Capital Allocation Project

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Abstract

This project aims to study the impact of assets dependency on the capital allocation in a portfolio. In particular, we will make a portfolio composed by 3 assets (U,V,X). Then, we will make a study on how the diversification benefit changes by changing the dependency model. The final result shows how the choice of dependency model is crucial to determine the diversification benefit.

1 THEORY

1.1 Setup

We consider a portfolio Z of three risks represented by the random variables U,V,X defined in a probability space (Ω, A, P) (their realizations will be denoted by u, v, x).

We first aggregate U and V: $U + V = Y$
Then we aggregate Y and X: $Y + X = Z$
This is done by considering 3 different situations:

- Independency between the 3 assets
- Dependency of the 3 assets through Clayton (Survival) copula
- Dependency of the 3 assets through Gaussian copula

1.2 Defining risk

"Risk is a concept that denotes the precise probability of specific eventualities. Technically, the notion of risk is independent from the notion of value and, as such, eventualities may have both beneficial and adverse consequences. However, in general usage the convention is to focus only on potential negative impact to some characteristic of value that may arise from a future event."

The notion of risk is often confused with uncertainty. The seminal work of Frank Knight (1921) makes a clear distinction between concept of risk and concept of uncertainty. He defines "risk" as the randomness with knowable probabilities (so, is a measurable uncertainty), while "uncertainty" is the randomness with unknowable probabilities (unmeasurable). In this sense, risk is measurable and thus manageable (hence the concept of risk management), the uncertainty does not.

Risk is also defined as the chance that an outcome or investment's actual gains will differ from an expected outcome or return. The relationship among risk and return is fundamental not only in financial terms but also in the insurance field. The greater the amount of risk an investor (or an insurer) is willing to take, the greater the potential return. From these definitions, we can understand that risk is not only a source of danger or loss but also of profit, especially profit (return) and therefore it is essential to find the means to measure it (which risk management does).

1.3 Risk measure

Risk can be measured in terms of probability distributions. However sometimes, it's useful to express it with one number. This number is represented by a measure form that can express the risk in terms of capital amount.

This measure should have the following properties:

1. Scalable (twice the risk should give a twice bigger measure)

2. Ranks risks correctly (bigger risks get bigger measure)

3. Allows for diversification (aggregated risks should have a lower measure). This leads to the concept of sub-additivity: if two risks are added together, the total risk should be at most equal to the sum of both.

$$\rho(\sum_{i=1}^{n} X_i) \leq \sum_{i=1}^{n} \rho(X_i) \tag{1}$$

4. Translation invariance (proper treatment of riskless cash flows).

5. Relevance (non-zero risks get non-zero risk measures)

The measure taken in account in risk management are:

- Standard Deviation: It is used to only account the average fluctuations.

- Value at Risk(VaR): it measures the position of alpha percentile (99% for example).

$$VaR_{\alpha} = Inf\{l \in R : P(l > L) \leq 1 - \alpha\} \tag{2}$$

- Expected Shortfall (ES or TVaR): it is the weighted average VaR beyond the alpha threshold

$$ES_{\alpha}(L) = \frac{1}{1-\alpha} \int_{\alpha}^{1} VaR_u du \tag{3}$$

Since we want to measure the extreme risks, VaR and TVaR are more appropriate.

1.4 Capital allocation

Before talking about the most appropriate capital allocation for our risk portfolio, it is interesting to understand what is really meant by capital allocated to a certain risk (pricing the portfolio). To begin to understand how the concept of allocated capital has evolved, we need to move into the insurance field, defining what is called the insurance premium. It is nothing more than

$$P = E[L] + \eta\rho(L) + e \tag{4}$$

Where L is the random variable representing the losses, ρ is the capital using a risk measure, η is the risk loading factor and e the costs. But this formula works as long as my premium does not depend on the loss. Moreover, it completely excludes the portfolio effect and the payout patterns of the losses.

At this point in the literature the point of view in pricing at risk has been changed: the basic idea is to price the expected profit rather than the loss, where the profit is defined as (introducing the time of money) the net present value of our Return

$$X = NPV(R) = NPV(P - L - e) = \sum_{i=0}^{n} \frac{P_i - L_i - e_i}{1 + r_i} \tag{5}$$

Where P is the premium to pay, L is the loss, e the expense, r is the risk free rate corresponding i time payment of the cash flow.

All these passages are useful because we got our new random variable X that we will consider in our model and now we can move on to capital allocation.

Let us consider a portfolio Z, consisting of a number of risks X_i s.t.:

$$Z = \sum_{i=0}^{m} X_i \tag{6}$$

What is an appropriate amount of profit? Below, we list the requirements that an appropriate amount of profit must have:

- The profit expectation, E[X], should be positive.
- It should cover the cost of capital to be paid back to investors
- It should cover the operation expense
- It should include a safety loading which should have the following properties:
    1. The higher the risk, the higher the loading
    2. The higher the dependence with the portfolio the higher the loading because, if the dependence is high, it will be more difficult to diversify risks.
    3. The longer it takes to develop ultimately (the last payment) the more capital is needed

Now, suppose that our portfolio is supported by a Risk-Adjusted-Capital. An allocation of capital K_i to the risk, requires a technical premium s.t.:

$$E[X_i] = \eta\tau_i K_i \tag{7}$$

where τ is the duration of the risk an $\eta$is the cost of capital. Ki is the capital that we want to allocate according to the principles listed above.

1.4.1 Euler Principle

For the capital allocation, Euler principle has a fundamental role in the risk management literature, because it steers the portfolio to optimization. If we want that the capital allocation makes the portfolio to be optimal, when we add a sub portfolio S to Z, the capital to allocate to S is the following:

$$K_s = \frac{d}{dt}|_{t=0} \quad \rho(Z + tS) \tag{8}$$

where tS is an infinitesimal part of S. It's a variational principle, because it consists in varying a little bit the portfolio and through the derivative, we get the capital for S. Moreover, this principle satisfies the following property: if the premium is higher (lower) than the technical one, then a small increase (decrease) of the participation in X will improve (lower) the return on RAC of the entire portfolio.

This is a nice principle, but we have it's not feasible, because we have to reprice the portfolio every time we add some risk. Luckily, in the literature there is a theorem from D. Tasche, that provide us, starting from the Euler Principle, a way to compute the capital allocation in a more feasible way. This is the **D. Tasche Theorem (1999)**

$$K_s = -\,E[S \mid Z \leq F_z^{-1}(\alpha)] \tag{9}$$

The allocated capital for S is equal to the S contribution to the Expected Shortfall (TVaR) of our portfolio Z. So, we allocate capital to a line of business according to its contribution to the bad performance of the whole portfolio.

1.5 A good capital allocation

In order to choose the best risk measure to use for our portfolio, we have to deal not only with the properties we listed above (scalability, sub-additivity etc.), but also with the principles of optimal capital allocation. These principles are:

- Full allocation: The different contributions $p(X_i \mid Z)$ should add up to the total risk $p(Z(u))$.
- Fairness: The contribution of the risk $p(X_i \mid Z)$ should never exceeded the stand-alone risk $p(Xi(u))$.
- Riskless allocation: Adding a risk free asset X_{n+1} such as cash, to the portfolio should not change the capital which are allocated to the risk-carrying business units $(X_1, ..., X_n)$.
- RoRAC compatible: If the RoRAC of risk i is larger than the RoRAC of the overall portfolio. Then this should imply the increasing the weight risk of risk i improves the overall performance of the portfolio.

Now, we can compare the two lists of principles and see which kind of relationships they have with each other. First of all, the "Ranks risks correctly" and the "Relevance" risk measure properties, can be compared with the "Riskless allocation" capital allocation principle. Indeed, if I add to my portfolio an asset that is risk-free, the portfolio allocated capital should not change. This is due to the fact that, since our allocated capital is based on the contribution of the added risk to the portfolio bad performance, the risk-free asset does not affect the total risk in any way. So, the capital will be the same. Moreover, if I add a risky asset (maybe with high dependence with the portfolio), my total risk will increase. Hence, as "Relevance" properties states, with bigger risk we will have a bigger measure and, consequentially, more capital to allocate.
The second comparison we can make, regards the "sub-additivity" property and the "fairness" principle. They state the same thing: the sum of risks in the portfolio can never exceed the singular risk portfolio. If not, it does not make any sense to aggregate them, because we do it in order to get some diversification benefit.
The last property, RoRAC compatibility (Return on Risk-Adjusted-Capital). It states that if I add risk whose RoRAC is higher than the one of the portfolio, it would makes sense to increase the weight of this risk within the

portfolio, because this would mean improving its performance. I have a return that is better comparing with the capital allocated.

1.6 Risk diversification and Copulas

When we talk about a portfolio of risks, we obviously mean a number of risks aggregated together. The aggregation of these risks is convenient because it produces the so-called risk diversification. It leads to a reduction of the total risk and thus of the risk-adjusted-capital which is allocated to the portfolio. If the risks are independent of each other, the diversification effect will be very high. Conversely, the greater the correlation (or dependency) between the individual risks, the lower the diversification effect will be. This is because if the worst-case scenario occurs for a particular risk, the same scenario could occur for another risk, since the correlation is high.

Let us consider a portfolio Z, composed by a certain number of risks X_i the diversification benefit is defined as:

$$D = 1 - \frac{\rho(Z}{\sum_{i=1}^{n} \omega_i \rho(X_i)} \tag{10}$$

where ρ is the risk measure used to compute the Risk Adjusted Capital and w is the weight associated to each asset.
However, risks are rarely completely independent. Now, let's inspect the concept of dependence between two risks. We know that there is a linear dependence and a non-linear one. The linear dependence is computed through Pearson Correlated Coefficients and is the following:

$$\rho(X, Y) = \frac{COV(X, Y}{\sqrt{\sigma^2(X)\sigma^2(Y)}} \tag{11}$$

where COV(X,Y) is the covariance between X and Y defined as:

$$E[XY] - E[X] * E[Y] \tag{12}$$

Unfortunately, the dependence is often non-linear. This is the case in which we can apply **copulas** (= generalized dependence structure).

A copula is a multivariate cumulative distribution function
$C : [0, 1]^d \to [0, 1]$ whose margins are standard uniform. It's a projection from the sphere to a line.

Sklar's Theorem: Let F be a joint distribution function with margins $F_1, \ldots, F_d$. Then, there exists a copula s.t. for all vectors of d random variables $(X1, \ldots, Xn) \in [-\infty, \infty]$, the distribution of this vector is equal to the copula of all the margins

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)) \tag{13}$$

Conversely, if C is a copula and $F_1, \ldots, F_d$ are uniform and univariate distributions the F defined above is a multivariate distribution function with marginals $F_1, \ldots, F_d$.

Copulas are great tools in order to describe the dependence structure of risks in a portfoflio, because the dependence structure influences greatly the needs for Risk Adjusted Capital and the diversification benefit that we get.

1.6.1 Rank Scatter plot

The rank scatterplot is a plot in which we have the cumulative distribution (or better, the ranks) of our random variables X and Y. It starts from the rank order correlation coefficient uses the ranking of the data, i.e. what position (rank) the data point takes in an ordered list from the minimum to maximum values, rather than the actual data values themselves. The rank scatter plot it's very important because, when there is a dependence in the tail, the value scatterplot looks an explosion and it's very hard to distinguish what is going on, while the rank one provides us with a better point of view about the dependence structure.

1.6.2 Clayton (Survival) Copula

The Clayton copula belongs to the Archimedean Copulas and it takes in account not u and v random variables values, but 1/u and 1/v because, otherwise, the ranks will concentrate on the bottom left part of the rank scatterplot.
The cumulative distribution function of the clayton copula is defined by θ that represents the degree of dependence in the copula. The higher is theta, the higher is the dependence. Hence, the lower the diversification benefit. The Clayton copula is defined as:

$$C(u, v) = max([u^{-\theta} + v^{-\theta} - 1]^{\frac{-1}{\theta}}, 0) \tag{14}$$

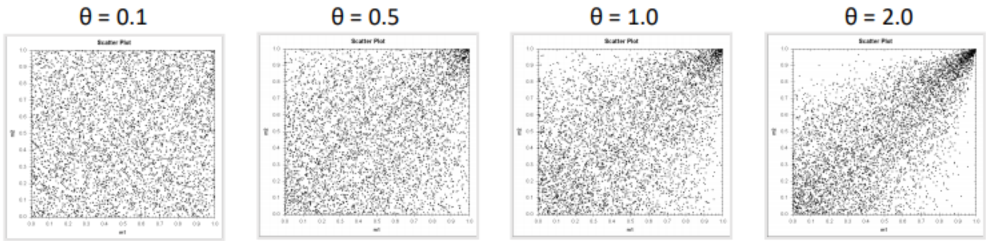


Figure 1: Rank scatter plots examples according to the dependence parameter θ.

where the generator of this copula is the following:

$$\Phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1) \tag{15}$$

and it provides us with the rank of the second random variable given the rank of the first one.
First of all, we draw variates using the conditional distribution approach to choose any margin. Then, we draw two independent uniform random variates u_1 and v_2 and we construct the u_2 (the one that is dependent) using this function:

$$u_2 = [u_1^{-\theta}(v_2^{\frac{-\theta}{1+\theta}} - 1) + 1]^{\frac{-1}{\theta}} \tag{16}$$

If we draw the cumulative distribution of u2, it will look like exactly v2 but it will be now dependent on u1, thanks to the Sklar's Theorem. The pair u_1 and u_2 represents draws from the Clayton copula of parameter θ.

1.6.3 Gaussian Copula

Gaussian copula belongs to the family of elliptical copulas, which gives an elliptical form of the distribution and also gives symmetry at the tails. As we can see, the first difference with Clayton Copula is that in the Gaussian one the points tends to concentrate on the diagonal. Instead, of computing the correlation among the random variables, we compute the correlation among ranks. So, another difference is that this time the size of our loss does not play any role. It is defined as follows:

$$C_R(u_1, ..., u_n) = \frac{1}{|R^{\frac{1}{2}}|} \exp(-\frac{S^T}{2} * (R^{-1} - I) * S) \tag{17}$$

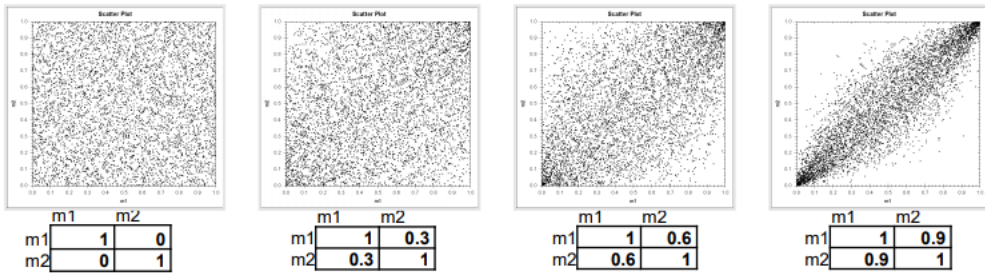


Figure 2: Rank scatter plot examples according to the correlation level

How to generate it? It may be generated by first obtaining a set of correlated normally distributed variates v_1 and v_2 using Choleski's decomposition, and then transforming these to uniform variables $u_1 = \Phi(v_1)$ and $u_2 = \Phi(v_2)$, where Φ is the cumulative standard normal. Then, the pair $(u_1, u_2\ )$ represents draws from the Gaussian copula.

2 Problem

The portfolio Z is composed by the random variables u,v and x, which are lognormally distributed. The portfolio is structured in an hierarchical way, such that the variable Y is the composed by U and V. Then Z is created by unifying X and Y. Given the random variables u,v and x, and given the level of confidence $\alpha = 0.01$, the random variable parameters are set:

U: µ= 2.90, σ= 0.4

V: µ= 2.80, σ= 0.5

X: µ= 2.70, σ= 0.6

In order to parametrize these variables in the correct way, we have to transform µand σaccording to the following equations:

$$\mu_{transf} = \frac{1}{2} \ln \frac{\mu^4}{\mu^2 + \sigma^2} \tag{18}$$

$$\sigma_{transf} = \sqrt{\ln 1 + \frac{\sigma^2}{\mu^2}} \tag{19}$$

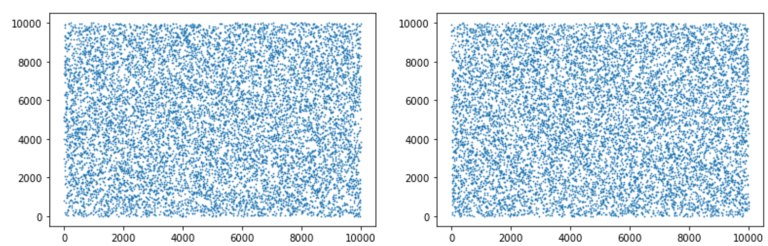
We do that for every couple and we generate the realizations for our 3 random variables X,U,V.

2.1 Independence

The first case it the one in which we have no dependence between our risks. The results in terms of ES standalone, Conditional ES and Average, for U, V, X and Z are showed in the following tabled:

Risks	ES standalone	Conditional ES	Average
U	16.32	10.80	1.77
V	14.29	6.86	1.65
X	13.33	6.72	1.61
Z	24.39	24.39	5.02

Since we are dealing with the case of independence, we don't see any areas where data points are concentrated in the following rank scatterplots.



(a) rank U and V

(b) rank X and Y

Figure 3: Independence Rank scatter plot

In the case of independency, the diversification benefit reaches its maximum level. The formula to compute the diversification benefit is the following:

$$D_\alpha = 1 - \frac{\rho(Z)}{\sum_{i=1}^{n} \omega_i \rho(X_i)} \tag{20}$$

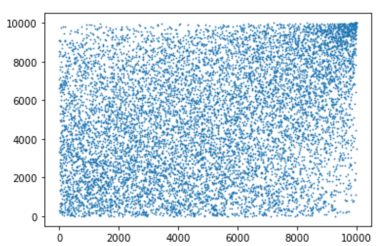
By exploiting this formula and using the $xTVaR_{99\%}$ as risk measure in this case we got a diversification benefit equal to $50,2\%$

2.2 Clayton (Survival) copula

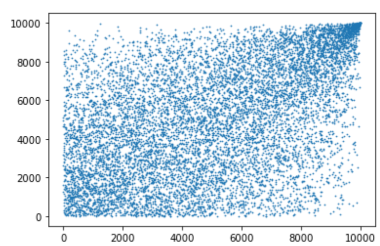
In this scenario we aggregate firstly U and V ($\theta = 0.5$) and then Y and X ($\theta = 1$) via a Survival Clayton copula. We compute the same analysis that we have explained above. The results are the following:

Risks	ES standalone	Conditional ES	Average
U	17.58	15.13	1.76
V	15.64	13.44	1.70
X	13.50	12.14	1.63
Z	40.71	40.71	5.02

As we can see, even though random variable means are still quite the same of the previous case (independence), the Conditional Expected Shortfall is higher. The reason why is that the variables are linked on the extreme tail thanks to mirrored Clayton Copula. This means that if a certain value of the r.v. U is high, the probability that a value of V is high as well, is high. This is the reason why the values in the tail are much larger than before for both the variables U, V and X, and for the portfolio Z. In the rank scatter plots below, we see that the data points are concentrated in the upper right-hand corner, because the Clayton Copula models the dependencies in the tail of the distribution. Moreover, the scatterplot between X and Y shows a higher accumulation of points on the top-right corner. This is due to the higher value of the θ parameter with respect to the left scatterplot.



(a) rank U and V



(b) rank X and Y

Figure 4: Clayton Copula Rank scatter plot

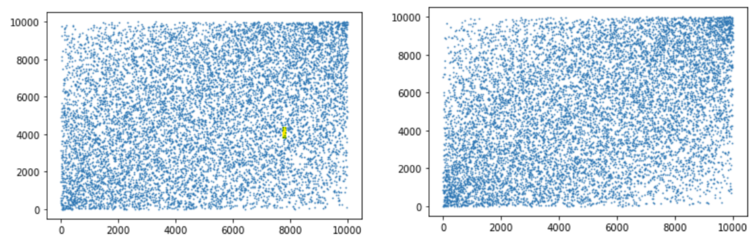
As we know, by introducing dependency between variables, the diversification benefit decreases. This is because when the worst-case scenario occurs for a given variable, it is much more likely, compared to independence or lower dependence, that the dependent variable will also be in the same scenario. Indeed, our diversification benefit in this case is 14.43%.

2.3 Gaussian Copula (Rank Correlation)

In this scenario we aggregate the parameters U and V by using the Spearman's correlation between the variables U and V of the previous scenario (once linked via survival Clayton copula). We do the same for X and Y (using the Spearman's correlation matrix computed from the variables X and Y of the survival scenario). Then we compute the same analysis that we have explained above. The results are the following:

Risks	ES standalone	Conditional ES	Average
U	19.44	11.86	1.83
V	14.32	7.71	1.67
X	25.09	19.48	2.62
Z	32.98	32.98	6.07

Modelling dependencies with the Gaussian Copula is not exactly like the case with the Clayton Copula. The Gaussian tends to model dependencies on both the tail and the head of the distribution. For this reason, the diversification benefit will certainly never be as high as in the dependency case, but not as low as in the Survival case either. Indeed, moreover, we can notice that, with respect to the Survival case, conditional expected shortfall values are much lower than the standalone ES values. Let's check our rank scatter plot (Figure 5).
From these plots, we can denote a little accumulation of point both at the top-right and at the bottom-left part of the plot. The correlation parameter that we used to model the gaussian copula was not that high (more ore less 0.29). In right rank scatter plot, however we modeled X and Y with a parameter equal to 0.49, and there are clear differencies with respect to left one. Indeed, points are much more accumulated. In this case we got a diversification benefit of 37.5%



(a) rank U and V

(b) rank X and Y

Figure 5: Gaussian Copula Rank scatter plot

3 Conclusions

We have shown how important the choice of dependency model is for aggregating risks in order to allocate capital. Indeed, the capital to be allocated to our portfolio Z changes considerably (here we have simulated data, let us imagine that we have to deal with a real portfolio of risks with real capital). The more dependent the risks are, the higher the overall risk level (of the portfolio). Consequently, the riskier the portfolio, the higher the capital to be allocated. In this study the riskiest scenario was the Survival one, and the capital allocation for Z (that is the conditional ES), is equal to 40.71 (the one for the Gaussian case is equal to 39.05 and the one for the independency scenario is 24.39). At the same time, the Diversification benefit decreases as the portfolio gets riskier. In the Survival case the diversification benefit was equal to 14.43%, much lower than the other 2 cases (37.5% for the Gaussian and 50.2% for the independence). All this evidence proves that it is necessary understanding what kind of relationship lays under the risks with which an insurance company operates.