A Constructive Approach to Lagrange's Multipliers in Analytical Mechanics

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1 Introduction

The derivation of the Euler-Lagrange's equations for a system without constraints relies heavily on the independence of the virtual displacements of the coordinates with respect to which the Lagrangian is written, this prevents the technique used for the unconstrained case to be directly applied to a system with constraints. There are two standard ways of overcoming this difficulty: to use the constraints to write the Lagrangian in new independent[1] coordinates, and the method of Lagrange's Multipliers. The latter is usually presented as just a useful modification of the original Lagrangian that allows to eliminate dependent coordinates; my objective in this article is to explore in further detail the method and show how it can be deduced in a more motivated and constructive way. For this I'll start with the case of a Lagrangian $L(q, \dot{q}, t)$ of n coordinates and 1 constraint, then show that the solution to this variational problem is a modified Euler-Lagrange's equation that can be obtained adding a simple term to the Lagrangian, last an arbitrarily number of constraints are introduced to the problem.

2 Solution for 1 constraint

2.1 Introduction of the constraint in the problem

Let $L(q, \dot{q}, t)$ be a Lagrangian in n variables and $f_1(q, t) = 0$ an holonomic constraint. Taking ϵ to be the variation parameter we write the first variation of the action:

$$\delta S = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q_j}} \right] \frac{\partial q_j}{\partial \epsilon} \, dt d\epsilon \tag{1}$$

[1]It is always possible for holonomic and semiholonomic constraints, for other types it may or not be possible.

In absence of constraints the stationary value condition (i.e. $\delta S = 0$) would give that the square brackets are identically 0 for all j due to the coordinates being independent; the presence of the constraint prohibits this, as now only $n-1$ are independent. This difficulty can be easily overcome as is shown for the case $n = 2$ in [2]; without loss of generality assume that removing q_1 from q makes the set independent, then we want to consider the variation of the constraint and use it to write $\frac{\partial q_1}{\partial \epsilon}$ in terms of the independent coordinates, this will give us equation (1) in a form to which an independence argument can be applied.

Obtaining $\frac{\partial q_1}{\partial \epsilon}$ from the first variation of the constraint:

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \frac{\partial f_1}{\partial q_j} \frac{\partial q_j}{\partial \epsilon} \\ \frac{\partial q_1}{\partial \epsilon} &= \left(-\frac{\partial f_1}{\partial q_1} \right)^{-1} \sum_{j=2}^{n} \frac{\partial f_1}{\partial q_j} \frac{\partial q_j}{\partial \epsilon} \end{aligned}$$

Now substituting in (1) and factoring $\frac{\partial q_j}{\partial \epsilon}$:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} \right) \frac{\partial q_1}{\partial \epsilon} + \sum_{j=2}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} \right] \frac{\partial q_j}{\partial q_j} \, dt d\epsilon$$

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} \right) \left(-\frac{\partial f_1}{\partial q_1} \right)^{-1} \sum_{j=2}^{n} \frac{\partial f_1}{\partial q_j} \frac{\partial q_j}{\partial \epsilon} + \sum_{j=2}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} \right] \frac{\partial q_j}{\partial q_j} \, dt d\epsilon$$

$$\delta S = \int_{t_1}^{t_2} \sum_{j=2}^{n} \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} \right) \left(-\frac{\partial f_1}{\partial q_1} \right)^{-1} \frac{\partial f_1}{\partial q_j} + \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} \right) \right] \frac{\partial q_j}{\partial \epsilon} \, dt d\epsilon$$

Now we set $\delta S = 0$ and due to the sum being over independent variables, it is obtained that for every j:

$$0 = \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} \right) \left(-\frac{\partial f_1}{\partial q_1} \right)^{-1} \frac{\partial f_1}{\partial q_j} + \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} \right)$$

$$\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} \right) \left(\frac{\partial f_1}{\partial q_1} \right)^{-1} = \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} \right) \left(\frac{\partial f_1}{\partial q_j} \right)^{-1}$$

Here comes the introduction of the Lagrange's Multiplier: define a function $\lambda_1(t)$ that is equal to both sides of the last equation, this plus the constraint equation give us the set of equations that describe the solution to our problem.

$$\begin{aligned} \lambda_1(t) &= \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} \right) \left(\frac{\partial f_1}{\partial q_j} \right)^{-1} \\ \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} &= \frac{\partial f_1}{\partial q_j} \lambda_1(t) \qquad (2) \end{aligned}$$

Then the complete solution of the problem requires to solve a set of $n + 1$ equations ((2) for every j + the constraint), finding all of the $q_j(t)$ and $\lambda_1(t)$.

2.2 Modified Lagrangian

Notice that the extra parameter $\lambda_1(t)$ acts as a new variable describing the system, on the same level as the $q_j(t)$. This motivates the idea of considering a new Lagrangian $L'(q, \dot{q}, \lambda_1(t), t)$ to describe the system. We will consider the following:

$$L'(q, \dot{q}, \lambda_1(t), t) = L(q, \dot{q}, t) + \lambda_1(t) f_1(t, q) \tag{3}$$

Here $\lambda_1(t)$ is only an arbitrary differentiable function and has not yet been defined to obey any resctriction. Consider now the first variation of the action with this lagrangian, keeping in mind that now we must variate λ_1:

$$\delta S' = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \lambda_1 \frac{\partial f_1}{\partial q_j} \right] \frac{\partial q_j}{\partial \epsilon} + \left(\frac{\partial (f_1 \lambda_1)}{\partial \lambda_1} - \frac{d}{dt}\frac{\partial (f_1 \lambda_1)}{\partial \dot{\lambda}_1} \right) \frac{\partial \lambda_1}{\partial \epsilon}\, dt d\epsilon$$

$$\delta S' = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \lambda_1 \frac{\partial f_1}{\partial q_j} \right] \frac{\partial q_j}{\partial \epsilon} + f_1 \frac{\partial \lambda_1}{\partial \epsilon}\, dt d\epsilon$$

$$\delta S' = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \lambda_1 \frac{\partial f_1}{\partial q_j} \right] \frac{\partial q_j}{\partial \epsilon}\, dt d\epsilon$$

Now restric λ_1 so that it must cancel the dependent term in the sum, as is suggested in [1]; this causes than when the stationary condition is introduced, the independence of the variables summed over implies the set of equations (2). The physical meaning of the extra term is to account for the constraint forces as is shown in [1]; this is specially easy to see in the case $n = 2$ for the lagrangian of a single particle and a scleronomic constraint: suppose that the lagrangian is written in cartesian coordinates, then we recover Newton's second law:

$$m\vec{a} = -\nabla V + \lambda_1 \nabla f_1 \tag{4}$$

Clearly the particle most move on the curve $f_1(x, y) = 0$; the term with the Lagrange's multiplier is perpendicular to the trayectory and must be a force of constraint.

3 Introduction of the remaining constraints

3.1 Introducing more constraints

Now the task is to generalize the aproach to multiple constraints; for this we will start from the modified form of the Lagrangian and then proceed to repeat the same process that allowed us to introduce the first constraint, as many times as required.

First a matter of notation to make the calculations less cumbersome and center our attention in the important details. Let there be m holonomic constraints noted as $f_\alpha(q,t)$ and also let $\mathscr{L}^j$ be defined as:

$$\mathscr{L}^j = \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \lambda_1 \frac{\partial f_1}{\partial q_j} \tag{5}$$

Now consider the variation of the action:

$$\delta S = \int_{t_1}^{t_2} \sum_{j=1}^{n} \mathscr{L}^j \frac{\partial q_j}{\partial \epsilon}\, dt d\epsilon \tag{6}$$

We repeat the same process done in section 2.1 that allowed to introduce a constraint; now we assume that if we remove both q_1 and q_2 the set q will be independent and we use f_2 to substitute $\frac{\partial q_2}{\partial \epsilon}$ in (6):

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \frac{\partial f_2}{\partial q_j}\frac{\partial q_j}{\partial \epsilon} \\ \frac{\partial q_2}{\partial \epsilon} &= \left(-\frac{\partial f_2}{\partial q_2}\right)^{-1}\left(\frac{\partial f_2}{\partial q_1}\frac{\partial q_1}{\partial \epsilon} + \sum_{j=3}^{n}\frac{\partial f_2}{\partial q_j}\frac{\partial q_j}{\partial \epsilon}\right) \end{aligned}$$

Substitute in (6):

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \mathscr{L}^1 \frac{\partial q_1}{\partial \epsilon} + \mathscr{L}^2\left(-\frac{\partial f_2}{\partial q_2}\right)^{-1}\left(\frac{\partial f_2}{\partial q_1}\frac{\partial q_1}{\partial \epsilon} + \sum_{j=3}^{n}\frac{\partial f_2}{\partial q_j}\frac{\partial q_j}{\partial \epsilon}\right) + \sum_{j=3}^{n}\mathscr{L}^j\frac{\partial q_j}{\partial \epsilon}\, dt d\epsilon \\ \delta S &= \int_{t_1}^{t_2} \left(\mathscr{L}^1 + \mathscr{L}^2\left(-\frac{\partial f_2}{\partial q_2}\right)^{-1}\frac{\partial f_2}{\partial q_1}\right)\delta q_1 + \sum_{j=3}^{n}\mathscr{L}^j + \frac{\partial f_2}{\partial q_j}\left(-\frac{\partial f_2}{\partial q_2}\right)^{-1}\mathscr{L}^2 \delta q_j\, dt \end{aligned} \tag{7}$$

Now define λ_1 to eliminate the first term and use and independence argument, that will give us a new modified equation that resembles to what we obtained in section (2.1):

$$\mathscr{L}^j\left(\frac{\partial f_2}{\partial q_j}\right)^{-1} = \mathscr{L}^2\left(\frac{\partial f_2}{\partial q_2}\right)^{-1} \tag{8}$$

Naturally the following step must be defining λ_2 as being equal to both sides of (8) and conclude:

$$\frac{\partial L}{\partial q_j} + \lambda_1 \frac{\partial f_1}{\partial q_j} + \lambda_2 \frac{\partial f_2}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \tag{9}$$

From here it is easily seen that this process of defining new paremeters to eliminate dependent variables can be repeated as many times as possible; is also intuitive now that the appropiate modification of the Lagrangian in order to introduce the constraints must be:

$$L' = L(q, \dot{q}, t) + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \tag{10}$$

As suggested in [1][2].

References

[1] Herbert Goldstein, Charles Poole, and John Safko. Classical mechanics, 2002.

[2] Jerry B Marion. *Classical dynamics of particles and systems.* Academic Press, 2013.