A QUICK GUIDE TO THE FORMULAS OF MULTIVARIABLE CALCULUS

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1. Analytic Geometry

1.1. Definition of a Vector. A *vector* $\mathbf{v}$ is an n-tuple of real numbers:

$$\mathbf{v} = (v_1, \dots, v_n).$$

Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$, addition and multiplication with a scalar $t \in \mathbf{R}$ are defined by

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (v_1, \dots, v_n) + (w_1, \dots, w_n) = (v_1 + w_1, \dots, v_n + w_n) \\ t \cdot \mathbf{v} &= t \cdot (v_1, \dots, v_n) = (tv_1, \dots, tv_n). \end{aligned}$$

From the definitions, it follows immediately that addition and scalar multiplication of vectors are:

(1) *distributive*: $t \cdot (\mathbf{v} + \mathbf{w}) = t \cdot \mathbf{v} + t \cdot \mathbf{w}$
(2) *associative:* $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
(3) *commutative:* $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

1.2. Scalar Product. The *scalar product* or *dot product* $\mathbf{v} \cdot \mathbf{w}$ is defined by

$$\mathbf{v} \cdot \mathbf{w} = (v_1, \dots, v_n) \cdot (w_1, \dots, w_n) = \sum_{i=1}^{n} v_i w_i.$$

1.3. Properties of the Scalar Product. It follows from the definition that the scalar product is

(1) *linear:* $t \cdot (\mathbf{v} \cdot \mathbf{w}) = (t \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (t \cdot \mathbf{w})$, and $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
(2) *commutative:* $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

1.4. Length and Unit Vectors. The *length* of a vector $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Vectors of length 1 are called *unit vectors*.

1.5. Angle. The *angle* θ between two vector $\mathbf{v}$ and $\mathbf{w}$ is defined implicitly by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta.$$

For two vectors $\mathbf{v}$, $\mathbf{w}$ with $\|\mathbf{v}\|, \|\mathbf{w}\| \neq 0$, the enclosed angle θ is hence given by

$$\theta = \cos^{-1} \frac{\sum_{i=1}^{n} v_i w_i}{\left(\sum_{i=1}^{n} v_i^2\right)^{1/2} \left(\sum_{i=1}^{n} w_i^2\right)^{1/2}}$$

This yields a geometric interpretation of the scalar product: to get $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{w}$ is projected orthogonally onto $\mathbf{v}$, and the length $\|\mathbf{w}\| \cos\theta$ of the projected vector is multiplies with $\|\mathbf{v}\|$.

1.6. **Parallel and Perpendicular Vectors.** Two vectors $\mathbf{v}, \mathbf{w}$ are called

(1) *perpendicular* or *orthogonal* if $\mathbf{v} \cdot \mathbf{w} = 0$. We denote this by $\mathbf{v} \perp \mathbf{w}$.
(2) *parallel* if $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|$. We denote this by $\mathbf{v}\|\mathbf{w}$.

Note that two vectors $\mathbf{v}, \mathbf{w}$ are parallel if and only if there is a scalar $t \in \mathbf{R}$ such that $\mathbf{v} = t \cdot \mathbf{w}$.

The zero vector $\mathbf{0} = (0, \dots, 0)$ is by definition parallel *and* perpendicular to every vector.

1.7. **Lines.** Let $\mathbf{u}, \mathbf{v}$ be two vectors with $\mathbf{v} \neq \mathbf{0}$. Then

$$\mathbf{u} + t \cdot \mathbf{v} = (u_1 + tv_1, \dots, u_n + tv_n);\ t \in \mathbf{R}$$

yields a line in $\mathbf{R}^n$. We say that $\mathbf{v}$ *spans* the line.

1.8. **Planes.** let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ be vectors where $\mathbf{v}$ and $\mathbf{w}$ are *not* parallel. Then

$$\mathbf{u} + s \cdot \mathbf{v} + t \cdot \mathbf{w} = (u_1 + sv_1 + tw_1, \dots, u_n + sv_n + tw_n)$$

We say that $\mathbf{v}$ and $\mathbf{w}$ *span* the plane.

1.9. **Cross Product.** At a point p of any plane in $\mathbf{R}^3$ there is exactly one line perpendicular to the plane. If $\mathbf{v}$ and $\mathbf{w}$ span the plane, a vector spanning this perpendicular line is given by the cross product:

For two vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ the *cross product* $\mathbf{v} \times \mathbf{w}$ is defined by

$$\mathbf{v} \times \mathbf{w} = (v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

Alternatively, the cross product is given by the 3×3 determinant $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

- $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$
- $\mathbf{v}, \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ are oriented according to the right hand rule
- if θ is the angle between$\mathbf{v}$ and $\mathbf{w}$, then $\|\mathbf{v} \times \mathbf{w}| = \|\mathbf{v}\|\|\mathbf{w} \sin\theta$.

The length of the $\mathbf{v} \times \mathbf{w}$ is equal to the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

1.10. **Properties of the Cross Product.**

(1) *linear*: $t \cdot (\mathbf{v} \times \mathbf{w}) = (t \cdot \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (t \cdot \mathbf{w})$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
(2) *anti-commutative:* $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

Another remarkable property of the cross product is the following:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot w.$$

Geometrically, $\|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\|$ is the volume of the parallelepiped given by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.
The notion of a perpendicular vector yields another form of representing a plane in $\mathbf{R}^3$:
Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ be a fixed point in the plane and $\mathbf{n}$ a vector perpendicular to the plane. An arbitrary point $\mathbf{r} = (x, y, z)$ on the plane satisfies

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

Moreover, the equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ defines a plane in $\mathbf{R}^3$ for *any* fixed $\mathbf{n}, \mathbf{v} \in \mathbf{R}^3$, provided $\mathbf{n} \neq \mathbf{0}$.

1.11. **Distances of Points from a Plane in $\mathbf{R}^3$.** Let $\mathbf{n}, \mathbf{r}_0 \in \mathbf{R}^3$ with $\|\mathbf{n}\| = 1$ be given. Then $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)$ yields the distance from the point $\mathbf{r} = (x, y, z)$ to the plane which passes through $\mathbf{r}_0$ and is perpendicular to $\mathbf{n}$. Note that the plane's distance from the oriin in this case is given by $\mathbf{n} \cdot \mathbf{r}_0$.

1.12. **Other Representations of Lines and planes in $\mathbf{R}^3$.** In $\mathbf{R}^3$, the representation of a line $\ell(t) = (x, y, z) = \mathbf{u} + t \cdot \mathbf{v}$ can be solved for t in each coordinate if none of the values v_1, v_2, v_3 are zero:

$$t = \frac{x - u_1}{v_1} \qquad t = \frac{y - u_2}{v_2} \qquad t = \frac{z - u_3}{v_3}$$

This yields another representation of a line in $\mathbf{R}^3$, which is called a *symmetric equation*:

$$\frac{x - u_1}{v_1} = \frac{y - u_2}{v_2} = \frac{z - u_3}{v_3}.$$

The parametric equation representing a plane can be transformed similarly giving the standard equation of a plane:

$$ax + by + cz = d,$$

where the normal vector $\mathbf{n}$ is (a, b, c).

1.13. **Spheres and Cylinders.** The vector notation allows for an elegant description of other geometric objects such as spheres and cylinders.

- For a scalar $r \geq 0$ and a vector $\mathbf{c} \in \mathbf{R}^3$, the equation $\|\mathbf{x} - \mathbf{c}\|^2 = r^2$ yields a *sphere* of radius r centered at $\mathbf{c}$.
- For a scalar $r \geq 0$ and vectors $\mathbf{c}, \mathbf{n} \in \mathbf{R}^3$ with $\|\mathbf{n}\| = 1$, the equation $\|\mathbf{n} \times (\mathbf{x} - \mathbf{c})\|^2 = r^2$ yields a *cylinder* of radius r centered around the line given by $\mathbf{c} + t \cdot \mathbf{n}$.

2. Parameterized Curves in $\mathbf{R}^3$

2.1. **Velocity, Speed, and Acceleration.** Given three differentiable functions $x, y, z : [a, b] \longrightarrow \mathbf{R}$, the vector

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

can be understood as describing a particle moving through space; the particle's position depends on the (time) parameter t. This perception gives rise to the following definitions. We define the

- *velocity* of $\mathbf{r}$ at time t to be $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$
- *speed* of $\mathbf{r}$ at time t to be $\|\mathbf{r}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$
- *acceleration* of $\mathbf{r}$ at time t to be $\mathbf{r}''(t) = (x''(t), y''(t), z''(t))$.

2.2. **Velocity of curves with $\|\mathbf{r}(t)\| = 1$.** Suppose that $\mathbf{r}(t)\| = 1$ for all t for curve $\mathbf{r}(t)$. Differentiating $\|\mathbf{r}(t)\| = 1$, we observe that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t:

$$\|\mathbf{r}(t)\| = 1 \Rightarrow \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \text{ for all } t.$$

2.3. **Smooth Curves.** A curve $\mathbf{r} : [a, b] \longrightarrow \mathbf{R}^3$ is called *smooth* if its speed vanishes at most at the endpoints:

$$|\mathbf{r}'(t)| \neq 0 \text{ for all } t \in (a, b).$$

2.4. **Arclength.** The *arclength* $\ell(a, b)$ of a curve $\mathbf{r} : [a, b] \longrightarrow \mathbf{R}^3$ is given by

$$\ell(a, b) = \int_a^b \|\mathbf{r}'(t)\| \mathrm{d}t$$

A curve$\mathbf{r} : [a, b] \longrightarrow \mathbf{R}$ is called *parameterized with respect to arclength* if $\|\mathbf{r}'(t)\| = 1$ for all $t \in [a, b]$.

2.5. **Curvature.** Let $\mathbf{r} : [a, b] \longrightarrow \mathbf{R}^3$ be parameterized with respect to arclength. Then $\kappa(t) = \|\mathbf{r}''(t)\|$ is called the *curvature* of $\mathbf{r}$ at t. Differentiating the equation $\|\mathbf{r}'(t)\| = 1$ shows that $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are orthogonal for all t. If $\kappa(t) \neq 0$, the vector

$$\mathbf{n}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|}$$

is a well-defined unit vector.

2.6. **Normal and Binormal Vectors.** If $\kappa(t) \neq 0$, we can define the

- *Normal vector* of the curve at time t to be $\mathbf{n}(t)$
- *Binormal vector* of the curve at time t to be $\mathbf{b}(t) = \mathbf{r}'(t) \times \mathbf{n}(t)$

The binormal vector is obviously a unit vector, so we can apply the same reasoning as before to see that $\mathbf{b}(t)$ and $\mathbf{b}'(t)$ are orthogonal. On the other hand, differentiating $\mathbf{b}(t) = \mathbf{r}'(t) \times \mathbf{n}(t)$ we get:

$$\mathbf{b}'(t) = \mathbf{r}''(t) \times \mathbf{n}(t) + \mathbf{r}'(t) \times \mathbf{n}'(t) = \mathbf{r}'(t) \times \mathbf{n}'(t).$$

Hence $\mathbf{b}'(t)$ is parallel to $\mathbf{n}(t)$.

2.7. **Torsion.** The equation $\mathbf{b}'(t) = \tau(t) \cdot \mathbf{n}(t)$ defines the *torsion* τ of the curve at time t.

2.8. **Frenet Frame.** Whenever $\kappa(t) \neq 0$, the vectors $\mathbf{r}'(t), \mathbf{n}(t), \mathbf{b}(t)$ are mutually orthogonal unit vectors. They span the *Frenet Frame.*

2.9. **Curvature of Curves NOT parmeterized with respect to Arclength.** Let $\mathbf{r} : [a, b] \longrightarrow \mathbf{R}^3$ be *amy* smooth curve. Its curvature $\kappa(t)$ at time t is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

3. Surfaces in $\mathbf{R}^n$

3.1. **Domain and Range.** A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ assigns a unique real value $f(x_1, \ldots, x_n)$ to each point $(x_1, \ldots, x_n)$ of a set D in $\mathbf{R}^n$. The set D is called the *domain* of f, the set $\mathrm{Im} f = \{f(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in D\}$ is called the *image* or *range* of f.

3.2. **Limits.** A real number L is said to be the *limit* of f at $(a, b, \ldots)$ if *for all* sequences $(a_m, b_m, \ldots)$ with $\lim_{m\to\infty} a_m = a$, $\lim_{m\to\infty} b_m = b$, $\ldots$, the following holds:

$$\lim_{m\to\infty} f(a_m, b_m, \ldots) = L.$$

We denote this by

$$\lim_{(x_1, x_2, \ldots) \to (a, b, \ldots)} f(x_1, x_2, \ldots) = L.$$

Equivalently, if for every real number $\epsilon > 0$ there is another real number $\delta > 0$ such that

$$\|(x_1, x_2, \ldots) - (a, b, \ldots)\| < \delta \Rightarrow |f(x_1, x_2, \ldots) - L| < \epsilon.$$

3.3. **Continuity.** A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ with domain D is said to be *continuous* at $(a, b, \ldots) \in D$ if

$$\lim_{(x_1, x_2, \ldots) \to (a, b, \ldots)} f(x_1, x_2, \ldots) = f(a, b, \ldots)$$

3.4. **Partial Derivatives.** Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a function with domain D and $(x_1, \ldots, x_n) \in D$. The *partial derivative* of f at $(x_1, \ldots, x_n)$ *with respect to* x_i is given by the limit

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

3.5. **Differentiability.** A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ with domain D is called *differentiable* at $(x_1, \ldots, x_n) \in D$ if *all* partial derivatives exist *and are continuous* at $(x_1, \ldots, x_n)$.

3.6. **Directional Derivaitve.** Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a function with domain D and $\mathbf{r} \in D$. Suppose that $\mathbf{u}$ is a unit vector in $\mathbf{R}^n$. The *directional derivative* of f at $\mathbf{r}$ in the direction of $\mathbf{u}$ is given by

$$\left. \frac{d}{dt} f(\mathbf{r} + t\mathbf{u}) \right|_{t=0}.$$

3.7. **Gradient Vector.** Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a function with domain D that is differentiable at $\mathbf{r} \in D$. The *gradient vector* ∇f of f at $\mathbf{r}$ is given by

$$\nabla f\big|_{\mathbf{r}} = \left(\left.\frac{\partial f}{\partial x_1}\right|_{\mathbf{r}}, \ldots, \left.\frac{\partial f}{\partial x_n}\right|_{\mathbf{r}} \right)$$

3.8. **Directional Derivatives and the Gradient Vector.** Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a function with domain D that is differentiable at $\mathbf{r} \in D$. Using the chain rule, the directional derivative is given by

$$\frac{d}{dt} f(\mathbf{r} + t\mathbf{u})\Big|_{t=0} = \nabla f\big|_{\mathbf{r}} \cdot \mathbf{u}$$

3.9. **Higher Derivatives and Clairaut's Theorem.** Let $f : \mathbf{R}^n \longrightarrow R$ be a function with domain D and suppose the partial derivatives of f are themselves differentiable. Then differentiating $\dfrac{\partial f}{\partial x_i}$ with respect to x_j is the same as differentiating $\dfrac{\partial f}{\partial x_j}$ with respect to x_i:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

3.10. **Global and Local Extrema.** A function $f : \mathbf{R}^n \longrightarrow R$ with domain D has

- a *global maximum* at $\mathbf{r} \in D$ if $f(\mathbf{x}) \le f(\mathbf{r})$ for all $\mathbf{x} \in D$
- a *global minimum* at $\mathbf{r} \in D$ if $f(\mathbf{x}) \ge f(\mathbf{r})$ for all $\mathbf{x} \in D$
- a *local maximum* at $\mathbf{r} \in D$ if there is a disc R centered at $\mathbf{r}$ such that $f(\mathbf{x}) \le f(\mathbf{r})$ for all $\mathbf{x} \in R$
- a *local minimum* at $\mathbf{r} \in D$ if there is a disc R centered at $\mathbf{r}$ such that $f(\mathbf{x}) \ge f(\mathbf{r})$ for all $\mathbf{x} \in R$

3.11. **Critical Points.** Let $f : D \subset \mathbf{R}^n \longrightarrow \mathbf{R}$ be differentiable. We call a point $\mathbf{r} \in D$ a *critical point* if $\nabla f\big|_{\mathbf{r}} = \mathbf{0}$. If f has an extremum at $\mathbf{r}$, then $\mathbf{r}$ is critical, but the converse is not necessarily true.

3.12. **Functions of Two Variables – Second Derivative Test.** Let $f : D \subset \mathbf{R}^2 \longrightarrow \mathbf{R}$ and its derivatives be differentiable and let $(x_0, y_0) \in D$ be a critical point of f. Let

$$D(x_0, y_0) = \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right) \Bigg|_{(x_0,y_0)}$$

- if $D(x_0, y_0) > 0$ and $\dfrac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} < 0$, then f has a maximum at (x_0, y_0).
- if $D(x_0, y_0) > 0$ and $\dfrac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} > 0$, then f has a minimum at (x_0, y_0).
- if $D(x_0, y_0) < 0$, then f has a saddle at (x_0, y_0).
- if $D(x_0, y_0) = 0$, then the second derivative test gives no information about the nature of the critical point.

3.13. **The Double Integral.** Let $R = [a, b] \times [c, d]$ and let $f : R \longrightarrow \mathbf{R}$ be continuous. The *double integral* of f over R is defined to be

$$\iint_R f(x, y)\ \mathrm{d}A = \lim_{|P| \to 0} \sum_{i,j} (x_i - x_{i-1})(y_j - y_{j-1}) f(x_i^*, y_j^*)$$

where $P = P_{[a,b]} \times P_{[c,d]}$ is a partition of R, $|P| = |P_{[a,b]}| \cdot |P_{[c,d]}|$ is its norm, and $(x_i, y_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

3.14. **Fubini's Theorem.** Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be continuous and $R = [a, b] \times [c, d]$. The double integral is given by

$$\iint_R f(x, y)\ \mathrm{d}A = \int_a^b \int_c^d f(x, y)\ \mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\ \mathrm{d}x\mathrm{d}y$$

3.15. **Level Surfaces.** Let $f(x, y, z) : \mathbf{R}^3 \longrightarrow \mathbf{R}$ be a function of three variables. The *level set* $\{(x, y, z) \mid f(x, y, z) = c\}$ for a constant c generally yields a surface in $\mathbf{R}^3$.

3.16. **Tangent Planes of Level Surfaces.** Consider the level set $\{(x, y, z) \mid f(x, y, z) = c\}$ of a function $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$.

(1) The gradient vector $\nabla f(x_0, y_0, z_0)$ at a point (x_0, y_0, z_0) is perpendicular to the plane tangent to the level surface at (x_0, y_0, z_0).
(2) The tangent plane is given by the equation

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

3.17. **Using Lagrange Multipliers to Find Extrema with Constraints.** let $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ be differentiable. To find the maximum and minimum value of f subject to the constraint

$$g(x, y, z) = c,$$

the gradients ∇f and ∇g must be parallel. An algorithm to find the maximum and minimum values is hence given by:

(1) Find all points (x, y, z) such that $\nabla f = \lambda \nabla g$, for some $\lambda \in \mathbf{R}$, and $g(x, y, z) = c$.
(2) Evaluate f at these points. The largest (smallest) value is the maximum (minimum) of f subject to the constraint $g(x, y, z) = c$.

3.18. **The Triple Integral.** Let $g : \mathbf{R}^3 \longrightarrow \mathbf{R}$ be continuous and $R = [a, b] \times [c, d] \times [e, f]$. The *triple integral* of g over R is defined to be

$$\iiint_R g(x, y, z)\ \mathrm{d}V = \lim_{|P| \to 0} \sum_{i,j,k} (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) g(x_i^*, y_j^*, z_k^*)$$

3.19. **Chain Rule.** Let $\mathbf{r}(t)$ be a smooth curve in $\mathbf{R}^n$ and let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a function of several variables. Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \nabla f \cdot \mathbf{r}'(t).$$

More generally, suppose each of the variables x_i is a function of the variables $t_1, \dots, t_m$, then

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

3.20. **Change of Variables Formula in Two Dimensions.** Let $(x(u, v), y(u, v))$ be a *parameter transformation* with *Jacobian matrix*

$$J = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

which maps $S \subset \mathbf{R}^2$ into $R \subset \mathbf{R}^2$. Then

$$\iint_R f(x, y)\ \mathrm{d}x\mathrm{d}y = \iint_S f(u, v) |\det J|\ \mathrm{d}u\mathrm{d}v$$

3.21. **Polar Coordinates.** The parameter transformation $(x(r, \theta), y(r, \theta)) = (r \cos\theta, r \sin\theta)$ expresses (x, y) in *polar coordinates*. Then

$$\iint_R f(x, y)\ \mathrm{d}x\mathrm{d}y = \iint_S f(r, \theta)\ r\mathrm{d}r\mathrm{d}\theta$$

3.22. **Surface Area.** Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be differentiable. The *surface area* of f over D is given by

$$\iint_R \sqrt{1 + \left[\frac{\partial f}{\partial x}\right]^2 + \left[\frac{\partial f}{\partial y}\right]^2}\ \mathrm{d}x\mathrm{d}y$$

3.23. **Path Integrals.** Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be differentiable and let $\mathbf{r}(t)$ parameterize a smooth curve C in $\mathbf{R}^2$. The *path integral* of f along C is given by

$$\int_C f(\mathbf{r}(t)) \cdot \|\mathbf{r}'(t)\|\ \mathrm{d}t$$

4. Vector Fields

A *vector field* $\mathbf{F}$ assigns to each point in a domain $R \subset \mathbf{R}^n$ a vector in $\mathbf{R}^n$.

4.1. **Line Integrals.** Let $\mathbf{F}$ be a vector field on a domain $R \subset \mathbf{R}^n$ and let C be a smooth curve in R parameterized by $\mathbf{r}(t)$. The *line integral* of $\mathbf{F}$ along C is given by

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ \mathrm{d}t$$

4.2. **Conservative Vector Fields and the Potential Function.** If $\mathbf{F} = \nabla f$ for some function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, then $\mathbf{F}$ is called a *conservative vector field* with *potential function* f.

4.3. **Line Integrals of Conservative Vector Fields.** Let $\mathbf{F} = \nabla f$ be conservative on a domain $R \subset \mathbf{R}^n$. For any continuous curve C in R from $\mathbf{u}$ to $\mathbf{b}$ which is parameterized by $\mathbf{r}(t)$, we have

$$\int_C \mathbf{F} \cdot \mathbf{r}'(t)\ \mathrm{d}t = f(\mathbf{v}) - f(\mathbf{u}).$$

In particular

1. If C is closed then $\int_C \mathbf{F} \cdot \mathbf{r}'\ \mathrm{d}t = 0$.
2. if $\widetilde{C}$ is another continuous curve in R from $\mathbf{u}$ to $\mathbf{v}$ parameterized by $\mathbf{s}(t)$, then $\int_{\widetilde{C}} \mathbf{F} \cdot \mathbf{s}'\ \mathrm{d}t = \int_C \mathbf{F} \cdot \mathbf{r}'\ \mathrm{d}t$. The integral is said to be *path-independent*.

4.4. **Line Integrals of Vector Fields in $\mathbf{R}^2$.** Let $\mathbf{F} = (P, Q)$ be a vector field on a domain $R \subset \mathbf{R}^2$ and let C be a smooth curve in R given by $(x(t), y(t))$. Then

$$\int_a^b \mathbf{F}(P(x(t), y(t)), Q(x(t), y(t))) \cdot (x'(t), y'(t))\ \mathrm{d}t = \int_C P(x, y)\ \mathrm{d}x + \int_C Q(x, y)\ \mathrm{d}y$$

4.5. **Conservative Vector Fields in $\mathbf{R}^2$.** If $\mathbf{F} = (P, Q) = \nabla f$, then by Clairaut's theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

4.6. **Green's Theorem.** Let R be a simply connected region with positively-oriented boundary ∂R. Then

$$\int_{\partial R} P\ \mathrm{d}x + Q\ \mathrm{d}y = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ \mathrm{d}x\mathrm{d}y$$

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