Chapter 1

Two Fibonaccis

$fib(n)$:
  if $n = 0$ or $n = 1$ **then**
    return n
  else
    return $fib(n-1) + fib(n-2)$
  end if

We can state a recurrence for this algorithm:

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) + O(1) \\ &\geq fib(n-1) + fib(n-2) \\ &= fib(n) \end{aligned}$$

$fib2(n)$:
  if $n = 0$ **then**
    return 0
  end if
  create array $f[0..n]$
  $f[0] \leftarrow 0, f[1] \leftarrow 1$
  for $i \leftarrow 2$ to n **do**
    $f[i] \leftarrow f[i-1] + f[i-2]$
  end for
  return $f[n]$

Addition of two numbers in the preceding algorithm takes constant time until the values exceed the maximum value that can be stored in a word. After which, we need to consider how values of arbitrary length are added.

Chapter 2

Integer Multiplication

Let us consider an integer X which is composed of X_L which are the leftmost bits of X, and X_R which are the rightmost bits of X.

$$X = X_L|X_R$$

We can multiply integers X, Y as follows:

$$\begin{aligned} XY &= (2^{n/2}X_L + X_R)(2^{n/2}Y_L + Y_R) \\ &= 2^n X_L Y_L + 2^{n/2}X_L Y_R + 2^{n/2}X_R Y_L + X_R Y_R \\ &= 2^n X_L Y_L + 2^{n/2}(X_L Y_R + X_R Y_L) + X_R Y_R \end{aligned}$$

Which gives the recurrence

$$\begin{aligned} T(n) &= 4T(n/2) + O(n) \\ &\leq 4T(n/2) + cn \\ &\leq 4(4T(n/4) + cn/2) + cn \\ &\leq 4(4(4T(n/8) + cn/4) + cn/2) + cn \\ &\leq 64T(n/8) + cn(1 + 2 + 4) \\ &... \\ &\leq 4^i T(n/2^i) + cn(1 + 2 + ... + 2^{i-1}) \end{aligned}$$

Where i is the number of times we can divide n by 2, or $log_2 n$.

$$\begin{aligned} T(n) &\leq 4^{log_2 n}T(n/2^{log_2 n}) + cn(1 + 2 + ... + 2^{log_2 n - 1}) \\ &\leq n^{log_2 4}T(n/n^{log_2 2}) + cn\sum_{i=0}^{log_2 n - 1} 2^i \\ &\leq n^2 T(1) + cn2^{log_2 n} \\ &\leq n^2 T(1) + cnn^{log_2 2} \\ &\leq n^2 T(1) + cn^2 \\ &\leq n^2(T(1) + c) \\ &\leq n^2(O(1) + c) \\ &\leq O(n^2) \end{aligned}$$

Can we do better? Yes.

We need: $X_L Y_L, X_R Y_R$, and $X_L Y_R + X_R Y_L$

Observe:

$$\begin{aligned} &(X_L + X_R)(Y_L + Y_R) - X_L Y_L - X_R Y_R \\ &= X_L Y_L + X_R Y_L + X_L Y_R + X_R Y_R - X_L Y_L - X_R Y_R \\ &= X_R Y_L + X_L Y_R \end{aligned}$$

Since we must compute $X_L Y_L$ and $X_R Y_R$ anyway, this saves us an entire multiplication. Reducing our recurrence from $T(n) = 4T(n/2)+O(n)$ to $T(n) = 3T(n/2)+O(n)$.

We can solve this new recurrence as follows:

$$
\begin{aligned}
T(n) &= 3T(n/2) + O(n) \\
&\leq 3T(n/2) + cn \\
&\leq 3(3T(n/4) + cn/2) + cn \\
&\leq 3^i T(n/2^i) + cn(1 + 3/2 + ... + (3/2)^{i-1}) \\
&\leq 3^{log_2 n} T(n/2^{log_2 n}) + cn(1 + 2 + ... + (3/2)^{log_2 n - 1}) \\
&\leq n^{log_2 3} T(1) + cn \sum_{i=0}^{log_2 n - 1} (3/2)^i \\
&\leq n^{log_2 3} T(1) + cn(3/2)^{log_2 n} \\
&\leq n^{log_2 3} T(1) + cn n^{log_2 (3/2)} \\
&\leq n^{log_2 3} T(1) + cn n^{log_2 3 - log_2 2} \\
&\leq n^{log_2 3} T(1) + cn n^{log_2 3 - 1} \\
&\leq n^{log_2 3} T(1) + cn n^{log_2 3} n^{-1} \\
&\leq n^{log_2 3} (T(1) + cn/n) \\
&\leq n^{log_2 3} (T(1) + c) \\
&\leq O(n^{log_2 3})
\end{aligned}
$$

Chapter 3

Solving Recurrences

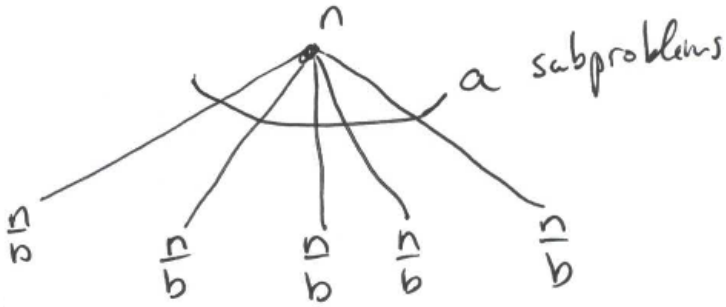
3.1 The Master Theorem

If $T(n) = aT(\frac{n}{b}) + O(n^d)$, then

$$T(n) = \begin{cases} O(n^d) & \text{if} d > log_b a \\ O(n^d logn) & \text{if} d = log_b a \\ O(n^{log_b a}) & \text{if} d < log_b a \end{cases}$$

3.2 Recursion Tree

We can reason about a recurrence of the form: $T(n) = aT(\frac{n}{b}) + f(n)$ where $a \geq 0, b > 0$ with the following recursion tree:



This tree has the following properties:

1. The number of nodes at level i: a^i
2. Work done at each node of level i: $f(\frac{n}{b^i})$
3. Number of levels: $log_b n$
4. Number of leaves: $n^{log_b a}$

We can use this information to solve the recurrence:

$$\begin{aligned} T(n) &= \sum_{i=0}^{log_b n} (\#\text{ nodes at level } i)(\text{work done at level } i) \\ &= \sum_{i=0}^{log_b n} a^i f(\frac{n}{b^i}) \end{aligned}$$

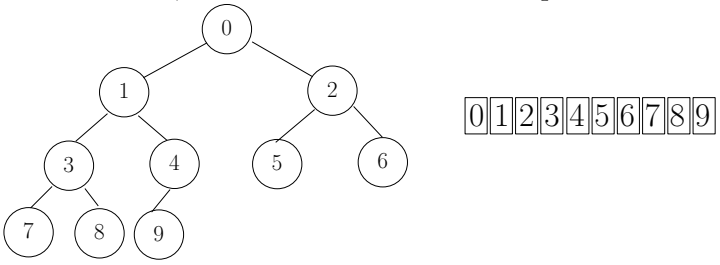
Chapter 4

Heaps

For the purposes of generality, instead of referring to elements that are "greater than" or "less than" others, we will simply say that they are "better than" or "worse than" others. For any particular ordering, the "best" element is desired first. A Heap is a binary tree that satisfies the following properties:

- The root of a heap is better than its two children (the heap property)
- The children of the root are also heaps
- A heap is a complete binary tree (only the last level may not be full, and all elements in the last level are on the left)

A heap differs from most binary trees in that it provides no particular ordering of the elements, but rather guarantees that the best element is at the root. Further, while heaps are usually discussed and defined as binary trees, they need not be implemented as such. A heap can in fact be implemented as an array with no performance reduction. To do so, we can implement a simple indexing. If an element is at the ith index in the array, it's left and right child are at the $2i + 1$th and $2i + 2$th indexes respectively. By placing the root at the 0th index, all others follow. This is depicted below.



Heaps are commonly used to implement priority queues, because they do the minimum amount of work required to keep track of the best element.

4.1 How to Implement a Heap

A heap must support the following operations

- $best()$: returns the best element in the heap
- $pop - best()$: removes the best element in the heap
- $insert(x)$: inserts x into the heap

From the definition of a heap, we know the we can easily implement $best()$ by returning the root of the heap, which should take $O(1)$ time. However the other operations are less obvious.

To $insert(x)$ recall that a heap must be complete, therefore, if a new element is added, it must be added to the left-most available space in the last row of the heap. However, the heap property may now be violated. If the new element x is worse than its parent y, the heap property is satisfied and we may stop. However if it is not satisfied we may swap x with y and recurse on x's new position. This works because we know that y is better than all of x's children, because the heap property was satisfied before x was added. Further, because x is better than y, it is also better than all of y's children. However x may still be better than its new parent, so we must recurse. If x is the new best element, it will eventually reach the root. Because heaps are complete, this operation will take $O(\log n)$ time, as this is the height of the heap.

To $pop - best()$, we may simply remove the root, however this completely destroys the entire heap. Instead, we will swap the root with the bottom-left-most element, y. Now removing the best element leaves us with a still complete tree. However the heap property has likely been violated once more. If y is better than both its children, then we may stop. However, if not, we shall swap y with its largest child and recurse on $y's$ new position. Because the element we swap with y is better than both y and the other child, the heap property has been satisfied for this sub-heap. However, the heap property may still be violated for $y's$ new sub-heap, so we must do this again, until y is the root of a valid heap. Once

more, this operation requires $O(\log n)$ time, as it must at worse traverse the entire height of the tree.

Therefore, a heap may support $best()$ in $O(1)$ time, $insert(x)$ in $O(\log n)$ time, and $pop - best()$ in $O(\log n)$ time.

4.2 Building a Heap

Now that we can support all the operations that a heap must implement, it would be nice to be able to actually construct one given a list of n elements. A naive approach is to simply call $insert(x)$ on every element in the list. However, since $insert(x)$ requires $O(\log n)$ time, this will require $O(n \log n)$ time. These seems pretty bad, considering one can find the best element in a list by brute force in $O(n)$ time. Can we achieve this? Instead of building the heap top down with $insert$, we can build it from the bottom up. Remark that a single element is a valid heap. If we were to try to build from the bottom up, we would first take the last $n/2$ elements in the list. All of these elements are their own valid heaps, and we therefore do not need to do anything to them. To add the next $n/4$ elements, we simply perform the procedure we did in $pop - best$ to bring the fix the fact that y might be violating the heap property, because we know all the elements below it are valid heaps. By repeating this process until we are just adding the last element, we will construct the entire heap.

Because we are doing very little work for the majority of the elements, we end up doing only $O(n)$ work over all, which is optimal, as this is the amount of time required to find the best element.

4.3 Heapsort

Another nice property of a heap is that once a heap has been implemented it provides a very simple procedure to be used to sort elements. A simple algorithm to do this to construct a heap on the list and then simply return and then pop the best element over and over until there are no more. In fact, since our algorithm for building the heap is in-place and takes $O(n)$ time, and our remove method leaves the best element at the end of the array, by simply building a heap on the input array and calling $pop - best()$ n times, we will be left with a reverse sorted array in $O(n \log n)$ time. This algorithm is particularly excellent because it requires no extra space, runs deterministically, and is worst-case optimal.

Chapter 5

Selection

Consider the following problem: given an array A of n elements, output the i-th smallest element of A.

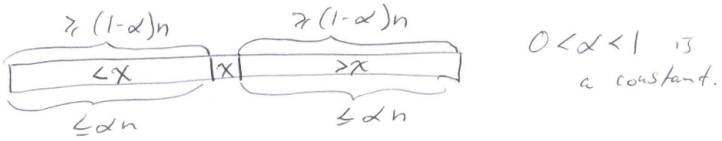
As a simple first solution, we can sort A and then return $A[i]$. Since sorting takes $O(nlogn)$ time and returning $A[i]$ takes $O(1)$ time, this solution takes $O(nlogn) + O(1) = O(nlogn)$ time.

But it should be easy to see that we can do better in specific cases like $i = 1$ or $i = n$. Simply iterate once over the array and store the minimum ($i = 1$) or maximum ($i = n$) value. Since looking at a particular element of the array takes $O(1)$ time, and we look at all n elements, this takes total $n \cdot O(1) = O(n)$ time.

We can also tell that this is optimal, because we know that to determine the i-th element, we need to look at all n elements in the array, so we have a lower bound of $\Omega(n)$ time.

But is this possible in general, for any value of i? Yes.

Suppose in linear time we can find element x such that



x is somewhere around the middle of the array, and is preceded only by elements smaller than x, and followed only by elements larger than x. We also know that there are $\geq (1-\alpha)n$ and $\leq \alpha n$ elements both before and after x.

We can calculate this x as follows:

1. Split A into groups of 5. There will be $\frac{n}{5}$ of these groups.
2. Compute the median m_j of each group M_j for $1 \leq j \leq \frac{n}{5}$.
3. Compute the median x of $m_1, m_2, ..., m_{n/5}$.

It should be clear that step 1 takes constant time, step 2 takes constant time for each group of constant size and $O(n)$ time total for all $\frac{n}{5}$ groups, and step 3 takes $T(n/5)$ time.

Claim 5.1. This x has the properties we needed above.

Proof. We know that $\frac{1}{2}$ of m_j are smaller than x, and since there are $\frac{n}{5} m_j$s, we know $\frac{n}{10}$ of m_j are $\leq x$.

So for each m_j where $m_j \leq x$

- there are 3 elements that are $\leq m_j$
- so 3 (or more) elements are $\leq x$

□

Now we must put x into its position in the array using partitioning. As a side note, partitioning is used in quicksort.

1. Find x, put it at the end
2. Partition elements around x
3. Put x into its proper position

We now have an x that satisfies the properties we needed, and it is properly located at position q in A. We are left with 3 cases:

1. If $i = q$: x is the i-th element of A.
2. If $i < q$: recurse on the subarray which is $< x$
3. If $i > q$: recurse on the subarray which is $> x$, and $i \leftarrow i - q$

This last step gives a recurrence of $T\left(\frac{7n}{10}\right)$ in the worst case because at least $\frac{3n}{10}$ elements in A are smaller than x.

5.1 Analysis

We now have the following recurrence:

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n)$$

Claim 5.2. $T(n) \leq cn$

Proof.

$$\begin{aligned} T(n) &= dn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \\ &\leq dn + c\frac{n}{5} + c\frac{7n}{10} \\ &= dn + \frac{9}{10}cn \\ &= cn\left(\frac{d}{c} + \frac{9}{10}\right) && \leq cn \end{aligned}$$

As long as

$$\begin{aligned} \frac{d}{c} + \frac{9}{10} &\leq 1 \\ \frac{d}{c} &\leq \frac{1}{10} \\ 10d &\leq c \end{aligned}$$

□

5.2 What is special about 5?

First of all, we need an odd number for there to be a median. Secondly, notice:

$$\frac{1}{5} + \frac{7}{10} = \frac{9}{10} < 1$$

Dividing into groups of 3 doesn't work because:

$$T(n) = O(n) + T(n/3) + T(2n/3) = \Theta(nlogn)$$

Dividing into groups of 7 actually does work because:

$$T(n) = O(n) + T(n/7) + T(5n/7) = \Theta(n)$$

Chapter 6

Union-Find

Chapter 7

Graphs

A graph is a set of vertices V and edges E. An edge is a pair of vertices which are said to be connected. For example:

$$V = \{A, B, C\}$$
$$E = \{(A, B), (B, C)\}$$

Which can be represented more visually:

Note that the circles are vertices and the lines are edges connecting vertices to one another.

Sometimes an edge is directional, meaning (A, B) connects A to B, but not B to A. We say such an edge is incoming on B and outgoing on A. This is represented visually by an edge with an arrow at one end, indicating the direction:

Generally when we say graph, we mean undirected graph and will specify directed graph or digraph.

The number of edges connecting a vertex v is called the degree of v or $deg(v)$. In a digraph, the number of incoming edges of v is the in-degree or $in - deg(v)$. Similarly, the number of outgoing edges is the out-degree or $out - deg(v)$.

Chapter 8

Depth-First Search