Numerical Solving of Navier-Stokes Equations

Finite Element Solver using deal.II for 2D/3D Benchmarks

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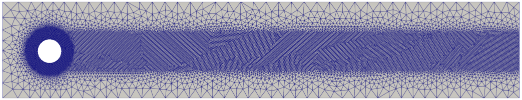
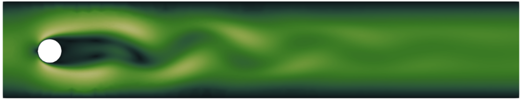
Project Overview

- **Objective:** Development of a C++ Finite Element solver for unsteady, incompressible Navier-Stokes equations.
- **Softwares:** `deal.II`, `MPI`, and `Trilinos`.
- **Benchmark:** "Flow past a cylinder" in 2D and 3D (Schäfer-Turek benchmark).
- **Analysis:** Computation of aerodynamic forces:
  - Drag coefficient (C_D) and Lift coefficient (C_L).
  - Frequency of vortex shedding (Strouhal number).



(a) Computational Domain



(b) Velocity Magnitude Field of 2D Test Case 3

The Navier-Stokes Equations

Given a domain $\Omega \subset \mathbb{R}^d$ and $t \in (0, T]$, find velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $p(\boldsymbol{x}, t)$:

Strong Formulation

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } \Omega, t > 0 \\ \nabla \cdot \boldsymbol{u} = 0 \qquad & \text{in } \Omega, t > 0 \end{aligned}$$

- **Boundary Conditions:**
  - **Dirichlet (Γ_D):** $\boldsymbol{u} = \boldsymbol{g}$ (Inflow and No-slip conditions).
  - **Neumann (Γ_N):** $\nu \nabla \boldsymbol{u} \cdot \boldsymbol{n} - p\boldsymbol{n} = \boldsymbol{h}$ (Outflow condition).
- **Viscosity (ν):** Governs the Reynolds number $Re = \frac{UD}{\nu}$.

Multiply by test functions $\boldsymbol{v} \in V$ and $q \in Q$, and integrate by parts:

Weak Formulation

Find $\boldsymbol{u} \in V$ and $p \in Q$ such that:

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V$$
$$b(\boldsymbol{u}, q) = 0 \quad \forall q \in Q$$

- **Diffusion:** $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nu \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\Omega$
- **Convection:** $c(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} \, d\Omega$
- **Pressure Coupling:** $b(\boldsymbol{v}, q) = -\int_\Omega q \nabla \cdot \boldsymbol{v} \, d\Omega$
- **Source Term:** $F(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\Omega + \int_{\Gamma_N} \boldsymbol{h} \cdot \boldsymbol{v} \, d\gamma$.

- **Space:** Based on Finite Element Method (FEM) with Taylor-Hood elements ($\mathbb{P}_2 - \mathbb{P}_1$) for Inf-Sup stability.
- **Time:** First-order BDF1 (Implicit Euler).
- **Non-linear Treatment:** Semi-implicit treatment of the non-linear convective term:

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \approx (\boldsymbol{u}^k \cdot \nabla)\boldsymbol{u}^{k+1}$$

Fully Discrete Scheme

Find $(\boldsymbol{u}_h^{k+1}, p_h^{k+1})$ such that:

$$\underbrace{\frac{1}{\Delta t}(\boldsymbol{u}_h^{k+1}, \varphi_i)}_{\text{Inertia}} + \underbrace{a(\boldsymbol{u}_h^{k+1}, \varphi_i)}_{\text{Diffusion}} + \underbrace{c(\boldsymbol{u}_h^{k}, \boldsymbol{u}_h^{k+1}, \varphi_i)}_{\text{Linearized Conv.}} + \underbrace{b(\varphi_i, p_h^{k+1})}_{\text{Pressure}} = \underbrace{F^{k+1}(\varphi_i) + \frac{1}{\Delta t}(\boldsymbol{u}_h^{k}, \varphi_i)}_{\text{Known Terms}}$$

The Algebraic System

The discretization leads to the **linear system**:

$$\begin{bmatrix} \frac{1}{\Delta t}M + \mathcal{A} + \mathcal{C}[\boldsymbol{u}^k] & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{k+1} \\ p^{k+1} \end{bmatrix} = \begin{bmatrix} F^{k+1} + \frac{1}{\Delta t}M\boldsymbol{u}^k \\ 0 \end{bmatrix}$$

- **M (Mass):** Time derivative contribution.
- $\mathcal{A}$ **(Stiffness):** Diffusion term.
- $\mathcal{C}[\boldsymbol{u}^k]$ **(Convection):** State-dependent linearized transport.
- **B** / B^T **(Divergence / Gradient):** Incompressibility constraint and pressure gradient.

Implementation with deal.II

- **Library:** Use of `deal.II` for mesh management and FEM.
- **Dimension Agnostic:** C++ Templates allow the same code for both 2D and 3D.
- **Parallelization:** MPI & Trilinos to handle the partition of the mesh and the distributed sparse matrices and vectors.

Parameter Management

- **ParameterHandler:** Built-in class to read simulation settings from external `.prm` files.
- **Flexibility:** Allows modifying Re, Δt, or the preconditioner without recompiling.

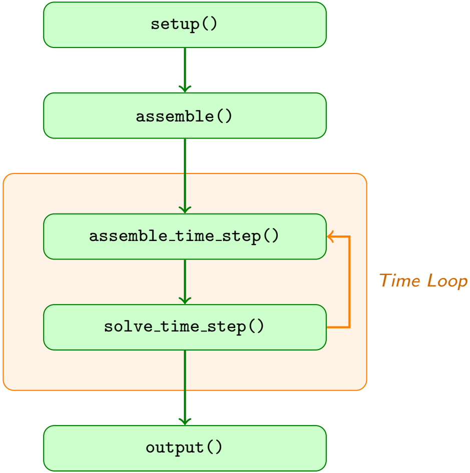
Example `parameters.prm`

```
subsection Physical properties
  set Viscosity          = 0.001
  set T final            = 10.0
  set Time step          = 0.01
end

subsection Numerical parameters
  set Velocity degree      = 2
  set Pressure degree      = 1
end

subsection Mesh
  set Mesh file = cylinder.msh
end
```

Solver Lifecycle

Algorithm Flow:

- `setup()`: Mesh, MPI & DoFs.
- `assemble()`: pre-compute M, A, B (constant) to optimize performance.
- assemble_time_step(): update $\mathcal{C}[u^k]$ and the RHS at each temporal iteration.
- `solve_time_step()`: execution of the GMRES solver using block-preconditioning
- `output()`: Save VTU files.



Preconditioners

- **The Challenge:** The Navier-Stokes saddle-point problem is highly ill-conditioned. The resulting linear system is solved using the GMRES algorithm.
- **The Solution:** Implementation of preconditioning strategies to accelerate convergence.

Implemented Preconditioners:

- **SIMPLE:**
  - Acts as an effective fractional-step solver within the Krylov subspace iteration.
  - Computationally cheaper per iteration, requiring a single GMRES solve for the momentum matrix.
- **Yosida:**
  - Provides an algebraic approximation of the exact Schur complement.
  - Requires two GMRES solves per application, making it potentially more expensive per iteration.
  - Offers enhanced robustness and a reduction in the total number of outer Krylov iterations.

2D Results, TC1

- **Parameters:** Steady flow, $Re = 20$, $U_m = 0.3$ m/s.
- **Inflow Condition:** Parabolic profile:

$$U(0, y) = \frac{4U_m y(H - y)}{H^2}, \quad V = 0$$

- **Flow Physics:**
  - Velocity increases around the cylinder (mass conservation).
  - Pressure drops along x-axis (viscous dissipation).
- **Coefficients (C_D, C_L):** Brief transitory phase → constant values (expected for low Re).
- **Validation:** Simulation successfully matches benchmark outcomes.

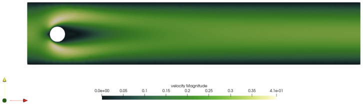


Figure: Velocity in test case 1

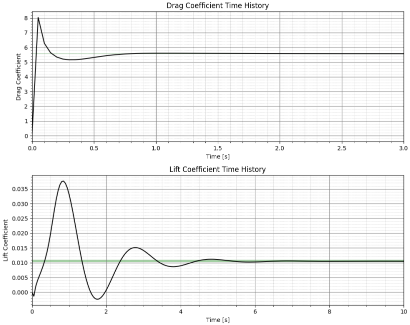


Figure: Test case 1 results

2D Results, TC2

- **Parameters:** Unsteady flow, $Re = 100$, $U_m = 1.5$ m/s, $T = 10$ s.
- **Inflow Condition:** Constant parabolic profile:

$$U(0, y, t) = \frac{4U_m y(H - y)}{H^2}, \quad V = 0$$

- **Flow Physics:**
  - Flow is unsteady, leading to the formation of von Kármán vortexes.
- **Coefficients (C_D, C_L):**
  - Transitory phase $\rightarrow$ periodic oscillation phase.
  - Lift magnitude increases until ≈ 4.5 s, then reaches a stable equilibrium.
- **Validation:** Results match benchmark bounds. Strouhal number (St) evaluated via FFT on the lift signal.

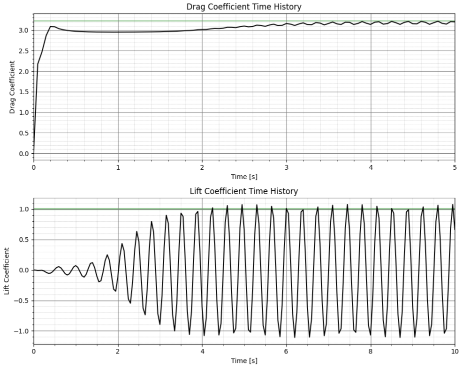


Figure: Test case 2 results

2D Results, TC2

- **Tolerance Bands:** Simulation results are compared against benchmark bounds, highlighted in green.
- **Strouhal Number (St):** Characterizes the oscillating flow mechanism:

$$St = \frac{Df}{U_m}$$

  where $D = 0.1$ m is the cylinder diameter and $U_m = 1.0$ m/s is the average velocity.
- **FFT Analysis:** A Fast Fourier Transform (FFT) is applied to the lift signal to pass from the time domain to the frequency domain.
- **Frequency Peak:** This operation extracts the exact separation frequency f. The resulting peak falls correctly within the bands calculated from the benchmark.

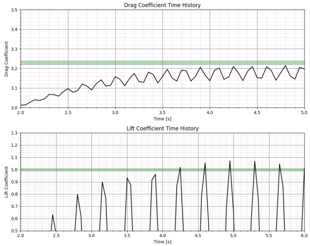


Figure: Zoom on tolerance bands

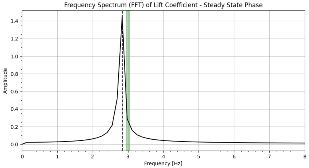


Figure: FFT of the lift coefficient

2D Results, TC3

- **Parameters:** Time-varying flow, $0 \leq Re(t) \leq 100$, $U_m = 1.5$ m/s, interval $0 \leq t \leq 8$ s.
- **Inflow Condition:** Sinusoidal velocity profile:

$$U(0, y, t) = \frac{4U_m y(H - y)\sin(\frac{\pi t}{8})}{H^2}, \quad V = 0$$

- **Flow Physics:** Variable inlet velocity leads to flow acceleration, followed by overall deceleration from $t = 4$ s.
- **Von Kármán** vortex street triggers only when instantaneous Re exceeds the critical shedding threshold.
- **Validation:** Maximum peaks (C_{Dmax}, C_{Lmax}) properly match the expected benchmark tolerance bands.

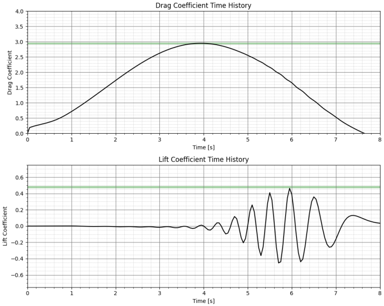


Figure: Test case 3 results

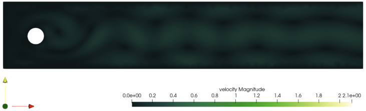


Figure: Velocity distribution at $t = 8\ s$

Error Analysis

- **Methodology:** Since the benchmark provides tolerance bands, the reference value (ϕ_{ref}) is taken as the arithmetic mean of the lower and upper bounds.
- **Relative Error Computation:**

$$Err_{\%} = \frac{|\phi_{sim} - \phi_{ref}|}{|\phi_{ref}|} \times 100$$

.

- **Lift & Strouhal Sensitivity:** Lift (C_L) and Strouhal number (St) are more challenging to predict accurately. They are highly sensitive to wake dynamics resolution.
- **Numerical Dissipation:** Any numerical dissipation dampens the von Kármán vortex street, leading to peak errors in TC2 ($\approx 7.5\%$ for C_L and $\approx 5.4\%$ for St).

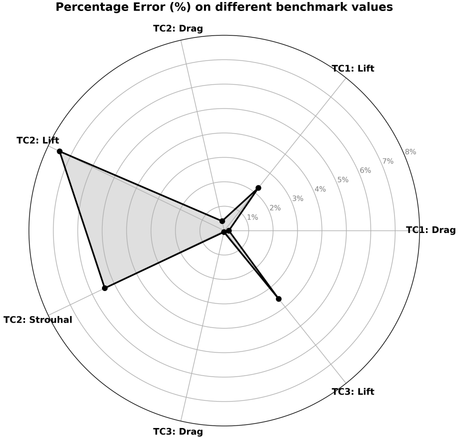


Figure: Radar error distribution