1 The Language of Quantum Information Theory

1.1 Density Operators

We begin by stating the a common axiomatization of Quantum Mechanics (QM) based on Hilbert Spaces following [5], we choose it for mathematical simplicity; for alternatives see e.g. [36].

1. To each physical system $\mathcal{S}$ there corresponds a Hilbert Space $\mathcal{H}$ and states of the system are described by positive and unit trace operators. The Hilbert Space of a composite system made up of $\mathcal{S}$ and $\mathcal{S}'$ is given by the tensor product of the Hilbert spaces i.e. $\mathcal{H} \otimes \mathcal{H}'$.

2. To each dynamical variable there corresponds a self-adjoint operator on $\mathcal{H}$, called an observable, whose possible values are given by its eigenvalues.

3. Given a system in a state ρ and some observable A of it, the probability of measuring A and obtaining the result λ is given by $\mathrm{Tr}[\rho P_\lambda]$ where P_λ is the eigen-projector into the subspace associated with λ. Furthermore the expectation value is $\mathrm{Tr}[\rho A]$.

4. After a measurement with result λ the state of the system becomes $\frac{P_\lambda \rho P_\lambda}{\mathrm{Tr}[P_\lambda \rho P_\lambda]}$.

5. The time evolution of the system in a time interval $(0, t)$ in which no measurement is done is given by some unitary operator U_t according to $\rho_t = U_t \rho U_t^\dagger$ where ρ is the state of the system at time $t = 0$.

Operators satisfying the properties required for a state are called ***Density Operators*** and in contrast to frameworks whose treatment of quantum states is merely as rays in $\mathcal{H}$, they describe statistical mixtures so imperfect state preparation can be handled. To see this consider the spectral resolution of some density operator:

$$\rho = \sum_n p_n \left|\psi_n\right\rangle\!\left\langle\psi_n\right| \tag{1-1}$$

by definition we have $p_n \geq 0, \sum_n p_n = 1$ any density operator can be seen as a convex sum of rays in $\mathcal{H}$ (provided we identify each one with its associated projector $|\psi\rangle\langle\psi|$) and from it an interpretation of ρ as an statistical mixture of rays is suggested: given a preparation process, there is a probability p_n for the system to be in the state $|\psi_n\rangle\langle\psi_n|$ after it, for this reason states of the form $|\psi\rangle\langle\psi|$ are called ***Pure*** while those who are not we refer to as ***Mixed***. This ***Ensemble Interpretation*** has serious conceptual challenges when one tries to use it outside a fixed preparation procedure due to the non-uniqueness of the decomposition into pure states [30], but is good enough for the porposes of the present work, for a comprehensive discussion of this topic the reader is refered to [39].

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