1 Quantum Estimation

In this chapter we identify the anatomy of the metrology problem and formulate it in the language of quantum channels and statistical inference. Once this is done we obtain a generic recipe to obtain POVM independent precision bounds.

1.1 Statistical Inference

1.1.1 Estimators

Any experiment one realizes has an underlying probability distribution over some set χ that representes the possible outcomes, specified by the laboratory conditions e.g. temperature, pressure, instrumental precision, initial state and represented by an element in some parameter space Θ; our objective is to study this unknown distribution from the experimental results, identifying it **as precisely as possible** i.e. we have a problem of statistical inference. Below we present the basic structure of the **local estimation framework** following [1, 3], its essence is the assumption that the distribution of study is a member of a known parametric family of probability distributions $\{p(x;\theta)\}_{\theta\in\Theta}$ from which we must identify the particular θ that corresponds to our experiment via samples. From this is clear that we need a rule to go from the sample to the parameter space and that its properties will allow us to study precision, this is the notion that the following definition seeks to capture.

Definition 1.1 *Given a family $\{p_\theta(x)\}_{\theta\in\Theta}$ of probability distributions over some set χ, we call an estimator for θ a sample of size n a function $T:\chi^n\to\Theta$. Assuming $\Theta\subseteq\mathbb{R}$:*

- *The difference $T-\theta$ is called **the error** of the estimator, note this is a random variable*
- *The expected value of the error is called **the bias**, and if it is zero we say the estimator is **unbiased**.*
- *Let $X_1,...,X_n\sim p_\theta$ be i.i.d random variables, $E[(T(X_1,...,X_n)-\theta)^2]$ is called the **Mean Square Error** (MSE) of the estimator.*
- *An estimator T_1 is said to **dominate** another one T_2 if its MSE is less than or equal for all $\theta\in\Theta$.*

The MSE is the figure of merit that classifies the estimator T, and if it is unbiased we can identify it with $\mathrm{Var}[T]$ so that our credence on $T(x_1, ...x_n)$ is codified in it. We give this last statement a concrete operational meaning through Chebyshev's inequality:

Theorem 1.1 *Let X be a random variable with finite non-zero variance σ^2 and expected value μ. For any real k the probability of the difference between X and μ being greater than $k\sigma$ is [2]:*

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

The smaller the variance of T the less likely it is for the difference between T and the actual value θ to be greater than $\mathrm{Var}[T]$.

1.1.2 The Fisher Information

When one characterizes a measurement apparatus for a quantity X a key property is how **sensible** it is i.e. given two values of X, x and x', what is the smallest $\Delta X = |x - x'|$ such that it can differentiate between the two. For estimators there is a similar connection between a notion of sensitivity and its variance, given by the **Fisher Information** and the **Cramér-Rao bound** respectevely.

Definition 1.2 *For a parametric family of probability distributions $\{p_\theta\}_{\theta\in\Theta}$ we define the* ***Fisher Information*** *as*

$$F(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log p_\theta(x)\right)^2\right] = \int dx\, \frac{(\partial_\theta p_\theta(x))^2}{p_\theta(x)} \tag{1.1}$$

Theorem 1.2 *The MSE of an unbiased estimator T of the parameter θ is bounded by*

$$Var[T] \geq \frac{1}{F(\theta)}. \tag{1.2}$$

This inequality is called the ***Cramé-Rao bound*** *(CRB).*

We say an estimator is **efficient** if it saturates the CRB.

Bibliography

[1] T. M. Cover and Joy A. Thomas. *Elements of information theory*. Hoboken, N.J. : Wiley-Interscience, 2006.

[2] William Feller et al. “An introduction to probability theory and its applications”. In: (1971).

[3] Matteo G. A. Paris. *Quantum estimation for quantum technology*. Aug. 25, 2009.