Class 12: Review

Schedule

Problem Set 5 is due **Friday at 6:29pm**.

Exam 1 is in class on **Thursday, 6 October**. See Notes 11 for details. Remember to bring a good writing instrument (sharp pencil is recommended) and your page of notes to the exam.

Exam Review

Below are some notations you should understand. We use variables P and Q to represent logical propositions (with **T** or **F** value), A and B and D (domain of discourse) to represent sets, and x represents any mathematical object.

	Logical Operators
P IMPLIES Q, $P \implies Q$	Logical implication: when P is **T**, Q must be **T**
P IFF Q, $P \iff Q$	Double implication: $P \implies Q \vee Q \implies P$
NOT(P), $\neg P$, $\overline{P}$	Logical negation
P AND Q, $P \wedge Q$	Logical conjunction (**T** when both P and Q are **T**)
P OR Q, $P \vee Q$	Logical disjunction (**T** when P or Q is **T** (or both)
P XOR Q, $P \oplus Q$	Exlusive or (**T** when either P or Q is **T**, but not both
	Quantifiers
$\forall x \in A.\, P(x)$	$P(x)$ for *every* element x of the set A
$\exists x \in A.\, P(x)$	$P(x)$ for *at least one* element x of the set A
	Set Operators
$x \in A$	Set membership, A contains the element x
$x \notin A$	Set non-membership, A does not contain x
$A \subseteq B$	A is a subset of B: $\forall x \in A. x \in B$.
$A = B$	set equality: $A \subseteq B \vee BsubseteqA$
$A \cup B$	Set Union: $\forall x. x \in A \cup B \iff x \in A \vee x \in B$.
$A \cap B$	Set Intersection: $\forall x. x \in A \cup B \iff x \in A \wedge x \in B$.
$A - B$	Set Difference: $\forall x. x \in A - B \iff x \in A \wedge x \notin B$.
$\overline{A}$	Set Complement: $\forall x. x \in D. x \in \overline{A} \iff x \notin A$.
$A \times B$	Cartesian Product: $\forall a \in A, b \in B. (a, b) \in A \times B$.
pow(A)	Power Set: $S \in A \iff S \subseteq A$.

Well-Ordering Principle

Remember the well-ordering principle template from Class 3:

To prove that $P(n)$ is true for all $n \in \mathbb{N}$:

1. Define the set of counterexamples, $C ::= \{n \in \mathbb{N} | NOT(P(n))\}$.
2. Assume for contradiction that C is non-empty.
3. By the well-ordering principle, there must be some smallest element, $m \in C$.
4. Reach a contradiction (this is the creative part!). One way to reach a contradiction would be to show $P(m)$. Another way is to show there must be an element $m' \in C$ where $m' < m$.
5. Conclude that C must be empty, hence there are no counter-examples and $P(n)$ always holds.

Since we didn't finish the winning strategy proof in class, I've written it out fully here.

Theorem. Player 1 has a winning strategy in Take-Away if the number of sticks, n is not divisible by 4.

For any induction proof, the first thing we need to do is write the theorem as a predicate on the natural numbers.

$$\forall n \in \mathbb{N}.\, P(n) ::= \text{ Player 1 has a winning strategy if } \exists a \in \{1,2,3\}, \exists k \in \mathbb{N} \text{ such that } n = 4k + a.$$

Base cases: $P(1)$, $P(2)$, $P(3)$.

$P(1)$: If there is 1 stick remaining, Player 1 wins by taking 1 stick.
$P(2)$: If there are 2 sticks remaining, Player 1 wins by taking 2 sticks.
$P(3)$: If there are 3 sticks remaining, Player 1 wins by taking 3 sticks.

Inductive case: Using strong induction, $\forall m \in \mathbb{N}, m \geq 4. (\forall k \in \mathbb{N}, k \leq m.P(k)) \implies P(m+1)$.

Since $m \geq 4$ we can write $m = 4k + b$ for some $k \in \mathbb{N}^{+}$ and $b \in \{0,1,2,3\}$.

We consider four cases for each value of b.

Case 3: $b = 3$. Since $m = 4k + 3$, $m + 1 = 4k + 4 = 4(k + 1)$. Since $m + 1$ is divisible by 4, $P(m + 1)$ holds because the predicate makes no claims when n is divisible by 4.

Cases 0, 1, 2: Since $m = 4k + b$ for $b \in \{0,1,2\}$, we know $m + 1 = 4k + c$ for $c \in \{1,2,3\}$ (since $c = b + 1$ to produce $m + 1$). We need to show $P(m + 1)$, which means showing that player 1 has a winning strategy for $n = 4k + c$, $c \in \{1,2,3\}$. Player 1 takes c sticks, leaving $4k$ sticks.

For the next turn, Player 2 can remove 1, 2, or 3 sticks, leaving $4k - d$ sticks, $d \in \{1,2,3\}$. This can be simplified to $4(k-1) + e$ sticks where $e = 4 - d$ since $4k - 4 + (4 - d) = 4k - d = 4(k - 1) + e$. Hence, after Player 2's turn it will be Player 1's turn with $4(k - 1) + e$ sticks, $e \in \{1,2,3\}$. We know $4(k - 1) + e < m$ and it is not divisible by 4. So, Player 1 has a winning strategy from $P(m + 1)$ since she has a move to make such that no matter what move player 2 makes, it leads to a number that is not divisible by 4 and is less than m, which we know is a position where Player 1 has a winning strategy but strong induction.

Note that $4(k - 1) + e$ is $m - 1$, $m - 2$ or $m - 3$, so we need to know $m \geq 4$ for this to be valid. That's why we needed three base cases!

