1 Quantum Estimation

In this chapter we identify the anatomy of the metrology problem and formulate it in the language of quantum channels and statistical inference. Once this is done we obtain a generic recipe to obtain POVM independent precision bounds.

1.1 Statistical Inference

1.1.1 Estimators

Any experiment one realizes has an underlying probability distribution over some set χ that representes the possible outcomes, specified by the laboratory conditions e.g. temperature, pressure, instrumental precision, initial state and represented by an element in some parameter space Θ; our objective is to study this unknown distribution from the experimental results, identifying it **as precisely as possible** i.e. we have a problem of statistical inference. Below we present the basic structure of the **local estimation framework** following [2, 5], its essence is the assumption that the distribution of study is a member of a known parametric family of probability distributions $\{p(x;\theta)\}_{\theta\in\Theta}$ from which we must identify the particular θ that corresponds to our experiment via samples. From this is clear that we need a rule to go from the sample to the parameter space and that its properties will allow us to study precision, this is the notion that the following definition seeks to capture.

Definition 1.1 *Given a family $\{p_\theta(x)\}_{\theta\in\Theta}$ of probability distributions over some set χ, we call an estimator for θ a sample of size n a function $T : \chi^n \to \Theta$. Assuming $\Theta \subseteq \mathbb{R}$:*

- *The difference $T-\theta$ is called* ***the error*** *of the estimator, note this is a random variable*
- *The expected value of the error is called* ***the bias****, and if it is zero we say the estimator is* ***unbiased****.*
- *Let $X_1, ..., X_n \sim p_\theta$ be i.i.d random variables, $E[(T(X_1, ..., X_n) - \theta)^2]$ is called the* ***Mean Square Error*** *(MSE) of the estimator.*
- *An estimator T_1 is said to* ***dominate*** *another one T_2 if its MSE is less than or equal for all $\theta \in \Theta$.*

The MSE is the figure of merit that classifies the estimator T, and if it is unbiased we can identify it with $\mathrm{Var}[T]$ so that our credence on $T(x_1, ...x_n)$ is codified in it. We give this last statement a concrete operational meaning through Chebyshev's inequality:

Theorem 1.1 *Let X be a random variable with finite non-zero variance σ^2 and expected value μ. For any real k the probability of the difference between X and μ being greater than $k\sigma$ is [3]:*

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

The smaller the variance of T the less likely it is for the difference between T and the actual value θ to be greater than $\mathrm{Var}[T]$.

1.1.2 The Fisher Information

When one characterizes a measurement apparatus for a quantity X a key property is how **sensible** it is i.e. given two values of X, x and x', what is the smallest $\Delta X = |x - x'|$ such that it can differentiate between the two. For estimators there is a similar connection between a notion of sensitivity and its variance, given by the **Fisher Information** and the **Cramér-Rao bound** respectevely.

Definition 1.2 *For a parametric family of probability distributions $\{p_\theta\}_{\theta\in\Theta}$ we define the **Fisher Information** (FI) as*

$$F(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log p_\theta(x)\right)^2\right] = \int dx\, \frac{(\partial_\theta p_\theta(x))^2}{p_\theta(x)} \tag{1.1}$$

Theorem 1.2 *The MSE of an unbiased estimator T of the parameter θ is bounded by*

$$Var[T] \geq \frac{1}{F(\theta)}. \tag{1.2}$$

*This inequality is called the **Cramé-Rao bound** (CRB).*

We say an estimator is **CR-efficient** if it saturates the CRB [2, 6]. Note that the CR bound depends on the family of probability distributions and its parametrization at a given point, not on the estimator. The FI **quantifies the amount of information about the parameter contained in the actual probability distribution** by describing the limits on the amount of creedence we could assign to **any** estimation of the parameter θ. The question now is whether the CRB is tight i.e. if there always exists a CR-efficient estimator, the next theorem shows this is the case asymptotycally.

Definition 1.3 *Given a probability distribution p_θ where θ is a parameter, we define the* ***likelihood*** *of a sample $\{x_1, ..., x_n\}$ as*

$$f_\theta(x_1, ..., x_n) = \prod_{k=1}^{n} p_\theta(x_k).$$

and the ***Maximum Likelihood Estimator*** *(MLE) $\hat{\theta}_{ML} : \chi^n \to \mathbb{R}$ as*

$$\hat{\theta}_{ML}(x_1, ..., x_n) = \underset{\theta}{arg\,max}\, f_\theta(x_1, ..., x_n)$$

essentially the likelihood measures how probable is to find a given sample provided a value of theta, and the MLE proposes as estimation the value of the parameter for which this sample is most likely.

Theorem 1.3 *The MLE saturates the CRB asymptotically [4].*

1.2 Characterization of Quantum Channels

Let ε_γ be a quantum channel with an unknown γ we seek to estimate through some measurement, for this we need to define an input (initial) state ρ_0 and perform some measurements on the output. The particular way we measure will correspond to choosing a POVM $\{\Pi_x\}$ that describes the statistics of the experiment. More concretely, the outcomes follow the distribution

$$\wp(x|\gamma) = \mathrm{Tr}\left[\Pi_x \varepsilon_\gamma(\rho_0)\right] \tag{1.3}$$

and from it we must infer the value of γ; from this it is clear that what we got at hand really is a problem about statistical inference, in which the family of distributions is induced by the POVM. This reasoning shows all the parts of a **Metrology Protocol**:

1. an initial state
2. a channel with an unknown parameter one wants to estimate, producing output states $\rho_\gamma = \varepsilon_\gamma(\rho_0)$
3. a measurement strategy, whose statistics are described by a POVM
4. an estimator producing an estimate $\tilde{\gamma}$.

in figure 1.1 this is represented.
The objective of quantum metrology is to leverage the nonclassical aspects of quantum theory to estimate as precisely as possible a parameter of interest, given a fixed amount of certain **resource** involved in the estimation.

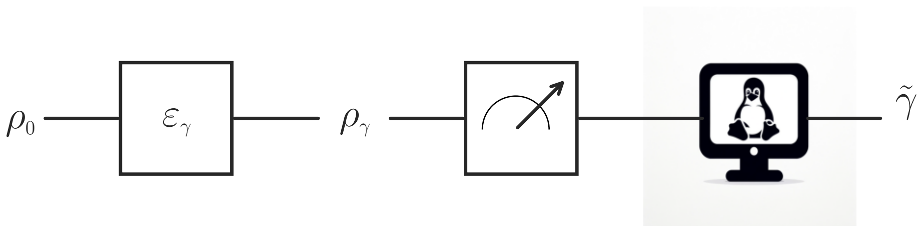

Figure 1.1: Schematic depiction of a metrology scheme

1.2.1 Optimal POVMs and Precision Bounds

Different choices of measurement strategy will lead to different distributions and in turn to different values of FI, this means that not all the POVMs are made equal. A good question now is wheter one can bound the FI for all possible POVMs, this is indeed the case via a procedure proposed the [1] which we outline below.

The Quantum Fisher Information

Consider a generic metrology scheme as outlined at the beginning of the section, its FI is easily bounded by:

$$F(\gamma) = \int dx \frac{\left(\mathrm{Tr}\left[\Pi_x \partial_\gamma \rho_\gamma\right]\right)^2}{\mathrm{Tr}\left[\Pi_x \rho_\gamma\right]} \leq \int dx \frac{\left|\mathrm{Tr}\left[\Pi_x \partial_\gamma \rho_\gamma\right]\right|^2}{\mathrm{Tr}\left[\Pi_x \rho_\gamma\right]}. \tag{1.4}$$

The integrand heavily resembles a logarithmic derivative so taking this as inspiration a new operator called the **Self-adjoint Logarithmic Derivative** (SLD) is introduced, we define it as a self-adjoint solution to the equation

$$\partial_\gamma \rho_\gamma = \frac{1}{2}\left(\Lambda_\gamma \rho_\gamma + \rho_\gamma \Lambda_\gamma\right), \tag{1.5}$$

later we'll come back to discuss its uniqueness and potential problems that may arise. Using this we find:

$$\mathrm{Tr}\left[\Pi_x \partial_\gamma \rho_\gamma\right] = \frac{1}{2}\left(\mathrm{Tr}\left[\Pi_x \rho_\gamma \Lambda_\gamma\right] + \mathrm{Tr}\left[\Pi_x \Lambda_\gamma \rho_\gamma\right]\right) \tag{1.6}$$

Bibliography

[1] Samuel L. Braunstein and Carlton M. Caves. "Statistical distance and the geometry of quantum states". In: *Physical Review Letters* 72.22 (May 30, 1994), pp. 3439–3443.

[2] T. M. Cover and Joy A. Thomas. *Elements of information theory*. Hoboken, N.J. : Wiley-Interscience, 2006.

[3] William Feller et al. "An introduction to probability theory and its applications". In: (1971).

[4] Steven M Kay. *Fundamentals of statistical signal processing: estimation theory*. Prentice-Hall, Inc., 1993.

[5] Matteo G. A. Paris. *Quantum estimation for quantum technology*. Aug. 25, 2009.

[6] Howard M. Wiseman and Gerard J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2010. 477 pp.