Class 2: Proof Methods

Schedule

Before **Friday (tomorrow), 6:29pm**:

- Read, print, and sign the Course Pledge. You should print the PDF version for signing, and submit a scan of your signed pledge using Collab.
- Read Jeremy Kun's *Habits of Highly Mathematical People*.
- Submit a Course Registration Survey (which includes some questions based on Kun's essay).

Next week:

- Before Tuesday's class: Read MCS Chapter 2
- Before Thursday's class: Read MCS Chapter 3
- Due Friday at 6:29pm: **Problem Set 1** (will be posted tomorrow)

Notes and Questions

An integer, z, is **even** if there exists an integer k such that $z = 2k$.

Is this a *definition*, *axiom*, or *proposition*?

An integer, z, is **odd** if there exists an integer k such that $z = 2k + 1$. (Note that there is no connection between the variables used here, and to define even above.)

To prove an implication, $P \implies Q$: 1. assume P. 2. Show chain of logical deductions that leads to Q.

Odd-Even Lemma: If an integer is not even, it is odd. Note: A *lemma* is just a name for a theorem, typically used for proving another theorem.

How should one decide what can be accepted as an axiom, and what must be proven?

What is the purpose of a *proof*? (in cs2102? in software development? in algorithm design?)

Proof from Class

Proposition: If the product of x and y is even, at least one of x or y must be even.

Our goal is to prove the implication, $P \implies Q$, where:

– P is “the product of x and y is even”
– Q is “at least one of x and y must be even”

We will prove the contrapositive. The inference rule,

$$\frac{NOT(Q) \implies NOT(P)}{P \implies Q}$$

will be the last step in our proof. To be able to use this rule, we need to make the antecedent true. So, our goal is to prove $NOT(Q) \implies NOT(P)$.

To prove an implication, we assume the left side and short the right side. So, our proof begins:

1. We prove the contrapositive.
2. Assume NOT(at least one of x and y must be even).
3. By the meaning of negation, this means both x and y are not even.
4. By the “Odd-Even Lemma”, both x and y are odd.
5. By the definition of *odd*, there exist integers k and m such that $x = 2k + 1$ and $y = 2m + 1$.
6. Substituting, $xy = (2k + 1)(2m + 1) = 4mk + 2m + 2k + 1 = 2(2mk + m + k) + 1$.
7. By the closure of addition and multiplication over the integers, there is some integer $r = 2mk + m + k$, so $xy = 2r + 1$.
8. By the definition of *odd*, xy is odd.
9. By the (not proven) “Even-Odd Lemma”, this means xy is not even.
10. So, we have shown NOT(the product of x and y is even).
11. Since we concluded this starting from the assumption (step 2), this proves the implication: NOT(at least one of x and y must be even) $\implies$ NOT(the product of x and y is even).
12. Using the contrapositive inference rule, we can conclude: the product of x and y is even $\implies$ at least one of x and y must be even. ■

Challenge Problem Opportunity. I wasn’t very satisfied with my proof of the "Odd-Even Lemma" in class. I hope a student will come up with a better proof. A proof that passes the "skeptical reader" test and would convince someone with no prior knowledge of odd and even numbers is worth up to 10 bonus points.

Proving "If and Only If"

Strategy: To prove P if and only if Q:

1. Prove $P \implies Q$.
2. Prove $Q \implies P$.

Definition. The *standard deviation* of a sequence of values $x_1, x_2, \ldots, x_n$ is

$$\sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \ldots + (x_n - \mu)^2}{n}}$$

where μ is the mean of the values:

$$\mu ::= \frac{x_1 + x_2 + \ldots + x_n}{n}$$

.

Theorem 1.6.1. The standard deviation of a sequence of values, $x_1, x_2, \ldots, x_n$ is 0 *if and only if* all of the x_i values are equal to the mean of $x_1, x_2, \ldots, x_n$.

The book proves this using a chain of iff implications; prove it using the two-implications strategy.

In physics, your solution should convince a reasonable person. In math, you have to convince a person who's trying to make trouble. Frank Wilczek

