Short Papers

Type-2 Fuzzy Sets as Functions on Spaces

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***Abstract*—For many readers and potential authors, type-2 (T2) fuzzy sets might be more readily understood if expressed by the use of standard mathematical notation and terminology. This paper, therefore, translates constructs associated with T2 fuzzy sets to the language of functions on spaces. Such translations may encourage researchers in different disciplines to investigate T2 fuzzy sets, thereby potentially broadening their application and strengthening the underlying theory.**

***Index Terms*—Embedded type-1 (T1) fuzzy sets, interval-valued fuzzy sets, type-2 (T2) fuzzy sets.**

I. Introduction

The International Fuzzy Systems Association (IFSA) 2009 conference in Lisbon, Portugal, featured a round-table session entitled "Type-2 or Not Type-2." Among the topics of animated discussion, considerable time was devoted to the airing of complaints by mathematically oriented members of the fuzzy community about the difficult notation used by the type-2 (T2) community. Indeed, ever since Zadeh [1] introduced T2 and higher order fuzzy sets, there have been multiple attempts to explain them to the nonspecialist reader. Mendel *et al*. have introduced a wealth of terminology, constructions, and notation [2]–[5]. At the same time, there have been more rigorous mathematical expositions, as well as applications of interval-valued and general T2 fuzzy sets [6]–[13], [20]. However, these papers do not simply explain T2 fuzzy-set concepts in the language of functions on spaces familiar to readers with a primarily mathematical background.

Mendel and John present T2 fuzzy sets in three different ways, all of which involve specialized constructs: First, a T2 fuzzy set on X is defined as a union over X of so-called "vertical slices." In fuzzy notation, a vertical slice is represented as $\int_{t\in J_x} \mu(x,t)/t$ for $x \in X$, where the integral sign denotes union. Here, J_x is a subset of the unit interval. A T2 fuzzy set is, thus, represented in full as follows:

$$\int_{x\in X}\left(\int_{t\in J_x} \mu\left(x,t\right)/t\right)\Big/x. \tag{1}$$

Alternatively, a T2 fuzzy set $\tilde{A}$ is thought of as a surface within the 3-D space $X \times [0,1] \times [0,1]$. In set notation, $\tilde{A} = \{(x,t), \mu(x,t) | x \in X, t \in J_x\}$. This is called a "3-D membership function" or 3-D MF [3], [4].

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Third, Mendel and John's representation theorem presents a T2 fuzzy set as the union of all its embedded T2 fuzzy sets. An embedded T2 fuzzy set is defined through an embedded type-1 (T1) fuzzy set. Specifically, an embedded T1 fuzzy set has membership function $u : X \to [0,1]$ such that $u(x) \in J_x$, and an embedded T2 fuzzy set is the subset $\{(x, u(x), \mu(x, u(x)))\}$ of the 3-D MF.

In an effort to improve communication between those versed in the language of T2 fuzzy sets and those familiar with the standard terminology of functions and spaces, this note translates terms from the former into the latter language. In this way, we tighten up the definition of primary membership and suggest that membership functions might be constrained to belong to particular classes or families.

II. Expressing Fuzzy-Set Concepts in Standard Mathematical Notation

Fuzzy sets are defined on a *universe*, which in other language is just a set. Membership functions are set functions from this set into the unit interval. However, it is common in mathematics to put structure onto sets and call them spaces: metric spaces, spaces with a measure, Borel spaces, etc. Functions are constrained to respect the structure. In our work with T2 fuzzy sets, we have explicitly considered spaces that have a measure, and forced membership functions to be measurable [14]. On real-valued universes, fuzzy numbers are defined by the use of constrained membership functions, and continuous piecewise linear membership functions have been employed in fuzzy reasoning [8], as have continuous piecewise planar functions in T2 fuzzy logic [15].

While the structure of universes is rarely used explicitly to constrain membership functions, the literature is dominated by examples in which the universe is a subset of the real line, and in which membership functions are Gaussian-like, trapezoidal, or similar simple functions. Such membership functions implicitly assume continuity and the usual metric on the real line and unit interval. When generalized to T2 fuzzy sets, these membership functions are usually "blurred" [3]. However, the embedded T1 and T2 sets are allowed to take any form and, in particular, do not have to be continuous. Practical use of the representation theorem is limited by the innumerable possible forms of embedded sets [3], [5].

When expressed in the language of functions and spaces, embedded T1 sets are naturally constrained by the structures of the underlying universe and the unit interval. This can be seen from Table I, which contains translations from fuzzy-set language. The first column lists terms used in various works by Mendel *et al*. The second column contains some of the corresponding formal definitions in fuzzy-set notation. The third and fourth columns contain a translation of the terms into conventional mathematical language. As discussed in Section III, the translations are not always technically exact, but the translated concept operates in practice in the same way as the original.

As a simple example of how such translations are extended and applied, consider the definition of the membership function $\mu_{\tilde{A}\cap\tilde{B}}(x)$ of the intersection of two T2 fuzzy sets $\tilde{A}$ and $\tilde{B}$, as expressed in [20, eq. (17)–(19)]: $\mu_{\tilde{A}\cap\tilde{B}}(x) = \int_{u\in J_x^u} \int_{w\in J_x^w} f_x(u) * g_x(w)/(u \wedge w)$, where "$*$" denotes a general t-norm $t(u,w)$, and intersection is defined by the minimum t-norm. $\mu_{\tilde{A}\cap\tilde{B}}(x)$ in this equation is, of course, implicitly also a function of $v = u \wedge w \equiv \min(u,w)$, i.e., we can write this membership function as $\mu_{\tilde{A}\cap\tilde{B}}(x,v)$, where v is the primary membership variable. Thus, by the use of Zadeh's extension principle [1], we may write $\mu_{\tilde{A}\cap\tilde{B}}(x,v)$ in standard mathematical

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TABLE I
EQUIVALENCES BETWEEN FUZZY SET NOTATION AND STANDARD MATHEMATICAL NOTATION

Fuzzy Set Term	Fuzzy Set Notation	Translated Terminology	Standard Mathematical Notation
Universe	X	Set or space with properties such as a metric or measure	X
(Primary) variable	$x \in X$	Variable, element, member, point	$x \in X$
Fuzzy set	$\sum_{1}^{n} u(x_i)/x_i$ or $\int_{x\in X} u(x)/x$ or $\{(x,u(x)) \mid x \in X\}$	Set function on X into $[0,1]$, possibly constrained to belong to a family such as. continuous functions.	$u : X \to [0,1]$
Grade of membership	$u(x)$	Function value	$u(x)$
Minimum t-norm	$u(x_1) \wedge u(x_2) \ldots \wedge u(x_n)$, $x_i \in X$	Function infimum, or function minimum when X is finite	$\inf\{u(x) \mid x \in X\}$
T2 fuzzy set or 3D membership function	$\int_{x\in X}\left(\int_{t\in J_x} \mu(x,t)/t\right)\Big/x$ or $\{(x,t),\mu(x,t) \mid x \in X,\, t \in J_x\}$	Bivariate function on the Cartesian product $X \times [0,1]$ into $[0,1]$	$\mu : X \times [0,1] \to [0,1]$
Primary membership	J_x (usually constrained to be an interval)	Support of $\mu_x : [0,1] \to [0,1]$, where $\mu_x(t) = \mu(x,t)$	$\{t \in [0,1] \mid \mu(x,t) > 0\}$
Secondary grade	$\mu_x(t)$	Function value	$\mu(x,t)$
Interval T2 fuzzy set, or interval-valued fuzzy set	$\int_{x\in X}\left(\int_{t\in J_x} 1/t\right)\Big/x$ or $\{(x,\mu(x)) \mid x \in X\}$ where $\mu : X \to D[0,1]$	Function on X into $D[0,1]$, where $D[0,1]$ is the set of closed subintervals of $[0,1]$	$\mu : X \to D[0,1]$
Vertical slice, or fuzzy grade, or secondary MF at $x \in X$	$\int_{t\in J_x} \mu(x,t)/t$ or $\mu(x)$ or $\mu(t \mid x)$	Restriction of function $\mu : X \times [0,1] \to [0,1]$ to $x \in X$	$\mu_x : [0,1] \to [0,1]$
Footprint of uncertainty; union of primary memberships	$\int_{x\in X} J_x \big/ x$	Two-dimensional support of μ	$\{(x,t) \in X \times [0,1] \mid \mu(x,t) > 0\}$
Lower MF and upper MF of FOU	Lower and upper bounding functions of FOU respectively denoted $\underline{\mu}(x)$ and $\overline{\mu}(x)$	Infimum and supremum of support of μ expressed as functions on X	$\underline{\mu}(x) = \inf\{t \mid t \in [0,1], \mu(x,t) > 0\}$ $\overline{\mu}(x) = \sup\{t \mid t \in [0,1], \mu(x,t) > 0\}$
Embedded T1 fuzzy set	$\int_{x\in X} u(x)/x$, $u(x) \in J_x$ or $\{(x,u(x)) \mid x \in X, u(x) \in J_x\}$	A function whose range is a subset of $[0,1]$ determined by μ	$u : X \to [0,1]$ satisfying $\mu(x,u(x)) > 0$ for all $x \in X$
Embedded T2 fuzzy set	$\int_{x\in X}\left(\mu(x,u(x))/u(x)\right)\Big/x,$ $u(x) \in J_x$	μ restricted to $\{(x,u(x)) \mid x \in X\}$ for some $u : X \to [0,1]$	$\mu_u : X \to [0,1]$ where $\mu_u(x) = \mu(x,u(x))$
Primary MF	A T1 FS from a set of T1 FSs produced by varying a parameter	Member of family of functions into unit interval which are parameterized by Ω	$\mu_\varphi : X \to [0,1]$, $\varphi \in \Omega$
Type-n fuzzy set	A T1 FS taking values in a Type-(n-1) fuzzy set.	Multivariate function on the Cartesian product $X \times [0,1]^{n-1}$ into $[0,1]$	$\mu : X \times [0,1]^{n-1} \to [0,1]$

notation as $\mu_{\tilde{A}\cap\tilde{B}}(x,v) = \sup_{u,w:\,\min(u,w)=v} t(f_x(u), g_x(w))$ where sup is supremum.

In Section III, we translate the centroid of T2 fuzzy sets to standard mathematical notation. Fuzzy logic applications, such as fuzzy controllers [11]–[13], combine intersections, centroids, and other operations in computer implementations that implicitly employ the mathematical expressions, albeit with discretized membership functions. Applications have also typically constrained the form of some membership functions, for example, inputs are assumed to be Gaussians. Replacement of fuzzy-set formulations with their mathematical translations would make reports of such applications more accessible to general readers.

III. Primary Membership

Zadeh's original presentation defined T2 fuzzy sets as $\tilde{A} = \int_{x\in X}(\int_{t\in[0,1]}\mu(x,t)/t)/x$. Thus, the secondary membership functions were defined on the entire unit interval so that J_x was always fixed to be the unit interval. This definition was used by Mizumoto and Tanaka [16] and adopted in earlier papers by Mendel and John, such as [2]. The subsequent introduction of variable primary memberships J_x was motivated by interval-valued fuzzy sets, defined independently by several authors in the mid-1970s [17], [18] as fuzzy sets whose membership functions take values in the set $D[0,1]$ of closed subintervals of $[0,1]$. Interval-valued fuzzy sets are equivalent to T2 fuzzy sets with membership $\mu(x,t) = 1$ when t is in an interval $J_x \in D[0,1]$ and with $\mu(x,t) = 0$ otherwise. T2 fuzzy sets defined as in (1) generalize such interval-valued fuzzy sets, allowing nonuniform preferences for elements in each interval J_x.

J_x plays a confusing and confounding role in the formalism of T2 fuzzy sets. As a subset of the unit interval, it is usually, but not always, constrained to be an interval. It includes the support of $\mu_x : [0,1] \to [0,1]$, where $\mu_x(t) = \mu(x,t)$. However, it can be a superset of the support, because $\mu_x(t)$ is an arbitrary membership function on J_x, and hence, it is possible that $\mu_x(t) = 0$ for $t \in J_x$. The confounding effect of introducing J_x is illustrated by the fact that the primary memberships of a T2 fuzzy set $\tilde{A} = \int_{z\in X}(\int_{t\in J_z}\mu(z,t)/t)/z$ can be changed without a meaningful change of $\tilde{A}$. To see this, suppose that for some $x \in X$, $J_x = [a,b]$, where $a > 0$. For any $r > 1$, construct a T2 fuzzy set $\tilde{A}' = \int_{z\in X}(\int_{t\in J'_z}\mu'(z,t)/t)/z$ that differs from $\tilde{A}$ only in that its primary membership interval at x extends J_x to $J'_x = [a/r, b] \subset [a,b]$. Set $\mu'(x,t) = 0$ when $t \in J'_x - J_x = [a/r, a)$, and otherwise, $\mu'(x,t) = \mu(x,t)$. By construction, $\tilde{A}'$ has the same support as $\tilde{A}$ for all its secondary membership functions, even though the primary membership differs at x. This suggests that primary membership be explicitly defined as the support, i.e., as $\{t|\mu(x,t) > 0\}$ or, perhaps, as the closure of the support. Adopting such a definition allows primary memberships J_x that are finite or infinite unions of intervals.

T2 fuzzy sets defined using (1) can be partitioned into equivalence classes. Given $\tilde{A} = \{(x,t), \mu(x,t)|x \in X, t \in J_x\}$ and $\tilde{B} = \{(x,t), \eta(x,t)|x \in X, t \in J'_x\}$, call $\tilde{A} \sim \tilde{B}$ if and only if the following condition is satisfied: Whenever $\mu(x,t) > 0$ or $\eta(x,t) > 0$, both functions are defined and equal at $(x,t) \in X \times [0,1]$. Members of an equivalence class differ in a nonessential way, because all their secondary memberships have the same support and are identical on this support. Each equivalence class of T2 fuzzy sets has one member of the form $\tilde{A}_\mu = \{(x,t), \mu(x,t)|x \in X, t \in [0,1]\}$, which is fully specified by the function $\mu : X \times [0,1] \to [0,1]$. In moving to a functional representation of T2 fuzzy sets, we represent each equivalence class by its unique member of the form $\tilde{A}_\mu$ and define the primary membership at x of the class to be the support of $\mu_x : [0,1] \to [0,1]$.

It does no harm to equate members of an equivalence class. For example, consider Karnik and Mendel's definition of the centroid [19]

$$\int_{\theta_1\in J_{x_1}} \cdots \int_{\theta_n\in J_{x_n}} [\mu_{x_1}(\theta_1) * \cdots * \mu_{x_n}(\theta_n)] \Bigg/ \frac{\sum_{x_i\in X} x_i\theta_i}{\sum_i \theta_i}.$$

Extended to the infinite case and rewritten in standard terminology with $\int$ denoting integration and with the minimum t-norm, the centroid of $\tilde{A}$ is the fuzzy set whose membership at

$$z \in \left\{ \frac{\int_X xu(x)dx}{\int_X u(x)dx} | u(x) \in J_x \right\} \subset [0,1]$$

is

$$\sup\left\{ \inf\{\mu(x, u(x))\,|x \in X\}\,|u(x) \in J_x, \frac{\int_X xu(x)dx}{\int_X u(x)dx} = z \right\}$$

and is otherwise zero. It is easy to see that the centroids of all T2 fuzzy sets in an equivalence class have the same support and the same membership function on this support. This is because wherever such fuzzy sets differ, $\mu(x, u(x)) = 0$, and hence, functions $u : X \to [0,1]$ which are nonzero outside the support of μ_x cannot contribute to the centroid-membership value. Consequently, we can replace J_x in the earlier expressions by the full interval $[0,1]$.

IV. Conclusion

We set out to translate T2 fuzzy-set constructs into the more general mathematical language of functions and spaces. In order to do so, we had to examine the role of the primary membership J_x as traditionally defined in the T2 literature. We showed that primary membership at x is more usefully defined simply in terms of the support $\{t|\mu(x,t) > 0\}$ of the membership function, which need not be an interval. The translation also suggests that T2 membership functions be constrained to belong to particular families, such as continuous functions.

There have been recent pleas [17], [18] for a unique terminology for interval-valued fuzzy sets to avoid the reinvention of results. It is probably neither possible nor desirable to standardize notation across the wide range of disciplines that could potentially benefit from T2 fuzzy-set modeling. However, translations such as Table I can increase recognition of different terminologies and notation and assist cross fertilization of ideas between disciplines.

References

[1] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-1," *Inf. Sci.*, vol. 8, pp. 99–249, 1975.

[2] R. I. John, "Type 2 fuzzy sets for knowledge representation and inferencing," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, 1998, pp. 1003–1008.

[3] J. M. Mendel and R. I. John, "Type-2 fuzzy sets made simple," *IEEE Trans. Fuzzy Syst.*, vol. 10, no. 2, pp. 117–127, Apr. 2002.

[4] J. M. Mendel, "Type-2 fuzzy sets and systems: An overview," *IEEE Comput. Intell. Mag.*, vol. 2, no. 1, pp. 20–29, Feb. 2007.

[5] S. Coupland and R. I. John, "Type 2 fuzzy logic and the modeling of uncertainty," in *Fuzzy Sets and Their Extensions*, H. Bustince, F. Herrera, J. Montero, and D. Dubois, Eds. New York: Springer-Verlag, 2007, pp. 3–22.

[6] I. B. Turksen, "Interval valued fuzzy sets based on normal forms," *Fuzzy Sets Syst.*, vol. 20, pp. 191–210, 1986.

[7] M. B. Gorzalczany, "A method of inference in approximate reasoning based on interval-valued fuzzy sets," *Fuzzy Sets Syst.*, vol. 28, pp. 45–53, 1988.

[8] P. M. Broek van den, "Fuzzy reasoning with continuous piecewise membership functions," in *Proc. Annu. Meeting North Amer. Fuzzy Inf. Process. Soc.*, 1997, pp. 371–376.

[9] D. Dubois and H. Prade, *Fundamentals of Fuzzy Sets*. New York: Springer-Verlag, 2000.

[10] H. Bustince, J. Montero, E. Barrenechea, and M. Pagola, "Laws for conjunctions and disjunctions in interval type 2 fuzzy sets," in *Proc. IEEE World Congr. Comput. Intell.*, 2008, pp. 1613–1618.

[11] H. Hagras, "Type-2 FLCs: A new generation of fuzzy controllers," *IEEE Comput. Intell. Mag.*, vol. 2, no. 1, pp. 30–43, Feb. 2007.

[12] O. Castillo and P. Melin, *Type-2 Fuzzy Logic Theory and Applications*. Berlin, Germany: Springer-Verlag, 2008.

[13] R. Sepulveda, O. Castillo, P. Melin, A. Rodriguez-Diaz, and O. Montiel, "Experimental study of intelligent controllers under uncertainty using type-1 and type-2 fuzzy logic," *Inf. Sci.*, vol. 177, pp. 2023–2048, 2007.

[14] J. T. Rickard, J. Aisbett, and G. Gibbon, "Subsethood of type-*n* fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 1, pp. 50–60, Feb. 2009.

[15] S. Coupland and R. I. John, "New geometric inference techniques for type-2 fuzzy sets," *Int. J. Approximate Reason.*, vol. 49, pp. 198–211, 2008.

[16] M. Mizumoto and T. Tanaka, "Some properties of fuzzy sets of type-2," *Inf. Control*, vol. 31, pp. 312–340, 1976.

[17] H. Bustince, J. Montero, M. Pagola, E. Barrenechea, and D. Gomez, "A survey of interval-valued fuzzy sets," in *Handbook of Granular Computing*, W. Pedrycz, A. Skowron, and V. Kreinovich, Eds. New York: Wiley, 2008, pp. 491–515.

[18] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, and H. Prade, "Terminological difficulties in fuzzy set theory—The case of 'Intuitionistic Fuzzy Sets,'" *Fuzzy Sets Syst.*, vol. 156, pp. 485–491, 2005.

[19] N. N. Karnik and J. M. Mendel, "Centroid of a type-2 fuzzy set," *Inf. Sci.*, vol. 132, pp. 195–220, 2001.

[20] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems*. Upper Saddle River, NJ: Prentice-Hall, 2001.