2 Markovian Open Systems

2.1 Dynamical Semigroups

The most accessible and studied type of open quantum systems are those which accept a description through a **markovian** and **time homogeneous** differential equation. Classical stochastic processes of this kind are characterized through the semigroup formed by the family of conditional probabilities [1], which is parametrized by the elapsed time, with this motivation one introduces the quantum analog called **Dynamical Semigroups**. The analogy is nuanced, as the time evolution of any probability distribution associated with an observable will not satisfy the Kolmogorov consistency condition, and thus no description as classical stochastic processes is avaliable [4], furthermore, the evolution of the state will still be deterministic.

Definition 2.1 *A differentiable parametric family of quantum operations* $\{\mathcal{E}_\tau\}_{\tau=0}^{\infty}$ *such that*

$$\mathcal{E}_\tau(\mathcal{E}_{\tau'}(\rho)) = \mathcal{E}_{\tau+\tau'}(\rho) \tag{2.1}$$

$$\mathcal{E}_0 = id \tag{2.2}$$

i.e. that has the semigroup property, is called a ***Dynamical Semigroup****. Strictly, one also demands some additional technical conditions on the continouity to treat the case of infinite dimensional Hilbert Spaces [3].*

In general these are irreversible; mathematically because the image is *contractive* and physically due to its positive entropy production [1], for this demanding a full group structure is too strong. The main result of this chapter is the classification in terms of generators due to Linblad, which is presented in the following theorem.

Theorem 2.1 *Given a dynamical semigroup* $\{\mathcal{E}_\tau\}_{\tau=0}^{\infty}$ *there exists a time independent super-operator* $\mathcal{L}$*, called* ***the generator****, such that:*

$$\partial_t \rho(t) = \mathcal{L}\rho(t) = -i[H,\rho] + \sum_k \gamma_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\} \right) \tag{2.3}$$

with L_k *operators and* H *a self-adjoint one [1, 2, 3, 5], futhermore, all equations of this form define a dynamical semigroup. Equation (2.3) is called* ***the Linblad Master Equation****, the traceless* L_k*'s the* ***Linblad operators****, and the* γ_k *are positive constants.*

[1] Called a propagator in [1].

2.2 Trajectory Interpretation of the Evolution

It is clear that at the very least (2.3) accepts a formal solution in terms of an exponential superator $\exp(t\mathcal{L})$, one can use this to form a generalized Dyson series expansion that will gives us a Kraus representation of it by defining a sort of interaction picture. Begin by defining an arbitrary decomposition of the generator in terms of two terms $\mathcal{L}_0, S$:

$$\mathcal{L} = \mathcal{L}_0 + S \tag{2.4}$$

and now introduce the auxiliary unnormalized state ρ'

$$\rho = e^{\mathcal{L}_0 t}\rho' \tag{2.5}$$

subsituting it in the first equality of the Linblad equation one obtains:

$$e^{\mathcal{L}_0 t}\partial_t\rho' + \mathcal{L}_0\rho = S\rho + \mathcal{L}_0\rho \tag{2.6}$$

$$\partial_t\rho' = e^{-\mathcal{L}_0 t}Se^{\mathcal{L}_0 t}\rho_1 \tag{2.7}$$

and integrating from 0 to t:

$$\rho' = \rho(0) + \int_0^t dt_1 e^{-\mathcal{L}_0 t_1}Se^{\mathcal{L}_0 t_1}\rho_1(t_1). \tag{2.8}$$

Iterating this equation:

$$\rho' = \rho(0) + \int_0^t dt_1 e^{-\mathcal{L}_0 t_1}Se^{\mathcal{L}_0 t_1}\left(\rho(0) + \int_0^{t_1} dt_2 e^{-\mathcal{L}_0 t_2}Se^{\mathcal{L}_0 t_2}\rho'(t_2)\right) \tag{2.9}$$

$$\rho' = \rho(0) + \int_0^t dt_1 e^{-\mathcal{L}_0 t_1}Se^{\mathcal{L}_0 t_1}\rho(0) + \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-\mathcal{L}_0 t_1}Se^{\mathcal{L}_0(t_1-t_2)}Se^{\mathcal{L}_0 t_2}\rho'(t_2) \tag{2.10}$$

$$\rho' = \rho(0) + \sum_{n=1}^{\infty}\int_0^t dt_n \int_0^{t_n} dt_{n-1}\ldots\int_0^{t_2} dt_1 e^{-t_n\mathcal{L}_0}Se^{(t_n-t_{n-1})\mathcal{L}_0}S\ldots e^{(t_2-t_1)\mathcal{L}_0}Se^{t_1\mathcal{L}_0}\rho(0). \tag{2.11}$$

Note that in the last line we inverted the order of the indexation to make it coincide with [2], so finally we have for the original state:

$$\rho = e^{t\mathcal{L}_0}\rho(0) + \sum_{n=1}^{\infty}\int_0^t dt_n \int_0^{t_n} dt_{n-1}\ldots\int_0^{t_2} dt_1 e^{(t-t_n)\mathcal{L}_0}Se^{(t_n-t_{n-1})\mathcal{L}_0}S\ldots e^{(t_2-t_1)\mathcal{L}_0}Se^{t_1\mathcal{L}_0}\rho(0). \tag{2.12}$$

This has a formally identical structure to that of the Dyson series in a pertubative expansion, suggesting that one can interpret the evolution of the system as being given by a perturbation S to the evolution e^{tL_0}. The main problem with this is that in general the splitting superoperators do not necessarily define a dynamical semigroup e.g. they could fail to map into a traceless operator, which is a consistency condition. To obtain a more suitable interpretation we choose a particular decomposition, coming from the second equality of (2.3) [2]:

$$\begin{aligned}\mathcal{L}_0 =& - i(\tilde{H}\rho - \rho\tilde{H}^\dagger) && (2.13)\\ S =& \sum_k \mathcal{L}_k && (2.14)\\ \tilde{H} =& H - \frac{i}{2}\sum_k \gamma_k L_k^\dagger L_k && (2.15)\\ \mathcal{L}_k =& J[\sqrt{\gamma_k}L_k]\rho && (2.16)\end{aligned}$$

where we used the notation $J[A]\rho = A\rho A^\dagger$. For the evaluation we introduce the following result:

Lemma 2.1 *For any superoperator of the form $\mathcal{A}\rho = A\rho + \rho A^\dagger$,*

$$e^{\tau\mathcal{A}}\rho = \exp(\tau A)\rho\exp(\tau A^\dagger) \tag{2.17}$$

Proof 2.1 *Taking the derivative one forms the equation*

$$\partial_t e^{\tau\mathcal{A}}\rho = A\rho + \rho A^\dagger \tag{2.18}$$

which by inspection one sees has the solution $e^{\tau\mathcal{A}}\rho = \exp(\tau A)\rho\exp(\tau A^\dagger)$.

allowing us to write (2.12) as in [5]:

$$\rho = J[e^{t\frac{1}{i}\tilde{H}}]\rho(0) + \sum_{n=1}^{\infty}\mathcal{K}_t^{(n)}\rho(0) \tag{2.19}$$

$$\mathcal{K}_t^{(n)} = \int_0^t dt_n \int_0^{t_n} dt_{n-1} ... \int_0^{t_2} dt_1 J[e^{(t-t_n)\frac{1}{i}\tilde{H}}]SJ[e^{(t_n-t_{n-1})\frac{1}{i}\tilde{H}}]...J[e^{(t_2-t_1)\frac{1}{i}\tilde{H}}]SJ[e^{t_1\frac{1}{i}\tilde{H}}] \tag{2.20}$$

$$\mathcal{K}_t^{(n)} = \sum_{k_1...k_n}\int_0^t dt_n \int_0^{t_n} dt_{n-1} ... \int_0^{t_2} dt_1 J\left[e^{(t-t_n)\frac{1}{i}\tilde{H}}\sqrt{\gamma_{k_n}}L_{k_n}e^{(t_n-t_{n-1})\frac{1}{i}\tilde{H}}...e^{(t_2-t_1)\frac{1}{i}\tilde{H}}\sqrt{\gamma_{k_1}}L_{k_1}e^{t_1\frac{1}{i}\tilde{H}}\right] \tag{2.21}$$

the integrands inside $\mathcal{K}_t^{(n)}$ are positive, and so they must be non-trace increasing for (2.19) to have the same trace in both sides. With this we conclude that (2.12) can be interpreted as a piecewise deterministic process, in which continous evolutions $\exp(\tau\mathcal{L}_0)$ are interrupted by environment induced transformations $J[L_k]$ at a rate γ_k. More precesely, the probability of the systems evolving during a time t without **jumps** i.e. only continously is:

$$P(R_0^t|\rho) = \mathrm{Tr}\left[e^{\mathcal{L}_0 t}\right] \tag{2.22}$$

and the probability of having n jumps $k_n, ..., k_1$ at respective times $t > t_n > ... > t_1$ is

$$P(R_n^{t>t_n>...>t_1}|\rho) = \mathrm{Tr}\left[J\left[e^{(t-t_n)\frac{1}{i}\tilde{H}} L_{k_n} e^{(t_n-t_{n-1})\frac{1}{i}\tilde{H}} ... e^{(t_2-t_1)\frac{1}{i}\tilde{H}} L_{k_1} e^{t_1\frac{1}{i}\tilde{H}}\right]\rho\right]. \tag{2.23}$$

This is the interpretation of the **Quantum Trajectories Method** [2, 5]

Bibliography

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