

Type-2 Hesitant Fuzzy Sets

Liu Feng, Fan Chuan-qiang & Xie Wei-he

To cite this article: Liu Feng, Fan Chuan-qiang & Xie Wei-he (2018) Type-2 Hesitant Fuzzy Sets, Fuzzy Information and Engineering, 10:2, 249-259, DOI: [10.1080/16168658.2018.1517977](https://doi.org/10.1080/16168658.2018.1517977)

To link to this article: <https://doi.org/10.1080/16168658.2018.1517977>



© 2018 The Author(s). Published by Taylor & Francis Group on behalf of the Fuzzy Information and Engineering Branch of the Operations Research Society of China & Operations Research Society of Guangdong Province.



Published online: 25 Oct 2018.



Submit your article to this journal [↗](#)



Article views: 70



View Crossmark data [↗](#)

Type-2 Hesitant Fuzzy Sets

Liu Feng^{a,b}, Fan Chuan-qiang^{a,c} and Xie Wei-he^d

^aInstitute of Systems Science, Northeastern University, Shenyang, People's Republic of China; ^bDalian Survey Group of the National Statistics Bureau, Dalian, People's Republic of China; ^cSchool of Science, Liaoning Shihua University, Fushun, People's Republic of China; ^dDepartment of Naval Gun, Dalian Naval Academy, Dalian, People's Republic of China

ABSTRACT

By using type-2 fuzzy sets and hesitant fuzzy sets, type-2 hesitant fuzzy sets are defined and their mathematical structure and characteristics are given. The relations of the structures of type-2 hesitant fuzzy sets and hesitant fuzzy sets are further studied. Consequently, we prove that type-2 hesitant fuzzy sets are the generalization of hesitant fuzzy sets. Type-2 hesitant fuzzy sets may deal with the problem that hesitant fuzzy sets can't have repeated memberships. Otherwise, a part of the special type-2 hesitant fuzzy sets can be changed into discrete type-2 fuzzy sets, but their operations have many differences. On dealing with the fact problems, type-2 hesitant fuzzy sets are the better methods to solve the problems and can be easy to get better results.

KEYWORDS

Type-2 fuzzy sets; hesitant fuzzy sets; type-2 hesitant fuzzy sets; discrete type-2 fuzzy sets

1. Introduction

In 1965, fuzzy sets [1–5] were firstly introduced by L. A. Zadeh and have been widely used in many areas of artificial intelligence and control. Since then several extensions have been developed and are applied to practice and got many good results, such as intuitionistic fuzzy sets [6–7], type-2 fuzzy sets [8–27], and hesitant fuzzy sets [28–32]. Especially, type-2 fuzzy sets [8–27] allow the membership of a given element as a fuzzy set and have been widely researched and application, for example, interval type-2 fuzzy systems [33–35] etc. By now the structures of type-2 fuzzy sets are very perfectly, and have made a lot of the actual application results. At the same time, hesitant fuzzy sets are new theory now, and also permits their membership having a set of possible values. Hesitant fuzzy sets are very similar to the fuzzy sets. However, hesitant fuzzy sets were firstly put forward in 2010, there are a lot of problems and new theories need to be researched, and they affect their practical application. Such as the new theory of type-2 hesitant fuzzy sets that are parallel to the type-2 fuzzy sets have never been bought up by now. Thus in order to in-depth study hesitant fuzzy sets and expand the application scopes of hesitant fuzzy sets, so according to the theory of type-2 fuzzy sets, the new theory of type-2 hesitant fuzzy sets are firstly proposed in this paper. For the next step of study interval type-2 hesitant fuzzy systems and their application, type-2 hesitant fuzzy sets are the first step and the necessary theory foundation.

CONTACT Liu Feng  77325344@qq.com

Hesitant fuzzy sets have been applied in some decision making problems and received some good practical effects [29–32] now. In order to widely use hesitant fuzzy sets, so it is necessary to further in-depth study the theory of hesitate to fuzzy sets. Firstly, On the basis of the basic theories of type-2 fuzzy sets and hesitant fuzzy sets, type-2 hesitant fuzzy sets are firstly defined and their mathematical structure and characteristics are also given. Then the relations of the structures of type-2 hesitant fuzzy sets and hesitant fuzzy sets are further studied. Because of the discrepancy of sets itself, the elements of hesitant fuzzy sets can't be repeated. But the fact has a lot of repeated possibilities. Type-2 hesitant fuzzy sets solve the above defect of hesitant fuzzy sets, thus the new type-2 hesitant fuzzy sets are a bit of promotion of old hesitant fuzzy sets. The formations of the definitions of type-2 hesitant fuzzy sets and discrete type-2 fuzzy sets are very similar but their basic operations have some different points, their basic operations can't be completely transformed each other except for very special circumstances. On dealing with the fact problems, type-2 hesitant fuzzy sets are closer to the fuzzy reality, and can cover more fuzzy information, and are the new methods to solve the problems and can be easy to get better results, and can open up a new path like type-2 fuzzy sets.

In order to do that, the remainders of the paper are organized as follows. In Section 2, the basic theories of hesitant fuzzy sets and fuzzy sets are introduced. In Section 3, type-2 hesitant fuzzy sets are defined and their mathematical structure and characteristics are also given. In section 4, the relations of the structures of type-2 hesitant fuzzy sets and hesitant fuzzy sets are further studied. Section 5 prove that a part of the special type-2 hesitant fuzzy sets can be changed into type-2 hesitant fuzzy sets. Section 6, type- n hesitant fuzzy sets are defined and their mathematical structure and characteristics are also given. Section 7 gives some concluding remarks.

2. Hesitant Fuzzy Sets and Discrete Type-2 Fuzzy Sets

In this section, we review the basic conceptions and operations of fuzzy sets and hesitant fuzzy sets, because they will be used later to discuss type-2 hesitant fuzzy sets.

Definition 2.1: [28] Given a reference set X , a fuzzy set A on X is represented in terms of a function $\mu : X \rightarrow [0,1]$, for $x \in X$.

Definition 2.2: [28] Let X be a reference set, then hesitant fuzzy sets are defined on X in terms of a function h that when applied to X returns a subset of $[0,1]$.

Definition 2.3: [28] Let $M = \{\mu_1, \mu_2, \mu_3, \dots, \mu_N\}$ be a set of N membership functions, then hesitant fuzzy sets associated with M , that is h_M , is defined as $h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}$.

Note: in this paper, we only discuss the kind of hesitant fuzzy sets that are given by definition 3.

Given a hesitant fuzzy set represented by its membership function h , its lower, upper, α -lower and α -upper bound are defined as follows:

- (1) lower bound: $h^-(x) = \min h(x)$;
- (2) upper bound: $h^+(x) = \max h(x)$;

- (3) α -lower bound: $h_{\alpha}^{-}(x) = \{h \in h(x) | h \leq \alpha\}$;
 (4) α -upper bound: $h_{\alpha}^{+}(x) = \{h \in h(x) | h \geq \alpha\}$.

Definition 2.4: [28] Given a hesitant fuzzy set represented by its membership function h , its complement as follows:

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}.$$

Definition 2.5: [28] Given two hesitant fuzzy sets represented by their membership function h_1 and h_2 , their union and intersection that are respectively represented by $h_1 \cup h_2$ and $h_1 \cap h_2$ are defined as follows:

- (1) $(h_1 \cup h_2)(x) = \{h \in (h_1(x) \cup h_2(x)) | h \geq \max(h_1^{-}, h_2^{-})\}$, or equivalently,
 (2) $(h_1 \cup h_2)(x) = \{(h_1(x) \cup h_2(x))_{\alpha}^{+}, \text{ for } \alpha = \max(h_1^{-}, h_2^{-})\}$;
 (3) $(h_1 \cap h_2)(x) = \{h \in (h_1(x) \cap h_2(x)) | h \leq \min(h_1^{+}, h_2^{+})\}$, or equivalently,
 (4) $(h_1 \cap h_2)(x) = \{(h_1(x) \cap h_2(x))_{\alpha}^{-}, \text{ for } \alpha = \min(h_1^{+}, h_2^{+})\}$.

Definition 2.6: [21, 22] A discrete type-2 fuzzy set is one whose primary variable and secondary membership functions are discrete (sampled), and it is represented as:

$$\tilde{A} = \sum_{x \in X} (\mu_{\tilde{A}}(x)/x) = \sum_{x \in X} \left(\left[\sum_{\mu \in J_x} f_x(\mu) \right] / x \right) = \sum_{x \in X} \left(\left[\sum_{k=1}^M f_x(\mu_{xk}) / \mu_{xk} \right] / x \right).$$

In this equation, $+$ also denotes union. Note that μ has been discretized into M values.

3. Type-2 Hesitant Fuzzy Sets

On the basis of the basic theories of type-2 fuzzy sets and hesitant fuzzy sets, type-2 hesitant fuzzy sets are firstly defined, then their mathematical structure and characteristics are also given.

In order to facilitate the following comparison and unify mathematical symbol, the following the size comparison of two-dimensional coordinate points are defined as follows:

Definition 3.1: Suppose (a,b) , (c,d) , (e,f) and $(l,t) \in [0,1] \times [0,1]$, then

- (1) $(a,b) \leq (c,d) \iff a \leq c \text{ and } b \leq d$;
 (2) $(a,b) \geq (c,d) \iff a \geq c \text{ and } b \geq d$;
 (3) $(a,b) = (c,d) \iff a = c \text{ and } b = d$;
 (4) $(e,f) = \max\{(a,b), (c,d)\} = (\max\{a, c\}, \max\{b, d\})$;
 (5) $(l,t) = \min\{(a,b), (c,d)\} = (\min\{a, c\}, \min\{b, d\})$.

Example 3.1: $(3,4) \leq (3,5)$, $(5,7) \geq (4,4)$, $\max\{(3,4), (4,3)\} = (4,4)$, $\min\{(3,4), (4,3)\} = (3,3)$.

Then imitating the definitions and operations of hesitant fuzzy sets, the corresponding conceptions of type-2 hesitate fuzzy sets are step by step given.

Definition 3.2: Let X be a reference set, then type-2 hesitant fuzzy sets are defined on X in terms of a function h that when applied to X returns a subset of $[0,1] \times [0,1]$.

Definition 3.3: Let $M^2 = \{(\mu_1, f_1(\mu_1)), (\mu_2, f_2(\mu_2)), (\mu_3, f_3(\mu_3)), \dots, (\mu_N, f_N(\mu_N))\}$ be a set of N dual membership functions, then type-2 hesitant fuzzy sets ${}^2h_{M^2}^A$ associated with M^2 , that is defined as ${}^2h_{M^2}^A(x) = \bigcup_{(\mu, f(\mu)) \in M^2} \{(\mu(x), f(\mu(x)))\}$.

Note: (1) empty set: ${}^2h_{M^2}^A(x) = \{(0,0)\}$ for all $x \in X$;

(2) full set: ${}^2h_{M^2}^A(x) = \{(1,1)\}$ for all $x \in X$.

Example 3.2: Let X be a reference set, $\forall x \in X$, ${}^2h_{M^2}^A(x) = \{(0.2, 0.6), (0.4, 0.5), (0.6, 0.3)\}$, ${}^2h_{M^2}^B(x) = \{(0.4, 0.7), (0.8, 1.0), (1.0, 0.9)\}$ are two the membership functions of type-2 hesitant fuzzy sets. Then ${}^2h_{M^2}^A, {}^2h_{M^2}^B$ are two type-2 hesitant fuzzy sets on X .

Given a type-2 hesitant fuzzy set represented by its membership function ${}^2h_{M^2}(x)$, and $M^2 = \{(\mu_1, f_1(\mu_1)), (\mu_2, f_2(\mu_2)), \dots, (\mu_N, f_N(\mu_N))\}$, then its lower, upper, α -lower and α -upper bound are defined as follows:

- (1) lower bound: $({}^2h_{M^2})^-(x) = (\mu^-(x), f(\mu^-(x))) = (\min \mu(x), \min f(\mu(x)))$;
- (2) upper bound: $({}^2h_{M^2})^+(x) = (\mu^+(x), f(\mu^+(x))) = (\max \mu(x), \max f(\mu(x)))$.
- (3) (α, β) -lower bound: $({}^2h_{M^2})_{(\alpha, \beta)}^-(x) = \{{}^2h_{M^2} \in {}^2h_{M^2}(x) = \{(\mu(x), f(\mu(x)))\} | \mu(x) \leq \alpha, f(\mu(x)) \leq \beta\}$;
- (4) (α, β) -upper bound: $({}^2h_{M^2})_{(\alpha, \beta)}^+(x) = \{{}^2h_{M^2} \in {}^2h_{M^2}(x) = \{(\mu(x), f(\mu(x)))\} | \mu(x) \geq \alpha, f(\mu(x)) \geq \beta\}$.

Example 3.3: Let X be a reference set, $\forall x \in X$, ${}^2h_{M^2}^A(x) = \{(0.2, 0.6), (0.4, 0.5), (0.6, 0.3)\}$, then $({}^2h_{M^2}^A)^-(x) = (0.2, 0.3)$, $({}^2h_{M^2}^A)^+(x) = (0.6, 0.6)$, $({}^2h_{M^2}^A)_{(0.4, 0.3)}^-(x) = \{(0.4, 0.5)\}$, $({}^2h_{M^2}^A)_{(0.4, 0.3)}^+(x) = \{(0.4, 0.5), (0.6, 0.3)\}$.

Definition 3.4: Given an hesitant fuzzy set represented by its membership function ${}^2h_{M^2}^A(x)$, and $M_2 = \{(\mu_1, f(\mu_1)), (\mu_2, f(\mu_2)), \dots, (\mu_N, f(\mu_N))\}$ its complement as follows:

$$({}^2h_{M^2}^A)^c(x) = \bigcup_{i=1}^N \{(1 - \mu_i(x), 1 - f_i(\mu_i(x)))\}.$$

Proposition 3.1: The complement is involutive, i.e.

$$({}^2h_{M^2}^A)^c)^c = {}^2h_{M^2}^A.$$

Proof: It is easy to be proved, so omitted. ■

Definition 3.5: Given two type-2 hesitant fuzzy sets represented by their membership function ${}^2h_{M^2}^1$ and ${}^2h_{M^2}^2$, their union and intersection that are respectively represented by ${}^2h_{M^2}^1 \cup {}^2h_{M^2}^2$ and ${}^2h_{M^2}^1 \cap {}^2h_{M^2}^2$ are defined as follows:

$(^2h_{M^2}^1 \cup ^2h_{M^2}^2)(x) = \{^2h_{M^2}(x) \in (^2h_{M^2}^1(x) \cup ^2h_{M^2}^2(x)) | ^2h_{M^2} \geq \max((^2h_{M^2}^1)^-, (^2h_{M^2}^2)^-)\}$,
or equivalently,

$$(^2h_{M^2}^1 \cup ^2h_{M^2}^2)(x) = \{(^2h_{M^2}^1(x) \cup ^2h_{M^2}^2(x))_{(\alpha, \beta)}^+, \text{ for } (\alpha, \beta) = \max((^2h_{M^2}^1)^-, (^2h_{M^2}^2)^-)\};$$

$(^2h_{M^2}^1 \cap ^2h_{M^2}^2)(x) = \{^2h_{M^2}(x) \in (^2h_{M^2}^1(x) \cap ^2h_{M^2}^2(x)) | ^2h_{M^2} \leq \min((^2h_{M^2}^1)^+, (^2h_{M^2}^2)^+)\}$,
or equivalently,

$$(^2h_{M^2}^1 \cap ^2h_{M^2}^2)(x) = \{(^2h_{M^2}^1(x) \cap ^2h_{M^2}^2(x))_{(\alpha, \beta)}^-, \text{ for } (\alpha, \beta) = \min((^2h_{M^2}^1)^+, (^2h_{M^2}^2)^+)\}.$$

Example 3.4: Let X be a reference set, $\forall x \in X$, $^2h_{M^2}^A(x) = \{(0.2, 0.6), (0.4, 0.5), (0.6, 0.3)\}$, $^2h_{M^2}^B(x) = \{(0.4, 0.7), (0.8, 1.0), (1.0, 0.9)\}$ are two the membership functions of type-2 hesitant fuzzy sets. Then $(^2h_{M^2}^1 \cup ^2h_{M^2}^2)(x) = \{(0.4, 0.7), (0.8, 1.0), (1.0, 0.9)\}$, $(^2h_{M^2}^1 \cap ^2h_{M^2}^2)(x) = \{(0.2, 0.6), (0.4, 0.5), (0.6, 0.3)\}$.

4. Type-2 Hesitant Fuzzy Sets and Hesitant Fuzzy Sets

Type-2 hesitant fuzzy sets are the generalization of hesitant fuzzy sets. At the same time, type-2 hesitant fuzzy sets keep the operation rules of original hesitant fuzzy sets unchanged.

Proposition 4.1: Hesitant fuzzy sets are the special cases of type-2 hesitant fuzzy sets.

Proof: Suppose the hesitant fuzzy sets associated with M , that is $h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}$, let $M^2 = \{(\mu_1, 1), (\mu_2, 1), (\mu_3, 1), \dots, (\mu_N, 1)\}$ be a set of N dual membership functions, then $^2h_{M^2}(x) = \bigcup_{\mu(x) \in M} \{\mu(x), 1\}$, then the hesitant fuzzy sets are the special case of type-2 hesitant fuzzy sets. ■

Proposition 4.2: Hesitant fuzzy sets are changed into type-2 hesitant fuzzy sets, then new type-2 hesitant fuzzy sets and the original hesitant fuzzy sets have the following operational formulas.

- (1) lower bound: $(^2h_{M^2})^-(x) = (\min \mu(x), 1) = (h^-(x), 1)$;
- (2) upper bound: $(^2h_{M^2})^+(x) = (\max \mu(x), 1) = (h^+(x), 1)$.
- (3) $(\alpha, 1)$ -lower bound: $(^2h_{M^2})_{(\alpha, 1)}^-(x) = \{^2h_{M^2} \in ^2h_{M^2}(x) = \{(\mu(x), 1) | \mu(x) \leq \alpha\} = h_{\alpha}^-(x) \times \{1\}$;
- (4) $(\alpha, 1)$ -upper bound: $(^2h_{M^2})_{(\alpha, 1)}^+(x) = \{^2h_{M^2} \in ^2h_{M^2}(x) = \{(\mu(x), 1) | \mu(x) \geq \alpha\} = h_{\alpha}^+(x) \times \{1\}$;
- (5) When $(^2h_{M^2})^c(x) = \bigcup_{k=1}^N (1 - \mu_k(x), 1)$, then $(^2h_{M^2})^c(x) = h^c(x) \times \{1\}$;
- (6) When $(^2h_{M^2}^1 \cup ^2h_{M^2}^2)(x) = \{(h_1 \cup h_2)(x)\} \times \{1\}$, then $(^2h_{M^2}^1 \cup ^2h_{M^2}^2)(x) = (^2h_{M^2}^1(x) \cup ^2h_{M^2}^2(x))_{(\alpha, 1)}^+ = (h_1(x) \cup h_2(x))_{\alpha}^+ \times \{1\}$;
- (7) When $(^2h_{M^2}^1 \cap ^2h_{M^2}^2)(x) = \{(h_1 \cap h_2)(x)\} \times \{1\}$, then $(^2h_{M^2}^1 \cap ^2h_{M^2}^2)(x) = (^2h_{M^2}^1(x) \cap ^2h_{M^2}^2(x))_{(\alpha, 1)}^+ = (h_1(x) \cap h_2(x))_{\alpha}^+ \times \{1\}$.

Proof: It is easy to be proved, so omitted. ■

Proposition 4.3: Type-2 hesitant fuzzy sets ${}^2h_{M^2}(x) = \bigcup_{\mu(x) \in M} \{(\mu(x), f((\mu(x))))\}$ can be changed into hesitant fuzzy sets $h(x) = \bigcup_{\mu(x) \in M} \{(\mu(x) \times f((\mu(x))))\}$.

Proof: As $M^2 = \{(\mu_1, f_1(\mu_1)), (\mu_2, f_2(\mu_2)), (\mu_3, f_3(\mu_3)), \dots, (\mu_N, f_N(\mu_N))\}$ is a set of N dual membership functions, let $M = \{\mu_1, \mu_2, \mu_3, \dots, \mu_N\}$, then let $g: M^2 \rightarrow M$, $(\mu_k, f(\mu_k)) \rightarrow \mu_k \times f(\mu_k)$, $k = 1, 2, \dots, N$, and $\forall i, j \in \{1, 2, \dots, N\}$, $g((\mu_i, f(\mu_i)) \cup (\mu_j, f(\mu_j))) = g((\mu_i, f(\mu_i))) \cup g((\mu_j, f(\mu_j)))$, then $g({}^2h_{M^2}) = h_M$.

So type-2 hesitant fuzzy sets can be changed into hesitant fuzzy sets. ■

Example 4.1: In some event, please six experts to appraise the qualities of the three articles, the symbols on behalf of the experts and their quantification of the position in the industry are denoted as $(f_1, 0.8)$, $(f_2, 0.6)$, $(f_3, 0.7)$, $(f_4, 0.5)$, $(f_5, 0.9)$, $(f_6, 0.6)$. In order to facilitate, three papers are recorded as 1, 2, 3, the back corresponding numerical values representative of the expert's overall evaluation scores of articles. Then six functions defined as follows:

$$f_1: 1 \rightarrow 0.5 \quad 2 \rightarrow 0.8 \quad 3 \rightarrow 0.9; \quad f_2: 1 \rightarrow 0.9 \quad 2 \rightarrow 0.6 \quad 3 \rightarrow 0.5;$$

$$f_3: 1 \rightarrow 0.6 \quad 2 \rightarrow 0.7 \quad 3 \rightarrow 0.5; \quad f_4: 1 \rightarrow 1.0 \quad 2 \rightarrow 1.0 \quad 3 \rightarrow 1.0;$$

$$f_5: 1 \rightarrow 0.6 \quad 2 \rightarrow 1.0 \quad 3 \rightarrow 0.3; \quad f_6: 1 \rightarrow 1.0 \quad 2 \rightarrow 0.4 \quad 3 \rightarrow 1.0,$$

And which is the domain of the functions $X = \{1, 2, 3\}$, then establishing a hesitant fuzzy set:

$$M^2 = \{(\mu_1, f_1(\mu_1)), (\mu_2, f_2(\mu_2)), (\mu_3, f_3(\mu_3)), (\mu_4, f_4(\mu_4)), (\mu_5, f_5(\mu_5)), (\mu_6, f_6(\mu_6))\},$$

then

$${}^2h_{M^2}^A(1) = \{(0.5, 0.8), (0.9, 0.6), (0.6, 0.7), (1.0, 0.5), (0.6, 0.9), (1.0, 0.6)\},$$

$${}^2h_{M^2}^A(2) = \{(0.8, 0.8), (0.6, 0.6), (0.7, 0.7), (1.0, 0.5), (1.0, 0.9), (0.4, 0.6)\},$$

$${}^2h_{M^2}^A(3) = \{(0.9, 0.8), (0.5, 0.6), (0.5, 0.7), (1.0, 0.5), (0.3, 0.9), (1.0, 0.6)\}.$$

Then the above type-2 hesitant fuzzy set can be changed into the hesitant fuzzy set is

$$h(1) = \{0.40, 0.54, 0.42, 0.50, 0.60\}, \quad h(2) = \{0.64, 0.6, 0.49, 0.5, 0.90, 0.24\},$$

$$h(3) = \{0.72, 0.30, 0.35, 0.5, 0.27, 0.60\}.$$

From the above type-2 hesitant fuzzy set and hesitant fuzzy set, it is obviously that paper 2 is better than paper 1 and paper 1 is better than paper 3.

5. Type-2 Hesitant Fuzzy Sets and Discrete Type-2 Fuzzy Sets

The formations of the definitions of type-2 hesitant fuzzy sets and discrete type-2 fuzzy sets are very similarly, but their basic operations are almost completely different, their basic operations can't be completely transformed each other except for very special circumstances.

Proposition 5.1: Type-2 hesitant fuzzy sets can be changed into discrete type-2 fuzzy sets in the form of the definitions.

Proof: Suppose the membership function of the type-2 hesitant fuzzy set is as follows:

$${}^2h_{M^2}(x) = \bigcup_{(\mu, f(\mu)) \in M} \{(\mu(x), f(\mu(x)))\},$$

then type-2 hesitant fuzzy set is

$${}^2h_{M^2} = \bigcup_{x \in X} \left(\bigcup_{(\mu, f(\mu)) \in M} \{(\mu(x), f(\mu(x)))\} \right) = \bigcup_{x \in X} \left(\frac{\left\{ \bigcup_{i=1}^N (\mu_i(x), f_i(\mu_i(x))) \right\}}{x} \right).$$

Then let \cup is replaced with \sum and $(\mu_i(x), f_i(\mu_i(x)))$ is replaced with $f_i(\mu_i(x))/\mu_i(x)$,
So type-2 hesitant fuzzy set is changed into the form:

$$\bigcup_{x \in X} \left(\frac{\left\{ \bigcup_{i=1}^N (\mu_i(x), f_i(\mu_i(x))) \right\}}{x} \right) \text{ can be changed into } \sum_{x \in X} \left(\left[\sum_{i=1}^N f_i(\mu_i)/\mu_i \right] / x \right).$$

Thus type-2 hesitant fuzzy sets can be changed into discrete type-2 fuzzy sets in the form of the definitions. ■

Example 5.1: Let X be a reference set, $\forall x \in X$, ${}^2h_{M^2}^A(x) = \{(0.2, 0.6), (0.4, 0.5), (0.6, 0.3)\}$, ${}^2h_{M^2}^B(x) = \{(0.4, 0.7), (0.8, 1.0), (1.0, 0.9)\}$ are two the membership functions of type-2 hesitant fuzzy sets. Then $({}^2h_{M^2}^1 \cup {}^2h_{M^2}^2)(x) = \{(0.4, 0.7), (0.8, 1.0), (1.0, 0.9)\}$, $({}^2h_{M^2}^1 \cap {}^2h_{M^2}^2)(x) = \{(0.2, 0.6), (0.4, 0.5), (0.6, 0.3)\}$. But the generated two discrete type - 2 fuzzy sets \tilde{A} and \tilde{B} , then $\tilde{A}(x) \cup \tilde{B}(x) = \frac{0.6 \wedge 0.7}{0.2 \vee 0.4} + \frac{0.5 \wedge 1.0}{0.8 \vee 0.4} + \frac{0.3 \wedge 0.9}{0.6 \vee 1.0} = \frac{0.6}{0.4} + \frac{0.5}{0.8} + \frac{0.3}{1.0}$, $\tilde{A}(x) \cap \tilde{B}(x) = \frac{0.6 \wedge 0.7}{0.2 \wedge 0.4} + \frac{0.5 \wedge 1.0}{0.8 \wedge 0.4} + \frac{0.3 \wedge 0.9}{0.6 \wedge 1.0} = \frac{0.6}{0.2} + \frac{0.5}{0.4} + \frac{0.3}{0.6}$. Then ${}^2h_{M^2}^1 \cup {}^2h_{M^2}^2$ and ${}^2h_{M^2}^1 \cap {}^2h_{M^2}^2$ are converted into discrete type -2 fuzzy sets, $(\tilde{A} \cup \tilde{B})(x) = \frac{0.7}{0.4} + \frac{1.0}{0.8} + \frac{0.9}{1.0}$, $(\tilde{A} \cap \tilde{B})(x) = \frac{0.6}{0.2} + \frac{0.5}{0.4} + \frac{0.3}{0.6}$, it is clearly that $(\tilde{A} \cup \tilde{B})(x) \neq \tilde{A}(x) \cup \tilde{B}(x)$, $(\tilde{A} \cap \tilde{B})(x) \neq \tilde{A}(x) \cap \tilde{B}(x)$.

Only when ${}^2h_{M^2}^A(x) = {}^1h_{M^2}^A(x)$, $(\tilde{A} \cup \tilde{B})(x) = \tilde{A}(x) \cup \tilde{B}(x)$, $(\tilde{A} \cap \tilde{B})(x) = \tilde{A}(x) \cap \tilde{B}(x)$.

So their basic operations are almost completely different, their basic operations can't be completely transformed each other except for very much special circumstances.

6. Type- n Hesitant Fuzzy Sets

Because of the discrepancy of set itself, the elements of hesitant fuzzy sets can't be repeated. But the fact has a lot of repeated possibilities. Type-2 hesitant fuzzy sets solve the mainly defect of hesitant fuzzy sets, thus the new type-2 hesitant fuzzy sets are the generalization of hesitant fuzzy sets. But for the same reason, the elements of type-2 hesitant fuzzy sets can't be repeated. So it needs to define a higher layer of the hesitant fuzzy set to solve this problem. Thus it is necessary to type- n hesitant fuzzy sets are defined below.

Example 6.1: Let h is a hesitant fuzzy set, $h = \frac{\{0.8, 0.6, 0.7\}}{a} + \frac{\{0.5, 0.6, 0.9\}}{b}$, it is obviously that membership functions of each element haven't duplicate values. If 0.8 and 0.6 all appear twice, let

${}^2h_{M^2} = \frac{\{(0.8, 1), (0.8, 0.9), (0.6, 1), (0.7, 1)\}}{a} + \frac{\{(0.5, 1), (0.6, 1), (0.6, 0.9), (0.9, 1)\}}{b}$, then the above problem is solved.

In the same way, the duplicate membership function value of type-2 hesitate fuzzy sets question can be solved through the type-3 hesitant fuzzy sets. Similarly, repeated problems can be solved through the way of upgrade the dimension of hesitate fuzzy sets.

In order to facilitate the following comparison and unify mathematical symbol, the following the size comparison of two-dimensional coordinate points are defined as follows:

Definition 6.1: Suppose $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n), (d_1, d_2, \dots, d_n)$ and $(c_1, c_2, \dots, c_n) \in [0, 1]^n$, then

- (1) $(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n) \iff a_i \leq b_i$ for $i = 1, 2, \dots, n$;
- (2) $(a_1, a_2, \dots, a_n) \geq (b_1, b_2, \dots, b_n) \iff a_i \geq b_i$ for $i = 1, 2, \dots, n$;
- (3) $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n) \iff a_i = b_i$ for $i = 1, 2, \dots, n$;
- (4) $(d_1, d_2, \dots, d_n) = \max\{(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)\} \iff (\max\{a_1, b_1\}, \max\{a_2, b_2\}, \dots, \max\{a_n, b_n\})$, for $i = 1, 2, \dots, n$;
- (5) $(c_1, c_2, \dots, c_n) = \min\{(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)\} \iff (\min\{a_1, b_1\}, \min\{a_2, b_2\}, \dots, \min\{a_n, b_n\})$, for $i = 1, 2, \dots, n$;

Definition 6.2: Let X be a reference set, then type- n hesitant fuzzy sets are defined on X in terms of a function h that when applied to X returns a subset of $[0, 1]^n$.

Definition 6.3: Let $M^n =$

$\{(\mu_1, f_{11}(\mu_1), f_{12}(f_{11}(\mu_1))), \dots, f_{1n}(f_{1(n-1)}(f_{1(n-2)}(f_{1(n-3)}(\dots(f_{11}(\mu_1)))))) \dots),$
 $(\mu_2, f_{21}(\mu_2), f_{22}(f_{21}(\mu_2))), \dots, f_{2n}(f_{2(n-1)}(f_{2(n-2)}(f_{2(n-3)}(\dots(f_{21}(\mu_2)))))) \dots),$
 $(\mu_3, f_{31}(\mu_3), f_{32}(f_{31}(\mu_3))), \dots, f_{3n}(f_{3(n-1)}(f_{3(n-2)}(f_{3(n-3)}(\dots(f_{31}(\mu_3)))))) \dots), \dots$
 $(\mu_N, f_{N1}(\mu_N), f_{N2}(f_{N1}(\mu_N))), \dots, f_{Nn}(f_{N(n-1)}(f_{N(n-2)}(f_{N(n-3)}(\dots(f_{N1}(\mu_N)))))) \dots)\}$ be a set of N n -dimensions membership functions, then type- n hesitant fuzzy sets ${}^n h_{M^2}^A$ associated with M^n , that is defined as ${}^n h_{M^2}^A(x) = \bigcup_{(\mu, f(\mu)) \in M^n} \{(\mu(x), f(\mu(x)))\}$.

Note: (1) empty set: ${}^n h_{M^2}^A(x) = \underbrace{\{(0, 0, \dots, 0)\}}_n$ for all $x \in X$;

(2) full set: ${}^n h_{M^n}^A(x) = \underbrace{(1, 1, \dots, 1)}_n$ for all $x \in X$.

Given a type- n hesitant fuzzy set represented by its membership function ${}^n h_{M^n}(x)$ and M^n , then its lower, upper, α -lower and α -upper bound are defined as follows:

- (1) lower bound: $({}^n h_{M^n})^-(x) = (\mu^-(x), f(\mu^-(x)), \dots, f(f(\dots f(\mu^-(x)) \dots))) = (\min \mu(x), \min f(\mu(x)), \dots, \min f(f(\dots f(\mu(x)) \dots)))$;
- (2) upper bound: $({}^n h_{M^n})^+(x) = (\mu^+(x), f(\mu^+(x)), \dots, f(f(\dots f(\mu^+(x)) \dots))) = (\max \mu(x), \max f(\mu(x)), \dots, \max f(f(\dots f(\mu(x)) \dots)))$;
- (3) $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ -lower bound: $({}^n h_{M^n})_{(\alpha_1, \alpha_2, \dots, \alpha_n)}^-(x) = \{(\mu(x), f(\mu(x)), \dots, f(f(\dots f(\mu(x)) \dots))) | \mu(x) \leq \alpha_1, f(\mu(x)) \leq \alpha_2, \dots, f(f(\dots f(\mu(x)) \dots)) \leq \alpha_n\}$
- (4) $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ -upper bound: $({}^n h_{M^n})_{(\alpha_1, \alpha_2, \dots, \alpha_n)}^+(x) = \{(\mu(x), f(\mu(x)), \dots, f(f(\dots f(\mu(x)) \dots))) | \mu(x) \geq \alpha_1, f(\mu(x)) \geq \alpha_2, \dots, f(f(\dots f(\mu(x)) \dots)) \geq \alpha_n\}$

Definition 6.4: Given a hesitant fuzzy set represented by its membership function ${}^n h_{M^n}^A(x)$ and M_n , its complement as follows:

$$({}^n h_{M^n}^A)^c(x) = \bigcup_{i=1}^N \{(1 - \mu_i(x), 1 - f_{i1}(\mu_i(x)), 1 - f_{i2}(f_{i1}(\mu_i(x))), \dots, 1 - f_{i(n-1)}f_{i(n-2)} \dots f_{i1}(\mu_i(x)) \dots)\}.$$

Proposition 6.1: The complement is involutive, i.e.

$$({}^n h_{M^n}^A)^{c,c} = {}^n h_{M^n}^A.$$

Definition 6.5: Given two type- n hesitant fuzzy sets represented by their membership function ${}^n h_{M^n}^1$ and ${}^n h_{M^n}^2$, their union and intersection that are respectively represented by ${}^n h_{M^n}^1 \cup {}^n h_{M^n}^2$ and ${}^n h_{M^n}^1 \cap {}^n h_{M^n}^2$ are defined as follows:

$$({}^n h_{M^n}^1 \cup {}^n h_{M^n}^2)(x) = \{{}^n h_{M^n}(x) \in ({}^n h_{M^n}^1(x) \cup {}^n h_{M^n}^2(x)) | {}^n h_{M^n} \geq \max(({}^n h_{M^n}^1)^-, ({}^n h_{M^n}^2)^-)\},$$

or equivalently,

$$\begin{aligned} ({}^n h_{M^n}^1 \cup {}^n h_{M^n}^2)(x) &= \left\{ ({}^n h_{M^n}^1(x) \cup {}^n h_{M^n}^2(x))_{(\alpha_1, \alpha_2, \dots, \alpha_n)}^+, \text{ for } (\alpha_1, \alpha_2, \dots, \alpha_n) \right\} \\ &= \max(({}^n h_{M^n}^1)^-, ({}^n h_{M^n}^2)^-); \end{aligned}$$

$$({}^n h_{M^n}^1 \cap {}^n h_{M^n}^2)(x) = \left\{ {}^n h_{M^n}(x) \in ({}^n h_{M^n}^1(x) \cap {}^n h_{M^n}^2(x)) | {}^n h_{M^n} \leq \min(({}^n h_{M^n}^1)^+, ({}^n h_{M^n}^2)^+) \right\},$$

or equivalently,

$$\begin{aligned} ({}^n h_{M^n}^1 \cap {}^n h_{M^n}^2)(x) &= ({}^n h_{M^n}^1(x) \cap {}^n h_{M^n}^2(x))_{(\alpha_1, \alpha_2, \dots, \alpha_n)}^-, \text{ for } (\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \min(({}^n h_{M^n}^1)^+, ({}^n h_{M^n}^2)^+). \end{aligned}$$

According to the need of solving practical problems, determine the level of the type of hesitant fuzzy sets. In general, type-2 hesitant fuzzy sets are enough to solve the problem.

7. Conclusions and Future Work

In this paper, type-2 hesitant fuzzy sets and their basic operations are firstly introduced. Then the relations of type-2 hesitant fuzzy sets and hesitant fuzzy sets are further studied. The relations of type-2 hesitant fuzzy sets and discrete type-2 fuzzy sets are also studied in the form. Because of the discrepancy of set itself, the elements of hesitant fuzzy sets can't be repeated, type-2 hesitant fuzzy sets solve the mainly defect, thus the new type-2 hesitant fuzzy sets are the generalization of hesitant fuzzy sets. In addition, type- n hesitant fuzzy sets and their basic operations are given.

In the future, the practical application and standard measures and distance of type-2 hesitate fuzzy sets will be an in-depth study in order to solve more and more practical problems. At last, the new interval type-2 hesitant fuzzy systems and their application will be further studied at once.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

- [1] Zadeh LA. Fuzzy sets information and control. [1965](#);8(3):338–353.
- [2] Zadeh LA. Fuzzy algorithm information and control. [1968](#);12(2):94–102.
- [3] Zadeh LA. A rationale for fuzzy control. *J Dyn Syst Meas Control*. [1972](#);94(Series G):3–4.
- [4] Zadeh LA. Outline of a new approach to the analysis complex systems and decision processes. *IEEE Transact Syst Man Cybernet*. [1973](#);3(1):28–44.
- [5] Dubois D, Prage H. Fuzzy sets systems. New York: Academic press; [1980](#).
- [6] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets Syst*. [1986](#);20(1):87–96.
- [7] Atanassov K, Gargov G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst*. [1989](#);31:343–349.
- [8] Castillo O, Melin P. Type-2 fuzzy logic theory and applications. Berlin: Springer-Verlag; [2008](#).
- [9] Gorzalczany MB. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets Syst*. [1987](#);21:1–17.
- [10] Hagrais H. Type-2 FLCs: a new generation of fuzzy controllers. *IEEE Comput Intell Mag*. [February 2007](#);2:30–43.
- [11] John R, Coupland S. Type-2 fuzzy logic: a historical view. *IEEE Comput Intell Mag*. [February 2007](#);2:57–62.
- [12] Karnik NN, Mendel JM. Operations on type-2 fuzzy sets. *Fuzzy Sets Syst*. [2001](#);122:327–348.
- [13] Karnik NN, Mendel JM. Centroid of a type-2 fuzzy set. *Inf Sci*. [2001](#);132:195–220.
- [14] Liang Q, Mendel JM. Interval type-2 fuzzy logic systems: theory and design. *IEEE Transact Fuzzy Syst*. [2000](#);8:535–550.
- [15] Liu F, Mendel JM. Aggregation using the fuzzy weighted average, as computed by the KM algorithms. *IEEE Transact Fuzzy Syst*. [February 2008](#);16:1–12.
- [16] Liu F, Mendel JM. Encoding words into interval type-2 fuzzy sets using an interval approach. *IEEE Transact Syst*. [December 2008](#);16:1503–1521.
- [17] Melgarejo M. Implementing interval type-2 fuzzy processors. *IEEE Comput Intell Mag*. [February 2007](#);2:63–71.
- [18] Mendel JM. Uncertain rule-based fuzzy logic systems: introduction and new directions. Upper-Saddle River (NJ): Prentice-Hall; [2001](#).
- [19] Mendel JM. Fuzzy sets for words: a new beginning. *Proc. IEEE FUZZ Conference*. St. Louis, MO, May 26–28, 2003, p. 37–42.
- [20] Mendel JM. Type-2 fuzzy sets and systems: an overview. *IEEE Comput Intell Mag*. [February 2007](#);2:20–29.

- [21] Mendel JM. Advances in type-2 fuzzy sets and systems. *Inf Sci.* **2007**;177:84–110.
- [22] Mendel JM, John RI. Type-2 fuzzy sets made simple. *IEEE Transact Fuzzy Syst.* **April 2002**;10:117–127.
- [23] Mendel JM, John RI, Liu F. Interval type-2 fuzzy logic systems made simple. *IEEE Transact Fuzzy Syst.* **December 2006**;14:808–821.
- [24] Rickard JT, Aisbett J, Gibbon G, et al. Fuzzy subsethood for type-n fuzzy sets, NAFIPS 2008, Paper # 60101, New York City, May 2008.
- [25] Wu D, Mendel JM. Uncertainty measures for interval type-2 fuzzy sets. *Inf Sci.* **2007**;177:5378–5393.
- [26] Wu D, Mendel JM. Aggregation using the linguistic weighted average and interval type-2 fuzzy sets. *IEEE Transact Fuzzy Syst.* **December 2007**;15:1145–1161.
- [27] Wu H, Mendel JM. Uncertainty bounds and their use in the design of interval type-2 fuzzy logic systems. *IEEE Transact Fuzzy Syst.* **October 2002**;10:622–639.
- [28] Torra V. Hesitant fuzzy sets. *Int J Intell Syst.* **2010**;25:529–539.
- [29] Xia MM, Xu ZS. Hesitant fuzzy information aggregation in decision making. *Int J Approx Reason.* **2011**;52:395–407.
- [30] Hu J, Zhang X, Chen X, et al. Hesitant fuzzy information measures and their applications in multi-criteria decision making. *Int J Syst Sci.* **2016**;47:62–76.
- [31] Xu Z, Xia M. Distance and similarity measures for hesitant fuzzy sets. *Inf Sci.* **2011**;181:2128–2138.
- [32] Bisht K, Kumar S. Fuzzy time series forecasting method based on hesitant fuzzy sets. *Expert Syst Appl.* **2016**;64:557–568.
- [33] Maldonado Y, Castillo O, Melin P. Particle swarm optimization of interval type-2 fuzzy systems for FPGA applications. *Appl Soft Comput.* **2013**;13(1):496–508.
- [34] Biglarbegian M, Melek WW, Mendel JM. **2008**. Stability analysis of type-2 fuzzy systems. In *Proceedings of IEEE FUZZ Conference*, Paper # FS0233. Hong Kong, China, June 2008.
- [35] Du X, Ying H. Derivation and analysis of the analytical structures of the interval type-2 fuzzy-PI and PD controllers. *IEEE Transact Fuzzy Syst.* **2010**;18(4):802–814.