Problem Set 3

Deliverable: Submit your responses as a single PDF file on the collab site before **6:29pm** on **Friday, 16 September**. The PDF you submit can be a scanned handwritten file (please check the scan is readable), or a typeset PDF file (e.g., generated by LaTeX or Word).

Collaboration Policy - Read Carefully

For this assignment, you should work in groups of *one* to *four* students of your choice. The only constraint on teams for PS3 is that if you worked with the same team for PS1 and PS2, you must not work with any of the same people for PS3.

The rest of the collaboration policy is identical to what it was on PS2, and is not repeated here.

Preparation

This problem set focuses on Chapter 3 (especially 3.4-3.6) of the MCS book, and Class 5 and Class 6 (which include some material not in Chapter 3).

Directions

Solve all [HOW MANY] problems on the next [HOW MANY] pages. For maximum credit, your answers should be correct, clear, well-written, and convincing. The problems marked with ($\star$) are believed to be challenging enough that it is no necessary to solve them well to get a "green-star level" grade on this assignment.

Quantified Formulas

For each pair of logical formula given, state if the the formula is *valid* and if it is *satisfiable*. You should provide a brief argument supporting your answer (enough to convince a reader you are not just guessing!), but do not need to provide a thorough proof.

$\mathbb{N}$ represents the non-negative integers = $\{0, 1, 2, \cdots\}$.

1. $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. x = y + 1$. Not valid $x = 1, y = 0$ Satisfiable: $x = 2, y = 1$
2. $\forall x \in \mathbb{N}. \forall y \in \mathbb{N}. \exists z \in \mathbb{N}. z = x + y$. Valid and Satisfiable: natural numbers are closed under addition
3. $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. \exists b \in \{0, 1\}. x = 2y + b$. Odd even Lemma, Valid and Satisfiable
4. $\forall F \in \text{CNF}. \exists G \in \text{3CNF}. F \equiv G$. (This means F and G are logically equivalent.) Valid and Satisfiable. Just add an $\vee F$ to each clause.

Conjunctive Normal Form

5. Write a logical formula in Conjuctive Normal Form that is equivalent to:

$$(A \vee B) \implies C$$

6. Write a logical formula in 3CNF form that is equivalent to:

$$A \vee \overline{B} \vee C \vee (\overline{D} \wedge E)$$

Use as few clauses as possible.

$$(A \vee \neg B \vee \neg y) \wedge (y \vee C \vee \neg z) \wedge (\neg D \vee y \vee z) \wedge (E \vee y \vee z)$$

SAT Solving

For each of the formula, either (1) give a satisfying assignment, or (2) state that it is not satisfiable. (If you follow a smart strategy, these can be solved by hand without a lot of tedious effort. But, you are welcome to use SAT solving programs to solve them, including the simple-sat solver from Class 6.)

7. $(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_2} \vee x_4)$

   $x_1 = True$

8. $(x1 \vee x2 \vee x3) \wedge (x1 \vee x2 \vee \overline{x3}) \wedge (x1 \vee \overline{x2} \vee x3) \wedge (x1 \vee \overline{x2} \vee \overline{x3}) \wedge (\overline{x1} \vee x2 \vee x3) \wedge (\overline{x1} \vee x2 \vee \overline{x3}) \wedge (\overline{x1} \vee \overline{x2} \vee x3) \wedge (\overline{x1} \vee \overline{x2} \vee \overline{x3})$

   Not satisfiable, has $2^3 = 8$ distinct clauses, which is all the possible combinations for three variables.

   So we can consider two cases. First, all the variables are the same: then either the first or last term will be false.

   Second: Two are the same and one is different. WLOG let $x_1, x_2 = True$ and $x_3 = False$. Then the clause $(\neg x_1 \vee \neg x_2 \vee x_3)$ will be false. And this clause will exist for any of these possible combinations because we are using all possible clauses.

Satisfaction

The length of the formula is the number of clauses, and no clause may be repeated. (The order of literals within a clause doesn't matter, so the clauses $(x_1 \vee x_2 \vee x_3)$ and $(x_3 \vee x_1 \vee x_2)$ would count as the same clauses, but $(x_1 \vee x_2 \vee x_3)$ and $(\overline{x_1} \vee x_2 \vee x_3$ are different clauses.)

9. What is the length of the *shortest* unsatisfiable formula involving 3 variables? (That is, each clause involves x_1, x_2, and x_3, and your goal is to show that there exists an unsatisfiable formula of length l, and any formula of length $< l$ is satisfiable.)

   Answer: 8

   There are 8 possible clauses with 3 variables. So as shown in the previous problem, it is impossible to have every combination of be true at once. The strategy when considering a formula of length 7 is to consider the clause that was removed. And set x_1, x_2, x_3 so that the removed clause is false.

   For example, if $(x_1 \vee x_2 \vee x_3)$ is removed, setting every x_i to False would satisfy the formula, because every other clause has at least one $\neg x_i$

   For removing a general clause the same idea applies. Each unique clause differs from every other clause by at least one value. But never zero. So the combination that would make one set of clause False, will never make another one False. Meaning every other one will be true. This means that a formula with 3 variables will always be solvable if it has < 8 clauses.

10. ($\star$) What is the length of the *longest* satisfiable formula involving v variables? An outstanding answer would include a convincing proof that there exists a satisfiable formula with v variables of length l, but no satisfiable formula with v variables of length $l+1$.

