Tema 1. Analiza Algoritmilor

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problema1.c > main()

```c
#include <stdio.h>

#define MAX_DIM 1001

int main() {
    int n, m[MAX_DIM][MAX_DIM];

    // Read matrix dimension
    printf("n = ");
    scanf("%d", &n);

    // Read matrix elements
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            scanf("%d", &m[i][j]);

    // Search for two consecutive even numbers sequence
    for (int current = 0; current < n * n && m[current / n][current % n] != 0; current++) {
        // Compute index for the first number
        int current_i = current / n;
        int current_j = current % n;

        // Compute index for the second number
        int next_i = (current + 1) / n;
        int next_j = (current + 1) % n;

        if (m[current_i][current_j] % 2 == 0 && m[next_i][next_j] % 2 == 0 &&
                m[next_i][next_j] != 0)
            printf("(%d %d) -> (%d %d)\n", current_i, current_j, next_i, next_j);
    }

    return 0;
}
```

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①

```
int main() {
                                         | Cost | Repetiții |
    int n, m[1001][1001];                | C1   | 1         |
|C2|X| for(int current=0; current<n*n && m[current/n][current%n]!=0; current++) {
        int current_i = current/n;       |C3|X-1|
        int current_j = current%n;       |C4|X-1|
        int next_i = (current+1)/n;      |C5|X-1|
        int next_j = (current+1)%n       |C6|X-1|
        if(m[current_i][current_j]%2==0 && m[next_i][next_j]%2==0
               && m[next_i][next_j]!=0)  |C7|X-1|
|C8|Y| printf("(%d %d) -> (%d %d)", current_i, current_j, next_i, next_j);
    }
    return 0;
}
```

$X \le n^*n+1$

$Y <= n^*n$

$T(n) = C_1 \cdot 1 + C_2 \cdot X + C_3 \cdot (X-1) + C_4 (X-1) + C_5 \cdot (X-1) + C_6 (X-1) + C_7 (X-1) + C_8 \cdot Y$

Cazul cel mai favorabil ($X=1$, $Y=0$):

$T(n) = C_1 + C_2 = \Theta(1)$

Cazul cel mai defavorabil ($X = n^*n+1$, $Y = n^*n$)

$T(n) = C_1 + C_2 (n^*n+1) + C_3 (n^*n) + C_4 (n^*n) + C_5 (n^*n) + C_6 (n^*n) + C_7 \cdot (n^*n) + C_8 (n^*n) = \Theta(n^*n)$

Cazul mediu $\left(X = \frac{n*n+1+1}{2} = \frac{n*n}{2}+1,\ Y = \frac{n*n+0}{2} = \frac{n*n}{2}\right)$

$$T(n) = C_1 + C_2 \cdot \left(\frac{n*n}{2}+1\right) + C_3\left(\frac{n*n}{2}\right) + C_4\left(\frac{n*n}{2}\right) + C_5\left(\frac{n*n}{2}\right) +$$
$$+ C_6\left(\frac{n*n}{2}\right) + C_7 \cdot \left(\frac{n*n}{2}\right) + C_8 \cdot \frac{n*n}{2} = \Theta\left(\frac{n*n}{2}\right) = \Theta(n*n)$$

Operații critice

Cazul cel mai favorabil

Operația critică este (1). $X = 1, Y = 0$

$T(n) = C_1 = \Theta(1)$

Cazul cel mai defavorabil

Operațiile critice sunt (3), (4), (5), (6), (8). $X = n*n+1,\ Y = n*n$

$T(n) = \Theta(n*n)$

Cazul mediu

Operațiile critice sunt (3), (4), (5), (6), $X = \frac{n*n}{2}+1,\ Y = \frac{n*n}{2}$

$T(n) = \Theta\left(\frac{n*n}{2}\right) = \Theta(n*n)$

② $\Theta(2n^3+3n^2)=\Theta(n^3)$

1) $\Theta(2n^3+3n^2)\in\Theta(n^3)$
2) $\Theta(n^3)\in\Theta(2n^3+3n^2)$ / Demonstrăm

1) $\Theta(2n^3+3n^2)\in\Theta(n^3)$ $\quad f(n)=(2n^3+3n^2)$, $g(n)=n^3$

Arătăm că $\exists\, c_1, c_2, n_0 \in \mathbb{R}_+^*$ a.î. $c_1\cdot g(n)\le f(n)\le c_2\cdot g(n)$, $(\forall) n\ge n_0 \Leftrightarrow$

$\Leftrightarrow c_1\cdot n^3\le 2n^3+3n^2\le c_2 n^3$

Fie $\begin{cases} c_1=2 \\ c_2=5 \end{cases}$, $n_0=1$ $\Rightarrow 2n^3\le 2n^3+3n^2\le 5n^3 \Leftrightarrow$

$\Leftrightarrow 0\le 3n^2\le 3n^3$, $(\forall) n\ge 1$ Adevărat $\Rightarrow 2n^3+3n^2\in\Theta(n^3)$ (1)

2) $\Theta(n^3)\in\Theta(2n^3+3n^2)$ $\quad f(n)=n^3$, $g(n)=2n^3+3n^2$

Arătăm că $\exists\, c_1, c_2, n_0\in\mathbb{R}_+^*$ a.î. $c_1\cdot g(n)\le f(n)\le c_2\cdot g(n)$, $(\forall) n\ge n_0 \Leftrightarrow$

$\Leftrightarrow c_1\cdot(2n^3+3n^2)\le n^3\le c_2(2n^3+3n^2)$, $(\forall) n\ge n_0$

Fie $\begin{cases} c_1=\frac{1}{4} \\ c_2=1 \end{cases}$, $n_0=2$ $\Rightarrow \frac{1}{2}n^3+\frac{3}{4}n^2\le n^3\le 2n^3+3n^2$, $(\forall) n\ge 2$

$\Rightarrow \begin{cases} \frac{3}{4}n^2\le\frac{1}{2}n^3, (\forall) n\ge 2 \text{ Adevărat} \\ n^3\le 2n^3+3n^2, (\forall) n\ge 2 \text{ Adevărat} \end{cases} \Rightarrow n^3\in\Theta(2n^3+3n^2)$ (2)

Din (1) și (2) $\Rightarrow$ $\Theta(2n^3+3n^2)=\Theta(n^3)$
prin dublă incluziune

③ $T(n) = \begin{cases} K_1, & n=1 \\ 2T([n/3]) + T([n/3]) + K_2 \cdot n, & n>1 \end{cases}$

a) Metoda iterativă

$T(n) = 3T([n/3]) + \theta(n)$

$T(1) = \theta(1)$

$$\cancel{T(n)} = 3\,\cancel{T\left(\frac{n}{3}\right)} + \theta(n) \;\Big|\cdot 3^0$$

$$\cancel{T\left(\frac{n}{3}\right)} = 3\,\cancel{T\left(\frac{n}{3^2}\right)} + \theta(n/3) \;\Big|\cdot 3^1$$

$$\cancel{T\left(\frac{n}{3^2}\right)} = 3\,\cancel{T\left(\frac{n}{3^3}\right)} + \theta(n/3^2) \;\Big|\cdot 3^2$$

$\vdots$

$$\cancel{T(n/3^k)} = 3 \cdot T\left(\frac{n}{3^{k+1}}\right) + \theta\left(\frac{n}{3^k}\right) \;\Big|\cdot 3^k$$

$\frac{n}{3^{k+1}} = 1 \Rightarrow n = 3^{k+1} \Rightarrow$

$(k+1)\,\cancel{\log 3} = \log_3 n$

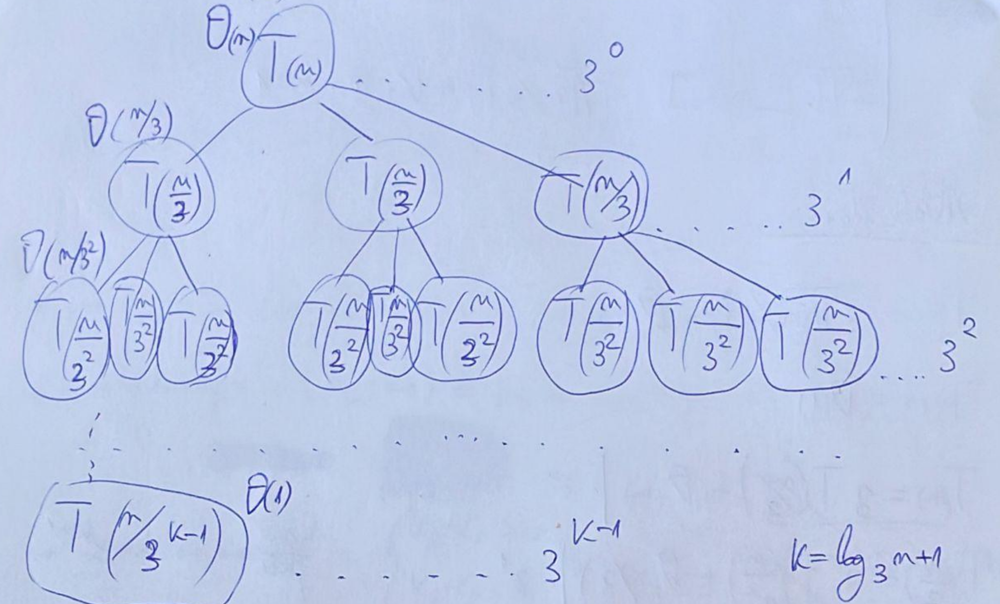
$k = \log_3 n - 1$

$$\oplus\quad T(n) = 3^{k+1}\, T\left(\frac{n}{2^{k+1}}\right) + \sum_{i=0}^{k} 3^i\, \theta\left(\frac{n}{3^i}\right) =$$

$$= n \cdot T(1) + (k+1)\,\theta(n) = n\,\theta(1) + \log_3 n \cdot \theta(n) =$$

$$= \theta(n + n\log n) = \theta(n \log n)$$

b) $T(n) = 3T\left(\frac{n}{3}\right) + \theta(n)$ <u>Metoda arborelui de recurență</u>



$k = \log_3 n + 1$

$$\Rightarrow T(n) = \sum_{i=0}^{k-1} 3^i \cdot \theta(n/3^i) = k \cdot \theta(n) = (\log_3 n + 1)\,\theta(n) = \theta(n \log n + n) =$$
$$= \theta(n \log n)$$

<u>c) Metoda substituției</u>

Presupunem $T(n) = \theta(n \log n + n)$

Arătăm că $\exists\, c_1, c_2 \in \mathbb{R}_+^*$, $\exists\, n_0 \in \mathbb{N}^*$ a.î. $c_1(n \log n + n) \leq T(n) \leq c_2(n \log n + n)$

Folosim inducția matematică pentru a demonstra inegalitățile.

Caz de bază:

$n=1$: $\boxed{c_1 \leq k_1 \leq c_2}$ (1)

Ipoteză de inducție

$$c_1\left(\frac{n}{3}\log\frac{n}{3}+\frac{n}{3}\right) \le T\left(\frac{n}{3}\right) \le c_2\left(\frac{n}{3}\log\frac{n}{3}+\frac{n}{3}\right)$$

$$3c_1\left(\frac{n}{3}\log\frac{n}{3}+\frac{n}{3}\right)+k_2 n \le 3T\left(\frac{n}{3}\right)+k_2 n \le 3c_2\left(\frac{n}{3}\log\frac{n}{3}+\frac{n}{3}\right)+k_2 n$$

$$c_1 n\log\frac{n}{3}+c_1 n+k_2 n \le T(n) \le c_2 n\log\frac{n}{3}+c_2 n+k_2 n$$

$$c_1 n\log n - c_1 n\cancel{\log_3 3}+c_1 n+k_2 n \le T(n) \le c_2 n\log n - c_2 n\log_3 3+c_2 n+k_2 n$$

$$c_1(n\log n+n) - n(k_2-c_1) \le T(n) \le c_2(n\log n+n)+n(k_2-c_2)$$

Impunem condițiile: $\begin{cases} k_2-c_1>0 \\ k_2-c_2<0 \end{cases} \to \begin{cases} k_2>c_1 \\ k_2<c_2 \end{cases} \to \boxed{c_1<k_2<c_2}$ (2)

$\to c_1(n\log n+n) \le T(n) \le c_2(n\log n+n)$

$\Rightarrow$ Dacă alegem $c_1=\min(k_1,k_2)$, $c_2=\max(k_1,k_2)\in\mathbb{R}^*$ și $n_0=1$

atunci $\exists\, c_1,c_2\in\mathbb{R}_+^*$ și $n_0=1\in\mathbb{N}^*$ a.î. $c_1(n\log n+n)\le T(n)\le c_2(n\log n+n)$ $(\forall)\, n\ge n_0$

$\Rightarrow \boxed{T(n)=\Theta(n\log n+n)=\Theta(n\log n)}$

d) Metoda Master

$T(n)=3T\left(\frac{n}{3}\right)+k_2\cdot n \Rightarrow \begin{cases} f(n)=k_2\cdot n \\ a=3 \\ b=3 \end{cases} \Rightarrow$

(Cazul 2)

$\Rightarrow n^{\log_b a} = n^{\log_3 3} = n$

$f(n) = \Theta(n^{\log_b a}) = \Theta(n) \Rightarrow T(n) = \Theta(n^{\log_b a} \cdot \log_3 n) \Rightarrow$

$\Rightarrow T(n) = \Theta(n \log_3 n) = \Theta(n \log_3 n + n))$

(5) $T(n) = \begin{cases} k_1, n = 1 \\ 3\,T(\lfloor n/4 \rfloor) + 2\,T(\lfloor n/4 \rfloor) + 4 k_2 n^2, n > 1 \end{cases}$

a) Metoda iterativă

$T(n) = 5\,T(\frac{n}{4}) + \Theta(n^2)$

$T(1) = \Theta(1)$

$T(n) = 5\,T(\lfloor \frac{n}{4} \rfloor) + \Theta(n^2) \;|\; 5^0$

$T(n/4) = 5\,T(\frac{n}{4^2}) + \Theta(n^2/4^2) \;|\; 5^1$

$T(n/4^2) = 5\,T(\frac{n}{4^3}) + \Theta(n^2/4^4) \;|\; 5^2$

$\vdots$

$T(n/4^k) = 5\,T(n/4^{k+1}) + \Theta(n^2/4^{2k}) \;|\; 5^k$

$\frac{n}{4^{k+1}} = 1 \Rightarrow n = 4^{k+1} \Rightarrow$

$\Rightarrow k+1 = \log_4 n \Rightarrow$

$\Rightarrow k = \log_4 n - 1$

(+) $T(n) = 5^{k+1}\,T(n/4^{k+1}) + \sum_{i=0}^{k} 5^i\,\Theta(\frac{n^2}{4^{2i}}) =$

$= 5^{\log_4 n}\,T(1) + \sum_{i=0}^{k} (\frac{5}{4^2})^i\,\Theta(n^2) =$

$= n^{\log_4 5}\,\Theta(1) + \sum_{i=0}^{k} (\frac{5}{16})^i\,\Theta(n^2)$

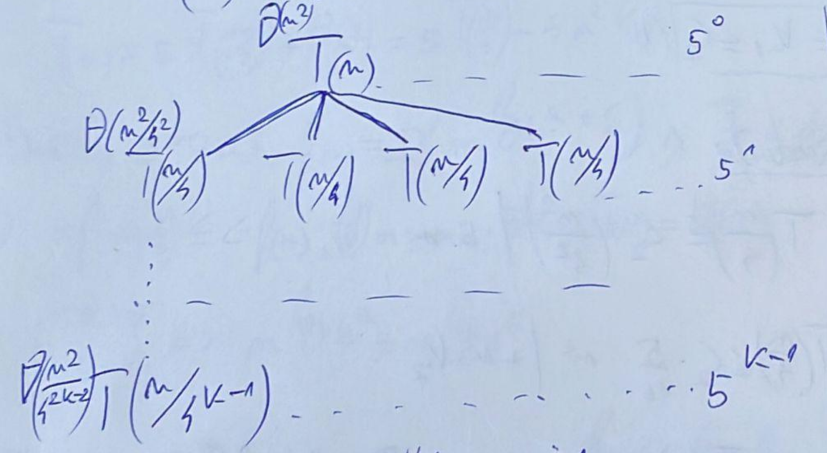
$$\Rightarrow T(n) = n^{\log_4 5}\,\Theta(1) + \left(\frac{\left(\frac{5}{16}\right)^{k+1} - 1}{\frac{5}{16} - 1}\right)\Theta(n^2) = \Theta(n^{\log_4 5}) + \Theta\left(n^2 \cdot \frac{\cancel{16}\,5}{\cancel{9}\,3}\right) \Rightarrow$$

$$\Rightarrow T(n) = \Theta(n^{\log_4 5}) + \Theta\left(\frac{5}{3} n^2\right) = \Theta\left(n^{\log_4 5} + \frac{5}{3} n^2\right) = \Theta(n^2) \Rightarrow$$

$$\Rightarrow T(n) = \Theta(n^2)$$

b) Metoda arborelui de recurență

$$T(n) = 5\,T(n/4) + \Theta(n^2)$$



$\frac{n}{4^{k-1}} = 1 \Rightarrow$

$\Rightarrow n = 4^{k-1} \Rightarrow$

$\Rightarrow \log_4 n = k-1 \Rightarrow$

$\Rightarrow k = \log_4 n + 1$

$$T(n) = \sum_{i=0}^{k-1} 5^i\,\Theta\left(\frac{n^2}{4^{2i-2}}\right) \cancel{= \sum_{i=0}^{k-1} 5\cdot\left(\frac{5}{4^2}\right)^{i-1}\Theta(n^2) = 5\,\Theta(n^2)\cdot\sum_{i=0}^{k-1}\left(\frac{5}{16}\right)^{i-1} =}$$

$$\cancel{= 5\,\Theta(n^2)\cdot\left(\frac{5}{16}\right)^k - 1} = \sum_{i=0}^{k-1} 4\left(\frac{5}{4^2}\right)^i \Theta(n^2) = 4\,\Theta(n^2)\cdot\sum_{i=0}^{k-1}\left(\frac{5}{16}\right)^i =$$

$$= 4\,\Theta(n^2)\cdot\frac{\left(\frac{5}{16}\right)^k - 1}{\frac{5}{16} - 1} = \frac{16}{3}\Theta(n^2) = \Theta(n^2) \Rightarrow T(n) = \Theta(n^2)$$

c) Metoda substituției

Presupunem $T(n) = \Theta(n^2)$

Arătăm că $\exists\ c_1, c_2 \in \mathbb{R}_+^*$, $\exists\ n_0 \in \mathbb{N}^*$, a.î.

$$c_1 n^2 \leq T(n) \leq c_2 n^2, \ (\forall)\ n \geq n_0$$

Folosim inducția matematică pentru a demonstra inegalitățile

Caz de bază

$n = 1: \ \boxed{c_1 \leq k_1 \leq c_2} \ (1)$

Ipoteza de inducție

$$c_1 \cdot \left(\frac{n^2}{4^2}\right) \leq T\left(\frac{n}{4}\right) \leq c_2 \left(\frac{n^2}{4^2}\right) \Big| \cdot 5$$

$$\frac{5}{16} c_1 n^2 \leq 5T\left(\frac{n}{4}\right) \leq c_2 \cdot \frac{5}{16} \cdot n^2 \ \Big| + n^2 \cdot k_2$$

$$\frac{5}{16} c_1 n^2 + n^2 k_2 \leq 5\,T\left(\frac{n}{4}\right) + n^2 \cdot k_2 \leq c_2 \cdot \frac{5}{16} \cdot n^2 + n^2 k_2$$

$$c_1 n^2 \left(1 - \frac{11}{16}\right) + n^2 k_2 \leq T(n) \leq c_2 \cdot n^2 \left(1 - \frac{11}{16}\right) + n^2 k_2$$

$$c_1 n^2 + n^2 \left(k_2 - \frac{11}{16} c_1\right) \leq T(n) \leq c_2 n^2 + n^2 \left(k_2 - \frac{11}{16} c_2\right)$$

Impunem condițiile: $\begin{cases} k_2 - \frac{11}{16} c_1 \geq 0 \\ k_2 - \frac{11}{16} c_2 \leq 0 \end{cases} \rightarrow \left\{ \boxed{c_1 \leq \frac{16}{11} k_2 \leq c_2} \right. (2)$

$\Rightarrow c_1 n^2 \leq T(n) \leq c_2 n^2 \ (A) \Rightarrow$

$\Rightarrow$ Dacă alegem $c_1 = \min(k_1, \frac{16}{11} k_2)$ și $n_0 = 1$ atunci

$c_2 = \max(k_1, \frac{16}{11} k_2)$

$\exists\, c_1, c_2 \in \mathbb{R}_+^*$ și $n_0 \in \mathbb{N}^*$ a.î. $c_1 n^2 \le T(n) \le c_2 n^2, \forall n \ge n_0 \Rightarrow$

$\Rightarrow T(n) = \Theta(n^2)$

d) Metoda Master

(Cazul 3)

$T(n) = 5T(n/4) + \Theta(n^2) = 5T(\frac{n}{4}) + 4n^2 k_2$

Dacă $\exists\, \varepsilon > 0$ a.î. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ și $\exists\, c \in (0,1), n_0 \in \mathbb{N}^*$ a.î. $a f(n/b) \le c f(n), \forall n \ge n_0$, atunci $T(n) = \Theta(f(n))$

$\begin{cases} a = 5 \\ b = 4 \end{cases} \Rightarrow n^{\log_b a} = n^{\log_4 5}$

$\exists\, c' \in \mathbb{R}_+^*, n_0' \in \mathbb{N}^*$ a.î. $c' n^{\log_4 5 + \varepsilon} \le f(n), \forall n \ge n_0'$

$c' n^{\log_4 5 + \varepsilon} \le k_2 n^2, \forall n \ge n_0$

Se alege $c' \le k_2 \Rightarrow 0 < \varepsilon \le 2 - \log_4 5 \Rightarrow c' n^{\log_4 5 + \varepsilon} \le k_2 n^2$

Fie $n_0 = 1$; $\Rightarrow \exists\, \varepsilon > 0$ a.î. $f(n) = k_2 n^2 = \Omega(n^{\log_4 5 + \varepsilon})$ (*)

$\exists\, c \in (0,1), n_0 \in \mathbb{N}^*$ a.î. $5 f(n/4) \le c \cdot f(n), \forall n \ge n_0 \Rightarrow$

$\Rightarrow 5 k_2 \frac{n^2}{16} \le c\, k_2 n^2 \Rightarrow \frac{5}{16} \le c \Rightarrow$ Fie $c = \frac{8}{16} = \frac{1}{2} \in (0,1)$ (*')

$\left.\begin{matrix}(*)\\(*')\end{matrix}\right\} \xRightarrow[M\ III]{} T(n) = \Theta(f(n)) = \Theta(n^2)$