Fault-Tolerant and Local Spanners Revisited

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1. Introduction

Euclidean graph and spanners. For a set P of points in $\mathbb{R}^d$, an *Euclidean graph* $G = (P, E)$ is an undirected graph with P as the set of vertices. An edge $pq \in E$ is naturally associated with the segment pq, and weight of the edge is the (Euclidean) length of the segment. Consider a pair of points $p, q \in P$. For a parameter $t \geq 1$, a path between p and q in G is a *t-path* if the length of the path is at most $t \|p - q\|$, where $\|p - q\|$ is the Euclidean distance between p and q. The graph G is a *t-spanner* of P if there is a t-path between any pair of points $p, q \in P$. Throughout the paper, n denotes the cardinality of the point set P, unless stated otherwise. We denote the length of the shortest path between $p, q \in P$ in the graph G by $\mathsf{d}(p, q)$.

Residual graphs. Let $\mathcal{F}$ be a family of regions in the plane. For a fault region $\mathscr{r} \in \mathcal{F}$ and a geometric graph G on a point set P, let $G - \mathscr{r}$ be the residual graph after removing from it all the points of P in $\mathscr{r}$. and all the edges that intersects $\mathscr{r}$. Formally, let

$$G - \mathscr{r} = \big(P \setminus \mathscr{r}, \{uv \in E \mid uv \cap \mathrm{int}(\mathscr{r}) = \emptyset\}\big),$$

where $\mathrm{int}(\mathscr{r})$ denotes the interior of $\mathscr{r}$. Similarly, let

$$G \cap \mathscr{r} = \big(P \cap \mathscr{r}, \{uv \in E \mid uv \subseteq \mathscr{r}\}\big).$$

be the residual graph after restricting G to the region $\mathscr{r}$.

Fault-tolerant and local spanners. A *fault-tolerant spanner* for $\mathcal{F}$, is a graph G, such that for any region $\mathscr{r}$ (i.e., the "attack"), the graph $G - \mathscr{r}$ is a t-spanner for all its vertices. Surprisingly, as shown by Abam *et al.* [AdBFG09], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have near linear number of edges.

In the same spirit, a graph G is a *local spanner* for $\mathcal{F}$, if for any region $\mathscr{r} \in \mathcal{F}$, we have that $G \cap \mathscr{r}$ is a t-spanner for all its vertices. The notion of local-spanner was defined by Abam and Borouny [AB21]. They showed how to construct such spanners for axis-parallel squares and vertical slabs. They also showed how to construct such spanners for disks, if one is allowed to add Steiner points. Abam and Borouny left the question of how to construct local spanners for disks as an open problem.

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Halfplanes	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AdBFG09]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[AB21]	$\mathcal{O}(\varepsilon^{-3} n \log n)$	Theorem 2.18
Vertical slabs	$\mathcal{O}(\varepsilon^{-2} n \log n)$	[AB21]		
Disks with Steiner points	$\mathcal{O}_\varepsilon(n)$	[AB21]		
Disks			$\mathcal{O}(\varepsilon^{-2} n \log \Phi)$	Theorem 2.12
			$\Omega(n \log \Phi)$	Lemma 2.15
δ-weak local $(1+\varepsilon)$-spanners				
Any bounded convex shape			$\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2}) n\big)$	Lemma 3.3
$(1-\delta)$-local $(1+\varepsilon)$-spanners				
Axis-parallel rectangles			$\mathcal{O}\big((\varepsilon^{-2} + \delta^{-2}) n \log^2 n\big)$	Theorem 4.6

Figure 1.1: Known and new results. The notation $\mathcal{O}_\varepsilon$ hides polynomial dependency on ε which is not specified in the original work.

Our results

We present a new construction of spanners, which surprisingly, is not only fault-tolerant for convex regions, but it also a local spanner for disks. This resolves the aforementioned open problem from Abam and Borouny [AB21]. Our construction is a variant of the original construction of Abam *et al.* [AdBFG09].

We then investigate various other constructions of local spanners, where one is allowed to slightly shrink the region.

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2. Local spanner for disks

Our purpose here is to build a local spanner for disks.

2.1. Preliminaries

2.1.1. Well separated pairs decomposition

For sets X, Y, let $X \otimes Y = \{\{x, y\} \mid x \in X,\ y \in Y, x \neq y\}$ be the set of all the (unordered) pairs of points formed by the sets B and C.

Definition 2.1 (Pair decomposition). For a point set P, a ***pair decomposition*** of P is a set of pairs

$$\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\},$$

such that (I) $X_i, Y_i \subseteq P$ for every i, (II) $X_i \cap Y_i = \emptyset$ for every i, and (III) $\bigcup_{i=1}^{s} X_i \otimes Y_i = P \otimes P$.

Definition 2.2. Given a pair decomposition $\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\}$ of a point set P, its ***weight*** is $\omega(\mathcal{W}) = \sum_{i=1}^{s} (|X_i| + |Y_i|)$.

Definition 2.3. The pair of sets $X, Y \subseteq \mathbb{R}^d$ is $(1/\varepsilon)$***-well-separated*** if

$$\max(\operatorname{diam}(X), \operatorname{diam}(Y)) \leq \varepsilon \cdot \mathsf{d}(X, Y),$$

Definition 2.4. For a point set P, a ***well-separated pair decomposition*** (***WSPD***) of P with parameter $1/\varepsilon$ is a pair decomposition of P with a set of pairs $\mathcal{W} = \left\{ \{B_1, C_1\}, \ldots, \{B_s, C_s\} \right\}$, such that, for any i, the sets B_i and C_i are $1/\varepsilon$-separated.

The ***closest pair*** distance of a set of points $P \subseteq \mathbb{R}^d$, is $\operatorname{cp}(P) = \min_{p,q \in P, p \neq q} \|p - q\|$. The ***diameter*** of P is $\operatorname{diam}(P) = \max_{p,q \in P} \|p - q\|$. The ***spread*** of P is $\Phi(P) = \operatorname{diam}(P)/\operatorname{cp}(P)$, which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD can be quadratic, if the spread is bounded, the weight is near linear.

Lemma 2.5 ([AH12]). *Let P be a set of n points in $\mathbb{R}^d$, with spread $\Phi = \Phi(P)$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$-WSPD $\mathcal{W}$ for P of total weight $\mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}\left(\varepsilon^{-d} \log \Phi\right)$ pairs. Namely, $\omega(\mathcal{W}) = O(\varepsilon^{-d} n \log \Phi)$.*

2.1.2. Semi separated pairs decomposition

Definition 2.6. Two sets of points B and C are $(1/\varepsilon)$***-semi-separated*** if

$$\min(\operatorname{diam}(B), \operatorname{diam}(C)) \leq \varepsilon \cdot \mathsf{d}(B, C),$$

where $\mathsf{d}(B, C) = \min_{q \in B, u \in C} \|q - u\|$.

For a point set P, a ***semi-separated pair decomposition*** (***SSPD***) of P with parameter $1/\varepsilon$, denoted by ε^{-1}-SSPD, is a pair decomposition of P formed by a set of pairs $\mathcal{W}$ such that all the pairs are $1/\varepsilon$-semi-separated.

Theorem 2.7 ([AH12, Har11]). *Let P be a set of n points in $\mathbb{R}^d$, and let $\varepsilon > 0$ be a parameter. Then, one can compute a $1/\varepsilon$-SSPD for P of total weight $\mathcal{O}\left(n\varepsilon^{-d} \log n\right)$. The number of pairs in the SSPD is $\mathcal{O}\left(n\varepsilon^{-d}\right)$, and the computation time is $\mathcal{O}\left(n\varepsilon^{-d} \log n\right)$.*

A *δ-double-wedge* is a region between two lines, where the angle between the two lines is at most ϑ.

Lemma 2.8. *Given a α-SSPD $\mathcal{W}$ of a set P of n points in $\mathbb{R}^d$ and a parameter $\beta \geq 2$, one can refine it, into a $\alpha\beta$-SSPD $\mathcal{W}'$, such that that $|\mathcal{W}'| = O(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\beta^d)$.*

Proof: The algorithm scans the pairs of $\mathcal{W}$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\operatorname{diam}(X) < \operatorname{diam}(Y)$. Let $\mathfrak{s}$ be the smallest axis-parallel cube containing X, with sidelength r. Let $r' = r / \left\lceil \sqrt{d}\beta \right\rceil$. Partition $\mathfrak{s}$ into a grid of cubes of sidelength r', and let T_Ξ be the resulting set of squares. The algorithm now add the set pairs

$$\{\{X \cap \mathfrak{t}, Y\} \mid \mathfrak{t} \in T_\Xi\}$$

to the output SSPD. Clearly, the resulting set is now $\alpha\beta$-semi separated, as we chopped the smaller part of each pair into β smaller portions. ■

Lemma 2.9. *Given a ε^{-1}-SSPD $\mathcal{W}$ of n points in the plane, one can refine it, into a ε^{-1}-SSPD $\mathcal{W}'$, such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε-double-wedge $\times_\Xi$, such that X and Y are contained in the two different faces of the double wedge $\times_\Xi$. We have that $|\mathcal{W}'| = O(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = O(\omega(\mathcal{W}')/\varepsilon)$. The construction time is proportional to the weight of $\mathcal{W}'$.*

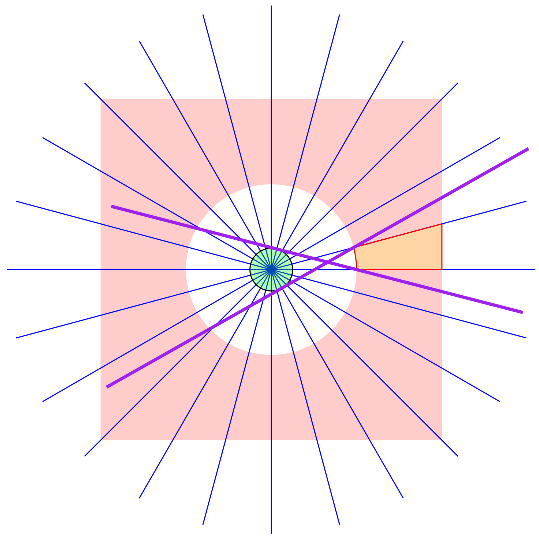

Figure 2.1

Proof: By using Lemma 2.8, we can assume that $\mathcal{W}$ is (say) $(10/\varepsilon)$-separated. Now, the algorithm scans the pairs of $\mathcal{W}$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$. Let $\bigcirc$ be the smallest axis-parallel square containing X, centered at point c. Partition the plane around c, by drawing around it $\mathcal{O}(1/\varepsilon)$ lines with the angle between any two consecutive lines being at most (say) $\varepsilon/4$, see Figure 2.1. This partition the plane into a set of cones $\mathcal{C}$. For a cone $C \in \mathcal{C}$, observe that there exists a ε-double-wedge that contains X on one side, and $Y \cap C$. To see that, take the double-wedge formed by the cross tangents between $\mathsf{ch}(X)$ and $\mathsf{ch}(Y \cap C)$, where $\mathsf{ch}(X)$ denotes the convex-hull of X. ■

2.1.3. Delaunay triangulation

We need the following well known property of Delaunay triangulation, which would play a center role in our construction.

Claim 2.10. *For a set of points $P \subseteq \mathbb{R}^2$ in general position, let $\mathcal{D} = \mathcal{DT}(P)$ denote its Delaunay triangulation. Then, for any close disk $\bigcirc$, we have $\mathcal{DT}(P) \cap \bigcirc$ is connected.*

Proof: We first prove that for any (close) disk $\bigcirc$ with two points $p, q \in P$ on its boundary, there is a path between p and q in $\mathcal{D} \cap \bigcirc$. The proof is by induction over the number m of points of P in the interior of $\bigcirc$:

- $m = 0$: The disk $\bigcirc$ contains no points of P in its interior, and thus pq is an edge of the Delaunay triangulation, as $\bigcirc$ testifies.
- $m > 0$: Let $u \in P$ be a point in the interior of $\bigcirc$. We move the center c of $\bigcirc$ in the direction of p, shrinking $\bigcirc$ in the process, so that the radius the disk is $\|c - p\|$, until we get a disk $\bigcirc' \subseteq \bigcirc$ such that u is on the boundary of $\bigcirc'$, see Figure 2.2. Observe that p and u are on the boundary of the new disk, and $|\mathrm{int}(\bigcirc') \cap P| < |\mathrm{int}(\bigcirc) \cap P|$. Thus, by induction, there is a path γ' between p and u in $\mathcal{D} \cap \bigcirc' \subseteq \mathcal{D} \cap \bigcirc$. Similarly, there must be a path γ'' between u and q, and concatenating the two paths results in a path between p and q in $\mathcal{D} \cap \bigcirc$.

Back to the original claim. For any two points $p, q \in \bigcirc \cap P$ one can get a disk $\bigcirc' \subseteq \bigcirc$ that contains p and q on its boundary. Indeed, shrink the radius of $\bigcirc$ till, say, p is on the boundary, and them move the

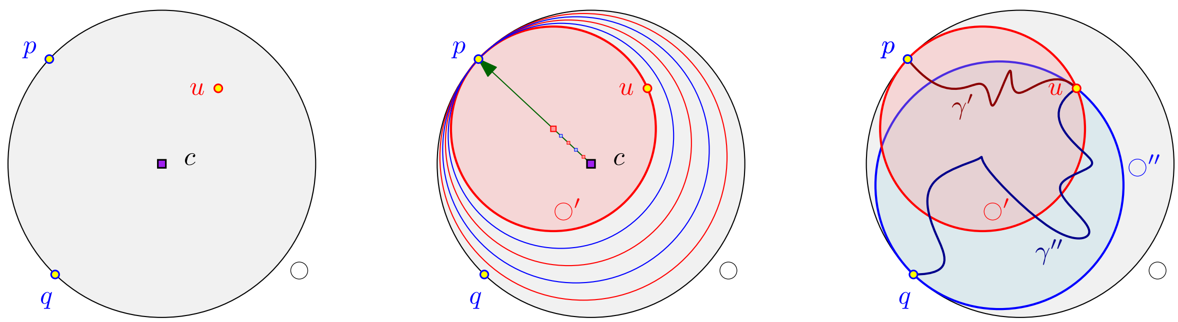

Figure 2.2

center of the disk towards p while shrinking the size of the disk to maintain p on the boundary, until q is also on the boundary of the shrank disk. ■

2.2. The construction of local spanners for disks

2.2.1. The construction

The input is a set P of n points in the plane (in general position) with $\Phi = \Phi(P)$, and a parameter $\varepsilon \in (0, 1)$.

The algorithm computes a $1/\vartheta$-WSPD $\mathcal{W}$ of P using the algorithm of Lemma 2.5, where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the Delaunay triangulation $\mathcal{D}_\Xi = \mathcal{DT}(X \cup Y)$. The algorithm adds all the edges in $\mathcal{D}_\Xi \cap (X \otimes Y)$ to the computed graph G.

2.2.2. Analysis

Size. For each pair $\Xi = \{X, Y\}$ in the WSPD, its Delaunay triangulation contains at most with $\mathcal{O}(|X| + |Y|)$ edges. As such, the number of edges in the resulting graph is bounded by

$$\sum_{\{X,Y\} \in \mathcal{W}} O\bigl(|X| + |Y|\bigr) = O(\omega(\mathcal{W})) = O\left(\frac{n \log \Phi}{\vartheta^2}\right),$$

by Lemma 2.5.

Construction time. The construction time is bounded by

$$\sum_{\{X,Y\} \in \mathcal{W}} O\bigl((|X| + |Y|) \log(|X| + |Y|)\bigr) = O(\omega(\mathcal{W}) \log n) = O\left(\frac{n \log \Phi \log n}{\vartheta^2}\right),$$

Local spanner property.

Lemma 2.11. *Let G be the graph constructed above for the point set P. Then, for any (close) disk $\bigcirc$, and any two points $x, y \in P \cap \bigcirc$, we have that $G \cap \bigcirc$ has a $(1+\varepsilon)$-path between x and y. That is, G is a $(1+\varepsilon)$-local spanner for disks.*

Proof: The proof is by induction on the distance between p and q (or more precisely, the rank of their distance among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and $y \in Y$.

For the base case, consider the case that x is the nearest-neighbor to y in P, and y is the nearest-neighbor to x in P. It must be, because of the separation property of Ξ, that X and Y are singletons. Indeed, if X contains another point, then y would not be the nearest-neighbor to x (this is true for $\vartheta < 0.5$). As such, $xy \in \mathcal{D}_\Xi$, $x, y \in \bigcirc$, and the edge $xy \in E(G)$, implying the claim.

For the inductive step, observe that , the claim follows if $xy \in \mathcal{D}_\Xi$, so assume this is not the case. By the connectivity of $\mathcal{D}_\Xi \cap \bigcirc$, see Claim 2.10, there must be points $x' \in X \cap \bigcirc$, $y' \in Y \cap \bigcirc$, such that $x'y' \in E(\mathcal{D}_\Xi)$. As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property, we have that

$$\max(\operatorname{diam}(X), \operatorname{diam}(Y)) \leq \vartheta \cdot \mathsf{d}(X, Y) \leq \vartheta \ell,$$

where $\ell = \|x - y\|$. In particular, $\|x' - x\| \leq \vartheta \ell$ and $\|y' - y\| \leq \vartheta \ell$. As such, by induction, we have $\mathsf{d}_G(x, x') \leq (1 + \varepsilon) \|x - x'\| \leq (1 + \varepsilon)\vartheta \ell$ and $\mathsf{d}_G(y, y') \leq (1 + \varepsilon) \|y - y'\| \leq (1 + \varepsilon)\vartheta \ell$. Furthermore, $\|x' - y'\| \leq (1 + 2\vartheta)\ell$. As $x'y' \in E(G)$, we have

$$\begin{aligned} \mathsf{d}_G(x, y) &\leq \mathsf{d}_G(x, x') + \|x' - y'\| + \mathsf{d}_G(y', y) \leq (1 + \varepsilon)\vartheta \ell + (1 + 2\vartheta)\ell + (1 + \varepsilon)\vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta)\ell \\ &= (1 + 6\vartheta)\ell \leq (1 + \varepsilon) \|x - y\|, \end{aligned}$$

if $\vartheta \leq \varepsilon/6$. ∎

The result.

Theorem 2.12. *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter. The above algorithm constructs a local $(1 + \varepsilon)$-spanner G for disks. The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any disk $\bigcirc$ in the plane, and any two points $p, q \in P \cap \bigcirc$, we have a $(1 + \varepsilon)$-path in $G \cap \bigcirc$.*

2.2.3. Applications and comments

Definition 2.13. Given a region R in the plane and a point set P, consider two points $p, q \in P$. The edge pq is *safe* in R, if there is a disk $\bigcirc$ such that $p, q \in \bigcirc \subseteq R$. Let $\mathcal{G}(P, R)$ be the graph formed by all the safe edges in P for R. Note, that his graph might have a quadratic number of edges in the worst case.

Observe that $\mathcal{G}(\mathbb{R}^2, P)$ is a clique.

Corollary 2.14. *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter, and let G be a local $(1 + \varepsilon)$-spanner for disks. Then, for R be an region in the plane, and consider the graph $H = \mathcal{G}(P, R)$. Then $G \cap R$ is a $(1 + \varepsilon)$-spanner for $H \cap R$. Formally, for any two points $p, q \in P \cap R$, we have that $\mathsf{d}_H(p, q) \leq (1 + \varepsilon)\mathsf{d}_G(p, q)$.*

In particular, for any convex region C, the graph $G - C$ is a $(1 + \varepsilon)$-spanner for $\mathcal{G}(\mathbb{R}^2, P) - C$.

Proof: Consider the shortest path $\pi = u_1 u_2 \ldots u_k$ between p and q in $\mathsf{d}_H(p, q)$. Every edge $e_i = u_i u_{i+1}$ has a disk $\bigcirc_i$ such that $u_i, u_{i+1} \in \bigcirc_i \subseteq R$. As such, there is a $(1 + \varepsilon)$-path between u_i and u_{i+1} in $G \cap \bigcirc_i \subseteq G \cap R$. Concatenating these paths directly yields the desired result.

The second claim follows by observing that the complement of C is the union of halfspaces, and halfspaces can be considered to be "infinite" radius disks. As such, the above argument applies verbatim. ∎

But why not SSPD? The result of Theorem 2.12 is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). A natural way is to try and emulate the construction of Abam *et al.* [AdBFG09] and use SSPD instead of WSPD. The total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε-separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are of different sides of the double wedge. The problem is that for the local disk $\bigcirc$, it might be the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \bigcirc$ is much longer than the two points of interest. This somewhat counter-intuitive situation is illustrated in Figure 2.3.

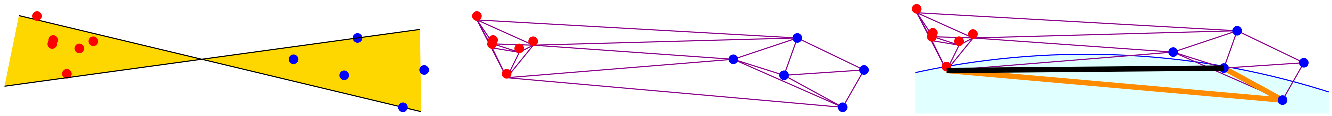

Figure 2.3: A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets are points are not well separated.

2.2.4. A lower bound for local spanner for disks

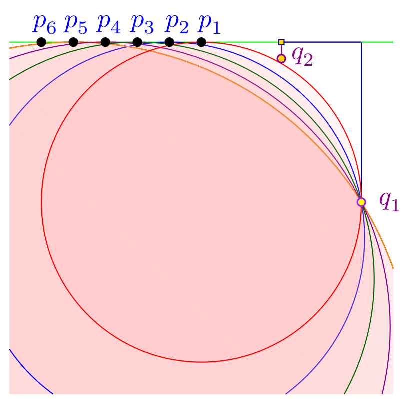

Figure 2.4

Lemma 2.15. *For $\varepsilon = 1/4$, and parameters n and $\Phi \geq 1$, there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $O(n\Phi)$, such that any local $(1+\varepsilon)$-spanner of P for disks, must have $\Omega(n \log \Phi)$ edges.*

Proof: Let $p_i = (-i, 0)$, for $i = 1, \ldots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$ and $q_1 = (n2^M, -1)$. For a point p on the x-axis, and a point q below the x-axis, to its right, let $\bigcirc_\downarrow^p(q)$ be the disk that its boundary passes through p and q, and its center has the same x-coordinate as p.

In the jth iteration, for $j = 2, \ldots, M-1$, consider the set of disks

$$D_j = \left\{ \bigcirc_\downarrow^{p_i}(q_{j-1}) \;\middle|\; i = 1, \ldots, n \right\},$$

see Figure 2.4. Let $x_j = n2^{M-j+1} = x(q_{j-1})/2$, and let $y_j < 0$ be the maximum y-coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the disks of $D_1 \cup \cdots \cup D_j$. Let $q_j = (x_j, 0.99y_j)$. Clearly, the point q_j lies outside all the disks of $D_1 \cup \ldots \cup D_j$. The construction now continues to the next value of j. Let $P = \{p_1, \ldots, p_n, q_2, \ldots, q_M\}$. We have that $|P| = n + M - 1$.

The minimum distance between any points in the construction is 1 (i.e., $\|p_1 - p_2\|$). Indeed $x(q_M) = 2n$ and thus $\|q_M - p_1\| \geq 2n$. The diameter of P is $\|p_1 - q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = O(n\Phi)$.

For any i and j, consider the disk $\bigcirc_{\downarrow}^{p_i}(q_j)$. This disk does not contain any point of $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ since its interior lies below the x-axis. By construction it does not contain any point $q_{j+1}, \ldots, q_{M-1}$. This disk potentially also contains the points $q_{j-1}, \ldots, q_1$, but observe that for any index $k \in [\![j-1]\!]$, we have that

$$\|p_i - q_k\| = \sqrt{\left(i + n2^{M-k+1}\right)^2 + \left(y(q_j)\right)^2},$$

which implies that $n2^{M-k+1} \leq \|p_i - q_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$\frac{\|p_i - q_k\|}{\|p_i - q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since $j \in [\![M-1]\!]$. Namely, the shortest path in G between p_i and q_j, can not use any of the points $q_1, \ldots q_{j-1}$. As such, the graph G must contain the edge $p_i q_j$. This implies that $|E(G)| \geq n(M-1)$, which implies the claim. ∎

2.3. A local spanner for axis parallel squares

One can modify the above construction for axis-parallel squares, and get a local spanner without dependency on the spread.

2.3.1. Construction

The input is a point set P of n points in the plane, and an approximation parameter $\varepsilon \in (0, 1/2)$. We assume that the input point set P is in general position. Specifically, no two points of P share a coordinate value, or appear in opposing corners of an axis-parallel square – this can be ensured by slightly perturbing the points if necessary.

One can define the Delaunay triangulation when the unit ball is replaced by the unit square. Formally, in this triangulation two points are connected $\iff$ there is a square that contains these two points on its boundary and no points in its interior. Let $\mathcal{D}_\Box$ denote the resulting *L_∞-Delaunay triangulation*.

Let $\vartheta = \varepsilon/20$. Instead of constructing a WSPD, the algorithm computes a $1/\vartheta$-SSPD $\mathcal{W}$, using the algorithm of Theorem 2.7. Increasing the weight and number of pairs by a factor of $\mathcal{O}(1/\vartheta)$, by using the algorithm of Lemma 2.9, one can assume that every pair $\{X, Y\} \in \mathcal{W}$ is not only semi-separated, but also there is an associated double wedge of angle $\leq \vartheta$. The algorithm now computes the "square" Delaunay triangulation for each such pair, and adds the edges of the triangulation to the resulting graph G.

2.3.2. Analysis

Size and running time. Computing the SSPD takes $\mathcal{O}\left(n\vartheta^{-2} \log n\right)$ time, and the refinement takes $\mathcal{O}\left(n\vartheta^{-3} \log n\right)$ time (which is also the weighted of the resulting SSPD). The number of edges of each L_∞-Delaunay triangulation for a pair is proportion to its weight, which implies that the total number of edges in the resulting graph G is $\mathcal{O}\left(\vartheta^{-3} n \log n\right)$. Computing all these Delaunay triangulations takes $\mathcal{O}\left(\vartheta^{-3} n \log^2 n\right)$ time.

Shrinking squares. We need the following lemma about shrinking of axis-parallel squares. Observe that this property definitely does not hold for disks, as illustrated in Figure 2.3.

Lemma 2.16. *(A) Let $\mathscr{s}$ be an axis parallel square in the plane, and let p, q be two arbitrary points in $\mathscr{s}$. Then, there is a square $\mathscr{t} \subseteq \mathscr{s}$ that contains p and q on its boundary.*

(B) Let X, Y be two point sets in the plane, such that $X' = X \cap \mathscr{s} \neq \emptyset$ and $Y' = Y \cap \mathscr{s} \neq \emptyset$. Let $x \in X, y \in Y$ be the two points realizing $\mathsf{d}_\infty(X', Y') = \min_{p \in X', q \in Y'} \|p - q\|_\infty$. Then, there is a square $\mathscr{t} \subseteq \mathscr{s}$ that contains x and y on its boundary, and $\mathscr{t}$ does not contain any other point of $X \cup Y$.

Proof: (A) Start shrinking $\mathscr{s}$ around its center till it contains one of the points (say p is on its boundary. Next, move the center of the square towards p till the boundary of the continuously shrinking square passes through q. If p and q lies on adjacent edges, then continue the shrinking process by moving the center towards the common corner of the shared edges – this process stops when one of the points is on the corner of the square. Clearly, the resulting square $\mathscr{t}$ is the desired square, see Figure 2.5.

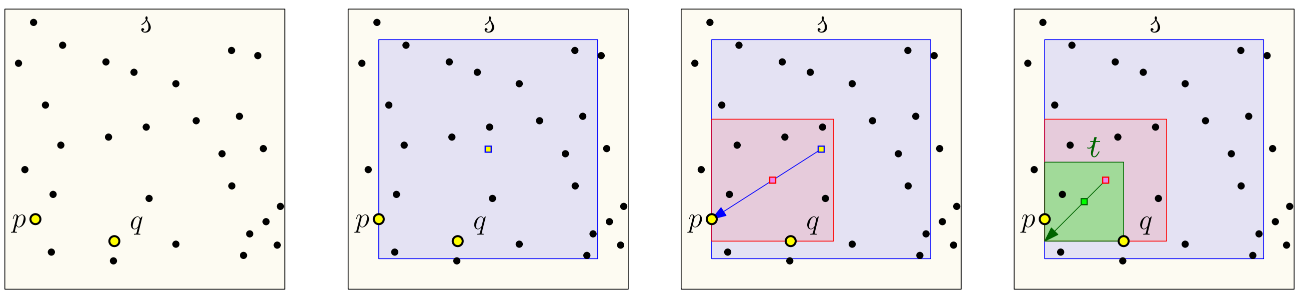

Figure 2.5

(B) Let $r = \mathsf{d}_\infty(X', Y')$. By (A), there is a square $\mathscr{t} \subseteq \mathscr{s}$ having x and y on apposing sides. As such, the sidelength of $\mathscr{t}$ is r. Assume for contradiction, that there is some other point $x' \in X \cap \mathscr{t}$. By our general position assumption, x' is in the interior of $\mathscr{t}$, and in particular, $\|x' - y\|_\infty < r$, which is a contradiction to the choice of x and y. ■

Local spanner property.

Lemma 2.17. *For any axis parallel square $\mathscr{s}$ in the plane, and any two points $p, q \in P \cap \mathscr{s}$, we have a $(1 + \varepsilon)$-path in $G \cap \mathscr{s}$.*

Proof: Consider two points $x, y \in P \cap \mathscr{s}$, where $\mathscr{s}$ is some arbitrary square. There exists a pair $\Xi = \{X, Y\} \in \mathcal{W}$ such that $x \in X$ and $y \in Y$, and this pair is ϑ^{-1}-semi separated and is also separated by a double wedge of angle $\leq \vartheta$. See Figure 2.6. Furthermore, assume that $\mathrm{diam}(X) < \mathrm{diam}(Y)$.

Let $X' = X \cap \mathscr{s}$ and $Y' = Y \cap \mathscr{s}$, and consider the two points $x' \in X'$ and $y' \in Y'$ realizing $r = \mathsf{d}_\infty(X', Y')$. By Lemma 2.16 there exists a square $\mathscr{t}$ containing x', y' on its boundary (on two apposing edges), such that $\mathscr{t} \subseteq \mathscr{s}$, and $\mathscr{t}$ contains no other points $X \cup Y$. By construction, we have that $x'y'$ is the L_∞-Delaunay triangulation of Ξ, and thus $x'y' \in G$. Since $\|x - x'\| \ll \|x - y\|$, we have by induction that $\mathsf{d}_G(x, x') \leq (1 + \vartheta) \|x - x'\|$.

Let $\ell = \|x' - y'\|$. By the semi-separation property and since $\mathrm{diam}(X) < \mathrm{diam}(Y)$. we have that

$$\|x - x'\| \leq \mathrm{diam}(X) \leq \vartheta \mathsf{d}_2(X, Y) = \vartheta\sqrt{2}\mathsf{d}_\infty(X, Y) \leq 2\vartheta\ell.$$

Since $\|x - x'\| \ll \|x - y\|$, and by induction we have that

$$\mathsf{d}_G(x, x') \leq (1 + \varepsilon) \|x - x'\| \leq (1 + \varepsilon)2\vartheta\ell \leq 4\vartheta\ell.$$

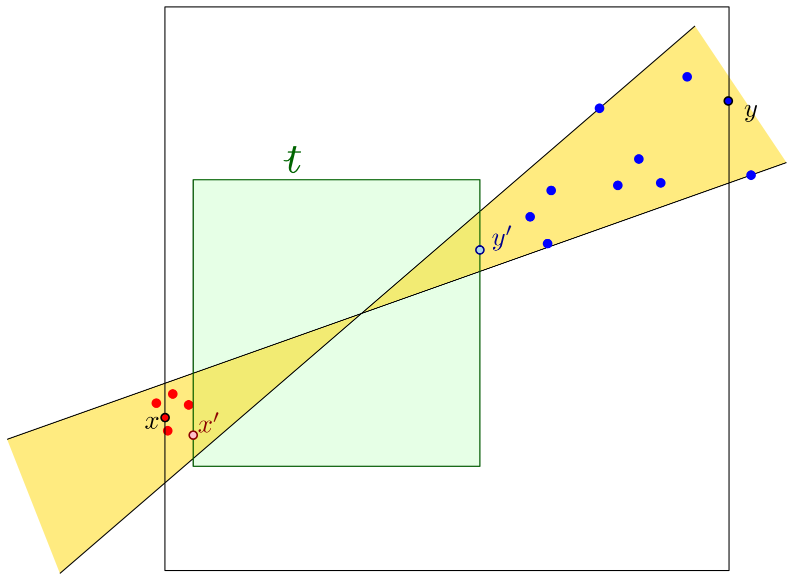

Figure 2.6

By the triangle inequality, we have

$$(1-2\vartheta)\ell \leq \|x'-y'\| - \|x-x'\| \leq \|x-y'\| \leq (1+2\vartheta)\ell.$$

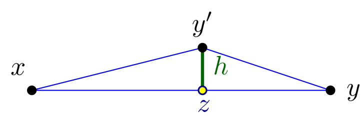

Figure 2.7

Consider the triangle $\triangle xy'y$, and observe that by the double-wedge property $\alpha = \angle y'xy \leq \vartheta$. Let z be the projection of y' to xy, and let

$$h = \|y'-z\| = \|x-y'\|\sin\alpha \leq \|x-y'\|\sin\vartheta \leq \|x-y'\|\,\vartheta \leq \vartheta(1+2\vartheta)\ell \leq 2\vartheta\ell,$$

as $\vartheta \in (0, 1/10)$, the monotonicity of sin in this range, and as $\sin\vartheta \leq \vartheta$.

We have that $\|x-z\| \leq \|x-y'\| \leq (1+2\vartheta)\ell$. Similarly, we have

$$\|x-z\| = \|x-y'\|\cos\alpha \geq (1-\alpha^2/2)\,\|x-y'\| \geq (1-\vartheta^2/2)(1-2\vartheta)\ell \geq (1-3\vartheta)\ell.$$

By the triangle inequality, we have that

$$\|y'-y\| \geq \|x-y\| - \|y'-x\| \geq \|x-y\| - (1+2\vartheta)\ell.$$

As for an upper bound, we have

$$\begin{aligned}\|y'-y\| &\leq \|z-y\| + h \leq \|x-y\| - \|x-z\| + 2\vartheta\ell \leq \|x-y\| - (1-3\vartheta)\ell + 2\vartheta\ell\\ &= \|x-y\| - (1-5\vartheta)\ell < \|x-y\|.\end{aligned}$$

As such, by induction $\mathsf{d}_G(y', y) \leq (1+\varepsilon)\,\|y'-y\|$.

We thus have that

$$\begin{aligned} \mathsf{d}_G(x,y) &\leq \mathsf{d}_G(x,x') + \|x'-y'\| + \mathsf{d}_G(y',y) \leq 4\vartheta\ell + \ell + (1+\varepsilon)\|y'-y\| \\ &\leq (1+4\vartheta)\ell + (1+\varepsilon)\big(\|x-y\| - (1-5\vartheta)\ell\big) \\ &= \big[1+4\vartheta - (1+\varepsilon)(1-5\vartheta)\big]\ell + (1+\varepsilon)\|x-y\| \\ &\leq (1+\varepsilon)\|x-y\|, \end{aligned}$$

for $\vartheta \leq \varepsilon/20$, as $1+4\vartheta-(1+\varepsilon)(1-5\vartheta) \leq 1+\varepsilon/5-(1+\varepsilon)(1-\varepsilon/4) = \varepsilon/5-(3/4)\varepsilon+\varepsilon^2/4 < 0$, as $\varepsilon < 1$. ■

Theorem 2.18. *Let P be a set of n points in the plane, and let $\varepsilon \in (0,1)$ be an approximation parameter. The above algorithm computes a local $(1+\varepsilon)$-spanner G for axis parallel squares. The construction time is $\mathcal{O}\big(\varepsilon^{-3} n \log^2 n\big)$, and the spanner G has $\mathcal{O}(\varepsilon^{-3} n \log n)$ edges.*

2.4. Result for other norms

2.4.1. The rest

Sariel: FILL IN *all* THE DETAILS – this is way too hand wavy – including refs, etc end Sariel

Using the same argument, we can extend the result for the case where $\mathcal{L}$ is the set of all scaled and translated copies, homothets, of a convex shape $\mathcal{C}$. While the Delaunay triangulation is not well defined for all convex shapes, the operation of creating edges between two points $p, q \in P$ such that there exist a homothet of $\mathcal{C}$ that contains only p and q and no other point of P is always well defined, and gives us a graph known as the $\mathcal{C}$-Delaunay graph of P, and denoted $\mathcal{DG}_{\mathcal{C}}(P)$. The above proof applies almost verbatim for any convex $\mathcal{C}$, and proves the connectivity of $\mathcal{DG}_{\mathcal{C}}(P)$ for any $L \in \mathcal{L}$.

We need only to define a suitable shrinking operation for convex region towards a point, which is possible, for example, by parameterizing the curve defining the region and leaving the desired point in the same coordinate of the smaller curve. So, we get a $(1+\varepsilon)-\mathcal{L}$ local spanner of size $\mathcal{O}(\varepsilon^{-3} n \log n)$ in $\mathcal{O}(\varepsilon^{-2} n \log n)$ time.(

3. Weak local spanners for regions with bounded aspect ratio

We would like to build local spanners (of subquadratic size) for axis-parallel rectangles, but as Figure 3.1 shows, there is no hope of achieving this. As such, we need to change the requirement somewhat.

One way to shrink a region is as a function of its diameter.

Definition 3.1. Given a convex region C, let

$$C_{\boxminus\delta} = \left\{ p \in C \;\middle|\; \mathsf{d}\big(p, \mathbb{R}^2 \setminus C\big) \leq \delta \mathrm{diam}(C) \right\}.$$

Formally, $C_{\boxminus\delta}$ is the Minkowski difference of C with a disk of radius $\delta\mathrm{diam}(C)$.

Definition 3.2. Consider a (bounded) set C in the plane. Let $r_{\mathrm{in}}(C)$ be the radius of the largest disk contained inside C. Similarly, $R_{\mathrm{out}}(C)$ is the smallest radius of a disk containing C.

The ***aspect ratio*** of a region C in the plane is $\mathsf{ar}(C) = R_{\mathrm{out}}(C)/r_{\mathrm{in}}(C)$. Given a family $\mathcal{F}$ or regions in the plane, its *aspect ratio* is $\mathsf{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \mathsf{ar}(C)$.

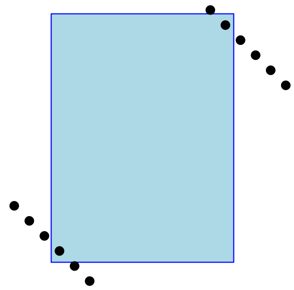

Figure 3.1: There are quadratic number of pairs of points that has to be connected in any local spanner for axis parallel rectangles. Indeed, for any point in the top diagonal and bottom diagonal, there is an axis parallel rectangle that contains only these two points. This holds even if we restrict ourselves to fat rectangles of similar size.

Note, that if a convex region C has bounded aspect ration, then $C_{\boxminus\delta}$ is similar to the result of scaling C by a factor of $1 - O(\delta)$. On the other hand, if C is long and skinny, say is has width smaller than $2\delta\mathrm{diam}(C)$, then $C_{\boxminus\delta}$ is empty.

Lemma 3.3. *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0,1)$. One can construct a graph G over P, in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}\big((\varepsilon^{-1} + \delta^{-2})n\big)$ edges, such that for any (bounded) convex C in the plane, we have that for any two points $p, q \in P \cap C_{\boxminus\delta}$ the graph $C \cap P$ has $(1+\varepsilon)$-path between p and q.*

Proof: Let $\vartheta = \min(\varepsilon, \delta^2)$. Construct, in $\mathcal{O}(\vartheta^{-1} n \log n)$ time, a standard $(1+\vartheta)$-spanner G for P using $\mathcal{O}(\vartheta^{-1} n)$ edges [AMS99].

So, consider any body $C \in \mathcal{F}$, and any two vertex $p, q \in P \cap C'$, where $C' = C_{\boxminus\delta}$. Let $\ell = \|p - q\|$. Let π be the shortest path between p and q in G. Let $\mathcal{E}$ be the loci of all points pc, such that $\|p - u\| + \|u - q\| \leq (1+\vartheta)\ell$. The region $\mathcal{E}$ is an ellipse that contains π. The furthest point from the segment pq in this ellipse is realized by the co-vertex of the ellipse. Formally, it is one of the two intersection points of the boundary of the ellipse with the line orthogonal to pq that passes through the middle point c of this segment, see Figure 3.2. Let z be one of these points.

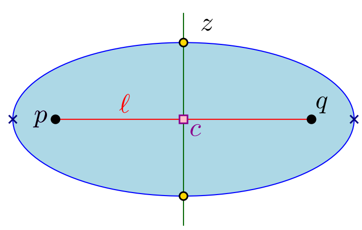

Figure 3.2

We have that $\|p - z\| = (1+\vartheta)\ell/2$. Setting $h = \|z - c\|$, we have that

$$h = \sqrt{\|p - z\|^2 - \|p - c\|^2} = \frac{\ell}{2}\sqrt{(1+\vartheta)^2 - 1} = \frac{\sqrt{\vartheta(2+\vartheta)}}{2}\ell \leq \sqrt{\vartheta}\ell \leq \sqrt{\vartheta}\mathrm{diam}(C).$$

as $\ell \leq \mathrm{diam}(C') \leq \mathrm{diam}(C)$.

For any point $x \in C'$, we have that $\mathsf{d}(x, \mathbb{R}^2 \setminus C') \leq \delta\mathrm{diam}(C)$. As such, to ensure that $\pi \subseteq \mathcal{E} \subseteq C$, we need that $\delta\mathrm{diam}(C) \geq h$, which holds if $\delta\mathrm{diam}(C) \geq \sqrt{\vartheta}\mathrm{diam}(C)$. This in turn holds if $\vartheta \leq \delta^2$. Namely, we have the desired properties if $\vartheta = \min(\varepsilon, \delta^2)$. ∎

4. Weak local spanners for axis-parallel rectangles

4.1. Quadrant separated pairs decomposition

For points $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ in $\mathbb{R}^d$, let $p \prec q$ denotes that q ***dominates*** p coordinate-wise. That is $p_i < q_i$, for all i. More generally, let $p <_i q$ denote that $p_i < q_i$. For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $\forall x \in X, y \in Y \quad x <_i y$. In particular X and Y are *i-coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is ***quadrant-separated***, if X and Y are i-coordinate separated, for $i = 1, \ldots, d$.

A ***quadrant-separated pairs decomposition*** of a point set $P \subseteq \mathbb{R}^d$, is a pairs decomposition (see Definition 2.1 $\mathcal{W} = \big\{\{X_1, Y_1\}, \ldots, \{X_s, Y_s\}\big\}$ of P, such that $\{X_i, Y_i\}$ are quadrant-separated for all i.

Lemma 4.1. *Given a set P of n points in $\mathbb{R}$, one can compute, in $\mathcal{O}(n \log n)$ time, a QSPD of P with $\mathcal{O}(n)$ pairs, and of total weight $\mathcal{O}(n \log n)$.*

Proof: If P is a singleton then there is nothing to do. If $P = \{p, q\}$, then the decomposition is the pair formed by the two singleton points.

Otherwise, let x be the median of P, such that $P_{\leq x} = \{p \in P \mid p \leq x\}$ contains exactly $\lceil n/2 \rceil$ points, and $P_{>x} = P \setminus P_{\leq x}$ contains $\lfloor n/2 \rfloor$ points. Construct the pair $\Xi = \{P_{\leq x}, P_{>x}\}$, and compute recursively a QSPDs $\mathcal{Q}_{\leq x}$ and $\mathcal{Q}_{>x}$ for $P_{\leq x}$ and $P_{>x}$, respectively. The desired QSPD is $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{>x} \cup \{\Xi\}$. The bounds on the size and weight of the desired QSPD are immediate. ■

Lemma 4.2. *Given a set P of n points in $\mathbb{R}^d$, one can compute, in $\mathcal{O}(n \log^d n)$ time, a QSPD of P with $\mathcal{O}(n \log^{d-1} n)$ pairs, and of total weight $\mathcal{O}(n \log^d n)$.*

Proof: The construction algorithm is recursive on the dimensions, using the algorithm of Lemma 4.1 in one dimension.

The algorithm computes a value α_d that partition the values of the points in dth coordinate roughly equally (and is distinct from all of them), and let h be a hyperplane parallel to the first $d-1$ coordinate axes, and having value α_d in the dth coordinate.

Let $P_\uparrow$ and $P_\downarrow$ be the subset of points of P that are above and below h, respectively. The algorithm computes recursively QSPDs $\mathcal{Q}_\uparrow$ and $\mathcal{Q}_\downarrow$ for $P_\uparrow$ and $P_\downarrow$, respectively. Next, the algorithm projects the points of P to h, and let P' be the resulting $d-1$ dimensional point set (after we ignore the dth coordinate). Compute recursively a QSPD $\mathcal{Q}'$ for P'.

For a point set $X' \subseteq P'$, let $\mathrm{lift}(X')$ be the subset of points of P that its projection to h is X'. The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\mathrm{lift}(X') \cap P_\uparrow, \mathrm{lift}(Y') \cap P_\downarrow\},\ \{\mathrm{lift}(X') \cap P_\downarrow, \mathrm{lift}(Y') \cap P_\uparrow\} \;\middle|\; \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is $\widehat{\mathcal{Q}} \cup \mathcal{Q}_\uparrow \cup \mathcal{Q}_\downarrow$.

To observe that this is indeed a QSPD, observe that all the pairs in $\mathcal{Q}_\uparrow, \mathcal{Q}_\downarrow$ are quadrant separated by induction. As for pairs in $\widehat{\mathcal{Q}}$, they are quadrant separated in the first $d-1$ coordinates by induction on the dimension, and separated in the d coordinate since one side of the pair comes from $P_\uparrow$, and the other side from $P_\downarrow$.

As for coverage, consider any pair of points $p, q \in P$, and observe that the claim holds by induction if they are both in $P_\uparrow$ or $P_\downarrow$. As such, assume that $p \in P_\uparrow$ and $q \ni P_\downarrow$. But then there is a pair

$\{X', Y'\} \in \mathcal{Q}'$ that separates the two projected points in h, and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates p and q as desired.

The number pairs in the decomposition is $N(n,d) = 2N(n,d-1) + 2N(n/2,d)$ with $N(n,1) = O(n)$. The solution to this recurrence is $N(n,d) = O(n \log^{d-1} n)$. The total weight of the decomposition is $W(n,d) = 2W(n,d-1) + 2W(n/2,d)$ with $W(n,1) = O(n \log n)$. The solution to this recurrence is $W(n,d) = O(n \log^d n)$. Clearly, this also bounds the construction time. ■

4.2. Weak local spanner for axis parallel rectangles

For a parameter $\delta \in (0,1)$, and an interval $I = [b,c]$, let $(1-\delta)I = [t-(1-\delta)r, t+(1-\delta)r]$ be the shrinking of I by a factor of $1-\delta$, where $t = (b+c)/2$, and $r = c-b$.

Let $\mathcal{R}$ be the set of all axis parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$, with $R = I \times J$, let $(1-\delta)R = (1-\delta)I \times (1-\delta)J$ denote the rectangle resulting from shrinking R by a factor of $1-\delta$.

Definition 4.3. Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0,1)$, a graph G is a *$(1-\delta)$-local $(1+\varepsilon)$-spanner* for rectangles, if for any axis-parallel rectangle R, we have that $G \cap R$ is a $(1+\varepsilon)$-spanner for all the points in $(1-\delta)R \cap P$.

Observe that rectangles in $\mathcal{R}$ might be quite "skinny", so the previous notion of shrinkage used before are not useful in this case.

4.2.1. Construction for a single quadrant separated pair

Consider a pair $\Xi = \{X, Y\}$ in a QSPD of P. The set X is quadrant-separated from Y. That is, there is a point c_Ξ, such that X and Y are contained in two opposing quadrants in the partition of the plane formed by the vertical and horizontal line through c_Ξ.

For simplicity of exposition, assume that $c_\Xi = (0,0)$, and $X \prec (0,0) \prec Y$. That is, the points of X are in the negative quadrant, and the points of Y are in the positive quadrant.

Consider a point $p \in X$. Its *set of clients* in Y, is

$$\mathsf{C}(p, Y) = \{ q \in Y \mid \|q - c_\Xi\|_1 \le \|p - c_\Xi\|_1 \}.$$

We construct a non-uniform grid $\mathsf{K}(p, \Xi)$ in the square $[0, x+y]^2$. To this end, we first partition it into four subrectangles

$B_{\nwarrow} = [0,x] \times [y, x+y]$	$B_{\nearrow} = [x, x+y] \times [y, x+y]$
$B_{\swarrow} = [0,x] \times [0,y]$	$B_{\searrow} = [x, x+y] \times [0,y]$.

Let $\tau = 4/\varepsilon + 4/\delta > 0$ be an integer number of specified shortly. We partition each of these rectangles into a $\tau \times \tau$ grid, where each cell is a scaled $1/\tau$ copy of itself. See Figure 4.1. This grid has $\mathcal{O}(\tau^2)$ cells. For a cell C in this grid, let $Y \cap \mathsf{C}$ be the points of Y contained in it. We connect p to the left-most and bottom-most points in $Y \cap \mathsf{C}$. This process generates two edges in the constructed graph for each grid cell, and $\mathcal{O}(\tau^2)$ edges overall.

The algorithm repeats this construction for all the points $p \in X$. It does the symmetric construction for all the points of Y.

4.2.2. The construction algorithm

The algorithm computes a QSPD $\mathcal{W}$ of P. For each pair $\Xi \in \mathcal{W}$, the algorithm generates edges for Ξ using the algorithm of Section 4.2.1 and adds them to the generated spanner G.

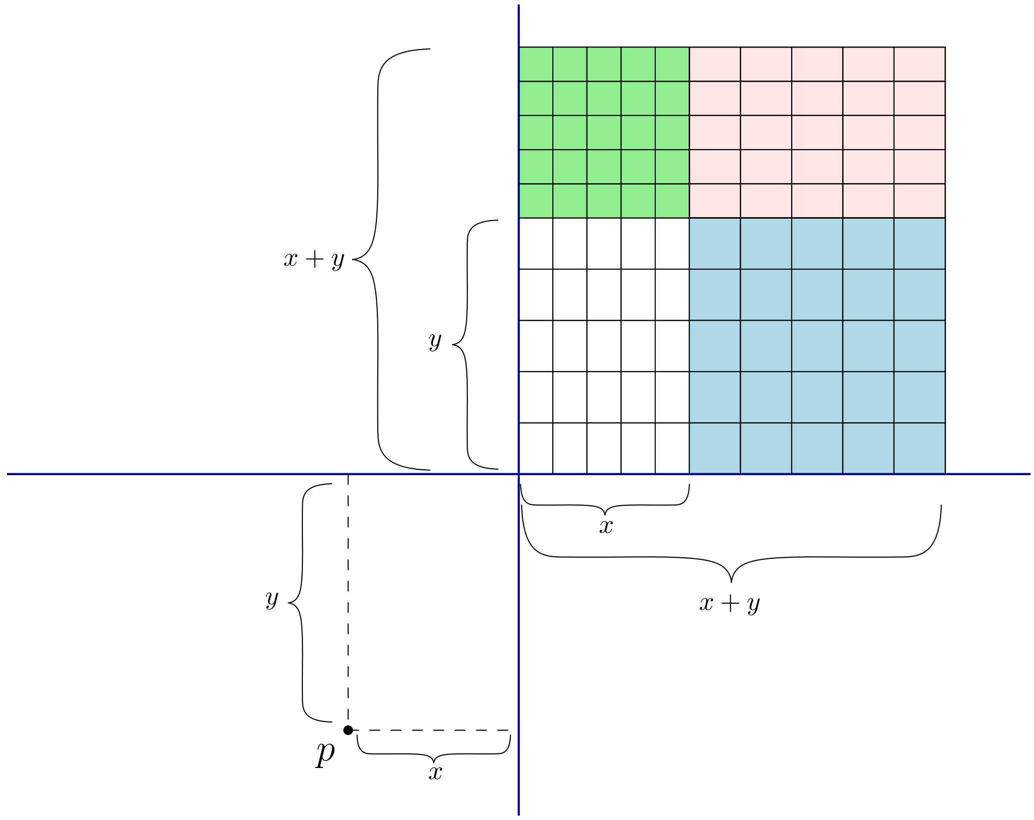

Figure 4.1: The construction of the grid $\mathsf{K}(p, \Xi)$ for a point $p = (x, y)$ and a pair Ξ. We are not showing the subrectangles of $B_{\nearrow}$ as they are not being used in the construction.

Figure 4.2

4.2.3. Correctness

For a rectangle R, let $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists (x', y) \in R\}$ be its expansion into a horizontal slab. Restricted to a rectangle R', the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure 4.2. Let $\updownarrow R = \{(x, y) \in \mathbb{R}^2 \mid \exists (x, y') \in R\}$

Lemma 4.4. *Assume that $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above construction, and a point $p = (x, y) \in X$, and its associated grid $\mathsf{K} = \mathsf{K}(p, \Xi)$, where $x \geq y$. Consider any axis parallel rectangle R, such that $p \in (1-\delta)R = I \times J$, and $(1-\delta)R$ intersects a cell $\mathsf{C} \in \mathsf{K}$. We have that:*

(I) If $\mathsf{C} \subseteq (1-\delta)R$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.

(II) $\mathrm{diam}(\mathsf{C}) \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.

(III) If $x \geq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.

(IV) If $x \leq y$ and $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\nwarrow}$ then $(1-\delta)^{-1}\mathsf{C} \subseteq R$.

(V) If $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$, then $(1-\delta)^{-1}\big(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\big) \subseteq R$.

(VI) If $x \leq y$ and $\mathsf{C} \subseteq R_{\searrow}$, then $(1-\delta)^{-1}\Big(\updownarrow\big((1-\delta)R\big) \cap \mathsf{C}\Big) \subseteq R$.

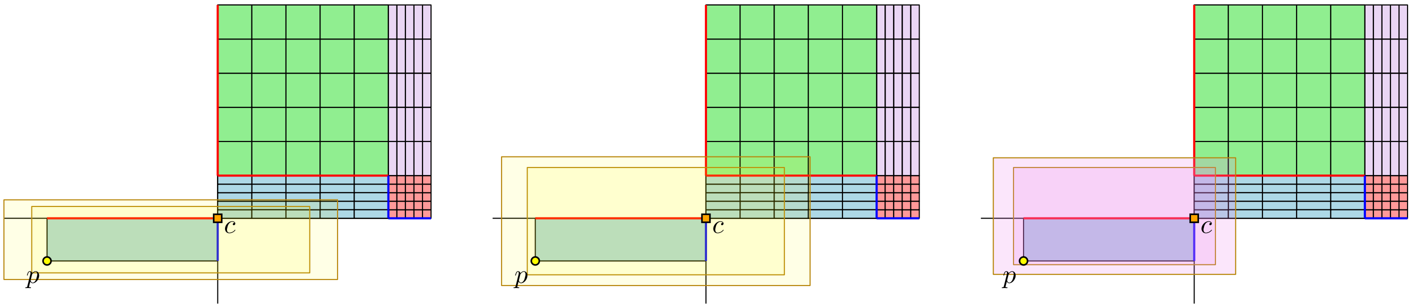

Figure 4.3

Proof: (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

(II) We have that $\mathrm{diam}(\mathsf{C}) \leq (x+y)/\tau \leq \|p\|_1/\tau \leq (\varepsilon/4)\mathsf{d}(p, \mathsf{C})$.

(III) The width of $(1-\delta)R$ is at least x, as it contains both p and the origin. As such,

$$\big(\mathsf{wd}(R) - \mathsf{wd}((1-\delta)R)\big)/2 \geq 2(x/\tau) \geq 2\mathsf{wd}(\mathsf{C}).$$

That is, the width of the "expanded" rectangle R is enough to cover C, and a grid cell adjacent to it to the right.

A similar argument about the height shows that R covers the region immediately above C – in particular, the vertical distance from C to the top boundary of R is at least the height of C. This implies that the expanded cell $(1-\delta)^{-1}\mathsf{C}$ is contained in R, as claimed, as $\delta < 1/2$.

(V) We decompose the claim to the two dimensions of the region. Let $B = \big(\overleftrightarrow{(1-\delta)R} \cap \mathsf{C}\big)$. Observe that containment in the x-axis follows by arguing as in (III). As for the y-interval of B, observe that is contained in the y-interval of $(1-\delta)R$, which implies that when expanded by $(1-\delta)^{-1}$, it would be contained in the y-interval of R. Combining the two implies the result. ∎

Lemma 4.5. *For any axis-parallel rectangle R, and any two points $p, q \in (1-\delta)R \cap P$, there exists a $(1+\varepsilon)$-path between p and q in G.*

Proof: The proof is the spirit of the "standard" recursive proof for spanners. Let $\Xi = \{X, Y\} \in \mathcal{W}$ be the pair in the QSPD that separates p and q, let c be the separation point of the pair, and assume for the simplicity of exposition that $p \in X$, $X \prec c \prec Y$, and $c = (0,0)$. Furthermore, assume that $\|p\|_1 \geq \|q\|_1$.

Let C be the grid cell of $\mathsf{K}(p, \Xi)$ that contains q, and let $p = (x, y)$. If $\mathsf{C} \subseteq (1-\delta)R$, then $(1-\delta)^{-1}\mathsf{C} \subseteq R$ by Lemma 4.4 (I). As such, let u be the leftmost point in $\mathsf{C} \cap P \cap R$. Both $q, u \in (1-\delta)^{-1}\mathsf{C}$, and by induction, there is an $(1+\varepsilon)$-path π between them in G (note, that the induction applies to the two points, and the "expanded" rectangle $(1-\delta)^{-1}\mathsf{C}$). Since pu is an edge of G, prefixing π by this edge results in an $(1+\varepsilon)$-path, as $\|q-u\| \leq (\varepsilon/4)\|p-q\|$, by Lemma 4.4 (II) (verifying this requires some standard calculations which we omit).

Otherwise, one need to apply the same argument using the appropriate case of Lemma 4.4. So assume that $x \geq y$ (the case that $y \geq x$ is handled symmetrically). If $\mathsf{C} \subseteq R_{\swarrow} \cup R_{\searrow}$, then (III) implies that $(1-\delta)^{-1}\mathsf{C} \subseteq R$. Which implies that induction applies, and the claim holds.

The remaining case is that $x \geq y$ and $\mathsf{C} \subseteq R_{\nwarrow}$. Let $D = \overleftrightarrow{(1-\delta)R} \cap \mathsf{C}$. By (V), we have $(1-\delta)^{-1}\big(D\big) \subseteq R$. Namely, $q \in (1-\delta)R \cap \mathsf{C} \subseteq D$, and let u be the lowest point in $\mathsf{C} \cap P$. By construction $pu \in E(G)$, $q, u \in D$, $(1-\delta)^{-1}D \subseteq R$. As such, we can apply induction to q, u, and $(1-\delta)^{-1}D$, and conclude that $\mathsf{d}_G(q, u) \leq (1+\varepsilon)\|q-u\|$. Plugging this into the regular machinery implies the claim. ∎

Theorem 4.6. *Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0,1)$ be parameters. The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2) n \log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2) n \log^2 n)$ edges. The graph G is a $(1-\delta)$-local $(1+\varepsilon)$-spanner for axis parallel rectangles. Formally, for any axis-parallel rectangle R, we have that $R \cap P$ is an $(1+\varepsilon)$-spanner for all the points of $\big((1-\delta)R\big) \cap P$.*

Proof: Computing the QSPD $\mathcal{W}$ takes $\mathcal{O}(n \log^2 n)$ time. For each pair $\{X, Y\}$ in the decomposition with $m = |X| + |PSY|$ points, we need to compute the lowest and leftmost points in $(X \cup Y) \cap \mathsf{C}$, for each cell in the constructed grid. This can readily be done using orthogonal range trees in $\mathcal{O}(\log^2 n)$ time per query (a somewhat faster query time should be possible by using that offline nature of the queries, etc). This yields the construction time. The size of the computed graph is $\mathcal{O}(\omega(\mathcal{W})\tau^2) = O\big((1/\delta^2 + 1/\varepsilon^2) n \log^2 n\big)$.

The desired local spanner property is provided by Lemma 4.5. ■

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