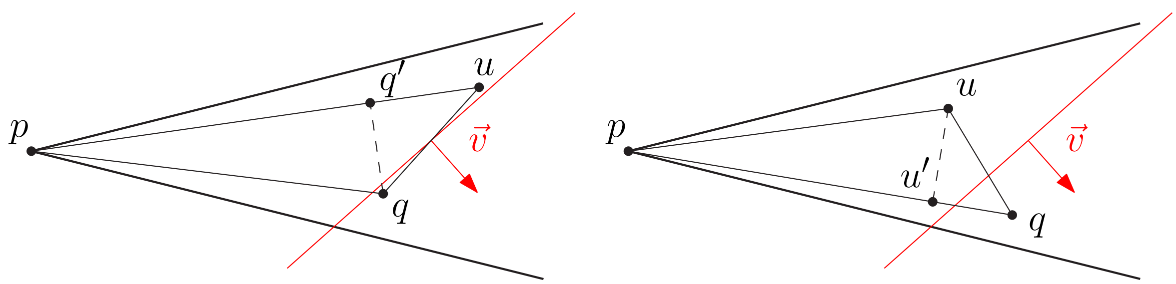

Figure 4.3: The two cases of Lemma 4.3. The vector used to determine the edge in the cone is shown in red, and marked $\vec{v}$

4.0.3. Analysis

We first prove a "shrinking lemma" similar to those in Section 2 and Section 3.

Lemma 4.4. *Let $\triangle$ be a triangle in the plane, and let p, q be two arbitrary points in $\triangle$. Then, there is a triangle $\triangle' \subseteq \triangle$ that contains one of p and q on a vertex, and the other on the edge opposite to that vertex (possibly on a different vertex)*

Proof: Choose one of the edges of $\triangle$ and start "sliding" it, i.e. incrementally translating it while keeping its slope, shrinking $\triangle$ in the process. We do so until one of p or q is on the edge. We do the same for the other two edges, choosing the order arbitrarily. The triangle remaining at the end of the process is $\triangle'$, and since we have kept the slopes of all edges we have that $\triangle \sim \triangle'$. We must have that two of the edges were stopped by the same point, which therefore lies on a vertex v of the new triangle. The other point must lie on the edge opposite to v, since otherwise one of the edges can still be moved to shrink $\triangle'$. See Figure 4.4.

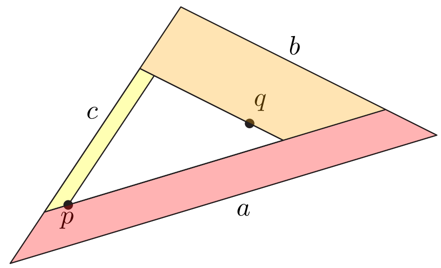

Figure 4.4: The edges are moved in the order a, b, c, resulting in the required similar triangle.

Local spanner property.

Lemma 4.5. *For a triangle $\triangle$, denote the graph constructed by the process above Section 4.0.1) by G. for any homothet of $\triangle$, and any two points $p, q \in P$ in the homothet, we have a $(1+\varepsilon)$-path in $G \cap \triangle$.*

Proof: We slightly abuse notation and denote the homothet by $\triangle$. We prove the existence of the $(1+\varepsilon)$-path by induction over the rank of the distance $\|p - q\|$. The closest pair p', q' must be connected in G, as otherwise we have that $\|p' - u'\| + \|q' - u'\| \geq 2\|p' - q'\|$ regardless of the angle of the cone, a contradiction to Lemma 4.3.

For any other pair $p, q \in PS \cap \triangle$ we have from Lemma 4.4 that there exists a homothet $\triangle' \subseteq \triangle$ with one of the two points, say p, at a vertex, and the other on the opposite edge. We therefore have a cone c with apex at p such that $q \in c \cap \triangle'$. If pq is an edge in G then we are done. Otherwise, we have a vertex $u \in c$ such that pq is an edge in G, and we therefore get from Lemma 4.3 that $\|q - u\| \leq \|p - q\|$ and therefore, by the induction hypothesis, we have that there exists a $(1+\varepsilon)$ path between u and q in G. From the same lemma it also follows that $\|p - u\| + (1+\varepsilon)\|q - u\| \leq (1+\varepsilon)\|p - q\|$, and thus we get the existence of a $(1+\varepsilon)$ path between p and q as well.

Size and running time. Let G be the graph constructed by the procedure described above.

Theorem 4.6. *Let P be a set of n points in the plane, and let $\varepsilon \in (0,1)$ be an approximation parameter. The above algorithm computes a local $(1+\varepsilon)$-spanner G for homothets of an α-fat triangle $\triangle$. The construction time is $O\left(\frac{1}{\alpha\varepsilon} n \log n\right)$, and the spanner G has $O\left(\frac{1}{\alpha\varepsilon} n\right)$ edges.*

Proof: The local-spanning property is proven in Lemma 4.5, and we are only left with bounding the size and the running time of the algorithm. The bound on the size is immediate from the construction, as every point p is the apex of $O\left(\frac{2\pi}{\varepsilon\alpha}\right)$ cones, each giving rise to a single edge incident to p. The construction time is bounded by the construction time for a θ-graph with cone size $\alpha\varepsilon$, which is $O\left(\frac{1}{\alpha\varepsilon} n \log n\right)$ ([Cla87]) ■

5. Weak local spanners for convex regions with bounded aspect ratio

We would like to build local spanners (of subquadratic size) for axis-parallel rectangles, but as Figure 5.1 shows, there is no hope of achieving this. As such, we need to somewhat change the requirements, and instead describe a weak local spanner for this case. The exact meaning of the weakness of the spanner, parameterized by δ is given below.

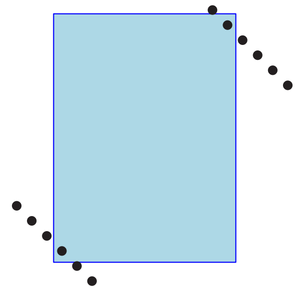

Figure 5.1: There are quadratic number of pairs of points that has to be connected in any local spanner for axis parallel rectangles. Indeed, for any point in the top diagonal and bottom diagonal, there is an axis parallel rectangle that contains only these two points. This holds even if we restrict ourselves to fat rectangles of similar size.

Definition 5.1. Given a convex region C, let

$$C_{\boxminus\delta} = \left\{ p \in C \;\middle|\; \mathsf{d}\big(p, \mathbb{R}^2 \setminus C\big) \geq \delta \cdot \mathrm{diam}(C) \right\}.$$

Formally, $C_{\boxminus\delta}$ is the Minkowski difference of C with a disk of radius $\delta \cdot \mathrm{diam}(C)$.

Definition 5.2. Consider a (bounded) set C in the plane. Let $r_{\mathrm{in}}(C)$ be the radius of the largest disk contained inside C. Similarly, $R_{\mathrm{out}}(C)$ is the smallest radius of a disk containing C.

The ***aspect ratio*** of a region C in the plane is $\mathsf{ar}(C) = R_{\mathrm{out}}(C)/r_{\mathrm{in}}(C)$. Given a family $\mathcal{F}$ or regions in the plane, its *aspect ratio* is $\mathsf{ar}(\mathcal{F}) = \max_{C\in\mathcal{F}} \mathsf{ar}(C)$.

Note, that if a convex region C has bounded aspect ratio, then $C_{\boxminus\delta}$ is similar to the result of scaling C by a factor of $1 - O(\delta)$. On the other hand, if C is long and skinny, say is has width smaller than $2\delta \cdot \mathrm{diam}(C)$, then $C_{\boxminus\delta}$ is empty.

Lemma 5.3. *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0, 1)$. One can construct a graph G over P, in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2})n\big)$ edges, such that for any (bounded) convex C in the plane, we have that for any two points $p, q \in P \cap C_{\boxminus\delta}$ the graph $C \cap P$ has $(1 + \varepsilon)$-path between p and q.*

Proof: Let $\vartheta = \min(\varepsilon, \delta^2)$. Construct, in $\mathcal{O}(\vartheta^{-1} n \log n)$ time, a standard $(1 + \vartheta)$-spanner G for P using $\mathcal{O}(\vartheta^{-1} n)$ edges [AMS99].

So, consider any body $C \in \mathcal{F}$, and any two vertices $p, q \in P \cap C'$, where $C' = C_{\boxminus\delta}$. Let $\ell = \|p - q\|$, let π be the shortest path between p and q in G, and let $\mathcal{E}$ be the locus of all points u, such that $\|p - u\| + \|u - q\| \leq (1 + \vartheta)\ell$. The region $\mathcal{E}$ is an ellipse that contains π. The furthest point from the segment pq in this ellipse is realized by the co-vertex of the ellipse. Formally, it is one of the two intersection points of the boundary of the ellipse with the line orthogonal to $\overline{pq}$ that passes through the middle point c of this segment, see Figure 5.2. Let z be one of these points.

We have that $\|p - z\| = (1 + \vartheta)\ell/2$. Setting $h = \|z - c\|$, we have that

$$h = \sqrt{\|p - z\|^2 - \|p - c\|^2} = \frac{\ell}{2}\sqrt{(1 + \vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2 + \vartheta)}}{2}\ell \leq \sqrt{\vartheta}\ell \leq \sqrt{\vartheta} \cdot \mathrm{diam}(C).$$

as $\ell \leq \mathrm{diam}(C') \leq \mathrm{diam}(C)$.

For any point $x \in C'$, we have that $\mathsf{d}(x, \mathbb{R}^2 \setminus C) \geq \delta \cdot \mathrm{diam}(C)$. As such, to ensure that $\pi \subseteq \mathcal{E} \subseteq C$, we need that $\delta \cdot \mathrm{diam}(C) \geq h$, which holds if $\delta \cdot \mathrm{diam}(C) \geq \sqrt{\vartheta} \cdot \mathrm{diam}(C)$. This in turn holds if $\vartheta \leq \delta^2$. Namely, we have the desired properties if $\vartheta = \min(\varepsilon, \delta^2)$. ∎

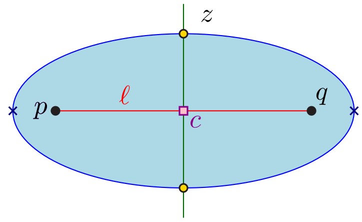

Figure 5.2

6. Weak local spanners for axis-parallel rectangles

6.1. Quadrant separated pair decomposition

For points $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ in $\mathbb{R}^d$, let $p \prec q$ denotes that q ***dominates*** p coordinate-wise. That is $p_i < q_i$, for all i. More generally, let $p <_i q$ denote that $p_i < q_i$. For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $\forall x \in X, y \in Y \quad x <_i y$. In particular X and Y are *i-coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is ***quadrant-separated***, if X and Y are i-coordinate separated, for $i = 1, \ldots, d$.

A ***quadrant-separated pair decomposition*** of a point set $P \subseteq \mathbb{R}^d$, is a pair decomposition (see Definition 2.1) $\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\}$ of P, such that $\{X_i, Y_i\}$ are quadrant-separated for all i.

Lemma 6.1. *Given a set P of n points in $\mathbb{R}$, one can compute, in $\mathcal{O}(n \log n)$ time, a QSPD of P with $\mathcal{O}(n)$ pairs, and of total weight $\mathcal{O}(n \log n)$.*

Proof: If P is a singleton then there is nothing to do. If $P = \{p, q\}$, then the decomposition is the pair formed by the two singleton points.

Otherwise, let x be the median of P, such that $P_{\leq x} = \{p \in P \mid p \leq x\}$ contains exactly $\lceil n/2 \rceil$ points, and $P_{>x} = P \setminus P_{\leq x}$ contains $\lfloor n/2 \rfloor$ points. Construct the pair $\Xi = \{P_{\leq x}, P_{>x}\}$, and compute recursively a QSPDs $\mathcal{Q}_{\leq x}$ and $\mathcal{Q}_{>x}$ for $P_{\leq x}$ and $P_{>x}$, respectively. The desired QSPD is $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{>x} \cup \{\Xi\}$. The bounds on the size and weight of the desired QSPD are immediate. ∎

Lemma 6.2. *Given a set P of n points in $\mathbb{R}^d$, one can compute, in $\mathcal{O}(n \log^d n)$ time, a QSPD of P with $\mathcal{O}(n \log^{d-1} n)$ pairs, and of total weight $\mathcal{O}(n \log^d n)$.*

Proof: The construction algorithm is recursive on the dimensions, using the algorithm of Lemma 6.1 in one dimension.

The algorithm computes a value α_d that partitions the values of the points' dth coordinates roughly equally (and is distinct from all of them), and let h be a hyperplane parallel to the first $d-1$ coordinate axes, and having value α_d in the dth coordinate.

Let $P_\uparrow$ and $P_\downarrow$ be the subset of points of P that are above and below h, respectively. The algorithm recursively computes QSPDs $\mathcal{Q}_\uparrow$ and $\mathcal{Q}_\downarrow$ for $P_\uparrow$ and $P_\downarrow$, respectively. Next, the algorithm projects the points of P on h, let P' be the resulting $d-1$ dimensional point set (after we ignore the dth coordinate), and recursively computes a QSPD $\mathcal{Q}'$ for P'.

For a point set $X' \subseteq P'$, let $\mathrm{lift}(X')$ be the subset of points of P whose projection on h is X'. The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\mathrm{lift}(X') \cap P_\uparrow, \mathrm{lift}(Y') \cap P_\downarrow\},\ \{\mathrm{lift}(X') \cap P_\downarrow, \mathrm{lift}(Y') \cap P_\uparrow\} \;\middle|\; \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is $\widehat{\mathcal{Q}} \cup \mathcal{Q}_\uparrow \cup \mathcal{Q}_\downarrow$.

To observe that this is indeed a QSPD, observe that all the pairs in $\mathcal{Q}_\uparrow, \mathcal{Q}_\downarrow$ are quadrant separated by induction. As for pairs in $\widehat{\mathcal{Q}}$, they are quadrant separated in the first $d-1$ coordinates by induction on the dimension, and separated in the d coordinate since one side of the pair comes from $P_\uparrow$, and the other side from $P_\downarrow$.

As for coverage, consider any pair of points $p, q \in P$, and observe that the claim holds by induction if they are both in $P_\uparrow$ or $P_\downarrow$. As such, assume that $p \in P_\uparrow$ and $q \ni P_\downarrow$. But then there is a pair $\{X', Y'\} \in \mathcal{Q}'$ that separates the two projected points in h, and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates p and q as desired.

The number pairs in the decomposition is $N(n,d) = 2N(n,d-1) + 2N(n/2,d)$ with $N(n,1) = O(n)$. The solution to this recurrence is $N(n,d) = O(n \log^{d-1} n)$. The total weight of the decomposition is $W(n,d) = 2W(n,d-1) + 2W(n/2,d)$ with $W(n,1) = O(n \log n)$. The solution to this recurrence is $W(n,d) = O(n \log^d n)$. Clearly, this also bounds the construction time. ■

6.2. Weak local spanner for axis parallel rectangles

For a parameter $\delta \in (0,1)$, and an interval $I = [b,c]$, let $(1-\delta)I = [t-(1-\delta)r, t+(1-\delta)r]$ be the shrinking of I by a factor of $1-\delta$, where $t = (b+c)/2$, and $r = (c-b)/2$.

Let $\mathcal{R}$ be the set of all axis parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$, with $R = I \times J$, let $(1-\delta)R = (1-\delta)I \times (1-\delta)J$ denote the rectangle resulting from shrinking R by a factor of $1-\delta$.

Definition 6.3. Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0,1)$, a graph G is a *$(1-\delta)$-local $(1+\varepsilon)$-spanner* for rectangles, if for any axis-parallel rectangle R, we have that $G \cap R$ is a $(1+\varepsilon)$-spanner for all the points in $(1-\delta)R \cap P$.

Observe that rectangles in $\mathcal{R}$ might be quite "skinny", so the previous notion of shrinkage used before are not useful in this case.

6.2.1. Construction for a single quadrant separated pair

Consider a pair $\Xi = \{X, Y\}$ in a QSPD of P. The set X is quadrant-separated from Y. That is, there is a point c_Ξ, such that X and Y are contained in two opposing quadrants in the partition of the plane formed by the vertical and horizontal line through c_Ξ.

For simplicity of exposition, assume that $c_\Xi = (0,0)$, and $X \prec (0,0) \prec Y$. That is, the points of X are in the negative quadrant, and the points of Y are in the positive quadrant.

Consider a point $p = (-x, -y) \in X$. Its *set of clients* in Y, is

$$\mathsf{C}(p, Y) = \left\{ q \in Y \mid \left\| q - c_\Xi \right\|_1 \leq \left\| p - c_\Xi \right\|_1 \right\}.$$

We construct a non-uniform grid $\mathsf{K}(p, \Xi)$ in the square $[0, x+y]^2$. To this end, we first partition it into four subrectangles

$B_\nwarrow = [0,x] \times [y, x+y]$	$B_\nearrow = [x, x+y] \times [y, x+y]$
$B_\swarrow = [0,x] \times [0,y]$	$B_\searrow = [x, x+y] \times [0,y]$.

Let $\tau \geq 4/\varepsilon + 4/\delta$ be an integer number. We partition each of these rectangles into a $\tau \times \tau$ grid, where each cell is a copy of the rectangle scaled by a factor of $1/\tau$. See Figure 6.1. This grid has $\mathcal{O}(\tau^2)$

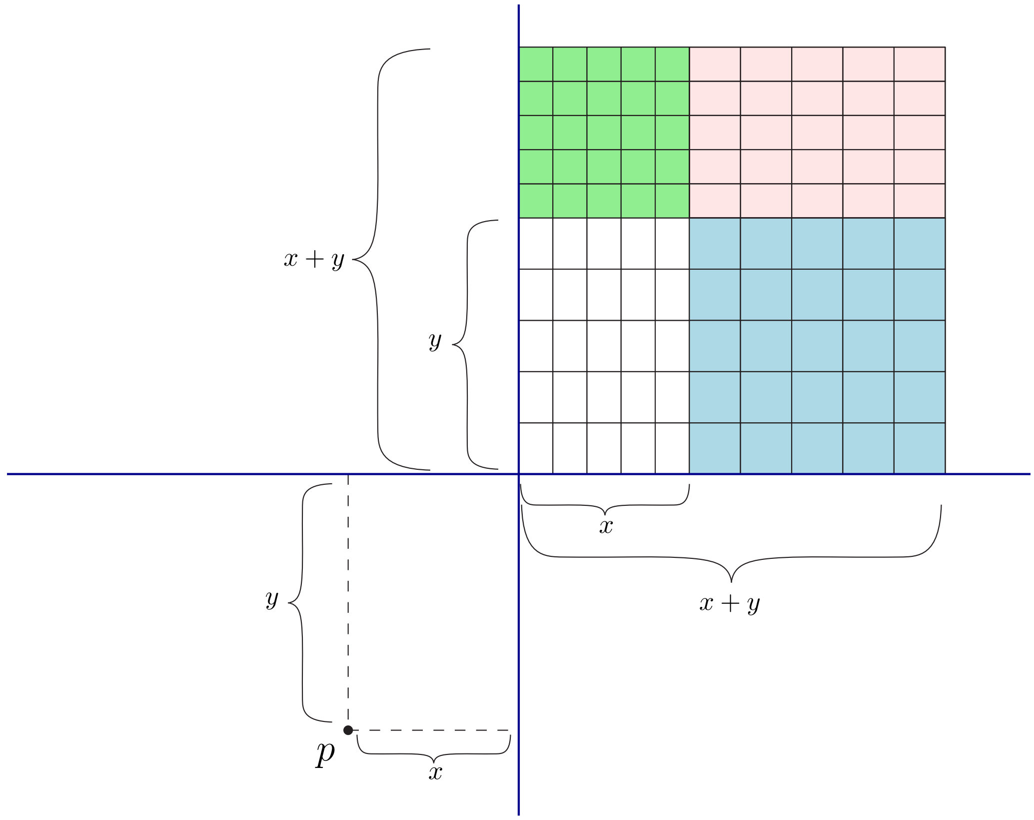

Figure 6.1: The construction of the grid $\mathsf{K}(p, \Xi)$ for a point $p = (-x, -y)$ and a pair Ξ.

Figure 6.2

cells. For a cell C in this grid, let $Y \cap \mathsf{C}$ be the points of Y contained in it. We connect p to the left-most and bottom-most points in $Y \cap \mathsf{C}$. This process generates two edges in the constructed graph for each grid cell, and $\mathcal{O}(\tau^2)$ edges overall.

The algorithm repeats this construction for all the points $p \in X$, and does the symmetric construction for all the points of Y.

6.2.2. The construction algorithm

The algorithm computes a QSPD $\mathcal{W}$ of P. For each pair $\Xi \in \mathcal{W}$, the algorithm generates edges for Ξ using the algorithm of Section 6.2.1 and adds them to the generated spanner G.

6.2.3. Correctness

For a rectangle R, let $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists (x', y) \in R\}$ be its expansion into a horizontal slab. Restricted to a rectangle R', the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure 6.2. Similarly, we denote $\updownarrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists (x, y') \in R\}$.

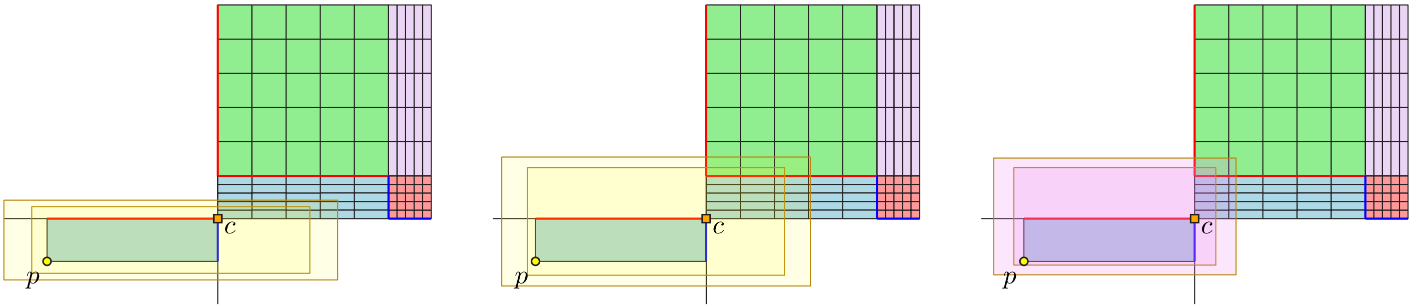

Figure 6.3

Lemma 6.4. *Assume that $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above construction, and a point $p = (-x, -y) \in X$, and its associated grid $\mathsf{K} = \mathsf{K}(p, \Xi)$. Consider any axis parallel rectangle R, such that $p \in (1-\delta)R = I \times J$, and $(1-\delta)R$ intersects a cell $\mathsf{C} \in \mathsf{K}$. We have that:*

(I) If $\mathsf{C} \subseteq (1-\delta)R$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.

(II) $\mathrm{diam}(\mathsf{C}) \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.

(III) If $x \geq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.

(IV) If $x \leq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\nwarrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.

(V) If $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$, then $(1-\delta)^{-1}\left(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\right) \subseteq R$.

(VI) If $x \leq y$ and $\mathsf{C} \subseteq R_{\searrow}$, then $(1-\delta)^{-1}\Bigl(\updownarrow\bigl((1-\delta)R\bigr) \cap \mathsf{C}\Bigr) \subseteq R$.

Proof: (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

(II) We have that $\mathrm{diam}(\mathsf{C}) \leq (x+y)/\tau \leq \|p\|_1/\tau \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.

(III) The width, denoted $\mathsf{wd}(\cdot)$, of $(1-\delta)R$ is at least x, as it contains both p and the origin. As such,

$$\bigl(\mathsf{wd}(R) - \mathsf{wd}((1-\delta)R)\bigr)/2 \geq 2(x/\tau) \geq 2\mathsf{wd}(\mathsf{C}).$$

That is, the width of the "expanded" rectangle R is enough to cover C, and a grid cell adjacent to it to the right.

A similar argument about the height shows that R covers the region immediately above C – in particular, the vertical distance from C to the top boundary of R is at least the height of C. This implies that the expanded cell $(1-\delta)^{-1}\mathsf{C}$ is contained in R, as claimed, as $\delta < 1/2$.

(V) We decompose the claim to the two dimensions of the region. Let $B = \left(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\right)$. Observe that containment in the x-axis follows by arguing as in (III). As for the y-interval of B, observe that it is contained in the y-interval of $(1-\delta)R$, which implies that when expanded by $(1-\delta)^{-1}$, it would be contained in the y-interval of R. Combining the two implies the result. ■

Lemma 6.5. *For any axis-parallel rectangle R, and any two points $p, q \in (1-\delta)R \cap P$, there exists a $(1+\varepsilon)$-path between p and q in G.*

Proof: The proof is the spirit of the "standard" recursive proof for spanners, and is done by induction over the size of R (i.e. area, width, or height). Let $\Xi = \{X, Y\} \in \mathcal{W}$ be the pair in the QSPD that separates p and q, let c be the separation point of the pair, and assume for the simplicity of exposition that $p \in X$, $X \prec c \prec Y$, and $c = (0, 0)$. Furthermore, assume that $\|p\|_1 \geq \|q\|_1$.

Let $p = (-x, -y)$, and let C be the grid cell of $\mathsf{K}(p, \Xi)$ that contains q. If $\mathsf{C} \subseteq (1-\delta)R$, then $(1-\delta)^{-1}\mathsf{C} \subseteq R$ by Lemma 6.4 (I). As such, let u be the leftmost point in $\mathsf{C} \cap P$. Both $q, u \in (1-\delta)^{-1}\mathsf{C}$, and by induction, there is an $(1+\varepsilon)$-path π between them in G (note that the induction applies to the two points, and the "expanded" rectangle $(1-\delta)^{-1}\mathsf{C}$). Since pu is an edge of G, prefixing π by this edge results in an $(1+\varepsilon)$-path, as $\|q-u\| \leq (\varepsilon/4)\|p-q\|$, by Lemma 6.4 (II) (verifying this requires some standard calculations which we omit).

Otherwise, one need to apply the same argument using the appropriate case of Lemma 6.4. So assume that $x \geq y$ (the case that $y \geq x$ is handled symmetrically). If $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$, then (III) implies that $(1-\delta)^{-1}\mathsf{C} \subseteq R$. Which implies that induction applies, and the claim holds.

The remaining case is that $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$. Let $D = \overleftrightarrow{(1-\delta)R} \cap \mathsf{C}$. By (V), we have $(1-\delta)^{-1}\bigl(D\bigr) \subseteq R$. Namely, $q \in (1-\delta)R \cap \mathsf{C} \subseteq D$, and let u be the lowest point in $\mathsf{C} \cap P$. By construction $pu \in E(G)$, $q, u \in D$, $(1-\delta)^{-1}D \subseteq R$. As such, we can apply induction to q, u, and $(1-\delta)^{-1}D$, and conclude that $\mathsf{d}_G(q,u) \leq (1+\varepsilon)\|q-u\|$. Plugging this into the regular machinery implies the claim. ∎

Theorem 6.6. *Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0,1)$ be parameters. The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n\log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n\log^2 n)$ edges. The graph G is a $(1-\delta)$-local $(1+\varepsilon)$-spanner for axis parallel rectangles. Formally, for any axis-parallel rectangle R, we have that $R \cap P$ is an $(1+\varepsilon)$-spanner for all the points of $\bigl((1-\delta)R\bigr) \cap P$.*

Proof: Computing the QSPD $\mathcal{W}$ takes $\mathcal{O}(n\log^2 n)$ time. For each pair $\{X, Y\}$ in the decomposition with $m = |X| + |Y|$ points, we need to compute the lowest and leftmost points in $(X \cup Y) \cap \mathsf{C}$, for each cell in the constructed grid. This can readily be done using orthogonal range trees in $\mathcal{O}(\log^2 n)$ time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is $\mathcal{O}(\omega(\mathcal{W})\tau^2) = O\bigl((1/\delta^2 + 1/\varepsilon^2)n\log^2 n\bigr)$.

The desired local spanner property is provided by Lemma 6.5. ∎

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