QFT Notes - Chapter 2

Stavros Klaoudatos

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1 2.3 The Klein Gordon Field as Harmonic Oscillations

The simplest type of field is the **Real Klein-Gordon Field**. We will start with a classical field theory and then quantize it.

Second Quantization
Reinterpret the Dynamic Variables as operators that obey canonical commutation relations

We will then solve the theory by **finding the eigenvalues and eigenstates of the Hamiltonian using the Harmonic Oscillator as an analogy.**

To move from classical to quantum field theory, we do what we do with all dynamical systems: **We promoted ϕ and π to operators** and we impose suitable commutation relations.

$$[q_i, p_j] = i\delta_{ij}$$
$$[p_i, p_j] = [q_i, q_j] = 0$$

For a continuous system, the generalization is quite natural. Because $\pi(x)$ is a the momentum **density**, we get a dirac delta rather than a Kronecker:

$$[\phi(x), \pi(y)] = i\delta^{(3)}(x-y)$$
$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0$$

((For now ϕ and π do not depend on time).
Now it is time for the **Hamiltonian**. First of all let's write the Klein-Gordon field in Fourier Space:

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \phi(p,t)$$

(Here we have $\phi^*(p) = -\phi(p)$ so that $\phi(x)$ is real.)

The Klein Gordon Equation becomes

$$[\frac{\partial^2}{\partial t^2} + (|p|^2 + m^2)]\phi(p,t) = 0$$

This is the same as the equation of motion for a simple harmonic oscillator with frequency $\omega = \sqrt{|p|^2 + m^2}$

SHO is something we know pretty well.

$$H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2$$

Now to find the eigenvalues of H_{SHO}, we write p and ϕ in terms of ladder operators.

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$$

$$p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

The Hamiltonian can now be written as

$$H_{SHO} = \omega(a^\dagger a + \frac{1}{2})$$

The state $|0\rangle$ is the ground-state with eigenvalue $\frac{1}{2}\omega$.
Furthermore, the Commutators $[H_{SHO}, a^\dagger] = \omega a^\dagger$ and $[H_{SHO}, a] = -\omega a$
From Quantum Mechanics we know that

$$|n\rangle = (a^\dagger)^n |0\rangle \quad with \;\; eigenstate \;\; (n + \frac{1}{2})\omega$$

We can find the spectrum of the Klein Gordon Hamiltonian with the same trick, but now, each Fourier mode of the field is treated as an independent oscillator with it's own creating and annihilation operators.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}}(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x})$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} - i\sqrt{\frac{\omega_p}{2}}(a_p e^{ip\cdot x} - a_p^\dagger e^{-ip\cdot x})$$

We will rearrange the above equation as they will be more useful to us in the future.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}}(a_p + a_{-p}^\dagger)e^{ip\cdot x} \tag{1}$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} - i\sqrt{\frac{\omega_p}{2}}(a_p - a_{-p}^\dagger)e^{ip\cdot x} \tag{2}$$

The commutation relation now becomes

$$[a_p, a_{p'}^\dagger] = i\delta^{(3)}(p - p')$$

Using these we can verify that

$$[\phi(x), \pi(x')] = \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{-i}{2}\sqrt{\frac{\omega_{p'}}{\omega_p}}([a_{-p'}^\dagger, a_{p'}] - [a_p, a_{-p'}^\dagger])e^{i(p\cdot x + p'\cdot x')} = i\delta^{(3)}(x - x')$$

Now, we can express the Hamiltonian in terms of ladder operators.

$$H = \int d^3x \int \frac{d^3p\ d^3p'}{(2\pi)^6} e^{i(p+p')\cdot x}\left\{ -\frac{\sqrt{\omega_p\omega_{p'}}}{4}(a_p - a_{-p}^\dagger)(a_{p'} - a_{-p'}^\dagger) + \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p\omega_{p'}}}(a_p + a_{-p}^\dagger)(a_{p'} + a_{-p'}^\dagger)\right\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\omega_p(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger])$$

To find this, all we do is plug the equations (1,20 to the Hamiltonian Expression:

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{3}$$

$$= \frac{1}{2}\int \frac{d^3x\ d^3p\ d^3q}{(2\pi)^6}\Big[-\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}{2}\Big(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\Big)\Big(a_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} - a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}}\Big)$$
$$+ \frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\Big(i\vec{p}a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - i\vec{p}a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\Big)\cdot\Big(i\vec{q}a_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} - i\vec{q}a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}}\Big)$$
$$+ \frac{m^2}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\Big(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\Big)\Big(a_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} + a_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{x}}\Big)\Big]$$

$$= \frac{1}{4}\int \frac{d^3p\ d^3q}{(2\pi)^3}\Big[-\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}\Big(a_{\vec{p}}a_{\vec{q}}\delta(\vec{p}+\vec{q}) - a_{\vec{p}}^\dagger a_{\vec{q}}\delta(-\vec{p}+\vec{q}) - a_{\vec{p}}a_{\vec{q}}^\dagger\delta(\vec{p}-\vec{q}) + a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger\delta(-\vec{p}-\vec{q})\Big)$$
$$+ \frac{1}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\Big(-\vec{p}\cdot\vec{q}a_{\vec{p}}a_{\vec{q}}\delta(\vec{p}+\vec{q}) + \vec{p}\cdot\vec{q}a_{\vec{p}}^\dagger a_{\vec{q}}\delta(-\vec{p}+\vec{q}) + \vec{p}\cdot\vec{q}a_{\vec{p}}a_{\vec{q}}^\dagger\delta(\vec{p}-\vec{q}) - \vec{p}\cdot\vec{q}a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger\delta(-\vec{p}-\vec{q})\Big)$$
$$+ \frac{m^2}{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\Big(a_{\vec{p}}a_{\vec{q}}\delta(\vec{p}+\vec{q}) + a_{\vec{p}}^\dagger a_{\vec{q}}\delta(-\vec{p}+\vec{q}) + a_{\vec{p}}a_{\vec{q}}^\dagger\delta(\vec{p}-\vec{q}) + a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger\delta(-\vec{p}-\vec{q})\Big)\Big]$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\Big[-\omega_{\vec{p}} a_{\vec{p}} a_{-\vec{p}} + \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}} + \omega_{\vec{p}} a_{\vec{p}} a^\dagger_{\vec{p}} - \omega_{\vec{p}} a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}}$$

$$+\frac{1}{\omega_{\vec{p}}}\vec{p}^2 a_{\vec{p}} a_{-\vec{p}} + \frac{1}{\omega_{\vec{p}}}\vec{p}^2 a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{\omega_{\vec{p}}}\vec{p}^2 a_{\vec{p}} a^\dagger_{\vec{p}} + \frac{1}{\omega_{\vec{p}}}\vec{p}^2 a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}}$$

$$+\frac{m^2}{\omega_{\vec{p}}} a_{\vec{p}} a_{-\vec{p}} + \frac{m^2}{\omega_{\vec{p}}} a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{m^2}{\omega_{\vec{p}}} a_{\vec{p}} a^\dagger_{\vec{p}} + \frac{m^2}{\omega_{\vec{p}}} a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}}\Big]$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\vec{p}}}\Big[(-\omega^2_{\vec{p}} + \vec{p}^2 + m^2) a_{\vec{p}} a_{-\vec{p}} + (-\omega^2_{\vec{p}} + \vec{p}^2 + m^2) a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}} + (\omega^2_{\vec{p}} + \vec{p}^2 + m^2) a_{\vec{p}} a^\dagger_{\vec{p}} + (\omega^2_{\vec{p}} + \vec{p}^2 + m^2) a^\dagger_{\vec{p}} a_{\vec{p}}\Big]$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\vec{p}}}\Big[(-\omega^2_{\vec{p}} + \vec{p}^2 + m^2)(a_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}}) + (\omega^2_{\vec{p}} + \vec{p}^2 + m^2)(a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}})\Big]$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}[a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}]$$

$$= \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}[a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]]$$

The term on the far right is proportional to $\delta(0)$, an infinite c-number. It is the sum over all modes of the zero-point energies $\frac{\omega_p}{2}$, so its presence is expected. The infinite energy shift cannot be detected experimentally, since experiments measure only energy differences from the ground state of H. Therefore, we will simply ignore this energy shift for now. *(look at epilogue)*

$$= \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}[a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)]$$