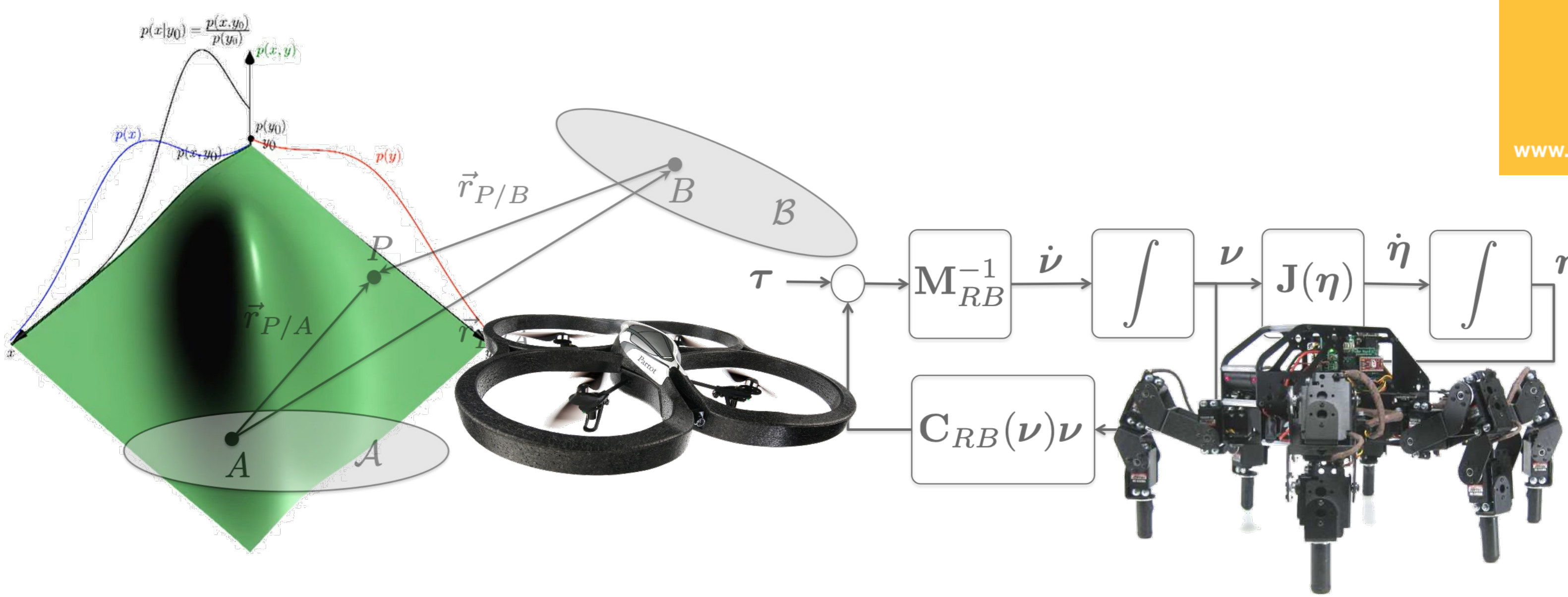
MCHA4100
Mechatronic Systems



THE UNIVERSITY OF NEWCASTLE AUSTRALIA

FACULTY OF ENGINEERING AND BUILT ENVIRONMENT

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Rotations and SO(3)

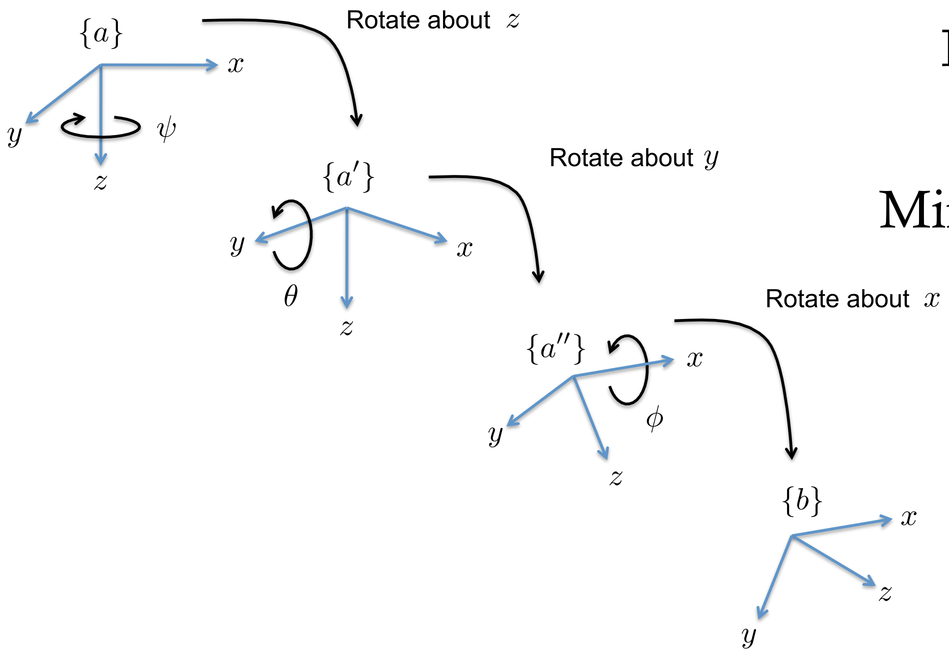
Pose and SE(3)

Vehicle kinematics

Rotations and SO(3)

Euler Angles (Yaw, Pitch, Roll)

The angle-axis representation is over-parameterised, alternatively we can use a set of Euler angles, which correspond to 3 consecutive rotations (Z-Y-X):



$$\begin{aligned}\mathbf{R}(\boldsymbol{\Theta}) &= \mathbf{R}_z(\psi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_x(\phi)\\ &= e^{\psi\mathbf{S}(\mathbf{e}_3)}e^{\theta\mathbf{S}(\mathbf{e}_2)}e^{\phi\mathbf{S}(\mathbf{e}_1)}\end{aligned}$$

Mind the order of multiplication

The vector of Euler angles that take $\{a\}$ into the orientation of $\{b\}$ will be denoted:

$$\boldsymbol{\Theta}_b^a \triangleq [\phi, \theta, \psi]^\mathsf{T}$$

From RPY Euler angles to SO(3)

Multiplying out the rotation matrices yields

$$
\begin{aligned}
\mathbf{R}(\boldsymbol{\Theta}) &= \mathbf{R}_z(\psi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_x(\phi) \\
&= e^{\psi \mathbf{S}(\mathbf{e}_3)} e^{\theta \mathbf{S}(\mathbf{e}_2)} e^{\phi \mathbf{S}(\mathbf{e}_1)} \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}
\end{aligned}
$$

$$
\mathbf{R}(\boldsymbol{\Theta}) = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}
$$

From SO(3) to RPY Euler angles

Let

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Then, one of the possible inverse maps from SO(3) to RPY Euler angles is given by

$$\phi = \operatorname{atan2}(R_{32}, R_{33}), \quad \theta = \operatorname{atan2}\left(-R_{31}, \sqrt{R_{32}^2 + R_{33}^2}\right), \quad \psi = \operatorname{atan2}(R_{21}, R_{11})$$

Note that we don't use $\theta = -\sin^{-1} R_{31}$, since $|R_{31}|$ may be numerically greater than one when $\theta \approx \pm\pi/2$. Mathematically, these are equivalent, since

$$\tan\theta = \frac{-R_{31}}{\sqrt{R_{32}^2 + R_{33}^2}} = \frac{\sin\theta}{\sqrt{\cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi}} = \frac{\sin\theta}{\cos\theta}$$

Other types of Euler angles

There are 12 non-trivial permutations of axis order

Proper Euler angles:

- ZXZ
- XYX
- YZY
- ZYZ (Robotics)
- XZX
- YXY

Tait-Bryan angles:

- XYZ
- YZX
- ZXY
- XZY
- ZYX (Nautical, aerospace)
- YXZ

The other axis permutations can't span SO(3), e.g., XXY→XY, YZZ→YZ, ZZZ→Z, since *consecutive rotations about the same axis commute*.

$$\mathbf{R}_i(\mu_1)\,\mathbf{R}_i(\mu_2) = \mathbf{R}_i(\mu_1 + \mu_2)$$

$$e^{\mu_1 \mathbf{S}(\mathbf{e}_i)} e^{\mu_2 \mathbf{S}(\mathbf{e}_i)} = e^{(\mu_1+\mu_2)\mathbf{S}(\mathbf{e}_i)}$$

Unit quaternions

We can construct unit quaternions from angle-axis parameters

$$\mathbf{q} = \begin{bmatrix} \eta \\ \boldsymbol{\epsilon} \end{bmatrix} = \begin{bmatrix} \cos\frac{\mu}{2} \\ \mathbf{n}\sin\frac{\mu}{2} \end{bmatrix} \qquad \mathbf{q}^\mathsf{T}\mathbf{q} = \eta^2 + \boldsymbol{\epsilon}^\mathsf{T}\boldsymbol{\epsilon} = \cos^2\frac{\mu}{2} + (\mathbf{n}^\mathsf{T}\mathbf{n})\sin^2\frac{\mu}{2} = 1$$

$$\begin{aligned}
\mathbf{R}(\mathbf{n},\mu) &= \mathbf{I} + \mathbf{S}(\mathbf{n})\sin\mu + \mathbf{S}^2(\mathbf{n})(1-\cos\mu) \\
&= \mathbf{I} + 2\,\underbrace{\mathbf{S}(\mathbf{n})\sin\frac{\mu}{2}}_{\mathbf{S}(\boldsymbol{\epsilon})}\underbrace{\cos\frac{\mu}{2}}_{\eta} + 2\mathbf{S}^2(\mathbf{n})(1-\underbrace{\cos^2\frac{\mu}{2}}_{\eta^2})
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}(\eta,\boldsymbol{\epsilon}) &= \mathbf{I} + 2\eta\mathbf{S}(\boldsymbol{\epsilon}) + 2\mathbf{S}^2(\boldsymbol{\epsilon}) \\
&= \boldsymbol{\epsilon}\boldsymbol{\epsilon}^\mathsf{T} + (1-\boldsymbol{\epsilon}^\mathsf{T}\boldsymbol{\epsilon})\mathbf{I} + 2\eta\mathbf{S}(\boldsymbol{\epsilon}) + \mathbf{S}^2(\boldsymbol{\epsilon}) \\
&= \boldsymbol{\epsilon}\boldsymbol{\epsilon}^\mathsf{T} + \eta^2\mathbf{I} + 2\eta\mathbf{S}(\boldsymbol{\epsilon}) + \mathbf{S}^2(\boldsymbol{\epsilon}) \\
&= \boldsymbol{\epsilon}\boldsymbol{\epsilon}^\mathsf{T} + (\eta\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}))^2 \\
&= \begin{bmatrix} -\boldsymbol{\epsilon} & \eta\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}) \end{bmatrix} \begin{bmatrix} -\boldsymbol{\epsilon} & \eta\mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon}) \end{bmatrix}^\mathsf{T}
\end{aligned}$$

There is a 2-to-1 mapping from quaternions to rotation matrices since **q** and –**q** represent the same rotation.

SO(3) parameterisations

Parameters	Parameter topology	Rotation matrix
Native	$\mathsf{SO}(3)$	$\mathbf{R}$
Axis-angle	$\mathsf{S}^2 \times \mathsf{S}^1$	$\mathbf{R}(\mathbf{n}, \mu) = \exp(\mu \mathbf{S}(\mathbf{n}))$
Exponential	$\mathbb{R}^3$	$\mathbf{R}(\mathbf{w}) = \exp(\mathbf{S}(\mathbf{w}))$
Euler angles (RPY)	$\mathsf{S}^1 \times \mathsf{S}^1 \times \mathsf{S}^1$	$\mathbf{R}(\phi, \theta, \psi) = e^{\psi \mathbf{S}(\mathbf{e}_3)} e^{\theta \mathbf{S}(\mathbf{e}_2)} e^{\phi \mathbf{S}(\mathbf{e}_1)}$
Cayley	$\mathbb{R}^3$	$\mathbf{R}(\mathbf{w}) = (\mathbf{I} - \mathbf{S}(\mathbf{w}))(\mathbf{I} + \mathbf{S}(\mathbf{w}))^{-1}$
Unit quaternion	S^3	$\mathbf{R}(\eta, \boldsymbol{\epsilon}) = [-\boldsymbol{\epsilon},\, \eta \mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon})][-\boldsymbol{\epsilon},\, \eta \mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon})]^\mathsf{T}$

- Minimal (3-parameter) representations suffer from singular alignment—the mathematical analog of physical gimbal lock
  - Euler (RPY): $\theta = (k \pm \frac{1}{2})\,\pi,\ \forall k \in \mathbb{Z}$
- Non-minimal representations require auxiliary constraints to be satisfied
  - Angle-axis: axis vector has unit length (1 constraint)
  - Quaternion: parameter vector has unit length (1 constraint)
  - Native: special orthogonal matrix (6 constraints)
- Other quirks
  - Angle-axis and quaternions are double covers of SO(3)

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Kinematic differential equation

Given $\mathbf{R} \in \mathsf{SO}(n)$, we have from orthogonality

$$\mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

Differentiating w.r.t. time

$$\begin{aligned}
\dot{\mathbf{R}}^\mathsf{T}\mathbf{R} + \mathbf{R}^\mathsf{T}\dot{\mathbf{R}} &= \mathbf{0} \\
\underbrace{(\mathbf{R}^\mathsf{T}\dot{\mathbf{R}})^\mathsf{T}}_{\boldsymbol{\Omega}^\mathsf{T}} + \underbrace{\mathbf{R}^\mathsf{T}\dot{\mathbf{R}}}_{\boldsymbol{\Omega}} &= \mathbf{0} \\
\boldsymbol{\Omega}^\mathsf{T} &= -\boldsymbol{\Omega}
\end{aligned}$$

This shows that $\mathbf{R}^\mathsf{T}\dot{\mathbf{R}} = \boldsymbol{\Omega} = \mathbf{S}(\boldsymbol{\omega}) \in \mathfrak{so}(n)$ is a skew symmetric matrix.

The KDE is given by

$$\boxed{\dot{\mathbf{R}} = \mathbf{R}\,\mathbf{S}(\boldsymbol{\omega})}$$

Kinematic differential equation

Express rotation dynamics in terms of rotation parameters

$$\dot{\mathbf{R}}_b^n(\mathbf{\Theta}) = \mathbf{R}_b^n(\mathbf{\Theta})\mathbf{S}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b) \implies \dot{\mathbf{\Theta}}_b^n = \mathbf{T}_K(\mathbf{\Theta}_b^n)\,\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b$$

RPY Euler angles:

$$\mathbf{\Theta} = \begin{bmatrix}\phi \\ \theta \\ \psi\end{bmatrix} \qquad \mathbf{T}_K(\mathbf{\Theta}) = \begin{bmatrix}1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta}\end{bmatrix}, \quad \cos(\theta) \neq 0$$

Quaternions:

$$\mathbf{\Theta} = \begin{bmatrix}\eta \\ \boldsymbol{\epsilon}\end{bmatrix} \qquad \mathbf{T}_K(\mathbf{\Theta}) = \frac{1}{2}\begin{bmatrix}-\boldsymbol{\epsilon} & \eta\mathbf{I} - \mathbf{S}(\boldsymbol{\epsilon})\end{bmatrix}^\mathsf{T}$$

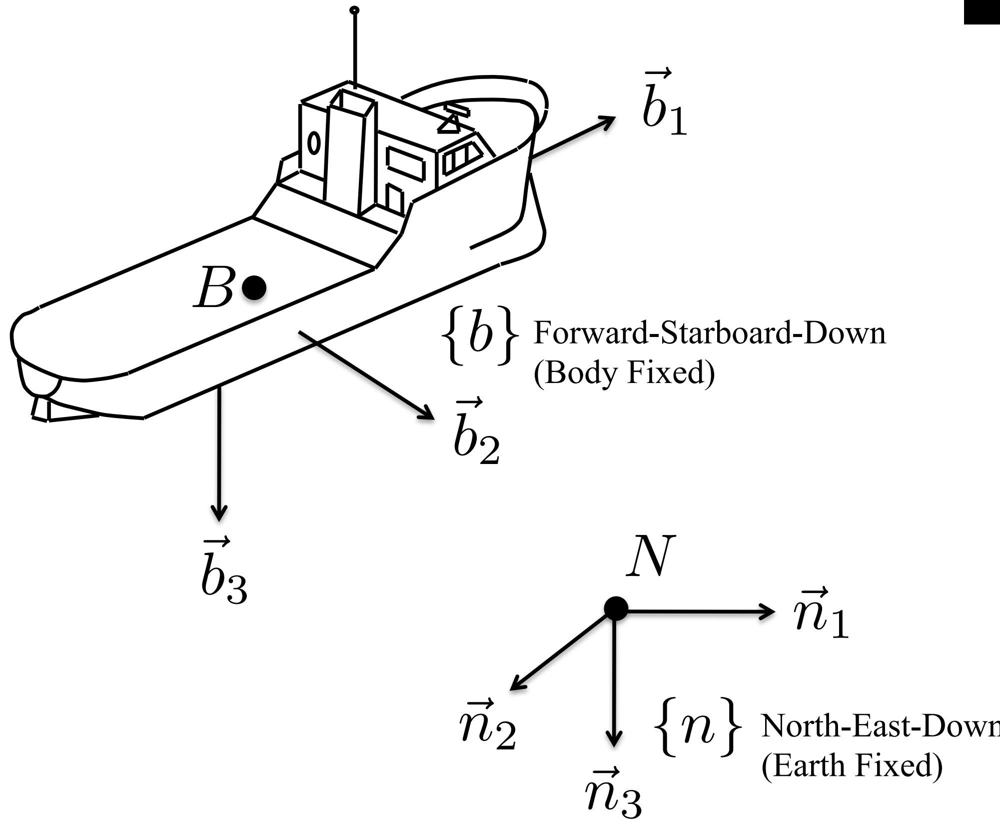
See Renton (2014, §A.11.5) for the derivation.

Vehicle pose and $SE(3)$

Vehicle position-orientation (pose)

$$\mathbf{r}^n_{B/N} \triangleq [N,\ E,\ D]^T$$

$$\boldsymbol{\Theta}^n_b \triangleq [\phi,\ \theta,\ \psi]^T$$



Position-orientation vector:

$$\boldsymbol{\eta} \triangleq \begin{bmatrix} \mathbf{r}^n_{B/N} \\ \boldsymbol{\Theta}^n_b \end{bmatrix} = [N,\ E,\ D,\ \phi,\ \theta,\ \psi]^T$$

Kinematic differential equation

SO(3) kinematic differential equation

$$\mathbf{R}_b^n \in \mathsf{SO}(3), \qquad \mathbf{S}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b) \in \mathfrak{so}(3)$$

$$\boxed{\dot{\mathbf{R}}_b^n = \mathbf{R}_b^n \mathbf{S}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b)}$$

SE(3) kinematic differential equation

$$\mathbf{T}_b^n = \begin{bmatrix} \mathbf{R}_b^n & \mathbf{r}_{B/N}^n \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \in \mathsf{SE}(3), \qquad \mathbf{V}^b = \begin{bmatrix} \mathbf{S}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b) & \mathbf{v}_{B/N}^b \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

$$\dot{\mathbf{T}}_b^n = \mathbf{T}_b^n \mathbf{V}^b$$

$$\begin{bmatrix} \dot{\mathbf{R}}_b^n & \dot{\mathbf{r}}_{B/N}^n \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_b^n & \mathbf{r}_{B/N}^n \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b) & \mathbf{v}_{B/N}^b \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}$$

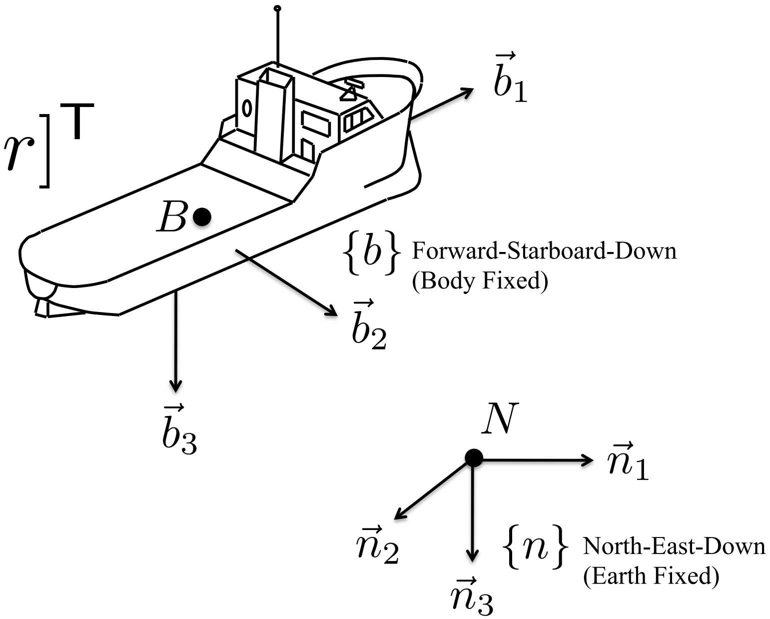
$$\dot{\mathbf{R}}_b^n = \mathbf{R}_b^n \mathbf{S}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^b)$$

$$\dot{\mathbf{r}}_{B/N}^n = \mathbf{R}_b^n \mathbf{v}_{B/N}^b$$

Vehicle body-fixed velocity

Body-fixed velocity vector:

$$\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}^b_{B/\mathcal{N}} \\ \boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = [u,\ v,\ w,\ p,\ q,\ r]^\mathsf{T}$$



Vehicle trajectory:

$$\boldsymbol{\eta}(t) = \boldsymbol{\eta}(t_0) + \int_{t_0}^{t} \dot{\boldsymbol{\eta}}(t')\, dt'$$

$$\dot{\boldsymbol{\eta}} = \mathbf{J}_K(\boldsymbol{\eta})\,\boldsymbol{\nu}$$

Vehicle KDE

$$\dot{\boldsymbol{\eta}} = \mathbf{J}_K(\boldsymbol{\eta})\,\boldsymbol{\nu}, \qquad \mathbf{J}_K(\boldsymbol{\eta}) = \begin{bmatrix} \mathbf{R}_b^n(\boldsymbol{\Theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_K(\boldsymbol{\Theta}) \end{bmatrix}$$

$$\mathbf{R}_b^n(\boldsymbol{\Theta}) = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi c_\phi s_\theta \\ s_\psi c_\theta & c_\psi c_\phi + s_\phi s_\theta s_\psi & -c_\psi s_\phi + s_\psi c_\phi s_\theta \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$

$$\mathbf{T}_K(\boldsymbol{\Theta}) = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & \frac{s_\phi}{c_\theta} & \frac{c_\phi}{c_\theta} \end{bmatrix}, \quad \cos(\theta) \neq 0$$

Summary of vehicle variables

Variable	Name	Frame	Units
N	North position	Earth-fixed	m
E	East position	Earth-fixed	m
D	Down position	Earth-fixed	m
ϕ	Roll angle	-	rad
θ	Pitch angle	-	rad
ψ	Yaw angle	-	rad
u	Surge speed	Body-fixed	m/s
v	Sway speed	Body-fixed	m/s
w	Heave speed	Body-fixed	m/s
p	Roll rate	Body-fixed	rad/s
q	Pitch rate	Body-fixed	rad/s
r	Yaw rate	Body-fixed	rad/s
$\mathbf{r}^n_{B/N} = [N,\ E,\ D]^\mathsf{T}$	Position vector	Earth-fixed	
$\mathbf{v}^b_{\mathcal{B}/\mathcal{N}} = {}^{\mathcal{N}}\dot{\mathbf{r}}^b_{\mathcal{B}/\mathcal{N}} = [u,\ v,\ w]^\mathsf{T}$	Linear-velocity vector	Body-fixed	
$\boldsymbol{\Theta} = [\phi, \theta, \psi]^\mathsf{T}$	Euler-angle vector	-	
$\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}} = [p, q, r]^\mathsf{T}$	Angular-velocity vector	Body-fixed	
$\boldsymbol{\eta} = [(\mathbf{r}^n_{B/N})^\mathsf{T}, \boldsymbol{\Theta}^T]^\mathsf{T}$	Position-orientation vector	-	
$\boldsymbol{\nu} = [(\mathbf{v}^b_{\mathcal{B}/\mathcal{N}})^\mathsf{T}, (\boldsymbol{\omega}^b_{\mathcal{B}/\mathcal{N}})^\mathsf{T}]^\mathsf{T}$	Body-fixed velocity vector	Body-fixed	