3.1) $Y_t = \beta_0 + \beta_1 t + X_t$

$$q(\hat\beta_0,\hat\beta_1) = \sum_{t=1}^{n}[Y_t - (\hat\beta_0 + \hat\beta_1 t)]^2$$

Goal:
minimize $q(\hat\beta_0,\beta_1)$ $= \sum_{t=1}^{n}(Y_t - \hat\beta_0 - \hat\beta_1 t)^2$

$$\frac{\partial q(\hat\beta_0,\hat\beta_1)}{\partial\hat\beta_0} = -2\sum_{t=1}^{n}(Y_t - \hat\beta_0 - \hat\beta_1 t)$$

$$\frac{\partial q(\hat\beta_0,\hat\beta_1)}{\partial\hat\beta_1} = -2\sum_{t=1}^{n}t(Y_t - \hat\beta_0 - \hat\beta_1 t)$$

set $\frac{\partial q(\hat\beta_0,\hat\beta_1)}{\partial\hat\beta_0}, \frac{\partial q(\beta_0,\beta_1)}{\partial\beta_1} = 0$

$$\frac{\partial q(\hat\beta_0,\hat\beta_1)}{\partial\hat\beta_0} = -\cancel{2}\sum_{t=1}^{n}(Y_t - \hat\beta_0 - \hat\beta_1 t) = 0$$

$= \sum_{t=1}^{n}(y_t - \hat\beta_0 - \hat\beta_1 t) = 0$ $\quad \bar X = \frac{1}{N}\sum_{t=1}^{n}X_i$

$= \sum_{t=1}^{n}y_t - \sum_{t=1}^{n}\hat\beta_0 - \sum_{t=1}^{n}\hat\beta_1 t = 0$ $\quad \sum_{t=1}^{n}x_i = n\bar X$

$= \cancel{n}\bar y - \cancel{n}\hat\beta_0 - \cancel{n}\hat\beta_1\bar t = 0$

$= \bar y - \hat\beta_0 - \hat\beta_1\bar t = 0$

$\Rightarrow \bar y = \hat\beta_0 + \hat\beta_1\bar t \rightarrow \boxed{\hat\beta_0 = \bar y - \hat\beta_1\bar t}$

$$\frac{\partial q(\hat\beta_0,\hat\beta_1)}{\partial\hat\beta_1} = -\cancel{2}\sum_{t=1}^{n}t(Y_t - \hat\beta_0 - \hat\beta_1 t) = 0$$

$= \sum_{t=1}^{n}tY_t - \sum_{t=1}^{n}\hat\beta_0 t - \sum_{t=1}^{n}\hat\beta_1 t^2 = 0$

$= \sum_{t=1}^{n}tY_t - n\hat\beta_0\bar t - \sum_{t=1}^{n}\hat\beta_1 t^2 = 0$

$= \sum_{t=1}^{n}tY_t - n(\bar y - \hat\beta_1\bar t)\bar t - \hat\beta_1\sum_{t=1}^{n}t^2 = 0$

$= \sum_{t=1}^{n}ty_t - n\bar y\bar t = \hat\beta_1\sum_{t=1}^{n}t^2 - n\hat\beta_1\bar t^2$

$$\hat\beta_1 = \frac{\sum_{t=1}^{n}ty_t - n\bar y\bar t}{\sum_{t=1}^{n}t^2 - n\bar t^2} = \boxed{\frac{\sum_{t=1}^{n}(y_t - \bar y)(t - \bar t)}{\sum_{t=1}^{n}(t - \bar t)^2}}$$

3.2) $Y_t = \mu + e_t - e_{t-1}$

Find $Var(\bar{Y}_t)$ → compare to $Var(\bar{Y}_t)$ when $Y_t = \mu + e_t$

First, find $\bar{Y}_t$.

$$\bar{Y}_t = \frac{1}{n}\sum_{t=1}^{n} Y_t = \frac{1}{n}\sum_{t=1}^{n}(\mu + e_t - e_{t-1})$$

$$= \mu + \frac{1}{n}\sum_{t=1}^{n}(e_t - e_{t-1})$$

$$= \mu + \frac{1}{n}(e_n - e_0)$$

$$Var(\bar{Y}_t) = Var(\mu + \frac{1}{n}(e_n - e_0))$$

$$= Var(\cancel{\mu}) + Var(\frac{1}{n}(e_n - e_0))$$

$$= \frac{1}{n^2}(\sigma^2 + \sigma^2)$$

$$= \frac{2\sigma^2}{n^2}$$

When $Y_t = \mu + e_t$

$$\bar{Y}_t = \frac{1}{n}\sum_{t=1}^{n} Y_t = \mu + \frac{1}{n}\sum_{t=1}^{n} e_t$$

$$Var(\bar{Y}) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

so, $Y_t = \mu + e_t - e_{t-1}$ | $Y_t = \mu + e_t$

$Var(\bar{Y}_t) = \frac{2\sigma^2}{n^2}$ | $Var(\bar{Y}) = \frac{\sigma^2}{n}$

Primary difference is in the denominator when $Y_t = \mu + e_t - e_{t-1}$, the sample mean has a significantly larger impact on the variance of the mean.

3.16)

$\{Y_t\}$ Stationary

$\rho(k) = \phi^k \quad k>0 \quad \phi: [-1,+1)$

a) Show that $V(\bar{Y}) = \frac{\gamma_0}{n}\left[\frac{1+\phi}{1-\phi} - \frac{2\phi}{n}\left(\frac{1-\phi^n}{(1-\phi)^2}\right)\right]$

Hint: $\sum_{k=0}^{n} \phi^k = \frac{1-\phi^{n+1}}{1-\phi}$

$\sum_{k=0}^{n} k\phi^{k-1} = \frac{d}{d\phi}\left(\sum_{k=0}^{n}\phi^k\right) = \frac{d}{d\phi}\left(\frac{1-\phi^{n+1}}{1-\phi}\right) \quad f'(\phi) = -(n+1)\phi^n \quad g'(\phi) = -1$

quotient rule: $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$\Rightarrow \frac{d}{d\phi}\left(\frac{1-\phi^{n+1}}{1-\phi}\right) = \frac{(-1)(n+1)\phi^n(1-\phi) - (1-\phi^{n+1})(-1)}{(1-\phi)^2}$

numerator: $(1-\phi)(-1)(n+1)\phi^n - (1-\phi^{n+1})(-1)$

$= (-1)(n+1)\phi^n + (n+1)\phi^{n+1} + 1 - \phi^{n+1}$

$= (-1)(n+1)\phi^n + n\phi^{n+1} + \cancel{\phi^{n+1}} + 1 - \cancel{\phi^{n+1}}$

$= -1(n+1)\phi^n + n\phi^{n+1} + 1$

$\frac{d}{d\phi}\left(\frac{1-\phi^{n+1}}{1-\phi}\right) = \frac{1 + n\phi^{n+1} - (n+1)\phi^n}{(1-\phi)^2}$

$Var(\bar{Y}) = \frac{\gamma_0}{n}\left(1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right)$

$= \frac{\gamma_0}{n}\left(1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\phi^k\right)$

$= \frac{\gamma_0}{n}\left(1 + 2\sum_{k=1}^{n-1}\phi^k - \sum_{k=1}^{n-1}\frac{k}{n}\phi^k\right)$

$= \frac{\gamma_0}{n}\left(1 + 2\sum_{k=1}^{n}\phi^k - \frac{2}{n}\sum_{k=1}^{n-1}k\phi^k\right)$

$= \frac{\gamma_0}{n}\left(1 + 2\left(\frac{1-\phi^n}{1-\phi}\right) - \frac{2}{n}\left(\frac{1 + (n-1)\phi^n - n\phi^{n-1}}{(1-\phi)^2}\right)\right)$

$= \frac{\gamma_0}{n}\left[\frac{1+\phi}{1-\phi} - \frac{2\phi}{n}\left(\frac{1-\phi^n}{(1-\phi)^2}\right)\right]$

b) for large n, $Var(\bar{Y}) \approx \frac{\gamma_0}{n}\left(\frac{1+\phi}{1-\phi}\right)$

because the $\frac{2\phi}{n}\left(\frac{1-\phi^n}{(1-\phi)^2}\right)$ term goes towards zero

this only leaves the first term