

1 Local Spanners Revisited

2 Author: Please provide author information

3 — Abstract —

4 For a set of points $P \subseteq \mathbb{R}^2$ and a family of regions \mathcal{F} , a *local t -spanner* of P is a sparse graph G over
5 P , such that for any region $r \in \mathcal{F}$ the subgraph restricted to r , denoted by $G \cap r$, is a t -spanner for
6 all the points of $r \cap P$.

7 We present algorithms for the construction of local spanners with respect to several families of
8 regions such as homothets of a convex region. Unfortunately, the number of edges in the resulting
9 graph depends logarithmically on the spread of the input point set. We prove that this dependency
10 can not be removed, thus settling an open problem raised by Abam and Borouny. We also show
11 improved constructions (with no dependency on the spread) of local spanners for fat triangles, and
12 regular k -gons. In particular, this improves over the known construction for axis parallel squares.

13 We also study notions of weaker local spanners where one is allowed to shrink the region a “bit”.
14 Surprisingly, we show a near linear size construction of a weak spanner for axis-parallel rectangles,
15 where the shrinkage is *multiplicative*. Any spanner is a weak local spanner if the shrinking is
16 proportional to the diameter of the region.

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1 Introduction

For a set P of points in \mathbb{R}^d , the *Euclidean graph* $\mathcal{K}_P = (P, \binom{P}{2})$ of P is an undirected graph. Here, an edge $pq \in E$ is associated with the segment pq , and its weight is the (Euclidean) length of the segment. Let $G = (P, E)$ and $I = (P, E')$ be two graphs over the same set of vertices (usually I is a subgraph of G). Consider two vertices $p, q \in P$, and parameter $t \geq 1$. A path π between p and q in I , is a *t-path*, if the length of π in I is at most $t \cdot d_G(p, q)$, where $d_G(p, q)$ is the length of the shortest path between p and q in G . The graph I is a *t-spanner* of G if there is a *t-path* in I , for every $p, q \in P$. Thus, for a set of points $P \subseteq \mathbb{R}^d$, a graph G over P is a *t-spanner* if it is a *t-spanner* of the Euclidean graph \mathcal{K}_P . There is a lot of work on building geometric spanners, see [11] and references there in.

Fault-tolerant spanners

An *\mathcal{F} -fault-tolerant spanner* for $P \subseteq \mathbb{R}^d$, is a graph $G = (P, E)$, such that for any region \mathcal{r} (i.e., the “attack”), the graph $G - \mathcal{r}$ is a *t-spanner* of $\mathcal{K}_P - \mathcal{r}$, where $G - \mathcal{r}$ denotes the graph after one deletes from G all the vertices in $P \cap \mathcal{r}$, and all the edges in G whose corresponding segments intersect \mathcal{r} (See Definition 1 for a formal definition of this notation). Surprisingly, as shown by Abam *et al.* [3], such fault-tolerant spanners can be constructed where the attack region is any convex set. Furthermore, these spanners have a near linear number of edges.

Fault-tolerant spanners were first studied with vertex and edge faults, meaning that some arbitrary set of maximum size k of vertices and edges has failed. Levkopoulos *et al.* [9] showed the existence of k -vertex/edges fault tolerant spanners for a set of points P in some metric space. Their spanner had $\mathcal{O}(kn \log n)$ edges, and weight, i.e. sum of edge weights, bounded by $f(k) \cdot wt(MST(P))$ for some function f . Lukovszki [10] later achieved a similar construction, improving the number of edges to $\mathcal{O}(kn)$, and was able to prove that the result is asymptotically tight.

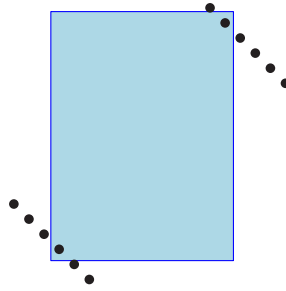
Local spanners

Recently, Abam and Borouny [2] introduced the notion of local spanners, which can be interpreted as having the complement property to being fault-tolerant. For a family of regions \mathcal{F} , a graph $G = (P, E)$ is a *\mathcal{F} -local t-spanner* for P , if for any $\mathcal{r} \in \mathcal{F}$, the subgraph of G induced on $P \cap \mathcal{r}$ is a *t-spanner*. Specifically, this induced subgraph $G \cap \mathcal{r}$ contains a *t-path* between any $p, q \in P \cap \mathcal{r}$ (note that we keep an edge in the subgraph only if both its endpoints are in \mathcal{r} , see Definition 1).

Abam and Borouny [2] showed how to construct such spanners for axis-parallel squares and vertical slabs. In this work, we further extend their results. They also showed how to construct such spanners for disks if one is allowed to add Steiner points, but left the question of how to construct local spanners for disks (without Steiner points) as an open problem.

To appreciate the difficulty in constructing local spanners, observe that unlike regular spanners, the construction has to take into account many different scenarios as far as which points are available to be used in the spanner. As a concrete example, a local spanner for axis-parallel rectangles requires a quadratic number of edges, see Figure 1.1.

Namely, regular spanners can rely on using midpoints in their path under the assurance that they are always there. For local spanners this is significantly harder as natural midpoints might “disappear”. Intuitively, a local spanner construction needs to use midpoints that are guaranteed to be present judging only from the source and destination points of the path.



■ **Figure 1.1** For any point in the top diagonal and bottom diagonal, there is a fat axis parallel rectangle that contains only these two points. Thus, a local spanner requires a quadratic number of edges in this case.

63 A good jump is hard to find

64 Most constructions for spanners can be viewed as searching for a way to build a path from
 65 the source to the destination by finding a “good” jump, either by finding a way to move
 66 locally from the source to a nearby point in the right direction, as done in the θ -graph
 67 construction, or alternatively, by finding an edge in the spanner from the neighborhood of the
 68 source to the neighborhood of the destination, as done in the spanner constructions using a
 69 well-separated pair decomposition (WSPD). Usually, one argues inductively that the spanner
 70 must have (sufficiently short) paths from the source to the start of the jump, and from the
 71 end of the jump to the destination, and then, combining these implies that the resulting
 72 new path is short. These ideas guide our constructions as well. However, the availability of
 73 specific edges depends on the query region, making the search for a good jump significantly
 74 more challenging. Intuitively, the constructions have to guarantee that there are many edges
 75 available, and that at least one of them is useful as a jump regardless of the chosen region
 76 (since slight perturbation in the region might make many of these edges unavailable).

77 Our results

78 ■ **Stav:** changed table after removing Section 4 — **end** — Our results are summa-
 79 rized in Table 1.1 ■ **Sariel:** Ref to the appendix — **end** —

80 Almost local spanners

81 We start by showing that regular geometric spanners are local spanners if one is required
 82 provide the spanner guarantee only to shrunken regions. Namely, if G is a $(1 + \varepsilon)$ -spanner of
 83 P , then for any convex region \mathcal{C} , the graph $G \cap \mathcal{C}$ is a spanner for $\mathcal{C}' \cap P$, where \mathcal{C}' is the set
 84 of all points in \mathcal{C} that are in distance at least $\delta \cdot \text{diam}(\mathcal{C})$ from its boundary, for $\delta = \Omega(\sqrt{\varepsilon})$ –
 85 see Lemma 12.

86 Homothets

87 A *homothet* of a convex region \mathcal{C} , is a translated and scaled copy of \mathcal{C} . In Section 3 we present
 88 a construction of spanners, which surprisingly, is not only fault-tolerant for all smooth convex
 89 regions, but is also a local spanner for homothets of a prespecified convex region. This
 90 in particular works for disks, and resolves the aforementioned open problem of Abam and
 91 Borouny [2]. Our construction is somewhat similar to the original construction of Abam
 92 *et al.* [3]. For a parameter $\varepsilon > 0$ the construction of a local $(1 + \varepsilon)$ -spanner for homothets

Region	# edges	Paper	New # edges	Location in paper
Local $(1 + \varepsilon)$ -spanners				
Halfplanes	$\mathcal{O}(\varepsilon^{-2}n \log n)$	[3]		
Axis-parallel squares	$\mathcal{O}_\varepsilon(n \log^6 n)$	[2]	$\mathcal{O}(\varepsilon^{-3}n \log n)$	Remark 27
Vertical slabs	$\mathcal{O}(\varepsilon^{-2}n \log n)$	[2]		
Disks+Steiner points	$\mathcal{O}_\varepsilon(n \log^2 n)$	[2]		
Disks			$\mathcal{O}(\varepsilon^{-2}n \log \Phi)$	Theorem 19
			$\Omega(n \log(1 + \frac{\Phi}{n}))$	Lemma 20
Homothets of a convex body			$\mathcal{O}(\varepsilon^{-2}n \log \Phi)$	Theorem 19
Homothets of α -fat triangles			$\mathcal{O}((\alpha\varepsilon)^{-1}n)$	Theorem 23
Homothets of triangles			$\Omega(n \log(1 + \frac{\Phi}{n}))$	Lemma 21
δ -weak local $(1 + \varepsilon)$ -spanners				
Bounded convex body			$\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$	Lemma 12
$(1 - \delta)$ -local $(1 + \varepsilon)$ -spanners				
Axis-parallel rectangles			$\mathcal{O}((\varepsilon^{-2} + \delta^{-2})n \log^2 n)$	In full version

■ **Table 1.1** Known and new results. The notation \mathcal{O}_ε hides polynomial dependency on ε which is not specified in the original work. The last result appears in the full version of this paper.

takes $\mathcal{O}(\varepsilon^{-2}n \log \Phi \log n)$ time, and the resulting spanner is of size $\mathcal{O}(\varepsilon^{-2}n \log \Phi)$, where Φ is the spread of the input point set P , and $n = |P|$.

The dependency on the spread Φ in the above construction is somewhat disappointing. However, the lower bound constructions, provided in Section 3.3, show that this is unavoidable for disks or homothets of triangles.

Thus, the natural question is what are the cases where one can avoid the “curse of the spread” – that is, cases where one can construct local spanners of near-linear size independent of the spread of the input point set.

The basic building block: \mathcal{C} -Delaunay triangulation

A key ingredient in the above construction is the concept of Delaunay triangulations induced by homothets of a convex body. Intuitively, one replaces the unit disk (of the standard L_2 -norm) by the provided convex region. It is well known [6] that such diagrams exist, have linear complexity in the plane, and can be computed quickly. In Section 3.1 we review these results, and restate the well-known property that the \mathcal{C} -Delaunay triangulation is connected when restricted to a homothet of \mathcal{C} . By computing these triangulations for carefully chosen subsets of the input point set, we get the results stated above.

Specifically, we use well-separated and semi-separated decompositions to compute these subsets.

Fat triangles

In Section 3.4 we give a construction of local spanners for the family \mathcal{F} of homothets of a given triangle Δ , and get a spanner of size $\mathcal{O}((\alpha\varepsilon)^{-1}n)$ in $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ time, where α is the smallest angle in Δ . This construction is a careful adaptation of the θ -graph spanner

construction to the given triangle, and it is significantly more technically challenging than the original construction.

k -regular polygons

It seems natural that if one can handle fat triangles, then homothets of k -regular polygons should readily follow by a simple decomposition of the polygon into fat triangles. Maybe surprisingly, this is not the case – a critical configuration might involve two points that are on the interior of two non-adjacent edges of a homothet of the input polygon. We overcome this by first showing that sufficiently narrow trapezoids provide us with a good jump somewhere inside the trapezoid, assuming one computes the Delaunay triangulation induced by the trapezoid, and that the source and destination lie on the two legs of the trapezoid. Next, we show that such a polygon can be covered by a small number of narrow trapezoids and fat triangles. By building appropriate graphs for each trapezoid/triangle in the collection, we get a spanner for homothets of the given k -regular polygon, with size that has no dependency on the spread. Of course, the size does depend polynomially on k . See Section 3.5 for details, and Theorem 26 for the precise result.

See Table 1.1 for a summary of known results and comparisons to the results of this paper.

2 Preliminaries

Residual graphs

► **Definition 1.** Let \mathcal{F} be a family of regions in the plane. For a fault region $r \in \mathcal{F}$ and a geometric graph G on a point set P , let $G - r$ be the residual graph after removing from it all the points of P in r and all the edges whose corresponding segments intersect r . Similarly, let $G \cap r$ denote the graph restricted to r . Formally, let

$$G - r = (P \setminus r, \{uv \in E \mid uv \cap \text{int}(r) = \emptyset\}) \quad \text{and} \quad G \cap r = (P \cap r, \{uv \in E \mid uv \subseteq r\}).$$

where $\text{int}(r)$ denotes the interior of r ,

2.1 On various pair decompositions

For sets X, Y , let $X \otimes Y = \{\{x, y\} \mid x \in X, y \in Y, x \neq y\}$ be the set of all the (unordered) pairs of points formed by the sets X and Y .

► **Definition 2** (Pair decomposition). For a point set P , a **pair decomposition** of P is a set of pairs

$$\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\},$$

such that (I) $X_i, Y_i \subseteq P$ for every i , (II) $X_i \cap Y_i = \emptyset$ for every i , and (III) $\bigcup_{i=1}^s X_i \otimes Y_i = P \otimes P$. Its **weight** is $\omega(\mathcal{W}) = \sum_{i=1}^s (|X_i| + |Y_i|)$.

The **closest pair** distance of a set of points $P \subseteq \mathbb{R}^d$, is $\text{cp}(P) = \min_{p, q \in P, p \neq q} \|pq\|$. The **diameter** of P is $\text{diam}(P) = \max_{p, q \in P} \|pq\|$. The **spread** of P is $\Phi(P) = \text{diam}(P) / \text{cp}(P)$, which is the ratio between the diameter and closest pair distance. While in general the weight of a WSPD (defined below) can be quadratic, if the spread is bounded, the weight is near linear. For $X, Y \subseteq \mathbb{R}^d$, let $d(X, Y) = \min_{p \in X, q \in Y} \|pq\|$ be the **distance** between the two sets.

153 ► **Definition 3.** Two sets $X, Y \subseteq \mathbb{R}^d$ are

154 $1/\varepsilon$ -**well-separated** if $\max(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot d(X, Y)$,
 155 and $1/\varepsilon$ -**semi-separated** if $\min(\text{diam}(X), \text{diam}(Y)) \leq \varepsilon \cdot d(X, Y)$.
 156

157 For a point set P , a **well-separated pair decomposition (WSPD)** of P with parameter
 158 $1/\varepsilon$ is a pair decomposition of P with a set of pairs $\mathcal{W} = \{\{B_1, C_1\}, \dots, \{B_s, C_s\}\}$, such
 159 that for all i , the sets B_i and C_i are $(1/\varepsilon)$ -separated. The notion of $1/\varepsilon$ -SSPD (a.k.a
 160 **semi-separated pairs decomposition**) is defined analogously.

161 ► **Lemma 4** ([1]). Let P be a set of n points in \mathbb{R}^d , with spread $\Phi = \Phi(P)$, and let
 162 $\varepsilon > 0$ be a parameter. Then, one can compute a $(1/\varepsilon)$ -WSPD \mathcal{W} for P of total weight
 163 $\omega(\mathcal{W}) = \mathcal{O}(n\varepsilon^{-d} \log \Phi)$. Furthermore, any point of P participates in at most $\mathcal{O}(\varepsilon^{-d} \log \Phi)$
 164 pairs.

165 ► **Theorem 5** ([1, 8]). Let P be a set of n points in \mathbb{R}^d , and let $\varepsilon > 0$ be a parameter. Then,
 166 one can compute a $(1/\varepsilon)$ -SSPD for P of total weight $\mathcal{O}(n\varepsilon^{-d} \log n)$. The number of pairs in
 167 the SSPD is $\mathcal{O}(n\varepsilon^{-d})$, and the computation time is $\mathcal{O}(n\varepsilon^{-d} \log n)$.

168 The following claim is straightforward.

169 ► **Lemma 6.** Given an α -SSPD \mathcal{W} of a set P of n points in \mathbb{R}^d and a parameter $\beta \geq 2$, one
 170 can refine \mathcal{W} into an $\alpha\beta$ -SSPD \mathcal{W}' , such that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\beta^d)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\beta^d)$.

171 ► **Definition 7.** An ε -**double-wedge** is a region between two lines, where the angle between
 172 the two lines is at most ε .

173 Two point sets X and Y that each lie in their own face of a shared ε -double-wedge are
 174 ε -**angularly separated**.

175 ► **Lemma 8** (Proof in [Appendix C.1](#)). Given a $(1/\varepsilon)$ -SSPD \mathcal{W} of n points in the plane, one
 176 can refine \mathcal{W} into a $(1/\varepsilon)$ -SSPD \mathcal{W}' , such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a
 177 ε -double-wedge \times_Ξ , such that X and Y are contained in the two different faces of the double
 178 wedge \times_Ξ . We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time
 179 is proportional to the weight of \mathcal{W}' .

180 ► **Corollary 9.** Let P be a set of n points in the plane, and let $\varepsilon > 0$ be a parameter. Then,
 181 one can compute a $(1/\varepsilon)$ -SSPD for P such that every pair is ε -angularly separated. The total
 182 weight of the SSPD is $\mathcal{O}(n\varepsilon^{-3} \log n)$, the number of pairs in the SSPD is $\mathcal{O}(n\varepsilon^{-3})$, and the
 183 computation time is $\mathcal{O}(n\varepsilon^{-3} \log n)$.

184 2.2 Weak local spanners for fat convex regions

185 ► **Definition 10.** Given a convex region C , let

$$186 \quad C_{\boxminus \delta} = \{p \in C \mid d(p, \mathbb{R}^2 \setminus C) \geq \delta \cdot \text{diam}(C)\}.$$

187 In other words, $C_{\boxminus \delta}$ is the Minkowski difference of C with a disk of radius $\delta \cdot \text{diam}(C)$.

188 ► **Definition 11.** Consider a (bounded) set C in the plane. Let $r_{\text{in}}(C)$ be the radius of the
 189 largest disk contained inside C . Similarly, $R_{\text{out}}(C)$ is the smallest radius of a disk containing
 190 C .

191 The **aspect ratio** of a region C in the plane is $\text{ar}(C) = R_{\text{out}}(C)/r_{\text{in}}(C)$. Given a family
 192 \mathcal{F} of regions in the plane, its aspect ratio is $\text{ar}(\mathcal{F}) = \max_{C \in \mathcal{F}} \text{ar}(C)$.

Note, that if a convex region C has bounded aspect ratio, then $C_{\Box\delta}$ is similar to the result of scaling C by a factor of $1 - \mathcal{O}(\delta)$. On the other hand, if C is long and skinny then this region is much smaller. Specifically, if C has width smaller than $2\delta \cdot \text{diam}(C)$, then $C_{\Box\delta}$ is empty.

► **Lemma 12** (Proof in [Appendix C.2](#)). *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0, 1)$. One can construct a graph G over P , in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$ edges, such that for any (bounded) convex C in the plane, we have that for any two points $p, q \in P \cap C_{\Box\delta}$ the graph $G \cap P$ has a $(1 + \varepsilon)$ -path between p and q .*

3 Local spanners of homothets of convex region

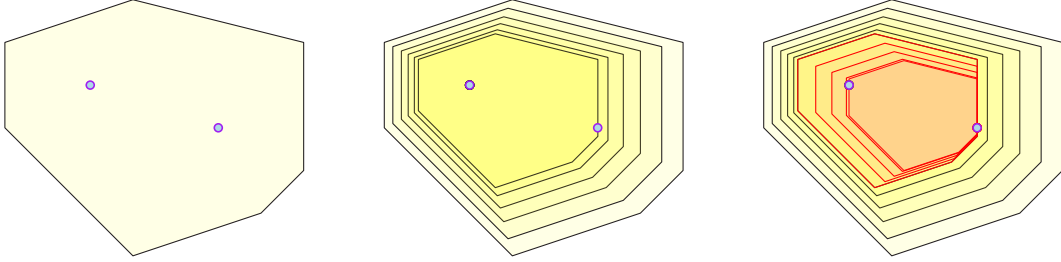
Let \mathcal{C} be a bounded convex and closed region in the plane (e.g., a disk). A *homothet* of \mathcal{C} is a scaled and translated copy of \mathcal{C} . A point set P is in *general position* with respect to \mathcal{C} , if no four points of P lie on the boundary of a homothet of \mathcal{C} , and no three points are colinear.

A graph $G = (P, E)$ is a *\mathcal{C} -local t -spanner* for P if for any homothet \mathcal{r} of \mathcal{C} we have that $G \cap \mathcal{r}$ is a t -spanner of $\mathcal{K}_P \cap \mathcal{r}$.

3.1 Delaunay triangulation for homothets

► **Definition 13** ([6]). *Given \mathcal{C} as above, and a point set P in general position with respect to \mathcal{C} , the \mathcal{C} -Delaunay triangulation of P , denoted by $\mathcal{D}_{\mathcal{C}}(P)$, is the graph formed by edges between any two points $p, q \in P$ such that there is a homothet of \mathcal{C} that contains only p and q and no other point of P .*

► **Theorem 14** ([6]). *For any convex body \mathcal{C} and set of points P , $\mathcal{D}_{\mathcal{C}}(P)$ can be computed in $\mathcal{O}(n \log n)$ time. Furthermore, the triangulation $\mathcal{D}_{\mathcal{C}}(P)$ has $\mathcal{O}(n)$ edges, vertices, and faces.*



■ **Figure 3.1** Shrinking of a homothet so that two specific points would lie on its boundary.

► **Lemma 15** (Proof in [Appendix C.3](#)). *Let \mathcal{C} be a convex bounded body, and let P be a set of points in general position with respect to \mathcal{C} . Then, if C is a homothet of \mathcal{C} that contains two points $p, q \in C \cap P$, then there exists a homothet $C' \subseteq C$ of \mathcal{C} such that $p, q \in \partial C'$.*

The following standard claim, usually stated for the standard Delaunay triangulations, also holds for homothets.

► **Claim 16** (Proof in [Appendix C.4](#)). *Let \mathcal{C} be a compact (bounded and closed) convex body. Given a set of points $P \subseteq \mathbb{R}^2$ in general position with respect to \mathcal{C} , let $\mathcal{D} = \mathcal{D}_{\mathcal{C}}(P)$ be the \mathcal{C} -Delaunay triangulation of P . For any homothet C of \mathcal{C} , we have that $\mathcal{D} \cap C$ is connected.*

3.2 The generic construction

The input is a set P of n points in the plane (in general position) with spread $\Phi = \Phi(P)$, a parameter $\varepsilon \in (0, 1)$, and a convex body \mathcal{C} that defines the “unit” ball. The task is to construct \mathcal{C} -local spanner.

The algorithm computes a $(1/\vartheta)$ -WSPD \mathcal{W} of P using the algorithm of Lemma 4, where $\vartheta = \varepsilon/6$. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, the algorithm computes the \mathcal{C} -Delaunay triangulation $\mathcal{D}_\Xi = \mathcal{D}_\mathcal{C}(X \cup Y)$, and adds all the edges in $\mathcal{D}_\Xi \cap (X \otimes Y)$ to the computed graph G .

► **Remark 17.** In the above algorithm, the idea of computing a triangulation for each WSPD pair seems to be new.

3.2.1 Analysis

Size. For each pair $\Xi = \{X, Y\}$ in the WSPD, its \mathcal{C} -Delaunay triangulation contains at most $\mathcal{O}(|X| + |Y|)$ edges. As such, the number of edges in the resulting graph is bounded by $\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}(|X| + |Y|) = \mathcal{O}(\omega(\mathcal{W})) = \mathcal{O}(n\vartheta^{-2} \log \Phi)$, by Lemma 4.

Construction time. The construction time is bounded by

$$\sum_{\{X, Y\} \in \mathcal{W}} \mathcal{O}((|X| + |Y|) \log(|X| + |Y|)) = \mathcal{O}(\omega(\mathcal{W}) \log n) = \mathcal{O}(n\vartheta^{-2} \log \Phi \log n).$$

► **Lemma 18 (Local spanner property).** For $P, \mathcal{C}, \varepsilon$ as above, let G be the graph constructed above for the point set P . Then, for any homothet C of \mathcal{C} and any two points $x, y \in P \cap C$, we have that $G \cap C$ has a $(1 + \varepsilon)$ -path between x and y . That is, G is a \mathcal{C} -local $(1 + \varepsilon)$ -spanner.

Proof. Fix a homothet C of \mathcal{C} , and consider two points $p, q \in P \cap C$. The proof is by induction on the distance between p and q (or more precisely, the rank of their distance among the $\binom{n}{2}$ pairwise distances). Consider the pair $\Xi = \{X, Y\}$ such that $x \in X$ and $y \in Y$.

If $xy \in \mathcal{D}_\Xi$ then the claim holds, so assume this is not the case. By the connectivity of $\mathcal{D}_\Xi \cap C$, see Claim 16, there must be points $x' \in X \cap C$, $y' \in Y \cap C$, such that $x'y' \in E(\mathcal{D}_\Xi)$. As such, by construction, we have that $x'y' \in E(G)$. Furthermore, by the separation property, we have that

$$\max(\text{diam}(X), \text{diam}(Y)) \leq \vartheta d(X, Y) \leq \vartheta \ell,$$

where $\ell = \|xy\|$. In particular, $\|x'x\| \leq \vartheta \ell$ and $\|y'y\| \leq \vartheta \ell$. As such, by induction, we have $d_G(x, x') \leq (1 + \varepsilon) \|xx'\| \leq (1 + \varepsilon) \vartheta \ell$ and $d_G(y, y') \leq (1 + \varepsilon) \|yy'\| \leq (1 + \varepsilon) \vartheta \ell$. Furthermore, $\|x'y'\| \leq (1 + 2\vartheta) \ell$. As $x'y' \in E(G)$, we have

$$\begin{aligned} d_G(x, y) &\leq d_G(x, x') + \|x'y'\| + d_G(y', y) \leq (1 + \varepsilon) \vartheta \ell + (1 + 2\vartheta) \ell + (1 + \varepsilon) \vartheta \ell \leq (2\vartheta + 1 + 2\vartheta + 2\vartheta) \ell \\ &= (1 + 6\vartheta) \ell \leq (1 + \varepsilon) \|xy\|, \end{aligned}$$

if $\vartheta \leq \varepsilon/6$. ◀

The result. We thus get the following.

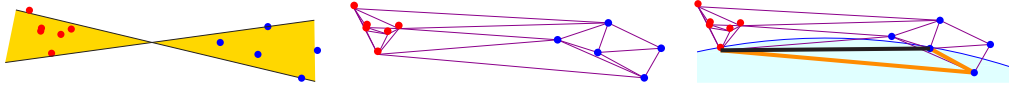
► **Theorem 19.** Let \mathcal{C} be a bounded convex body in the plane, let P be a given set of n points in the plane (in general position), and let $\varepsilon \in (0, 1/2)$ be a parameter. The above algorithm constructs a \mathcal{C} -local $(1 + \varepsilon)$ -spanner G . The spanner has $\mathcal{O}(\varepsilon^{-2} n \log \Phi)$ edges, and the construction time is $\mathcal{O}(\varepsilon^{-2} n \log \Phi \log n)$. Formally, for any homothet C of \mathcal{C} , and any two points $p, q \in P \cap C$, we have a $(1 + \varepsilon)$ -path in $G \cap C$.

3.3 Lower bounds

3.3.1 A lower bound for local spanner for disks

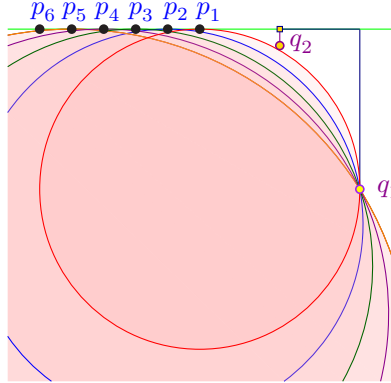
The result of [Theorem 19](#) is somewhat disappointing as it depends on the spread of the point set (logarithmically, but still). Next, we show a lower bound proving that this dependency is unavoidable, even in the case of disks.

Some intuition. A natural way to attempt a spread-independent construction is to try and emulate the construction of Abam *et al.* [3] and use a SSPD instead of a WSPD, as the total weight of the SSPD is near linear (with no dependency on the spread). Furthermore, after some post processing, one can assume every pair $\Xi = \{X, Y\}$ is angularly ε -separated – that is, there is a double wedge with angle $\leq \varepsilon$, such that X and Y are on different sides of the double wedge. The problem is that for a disk \circ , it might be that the bridge edge between X and Y that is in $\mathcal{D}_\Xi \cap \circ$ is much longer than the distance between the two points of interest. This somewhat counter-intuitive situation is illustrated in [Figure 3.2](#).



■ **Figure 3.2** A bridge too far – the only surviving bridge between the red and blue points is too far to be useful if the sets of points are not well separated.

► **Lemma 20.** For $\varepsilon = 1/4$, and parameters n and Φ , there is a point set P of $n + \lceil \log \Phi \rceil$ points in the plane, with spread $\mathcal{O}(n\Phi)$, such that any local $(1 + \varepsilon)$ -spanner of P for disks, must have $\Omega(n(1 + \log \frac{\Phi}{n}))$ edges, as long as $\sqrt{n} \leq \Phi \leq n2^n$.



■ **Figure 3.3** The set of disks D_1 , and the construction of q_2 .

Proof. Let $p_i = (-i, 0)$, for $i = 1, \dots, n$. Let $M = 1 + \lceil \log_2 \Phi \rceil$ and $q_1 = (n2^M, -1)$. For a point p on the x -axis, and a point q below the x -axis and to the right of p , let $\circ_{\downarrow}^p(q)$ be the disk whose boundary passes through p and q , and its center has the same x -coordinate as p .

In the j th iteration, for $j = 2, \dots, M - 1$, let $x_j = n2^{M-j+1} = x(q_{j-1})/2$, and let $y_j < 0$ be the maximum y -coordinate of a point that lies on the intersection of the vertical line $x = x_j$ and the union of disks $D_1 \cup \dots \cup D_j$ where

$$D_j = \left\{ \circ_{\downarrow}^{p_i}(q_{j-1}) \mid i = 1, \dots, n \right\},$$

see Figure 3.3 for an illustration of D_1 .

Let $q_j = (x_j, 0.99y_j)$.

Clearly, the point q_j lies outside all the disks of $D_1 \cup \dots \cup D_j$. The construction now continues to the next value of j . Let $P = \{p_1, \dots, p_n, q_2, \dots, q_M\}$. We have that $|P| = n + M - 1$.

The minimum distance between any points in the construction is 1 (i.e., $\|p_1 p_2\|$). Indeed $x(q_{M-1}) = 4n$ and thus $\|q_{M-1} p_1\| \geq 2n$. The diameter of P is $\|p_1 q_1\| = \sqrt{(n + n2^M)^2 + 1} \leq 2n2^M$. As such, the spread of P is bounded by $\leq n2^{M+1} = \mathcal{O}(n\Phi)$.

For any i and j , consider the disk $\odot_{\downarrow}^{p_i}(q_j)$. This disk does not contain any point of $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n$ since its interior lies below the x -axis. By construction it does not contain any point q_{j+1}, \dots, q_{M-1} . This disk potentially contains the points q_{j-1}, \dots, q_1 , but observe that for any index $k \in \llbracket j-1 \rrbracket$, we have that

$$\|p_i q_k\| = \sqrt{(i + n2^{M-k+1})^2 + (y(q_j))^2},$$

which implies that $n2^{M-k+1} \leq \|p_i q_k\| < n(2^{M-k+1} + 2)$. We thus have that

$$\frac{\|p_i q_k\|}{\|p_i q_j\|} \geq \frac{n2^{M-k+1}}{n(2^{M-j+1} + 2)} = \frac{2^{M-j} \cdot 2^{j-k}}{2^{M-j} + 1} = \frac{2^{j-k}}{1 + 1/2^{M-j}} \geq \frac{2}{1 + 1/2} = \frac{4}{3} > 1 + \varepsilon,$$

since $j \in \llbracket M-1 \rrbracket$. Namely, the shortest path in G between p_i and q_j , can not use any of the points q_1, \dots, q_{j-1} . As such, the graph G must contain the edge $p_i q_j$. This implies that $|E(G)| \geq n(M-1)$, which implies the claim. \blacktriangleleft

3.3.2 A lower bound for triangles

► **Lemma 21.** (Proof in Appendix A.1.) *For any $n > 0$, and $\Phi = \Omega(n)$, one can compute a set P of $n + \mathcal{O}(\log \Phi)$ points, with spread $\mathcal{O}(\Phi n)$, and a triangle Δ , such that any Δ -local $(3/2)$ -spanner of P requires $\Omega(n \log(1 + \frac{\Phi}{n}))$ edges.*

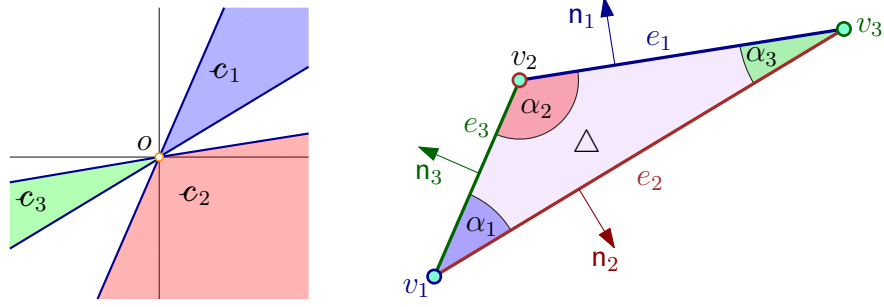
3.4 Local spanners for fat triangles

While local spanners for homothets of an arbitrary convex body are costly, if we are given a triangle Δ with the single constraint that Δ is not too “thin”, then one can construct a Δ -local t -spanner with a number of edges that does not depend on the spread of the points. See Figure A.1 for an illustration of a construction showing that dependency if “thin” triangles are allowed.

► **Definition 22.** *A triangle Δ is α -fat if the smallest angle in Δ is at least α .*

3.4.1 Construction

The input is a set P of n points in the plane, an α -fat triangle Δ , and an approximation parameter $\varepsilon \in (0, 1)$. Let v_i denote the i th vertex of Δ , α_i be the adjacent angle, and let e_i denote the opposing edge, for $i \in \llbracket 3 \rrbracket$. Let $c_i = \{(p - v_i)t \mid p \in e_i \text{ and } t \geq 0\}$ denote the cone with an apex at the origin induced by the i th vertex of Δ . Let \mathbf{n}_i be the outer normal of Δ orthogonal to e_i . See Figure 3.4 for an illustration. Let \mathcal{C}_i be a minimum size partition of c_i into cones each with angle in the range $[\beta/2, \beta]$, where $\beta = \varepsilon\alpha/\gamma$, and $\gamma > 1$ is a constant to be determined shortly. For each point $p \in P$, and a cone $c \in \mathcal{C}_i$, let $\text{nn}_i(p, c)$ be the first point in $(P - p) \cap (p + c)$ ordered by the direction \mathbf{n}_i (it is the “nearest-neighbor” to p in $p + c$ with respect to the direction \mathbf{n}_i).



■ **Figure 3.4** For the triangle \triangle with angles α_1, α_2 , and α_3 we create the cones c_1, c_2 , and c_3 .

325 The construction

326 Let G be the graph over P formed by connecting every point $p \in P$ to $\text{nn}_i(p, c)$, for all
 327 $i \in [3]$ and $c \in \mathcal{C}_i$. We get the following result (see Appendix D for details).

328 ► **Theorem 23.** *Let P be a set of n points in the plane, and let $\varepsilon \in (0, 1)$ be an approximation
 329 parameter. The above algorithm computes a \triangle -local $(1 + \varepsilon)$ -spanner G for an α -fat triangle
 330 \triangle . The construction time is $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$, and the spanner G has $\mathcal{O}((\alpha\varepsilon)^{-1}n)$ edges.*

331 **Proof.** The local-spanning property is proven in Lemma 36, and we are only left with
 332 bounding the size and the running time of the algorithm. The bound on the size is immediate
 333 from the construction, as every point p is the apex of $\mathcal{O}(\frac{2\pi}{\varepsilon\alpha})$ cones, each giving rise to a
 334 single edge incident to p . The construction time is bounded by the construction time for a
 335 θ -graph with cone size $\alpha\varepsilon$, which is $\mathcal{O}((\alpha\varepsilon)^{-1}n \log n)$ [7]. ◀

336 3.5 A local spanner for nice polygons

337 3.5.1 A good jump for narrow trapezoids

338 As a reminder, a trapezoid is a quadrilateral with two parallel edges, known as its *bases*. The
 339 other two edges are its *legs*. For $\varepsilon \in (0, 1/4)$, a trapezoid T is ε -*narrow* if the length of each
 340 of its legs is at most $\varepsilon \cdot \text{diam}(T)$.

341 ► **Lemma 24.** (Proof in Appendix A.2.) *Let $\varepsilon \in (0, 1)$ be some parameter, and $\vartheta = \varepsilon/16$.
 342 Let X, Y be two point sets that are $(1/\vartheta)$ -semi separated and ϑ -angularly separated (see
 343 Definition 7), and let T be a ϑ -narrow trapezoid, with two points $p \in X$ and $q \in Y$ lying on
 344 the two legs of T . Then, one can compute a homothet $T' \subseteq T$ of T , such that:*

- 345 (I) *There are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay
 346 triangulation of $X \cup Y$.*
 347 (II) *We have that $(1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|$.*

348 3.5.2 Breaking a nice polygon into narrow trapezoids

349 For a convex polygon \mathcal{C} , its *sensitivity*, denoted by $\text{sen}(\mathcal{C})$, is the minimum distance between
 350 any two non-adjacent edges (this quantity is no bigger than the length of the shortest edge in
 351 the polygon). A convex polygon \mathcal{C} is t -*nice*, if the outer angle at any vertex of the polygon
 352 is at least $2\pi/t$, and the length of the longest edge of \mathcal{C} is $\mathcal{O}(\text{sen}(\mathcal{C}))$. As an example, a
 353 k -regular polygon is k -nice.

354 ► **Lemma 25.** (Proof in [Appendix A.3](#).) *Let t be a positive integer. Given a t -nice polygon*
 355 *\mathcal{C} , and a parameter ϑ , one can cover it by a set \mathcal{T} of $\mathcal{O}(t^4/\vartheta^3)$ ϑ -narrow trapezoids, such that*
 356 *for any two points $p, q \in \partial\mathcal{C}$ that belong to two edges of \mathcal{C} that are not adjacent, there exists*
 357 *a narrow trapezoid $T \in \mathcal{T}$, such that p and q are located on two different short legs of T .*

358 3.5.3 Constructing the local spanner for nice polygons

359 ► **Theorem 26.** (Proof in [Appendix A.4](#).) *Let \mathcal{C} be a k -nice convex polygon, P be a set of*
 360 *n points in the plane, and let $\varepsilon \in (0, 1)$ be a parameter. Then, one can construct a \mathcal{C} -local*
 361 *$(1 + \varepsilon)$ -spanner of P . The construction time is $\mathcal{O}((k^4/\varepsilon^6)n \log^2 n)$, and the resulting graph*
 362 *has $\mathcal{O}((k^4/\varepsilon^6)n \log n)$ edges. In particular these bounds hold if \mathcal{C} is a k -regular polygon.*

363 ► **Remark 27.** For axis-parallel squares [Theorem 26](#) implies a local spanner with $\mathcal{O}(\varepsilon^{-6}n \log n)$
 364 edges. However, for this special case, the decomposition into narrow trapezoid can be skipped.
 365 In particular, in this case, the resulting spanner has $\mathcal{O}(\varepsilon^{-3}n \log n)$ edges. We do not provide
 366 the details here, as it is only a minor improvement over the above, and requires quite a bit
 367 of additional work – essentially, one has to prove a version of [Lemma 24](#) for squares. We
 368 leave the question of whether this bound can be further improved as an open problem for
 369 further research.

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A

 Proofs

A.1 Proof of Lemma 21

Proof. Let $h = \lceil \log_2 \Phi \rceil$. Let \triangle be the triangle formed by the points $(0, 0)$, $(0, 1)$ and $(8\Phi h, 0)$. The hypotenuse of this triangle lies on the line $\ell \equiv \frac{1}{8\Phi h}x + y = 1$, and let $v = (\frac{1}{8\Phi h}, 1)$ be the vector orthogonal to this line.

For $i \in \llbracket h \rrbracket$ and $j \in \llbracket n \rrbracket$, let

$$q_i = (2^{i+1}, 1 - i/h) \quad \text{and} \quad u_j = (\frac{j}{n} - 1, -\frac{j}{n}),$$

and let $P = \{q_1, \dots, q_h, u_1, \dots, u_n\}$, see Figure A.1. Observe that $\text{cp}(P) = \|u_1 u_2\| = \sqrt{2}/n$, and as such we have that $\Phi(P) = n \cdot \text{diam}(P)/\sqrt{2} \leq n(4\Phi + 2n) \leq 8\Phi n$, as $\Phi \geq n$. Observe that

$$\langle q_{i+1} - q_i, v \rangle = \langle (2^{i+1}, -\frac{1}{h}), (\frac{1}{8\Phi h}, 1) \rangle \leq \frac{4\Phi}{8\Phi h} - \frac{1}{h} < 0.$$

That is, the points q_1, \dots, q_i are increasing in distance from ℓ .

Let $\triangle_{i,j}$ be the homothet of \triangle , that has its bottom left corner at u_j , and its hypotenuse passes through q_i . By the above, $P(i, j) = \triangle_{i,j} \cap P = \{u_j, q_i, q_{i+1}, \dots, q_h\}$. Any $(1 + \varepsilon)$ -spanner for $P(i, j)$ must contain the edge $u_j q_i$. Indeed, we have, for any k , that $2^{k+1} \leq \|u_j q_k\| \leq 2^{k+1} + 3$. As such, any path on a graph induced on $P(i, j)$ from u_j to q_i that uses (say) a midpoint q_k , for $k > i$, must have dilation at least

$$\frac{\|u_j q_k\| + \|q_k q_i\|}{\|u_j q_i\|} \geq \frac{2^{k+1} + 2^k}{2^{i+1} + 3} \geq \frac{3 \cdot 2^{i+1}}{(1 + 3/4)2^{i+1}} = \frac{12}{7} > \frac{3}{2}.$$

Thus, any \triangle -local $3/2$ -spanner for homothets of \triangle , must contain the edge $u_j q_i$, for any $i \in \llbracket h \rrbracket$ and $j \in \llbracket n \rrbracket$. Thus, such a spanner must have $\Omega(n \log \Phi)$ edges, as claimed. \blacktriangleleft

A.2 Proof of Lemma 24

Proof. Let $\mathcal{D} = \mathcal{D}_T(X \cup Y)$. Claim 16 implies that $\mathcal{D} \cap T$ is connected. Thus, there is a path in $\mathcal{D} \cap T$ between p and q , and therefore, there must be an edge $p'q'$ along this path with $p' \in X$ and $q' \in Y$. This implies part (I).

Let $\ell = \|p'q'\|$. Assume for concreteness that $\|pp'\| \leq \text{diam}(X) \leq \vartheta \text{d}(X, Y) \leq \vartheta \ell \leq \vartheta \text{d}$, where $\text{d} = \text{diam}(T)$. Let q'' be the closest point on pq to q' .

We first consider the case that $q'' \in \text{int}(pq)$. We have that

$$\|pq''\| = \|pq'\| \cos \angle q'pq \geq (\|p'q'\| - \|pp'\|) \cos \angle q'pq \geq (1 - \vartheta)\ell \cdot (1 - \vartheta^2/2) \geq (1 - 2\vartheta)\ell,$$

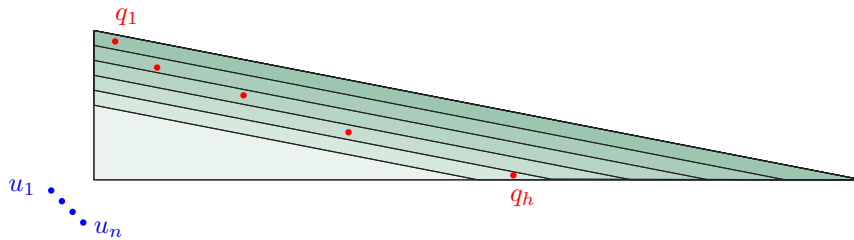
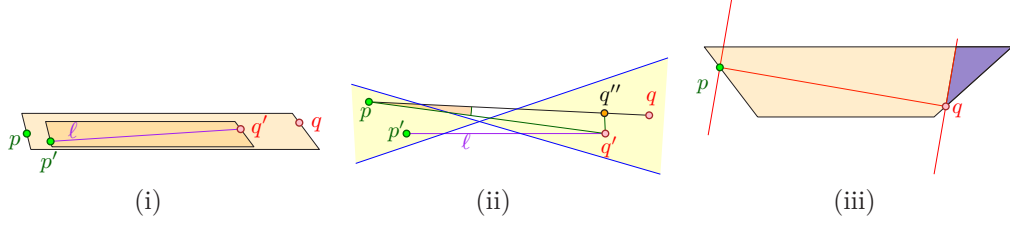


Figure A.1 An Illustration of the construction of Lemma 21.



■ **Figure A.2** Illustration of the settings in the proof of Lemma 24. Left: A ϑ -narrow trapezoid with p and q on its legs. Center: p and q are ϑ -semi separated and ϑ -angularly separated. Right: The triangle of all the points of the trapezoids whose nearest point on pq is q .

437 since $\cos \vartheta \geq 1 - \vartheta^2/2$, for $\vartheta < 1/2$. Similar arguments imply that $\|pq''\| \leq (1 + \vartheta)\ell$. As
 438 such, we have

$$439 \quad \|q'q''\| \leq (1 + \vartheta)\ell \sin \angle p'pq' \leq 2\vartheta\ell.$$

440 Thus, we have that

$$441 \quad \|qq'\| \leq \|qq''\| + \|q''q'\| \leq \|pq\| - \|pq''\| + 2\vartheta\ell \leq \|pq\| - (1 - 2\vartheta)\ell + 2\vartheta\ell \leq \|pq\| - \ell.$$

442 and finally,

$$443 \quad (1 + \varepsilon)\|pp'\| + \|p'q'\| + (1 + \varepsilon)\|q'q\| \leq (1 + \varepsilon)\vartheta\ell + \ell + (1 + \varepsilon)(\|pq\| - \ell) \\ 444 \quad = (1 + \varepsilon)\|pq\| + (1 + \varepsilon)\vartheta\ell + \ell - (1 + \varepsilon)\ell \leq (1 + \varepsilon)\|pq\|, \\ 445$$

446 for $\vartheta \leq \varepsilon/2$. Which establishes the claim in this case.

447 The case $q'' = p$ is impossible because of the angular separation property, and so, the only
 448 remaining possibility is that $q'' = q$. This however implies that q' must be in the triangle of
 449 all the points of the trapezoid whose nearest point on pq is q . The diameter of this triangle is
 450 bounded by the length of the leg of the trapezoid, which is bounded by ϑd . Namely, we have
 451 $\|qq'\| \leq \vartheta d$. Similarly, we have $(1 - 2\vartheta)d \leq \|pq\| \leq (1 + 2\vartheta)d$. Since $\|pp'\|, \|qq'\| \leq \vartheta d$, it
 452 follows that

$$453 \quad (1 - 4\vartheta)d \leq \ell \leq (1 + 4\vartheta)d.$$

454 As such, for $\vartheta \leq \varepsilon/8$ and $\varepsilon \leq 1$, we have

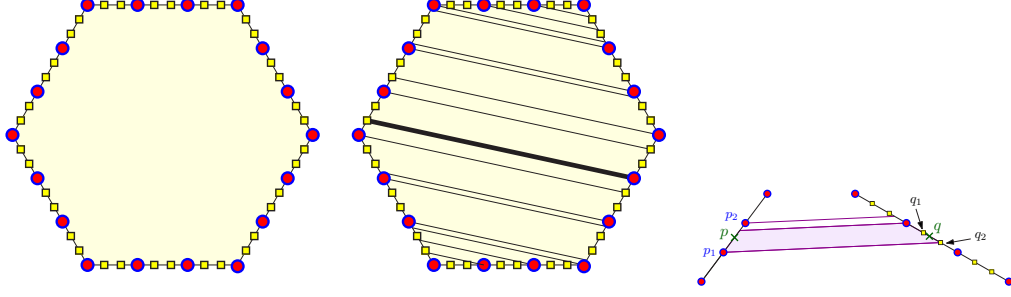
$$455 \quad (1 + \varepsilon)\|pp'\| + \ell + (1 + \varepsilon)\|q'q\| \leq 4\vartheta d + (1 + 4\vartheta)d = (1 + 8\vartheta)d \leq (1 + \varepsilon)\|pq\|.$$

456

457 A.3 Proof of Lemma 25

458 **Proof.** We show a somewhat suboptimal but simple construction. A t -nice polygon has at
 459 most t edges. Let ψ be the sensitivity of \mathcal{C} , and place a minimum set of points P on the
 460 boundary of \mathcal{C} , which includes all the vertices of \mathcal{C} , and such that the distance between any
 461 consecutive pair of points is in the range $[c_1, 2c_1]$, where $c_1 = \vartheta\psi/c_2$, for some sufficiently
 462 large constant c_2 . In particular, let $M = \max_{e \in E(\mathcal{C})} \lceil \|e\|/c_1 \rceil = \mathcal{O}(1/\vartheta)$.

463 In addition, place $c_3 \cdot t$ equally spaced points between any two consecutive points of P ,
 464 where c_3 is a constant to be determined shortly. Let Q be the set resulting from P after
 465 adding all these points.

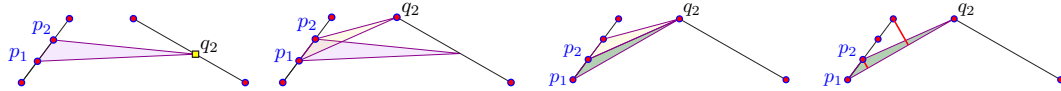


■ **Figure A.3** The points of P (round), and all the points added to P in order to create Q (square). On the right, a “vertical” decomposition induced by one of the directions of $P \times Q$.

466 We have that $|P| = \mathcal{O}(t/\vartheta)$ and $|Q| = \mathcal{O}(t^2/\vartheta)$. For a direction v , let \mathcal{T}_v be the
 467 decomposition into trapezoids formed by shooting rays from inside \mathcal{C} in the direction of v (or
 468 $-v$) from all the points of P , see Figure A.3. Let \mathcal{T}'_v be the set resulting from throwing away
 469 trapezoids with legs that lie on adjacent edges. It is easy to verify that all the trapezoids of
 470 \mathcal{T}'_v are ϑ -narrow. Let U be the set of all directions induced by pairs of points of $P \times Q$, and
 471 let $\mathcal{T} = \cup_{u \in U} \mathcal{T}'_u$. We have that $|\mathcal{T}| = \mathcal{O}(|P| \cdot |U|) = \mathcal{O}(|P|^2 |Q|) = \mathcal{O}(t^4/\vartheta^3)$.

472 Consider any two points p, q on non-adjacent edges of \mathcal{C} , and let p_1, p_2 be the two adjacent
 473 points of P such that $p \in p_1 p_2$. Now, let q_1, q_2 be the adjacent points of Q such that $q \in q_1 q_2$.
 474 We assume that p_1, p_2, q_1, q_2 are in this clockwise order along the boundary of \mathcal{C} .

475 Observe that when we project the interval $p_1 p_2$, to the line induced by $q_1 q_2$, in the
 476 direction $\overrightarrow{p_1 q_2}$, the projected interval contains $q_1 q_2$. The last claim is intuitively obvious,
 477 but requires some work to see formally. The minimum height of a triangle involving three
 478 vertices of \mathcal{C} is formed by three consecutive vertices. In the worst case, this is an isosceles
 479 triangle with sidelength ψ and base angle π/t . As such, the height of such a triangle is
 480 $h = \psi \sin(\pi/t) \geq \psi/t$.



■ **Figure A.4** The height of the triangle $\Delta p_1 p_2 q_2$ is minimized as q_2 and p_1 are moved to vertices of \mathcal{C} .

481 The height of the triangle $\Delta p_1 p_2 q_2$ is minimized when p_1 or p_2 is a vertex of \mathcal{C} , and
 482 q_2 is at a vertex of \mathcal{C} , see Figure A.4. Assume, for concreteness, that p_1 is a vertex of \mathcal{C} ,
 483 and observe that $\|p_1 p_2\| \geq \|e\|/M$, where e is the edge of \mathcal{C} containing this segment. Using
 484 similar triangles, it is straightforward to show that the height of this triangle is at least
 485 $h' = h/M = \Omega(\varepsilon\psi/t)$. The quantity h' is a lower bound on the length of the projection of
 486 $p_1 p_2$ on the line spanned by $q_1 q_2$. However, $\|q_1 q_2\| \leq 2c_1/c_3 t = \mathcal{O}(\vartheta\psi/c_3 t) < h'$, by picking
 487 c_3 to be a sufficiently large constant.

488 This readily implies that the trapezoid induced by the direction $u = \overrightarrow{p_1 q_2}$ in \mathcal{T}'_u that
 489 contains p on one of its leg, and q on the other. ◀

490 A.4 Proof of Theorem 26

491 **Proof.** Let $\vartheta = \varepsilon/c_4$, for c_4 sufficiently large constant. We construct Δ , a family of triangles
 492 induced by a vertex of \mathcal{C} , and an non-adjacent edge of \mathcal{C} . This family has $\mathcal{O}(k^2)$ triangles.

Each such triangle is $\Omega(1/k)$ -fat, and for each such triangle we construct the $(1 + \vartheta)$ -spanner of [Theorem 23](#) for P . Next, we cover \mathcal{C} by a set \mathcal{T} of $k' = \mathcal{O}(k^4/\vartheta^3)$ ϑ -narrow trapezoids using [Lemma 25](#).

We compute an ϑ -angular $(1/\vartheta)$ -SSPD \mathcal{W} decomposition of P using [Corollary 9](#) – the total weight of the decomposition is $w = \mathcal{O}(n\vartheta^{-3} \log n)$. For each pair $\{X, Y\} \in \mathcal{W}$, and each trapezoid $T \in \mathcal{T}$, we compute the T -Delaunay triangulation of $X \cup Y$.

Let G denote the union of all these graphs. We claim that it is the desired spanner. The construction time is

$$\mathcal{O}((k^3/\vartheta)n \log n + k'w \log n) = \mathcal{O}\left(\frac{k^3}{\vartheta}n \log n + \frac{k^4}{\vartheta^3} \cdot \frac{n}{\vartheta^3} \log n \cdot \log n\right) = \mathcal{O}\left(\frac{k^4}{\vartheta^6}n \log^2 n\right),$$

and the resulting graph has $\mathcal{O}((k^4/\vartheta^6)n \log n)$ edges.

As for correctness, consider a homothet \mathcal{C}' of \mathcal{C} that contains two points $p, q \in P$. By [Lemma 15](#), there is a homothet $\mathcal{C}'' \subseteq \mathcal{C}'$ of \mathcal{C} such that $p, q \in \partial\mathcal{C}''$. There are two possibilities:

(A) The point p is on a vertex of \mathcal{C}'' and q is on an edge. In this case, the vertex and the edge induce a fat triangle, that is a homothet of a triangle $\Delta \in \Delta$. Since the graph G contains a Δ -local $(1 + \varepsilon)$ -spanner for P , it follows readily that G is a $(1 + \varepsilon)$ -spanner for these points, and the path is strictly inside \mathcal{C}'' .

(B) The points p and q are on two non-adjacent edges of \mathcal{C}'' . Then, there is an ϑ -narrow trapezoid T' that has p and q on its two legs, and a homothet of T' , denoted by T , is in \mathcal{T} . There is a pair $\{X, Y\} \in \mathcal{W}$ that is $(1/\vartheta)$ -semi separated (and ϑ -angularly separated), such that $p \in X$ and $q \in Y$. By [Lemma 24](#), there are two points $p' \in X$ and $q' \in Y$, such that $p'q'$ is an edge of the T -Delaunay triangulation of $X \cup Y$, and by construction this edge is in G . We now use induction on the shortest paths from p to p' and from q to q' in G . By induction, and [Lemma 24](#), we have that

$$d(p, q) \leq d(p, p') + \|p'q'\| + d(q', q) \leq (1 + \varepsilon) \|pp'\| + \|p'q'\| + (1 + \varepsilon) \|q'q\| \leq (1 + \varepsilon) \|pq\|,$$

which implies that there is $(1 + \varepsilon)$ -path from p to q inside \mathcal{C}' . \blacktriangleleft

B Weak local spanners for axis-parallel rectangles

B.1 Quadrant separated pair decomposition

For the purpose of building the spanners in this section, we use a variation of a pair decomposition introduced by Agarwal *et al.* [4]. For two points $p = (p_1, \dots, p_d)$ and $q = (q_1, \dots, q_d)$ in \mathbb{R}^d , let $p \prec q$ denote that q *dominates* p coordinate-wise. That is $p_i < q_i$, for all i . More generally, let $p <_i q$ denote that $p_i < q_i$. For two point sets $X, Y \subseteq \mathbb{R}^d$, we use $X <_i Y$ to denote that $\forall x \in X, y \in Y \quad x <_i y$. In particular X and Y are *i -coordinate separated* if $X <_i Y$ or $Y <_i X$. A pair $\{X, Y\}$ is *quadrant-separated*, if X and Y are i -coordinate separated, for $i = 1, \dots, d$.

A *quadrant-separated pair decomposition* of a point set $P \subseteq \mathbb{R}^d$, is a pair decomposition (see [Definition 2](#)) $\mathcal{W} = \{\{X_1, Y_1\}, \dots, \{X_s, Y_s\}\}$ of P , such that $\{X_i, Y_i\}$ are quadrant-separated for all i .

► **Lemma 28.** *Given a set P of n points in \mathbb{R} , one can compute, in $\mathcal{O}(n \log n)$ time, a QSPD of P with $\mathcal{O}(n)$ pairs, and of total weight $\mathcal{O}(n \log n)$.*

Proof. If P is a singleton then there is nothing to do. If $P = \{p, q\}$, then the decomposition is the pair formed by the two singleton points.

Otherwise, let x be the median of P , such that $P_{\leq x} = \{p \in P \mid p \leq x\}$ contains exactly $\lceil n/2 \rceil$ points, and $P_{> x} = P \setminus P_{\leq x}$ contains $\lfloor n/2 \rfloor$ points. Construct the pair $\Xi = \{P_{\leq x}, P_{> x}\}$, and recursively compute QSPDs $\mathcal{Q}_{\leq x}$ and $\mathcal{Q}_{> x}$ for $P_{\leq x}$ and $P_{> x}$ respectively. The desired QSPD is $\mathcal{Q}_{\leq x} \cup \mathcal{Q}_{> x} \cup \{\Xi\}$. The bounds on the size and weight of the desired QSPD are immediate. \blacktriangleleft

► **Lemma 29.** *Given a set P of n points in \mathbb{R}^d , one can compute, in $\mathcal{O}(n \log^d n)$ time, a QSPD of P with $\mathcal{O}(n \log^{d-1} n)$ pairs, and of total weight $\mathcal{O}(n \log^d n)$.*

Proof. The construction algorithm is recursive on the dimensions, using the algorithm of Lemma 28 in one dimension.

The algorithm computes a value α_d that partitions the values of the points' d th coordinates roughly equally (and is distinct from all of them), and let h be a hyperplane parallel to the first $d - 1$ coordinate axes, and having value α_d in the d th coordinate.

Let P_{\uparrow} and P_{\downarrow} be the subset of points of P that are above and below h , respectively. The algorithm recursively computes QSPDs \mathcal{Q}_{\uparrow} and \mathcal{Q}_{\downarrow} for P_{\uparrow} and P_{\downarrow} respectively. Next, the algorithm projects the points of P on h , let P' be the resulting $d - 1$ dimensional point set (after we ignore the d th coordinate), and recursively computes a QSPD \mathcal{Q}' for P' .

For a point set $X' \subseteq P'$, let $\text{lift}(X')$ be the subset of points of P whose projection on h is X' . The algorithm now computes the set of pairs

$$\widehat{\mathcal{Q}} = \left\{ \{\text{lift}(X') \cap P_{\uparrow}, \text{lift}(Y') \cap P_{\downarrow}\}, \{\text{lift}(X') \cap P_{\downarrow}, \text{lift}(Y') \cap P_{\uparrow}\} \mid \{X', Y'\} \in \mathcal{Q}' \right\}.$$

The desired QSPD is $\widehat{\mathcal{Q}} \cup \mathcal{Q}_{\uparrow} \cup \mathcal{Q}_{\downarrow}$.

To observe that this is indeed a QSPD as all the pairs in $\mathcal{Q}_{\uparrow}, \mathcal{Q}_{\downarrow}$ are quadrant separated by induction, and pairs in $\widehat{\mathcal{Q}}$ are quadrant separated in the first $d - 1$ coordinates by induction on the dimension, and separated in the d coordinate since one side of the pair comes from P_{\uparrow} , and the other side from P_{\downarrow} .

As for coverage, consider any pair of points $p, q \in P$, and observe that the claim holds by induction if they are both in P_{\uparrow} or P_{\downarrow} . As such, assume that $p \in P_{\uparrow}$ and $q \in P_{\downarrow}$. But then there is a pair $\{X', Y'\} \in \mathcal{Q}'$ that separates the two projected points in h , and clearly one of the two lifted pairs that corresponds to this pair quadrant-separates p and q as desired.

The number pairs in the decomposition is $N(n, d) = 2N(n, d - 1) + 2N(n/2, d)$ with $N(n, 1) = \mathcal{O}(n)$. The solution to this recurrence is $N(n, d) = \mathcal{O}(n \log^{d-1} n)$. The total weight of the decomposition is $W(n, d) = 2W(n, d - 1) + 2W(n/2, d)$ with $W(n, 1) = \mathcal{O}(n \log n)$. The solution to this recurrence is $W(n, d) = \mathcal{O}(n \log^d n)$. Clearly, this also bounds the construction time. \blacktriangleleft

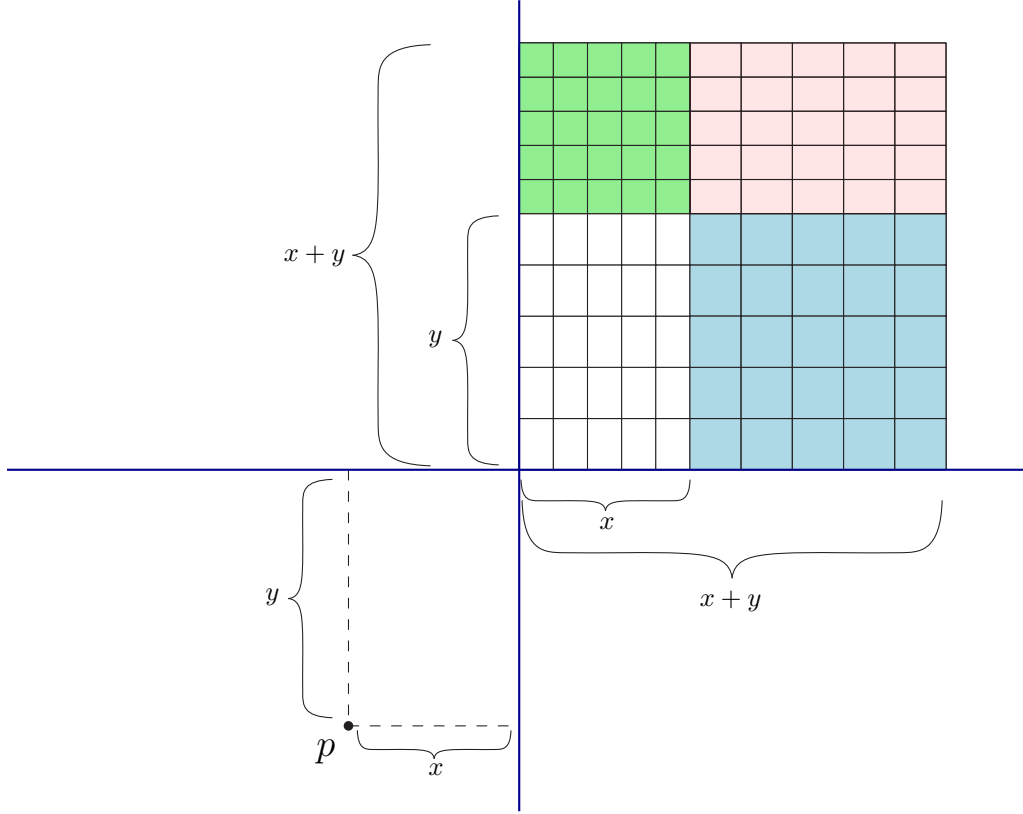
B.2 Weak local spanner for axis parallel rectangles

For a parameter $\delta \in (0, 1)$, and an interval $I = [b, c]$, let $(1 - \delta)I = [t - (1 - \delta)r, t + (1 - \delta)r]$, where $t = (b + c)/2$, and $r = (c - b)/2$, be the shrinking of I by a factor of $1 - \delta$.

Let \mathcal{R} be the set of all axis parallel rectangles in the plane. For a rectangle $R \in \mathcal{R}$ with $R = I \times J$, let $(1 - \delta)R = (1 - \delta)I \times (1 - \delta)J$ denote the rectangle resulting from shrinking R by a factor of $1 - \delta$.

► **Definition 30.** *Given a set P of n points in the plane, and parameters $\varepsilon, \delta \in (0, 1)$, a graph G is a $(1 - \delta)$ -local $(1 + \varepsilon)$ -spanner for rectangles, if for any axis-parallel rectangle R , we have that $G \cap R$ is a $(1 + \varepsilon)$ -spanner for all the points in $((1 - \delta)R) \cap P$.*

Observe that rectangles in \mathcal{R} might be quite “skinny”, so the previous notion of shrinkage used before is not useful in this case.



■ **Figure B.1** The construction of the grid $K(p, \Xi)$ for a point $p = (-x, -y)$ and a pair Ξ .

578 B.2.1 Construction for a single quadrant separated pair

579 Consider a pair $\Xi = \{X, Y\}$ in a QSPD of P . The set X is quadrant-separated from Y , that
 580 is, there is a point c_Ξ such that X and Y are contained in two opposing quadrants in the
 581 partition of the plane formed by the vertical and horizontal lines through c_Ξ .

582 For simplicity of exposition, assume that $c_\Xi = (0, 0)$, and $X \prec (0, 0) \prec Y$. That is, the
 583 points of X are in the negative quadrant, and the points of Y are in the positive quadrant.

584 For a point $p = (-x, -y) \in X$ we construct a non-uniform grid $K(p, \Xi)$ in the square
 585 $[0, x+y]^2$. To this end, we first partition it into four subrectangles

$$\begin{array}{c|c}
 B_{\swarrow} = [0, x] \times [y, x+y] & B_{\nearrow} = [x, x+y] \times [y, x+y] \\
 \hline
 B_{\swarrow} = [0, x] \times [0, y] & B_{\searrow} = [x, x+y] \times [0, y].
 \end{array}$$

587 Let $\tau \geq 4/\varepsilon + 4/\delta$ be an integer number. We partition each of these rectangles into a
 588 $\tau \times \tau$ grid, where each cell is a copy of the rectangle scaled by a factor of $1/\tau$. See Figure B.1.
 589 This grid has $\mathcal{O}(\tau^2)$ cells. For a cell C in this grid, let $Y \cap C$ be the points of Y contained in
 590 it. We connect p to the left-most and bottom-most points in $Y \cap C$. This process generates
 591 two edges in the constructed graph for each grid cell (that contains at least two points), and
 592 $\mathcal{O}(\tau^2)$ edges overall.

593 The algorithm repeats this construction for all the points $p \in X$, and does the symmetric
 594 construction for all the points of Y .



■ **Figure B.2** Left: The two rectangles R, R' . Right: In green $\overleftrightarrow{R} \cap R'$, the restriction of the slab \overleftrightarrow{R} to the rectangle R' .

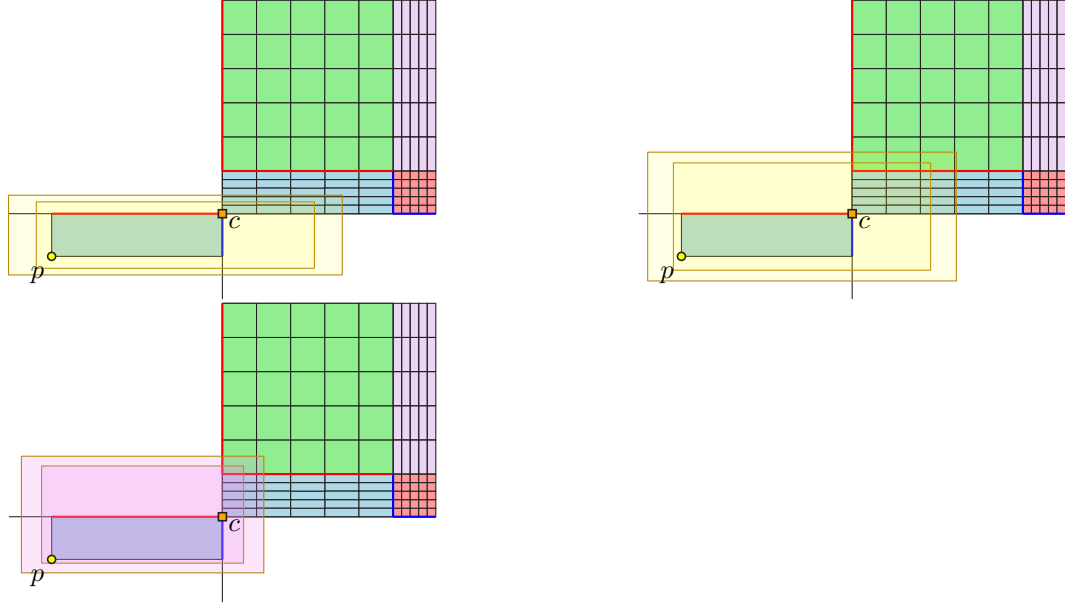
B.2.2 The construction algorithm

The algorithm computes a QSPD \mathcal{W} of P . For each pair $\Xi \in \mathcal{W}$, the algorithm generates edges for Ξ using the algorithm of Section B.2.1 and adds them to the generated spanner G .

B.2.3 Correctness

For a rectangle R , let $\overleftrightarrow{R} = \{(x, y) \in \mathbb{R}^2 \mid \exists(x', y) \in R\}$ be its expansion into a horizontal slab. Restricted to a rectangle R' , the resulting set is $\overleftrightarrow{R} \cap R'$, depicted in Figure B.2. Similarly, we denote

$$\overleftarrow{\downarrow} R = \{(x, y) \in \mathbb{R}^2 \mid \exists(x, y') \in R\}.$$



■ **Figure B.3** An illustration of $K(p, \Xi)$ with three rectangles and their shrunk version.

► **Lemma 31.** Assume that $\tau \geq \lceil 20/\varepsilon + 20/\delta \rceil$. Consider a pair $\Xi = \{X, Y\}$ in the above construction, and a point $p = (-x, -y) \in X$ with its associated grid $K = K(p, \Xi)$. Consider any axis parallel rectangle R , such that $p \in (1 - \delta)R = I \times J$, and $(1 - \delta)R$ intersects a cell $C \in K$. We have that:

(I) If $C \subseteq (1 - \delta)R$ then $(1 - \delta)^{-1}C \subseteq R$.

(II) $\text{diam}(C) \leq (\varepsilon/4)d(p, C)$.

609 (III) If $x \geq y$ and $C \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1 - \delta)^{-1}C \subseteq R$.

610 (IV) If $x \leq y$ and $C \subseteq R_{\swarrow} \cup R_{\searrow}$ then $(1 - \delta)^{-1}C \subseteq R$.

611 (V) If $x \geq y$ and $C \subseteq R_{\nwarrow}$, then $(1 - \delta)^{-1}(\overrightarrow{(1 - \delta)R} \cap C) \subseteq R$.

612 (VI) If $x \leq y$ and $C \subseteq R_{\swarrow}$, then $(1 - \delta)^{-1}(\uparrow((1 - \delta)R) \cap C) \subseteq R$.

613 **Proof.** (I) is immediate, (IV) and (VI) follows by symmetry from (III) and (V), respectively.

614 (II) We have that $\text{diam}(C) \leq (x + y)/\tau = \|p\|_1/\tau \leq (\varepsilon/4)d(p, C)$.

615 (III) The width, denoted $\text{wd}(\cdot)$, of $(1 - \delta)R$ is at least x , as it contains both p and the origin.

616 As such,

$$617 \quad (\text{wd}(R) - \text{wd}((1 - \delta)R))/2 \geq 2(x/\tau) \geq 2\text{wd}(C).$$

618 That is, the width of the “expanded” rectangle R is enough to cover C , and a grid cell
619 adjacent to it to the right.

620 A similar argument about the height shows that R covers the region immediately above C
621 – in particular, the vertical distance from C to the top boundary of R is at least the height of
622 C . This implies that the expanded cell $(1 - \delta)^{-1}C$ is contained in R , as claimed, as $\delta < 1/2$.

623 (V) We decompose the claim to the two dimensions of the region. Let $B = \overrightarrow{(1 - \delta)R} \cap C$.
624 Observe that containment in the x -axis follows by arguing as in (III). As for the y -interval
625 of B , observe that it is contained in the y -interval of $(1 - \delta)R$, which implies that when
626 expanded by $(1 - \delta)^{-1}$, it would be contained in the y -interval of R . Combining the two
627 implies the result. \blacktriangleleft

628 **► Lemma 32.** For any axis-parallel rectangle R , and any two points $p, q \in (1 - \delta)R \cap P$,
629 there exists a $(1 + \varepsilon)$ -path between p and q in G .

630 **Proof.** The proof is by induction over the size of R (i.e. area, width, or height). Let
631 $\Xi = \{X, Y\} \in \mathcal{W}$ be the pair in the QSPD that separates p and q , let c be the separation
632 point of the pair, and assume for the simplicity of exposition that $p \in X$, $X \prec c \prec Y$, and
633 $c = (0, 0)$. Furthermore, assume that $\|p\|_1 \geq \|q\|_1$.

634 Let $p = (-x, -y)$, and let C be the grid cell of $K(p, \Xi)$ that contains q . If $C \subseteq (1 - \delta)R$,
635 then $(1 - \delta)^{-1}C \subseteq R$ by Lemma 31 (I). As such, let u be the leftmost point in $C \cap P$. Both
636 $q, u \in (1 - \delta)^{-1}C$, and by induction, there is an $(1 + \varepsilon)$ -path π between them in G (note that
637 the induction applies to the two points, and the “expanded” rectangle $(1 - \delta)^{-1}C$). Since pu
638 is an edge of G , prefixing π by this edge results in an $(1 + \varepsilon)$ -path, as $\|qu\| \leq (\varepsilon/4)\|pq\|$, by
639 Lemma 31 (II) (verifying this requires some standard calculations which we omit).

640 Otherwise, one needs to apply the same argument using the appropriate case of Lemma 31.
641 So assume that $x \geq y$ (the case that $y \geq x$ is handled symmetrically). If $C \subseteq R_{\swarrow} \cup R_{\searrow}$, then
642 (III) implies that $(1 - \delta)^{-1}C \subseteq R$. Which implies that induction applies, and the claim holds.

643 The remaining case is that $x \geq y$ and $C \subseteq R_{\nwarrow}$. Let $D = \overrightarrow{(1 - \delta)R} \cap C$. By (V), we have
644 $(1 - \delta)^{-1}D \subseteq R$. Namely, $q \in (1 - \delta)R \cap C \subseteq D$, and let u be the lowest point in $C \cap P$. By
645 construction $pu \in E(G)$, $q, u \in D$, and $(1 - \delta)^{-1}D \subseteq R$. As such, we can apply induction
646 to q, u , and $(1 - \delta)^{-1}D$, and conclude that $d_G(q, u) \leq (1 + \varepsilon)\|qu\|$. Plugging this into the
647 regular machinery implies the claim. \blacktriangleleft

648 **► Theorem 33.** Let P be a set of n points in the plane, and let $\varepsilon, \delta \in (0, 1)$ be parameters.
649 The above algorithm constructs, in $\mathcal{O}((1/\varepsilon^2 + 1/\delta^2)n \log^2 n)$ time, a graph G with $\mathcal{O}((1/\varepsilon^2 +$
650 $1/\delta^2)n \log^2 n)$ edges. The graph G is a $(1 - \delta)$ -local $(1 + \varepsilon)$ -spanner for axis parallel rectangles.
651 Formally, for any axis-parallel rectangle R , we have that $R \cap P$ is an $(1 + \varepsilon)$ -spanner for all
652 the points of $((1 - \delta)R) \cap P$.

Proof. Computing the QSPD \mathcal{W} takes $\mathcal{O}(n \log^2 n)$ time. For each pair $\{X, Y\}$ in the decomposition with $m = |X| + |Y|$ points, we need to compute the lowest and leftmost points in $(X \cup Y) \cap \mathcal{C}$, for each cell in the constructed grid. This can readily be done using orthogonal range trees in $\mathcal{O}(\log^2 n)$ time per query (a somewhat faster query time should be possible by using the offline nature of the queries, etc). This yields the construction time. The size of the computed graph is $\mathcal{O}(\omega(\mathcal{W})\tau^2) = \mathcal{O}((1/\delta^2 + 1/\varepsilon^2)n \log^2 n)$.

The desired local spanner property is provided by [Lemma 32](#). ◀

C Some missing proofs

C.1 Proof of [Lemma 8](#)

Restatement of [Lemma 8](#). *Given a $(1/\varepsilon)$ -SSPD \mathcal{W} of n points in the plane, one can refine \mathcal{W} into a $(1/\varepsilon)$ -SSPD \mathcal{W}' , such that each pair $\Xi = \{X, Y\} \in \mathcal{W}'$ is contained in a ε -double-wedge \times_Ξ , such that X and Y are contained in the two different faces of the double wedge \times_Ξ . We have that $|\mathcal{W}'| = \mathcal{O}(|\mathcal{W}|/\varepsilon)$ and $\omega(\mathcal{W}') = \mathcal{O}(\omega(\mathcal{W})/\varepsilon)$. The construction time is proportional to the weight of \mathcal{W}' .*

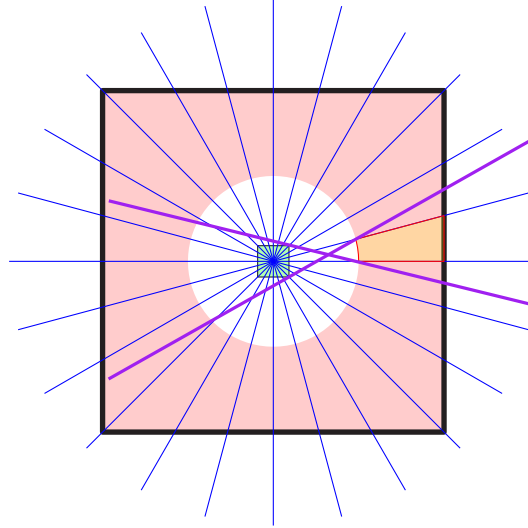
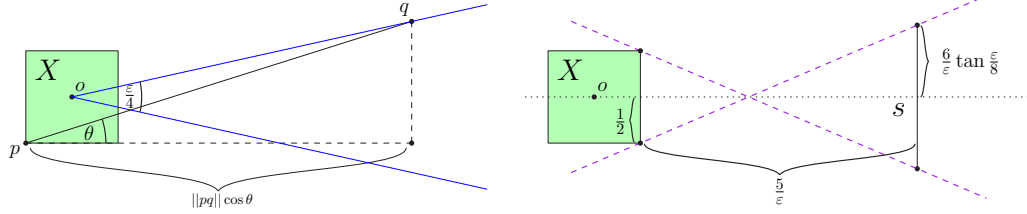


Figure C.1 An illustration of refining the pairs in a SSPD into pairs contained in opposite parts of an ε -double-wedge. X is contained in the green square \square , while Y is contained in the red square, and the white gap between them is a result of the separation property. The set of cones with the apex at the center of \square gives us the desired partition as demonstrated by the purple double-wedge.

Proof. By using [Lemma 6](#), we can assume that \mathcal{W} is (say) $(10/\varepsilon)$ -separated. For each pair $\Xi = \{X, Y\} \in \mathcal{W}$, assume that $\text{diam}(X) < \text{diam}(Y)$. Now, the algorithm scans the pairs of \mathcal{W} and performs the following procedure for each one. Let \square be the smallest axis-parallel square containing X , centered at point o . Partition the plane around o , by drawing $\mathcal{O}(1/\varepsilon)$ lines intersecting o with the angle between any two consecutive lines being at most (say) $\varepsilon/4$, see [Figure C.1](#). This partitions the plane into a set of cones \mathcal{C} . For a cone $c \in \mathcal{C}$, we show that there exists an ε -double-wedge that contains X in one side, and $Y \cap c$ in the other.

To see that, take the double-wedge formed by the cross tangents between $\text{ch}(X)$ and $\text{ch}(Y \cap c)$, where $\text{ch}(X)$ denotes the convex-hull of X . Assume w.l.o.g that \square has side length

1, and let c be a cone of angle $\varepsilon/4$ with apex o , whose angular bisector is a horizontal ray in the positive direction of the x axis. See figure Figure C.2 for an illustration.



■ Figure C.2 An illustration of the proof for Lemma 8

We would like to find a vertical segment s such that all points of Y lie to its right, with one endpoint on the upper line of c , and the other on the lower line of c . Using the segments' height and distance from the right side of \square we will be able to get a bound on the angle of the cross tangents. We first find a segment s with all points of Y to its right. A trivial bound on that distance is given by the segment from, say, the lower left corner of \square , denoted p , of length $10/\varepsilon$ with its right endpoint on the upper line of c , denote this point by q . We know that all points of Y lie to the right of q due to the $10/\varepsilon$ separation property of the SSPD. The segment pq creates an angle $\leq \pi/4$ with the x -axis (by the choice of the angle of c). We therefore get that the x -coordinate difference between \square and q is at most $10/\varepsilon \cdot \cos \frac{\pi}{4} - 1 \leq 7/\varepsilon - 1 \leq 6/\varepsilon$. So, let s' be a vertical segment between the upper and lower rays of c , with x -coordinate distance of $6/\varepsilon - \frac{1}{2}$ from \square (in order to make calculations easier). We get that s' is of length $2 \cdot \frac{6}{\varepsilon} \tan \frac{\varepsilon}{8}$. Finally, we take s to be a vertical segment of length $\frac{12}{\varepsilon} \tan \frac{\varepsilon}{8}$, with its center on the x -axis at a distance of $5/\varepsilon + \frac{1}{2}$ away from o . The angle of the x -axis and the segment between the lower end of the right side of \square and the upper end of s is now given by:

$$\arctan\left(\frac{\frac{6}{\varepsilon} \tan \frac{\varepsilon}{8} + \frac{1}{2}}{\frac{5}{\varepsilon}}\right) = \arctan\left(\frac{6}{5} \tan \frac{\varepsilon}{8} + \frac{\varepsilon}{10}\right) \leq \varepsilon$$

C.2 Proof of Lemma 12

Restatement of Lemma 12. *Given a set P of n points in the plane, and parameters $\delta, \varepsilon \in (0, 1)$. One can construct a graph G over P , in $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n \log n)$ time, and with $\mathcal{O}((\varepsilon^{-1} + \delta^{-2})n)$ edges, such that for any (bounded) convex C in the plane, we have that for any two points $p, q \in P \cap C_{\square\delta}$ the graph $C \cap P$ has a $(1 + \varepsilon)$ -path between p and q .*

Proof. The proof of the following claim is straightforward, and is included for the sake of completeness. Let $\vartheta = \min(\varepsilon, \delta^2)$. Construct, in $\mathcal{O}(\vartheta^{-1}n \log n)$ time, any standard $(1 + \vartheta)$ -spanner G for P using $\mathcal{O}(\vartheta^{-1}n)$ edges (e.g., [5]).

So, consider any body $C \in \mathcal{F}$, and any two points $p, q \in P \cap C'$, where $C' = C_{\square\delta}$, let $\ell = \|pq\|$, let π be the shortest path between p and q in G , and let \mathcal{E} be the locus of all points u , such that $\|pu\| + \|uq\| \leq (1 + \vartheta)\ell$. The region \mathcal{E} is an ellipse that contains π . The furthest point from the segment pq in this ellipse is realized by the co-vertex of the ellipse. Formally, it is one of the two intersection points of the boundary of the ellipse with the line orthogonal to pq that passes through the middle point c of this segment, see Figure C.3. Let z be one of these points.



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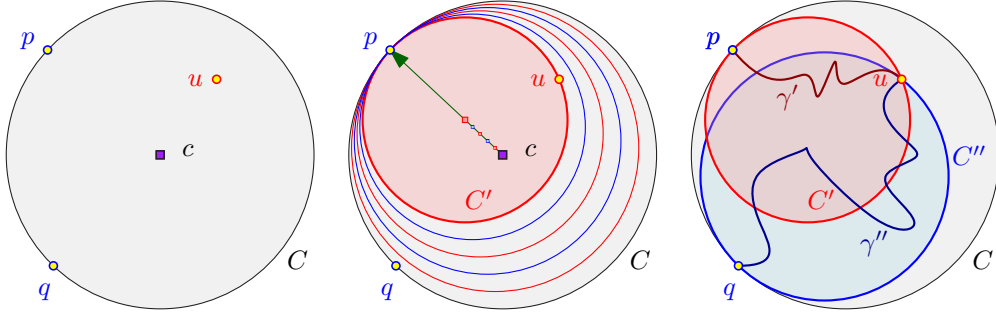
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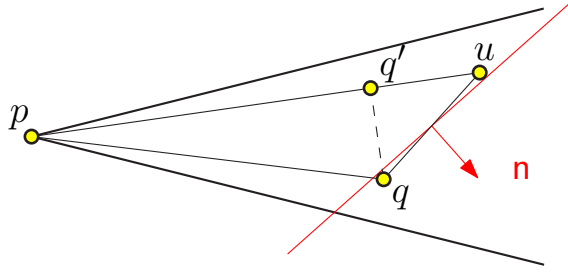
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■ **Figure C.4** An illustration of the proof of Claim 16 in the case that C is a disk.

D Missing details about spanner for fat triangles

► **Lemma 34.** Let $p \in P$, $c \in \mathcal{C}_i$, and $u = \text{nn}_i(p, c)$, and let q be a point in $(P \cap (p+c)) \setminus \{p, u\}$. We have that $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$ and $\|qu\| \leq \|pq\|$.



■ **Figure D.1** The case that $\|pq\| \leq \|pu\|$ in Lemma 34. The vector used to determine $\text{nn}_i(p, c)$ is shown in red, and denoted n

Proof. Consider the triangle Δpqu and denote the angles at p, q , and u by $\angle p, \angle q$, and $\angle u$ respectively. Since the angle of c is smaller than 60 degrees (for an appropriate choice of γ), we have that $\|qu\| \leq \max\{\|pu\|, \|pq\|\}$.

Consider the case that $\|pq\| \leq \|pu\|$, illustrated in Figure D.1. Observe that $\angle u \leq \angle q$. As such $\angle u \leq \pi/2$. Furthermore, $\angle u \geq \alpha \gg \varepsilon\alpha/\gamma = \beta \geq \angle p$. Similarly, $\angle q \in [\alpha, \pi - \alpha]$. By the 1-Lipshitz of \sin , and as $\sin x \approx x$, for small x , and for γ sufficiently large, we have that

$$\sin(\angle q + \angle p) \in [1 - \varepsilon/4, 1 + \varepsilon/4] \sin \angle q \quad \text{and} \quad \sin \angle p \leq (\varepsilon/4) \sin \angle u.$$

As such, by the law of sines, we have that $\frac{\|qu\|}{\sin \angle p} = \frac{\|pq\|}{\sin \angle u} = \frac{\|pu\|}{\sin \angle q}$. This implies that

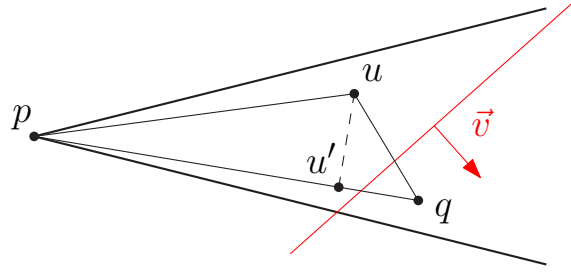
$$\|pu\| + (1 + \varepsilon) \|qu\| = \left(\frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \right) \|pq\|.$$

Observe, by the above that

$$\frac{\sin \angle q}{\sin \angle u} + (1 + \varepsilon) \frac{\sin \angle p}{\sin \angle u} \leq \frac{\sin \angle q}{\sin(\angle p + \angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq \frac{\sin \angle q}{(1 - \varepsilon/4) \sin(\angle q)} + (1 + \varepsilon) \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$

The other possibility, illustrated in Figure D.2, is that $\|pq\| > \|pu\|$. Let u' be the projection of u to pq . Observe that

$$\|uu'\| = \|pu'\| \tan \angle p \leq 2\beta \|pu'\| \leq (\varepsilon/8) \|pu'\|.$$



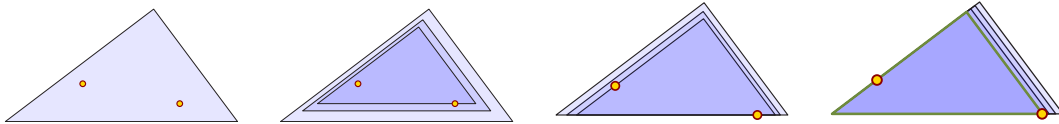
■ **Figure D.2** The case that $\|pq\| > \|pu\|$ in Lemma 34.

Observe that $\cos \angle p \geq 1 - (\angle p)^2/2 \geq 1 - \varepsilon^2/8$ as $\angle p$ is an angle smaller than (say) $\varepsilon/16$. As such $1/\cos \angle p \leq 1 + \varepsilon^2/4$. This implies that $\|pu\| \leq \|pu'\|/\cos \angle p \leq (1 + \varepsilon^2/4) \|pu'\|$. We thus have that

$$\begin{aligned} \|pu\| + (1 + \varepsilon) \|qu\| &\leq (1 + \varepsilon^2/4) \|pu'\| + (1 + \varepsilon) (\|uu'\| + \|u'q\|) \\ &\leq (1 + \varepsilon^2/4 + (1 + \varepsilon)\varepsilon/8) \|pu'\| + (1 + \varepsilon) \|u'q\| \leq (1 + \varepsilon) \|pq\|. \end{aligned}$$

► **Lemma 35.** Let \triangle be a triangle that contains two points p, q . Then, there is a homothet $\triangle' \subseteq \triangle$ of \triangle , such that one of these points is a vertex of \triangle' , and the other point lies on the edge of \triangle' facing that vertex.

Proof. This follows by the same shrinking argument as Lemma 15, with the addition of a single step. When a homothet \triangle' with $p, q \in \partial\triangle'$ is found, if neither point is on a vertex, we “push” the only edge that does not contain one of the points towards the vertex v opposite of it (this is the same mapping described in Lemma 15 with center v), until one of the points, say p lies on the edge. p now lies on two edges, meaning, at a vertex, while q lies on the only remaining edge which must be opposite of that vertex. See Figure D.3.



■ **Figure D.3** An illustration of the shrinking process of Lemma 35. The three left figures illustrates the process of Lemma 15, for the case that the convex region \mathcal{C} is a triangle, and the rightmost figure is the additional final step.

Local spanner property

► **Lemma 36.** Let \triangle' be a homothet of \triangle . For any two points $p, q \in P \cap \triangle'$, we have a $(1 + \varepsilon)$ -path in $G' = G \cap \triangle'$.

Proof. Consider the closest pair $p, q \in P \cap \triangle$. They must be connected directly in G' , as otherwise there is a point $u \in P' = P \cap \triangle'$ in the cone containing the segment pq , such that $pu \in E(G')$. But then, by Lemma 34, we have $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$, which implies that either pu or qu are the closest pair, which is a contradiction.

781 For any other pair $p, q \in P'$ we have from [Lemma 35](#) that there exists a homothet
 782 $\triangle'' \subseteq \triangle'$ with one of the two points, say p , at a vertex, and the other on the opposite edge.
 783 We therefore have a cone c with apex at p such that $q \in c \cap \triangle''$. If pq is an edge in G
 784 then we are done. Otherwise, we have a vertex $u \in c$ such that pu is an edge in G , and by
 785 [Lemma 34](#) we have $\|qu\| \leq \|pq\|$, which, by induction, means that there exists a $(1 + \varepsilon)$ path
 786 between u and q in G . [Lemma 34](#) now implies that $\|pu\| + (1 + \varepsilon) \|qu\| \leq (1 + \varepsilon) \|pq\|$. Thus,
 787 there is a $(1 + \varepsilon)$ path between p and q in G' , as stated. \blacktriangleleft