Class 3: Well-Ordering Principle

Schedule

This week, you should read MCS Chapter 2 and MCS Chapter 3 (at least through the end of Section 3.4).

Problem Set 1 is due **Friday at 6:29pm**.

Office hours started Monday. There are upcoming office hours: today (Tuesday) 3:30-5pm and 7:30-9:30pm; Wednesday 10am-1pm, 2:30-4pm, 6:30-9:30pm (all of these are in Rice 436), and Dave has office hours Wednesday 1-2pm (in Rice 507). See the course calendar for the full office hours schedule.

Notes and Questions

Definition. A set is *well-ordered* with respect to an ordering function (e.g., <), if any of its non-empty subsets has a minimum element.

What properties must a sensible ordering function have?

Trichotomy. A relation, $\prec$, satisfies the axiom of trichotomy if exactly one of these is true for all elements a and b: $a \prec b$, $b \prec a$ or $a = b$.

Which of these are well-ordered?

– The set of non-negative integers, comparator $<$.
– The set of integers, comparator $<$.
– The set of integers, comparator $|a| < |b|$.
– The set of integers, comparator if |a| = |b|: $a < b$, else: $|a| < |b|$.
– The set of national soccer teams, comparator winning games.

Prove the set of positive rationals is *not* well-ordered under $<$.

Provide a comparison function that can be used to well-order the positive rationals.

Template for Well-Ordering Proofs (Section 2.2)

To prove that $P(n)$ is true for all $n \in \mathbb{N}$:

1. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} | NOT(P(n))\}$.
2. Assume for contradiction that C is non-empty.
3. By the well-ordering principle, there must be some smallest element, $m \in C$.
4. Reach a contradiction (this is the creative part!). One way to reach a contradiction would be to show $P(m)$. Another way is to show there must be an element $m' \in C$ where $m' < m$.
5. Conclude that C must be empty, hence there are no counter-examples and $P(n)$ always holds.

Example: Betable Numbers. A number is *betable* if it can be produced using some combination of $2 and $5 chips. Prove that all integer values greater than $3 are betable.

Example: Division Property. Given integer a and positive integer b, there exist integers q and r such that: $a = qb + r$ and $0 \leq r < b$.

The whole problem with the world is that fools and fanatics are always so certain of themselves, and wiser people so full of doubts. Bertrand Russell

