Chapter 6
Type-2 Fuzzy Sets

6.1 The Concept of a Type-2 Fuzzy Set

Consider[1] the transition from ordinary sets to fuzzy sets. When the membership of an element in a set cannot be determined as either 0 or 1, type-1 fuzzy sets are used. Similarly, when the circumstances are so uncertain that it is difficult to determine the membership grade even as a crisp number in [0, 1] then fuzzy sets of type 2 can be used, a concept that was first introduced in Zadeh (1975).

When something is uncertain (e.g., a measurement), its exact value cannot be determined, so using type-1 sets makes more sense than using crisp sets. However, using exactly specified type-1 fuzzy set membership functions (MFs) seems counterintuitive—an early criticism of fuzzy sets. If the exact value of an uncertain quantity cannot be determined, then how can the exact membership of a fuzzy set be determined? Of course, this criticism applies to type-2 fuzzy sets as well, because even though their membership is fuzzy, it is still specified exactly, which again seems counterintuitive. Continuing to think along these lines, it follows that no finite-type fuzzy set can represent uncertainty "completely." So, ideally, a type-∞ fuzzy set has to be used to completely represent uncertainty. Of course, this cannot be done in practice, so some finite-type sets have to be used. This book deals just with type-1 and type-2 fuzzy sets. Higher type fuzzy sets could be examined, but the complexity of such fuzzy sets increases rapidly.

Uncertain MFs also occur when fuzzy sets are used as models for linguistic terms (words). When one wants to simultaneously model intra- and interlinguistic uncertainties (Sect. 5.2), this cannot be done using type-1 fuzzy sets, but it can be done by using type-2 fuzzy sets.

Example 6.1 The number of terms associated with a linguistic variable is free to be chosen. In Fig. 6.1 the linguistic variable temperature has been decomposed into five terms {*very low*, *moderately low*, *near zero*, *moderately high*, *very high*}.

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[1]The first two paragraphs in this section are adapted from Karnik and Mendel (1998, p. 2).

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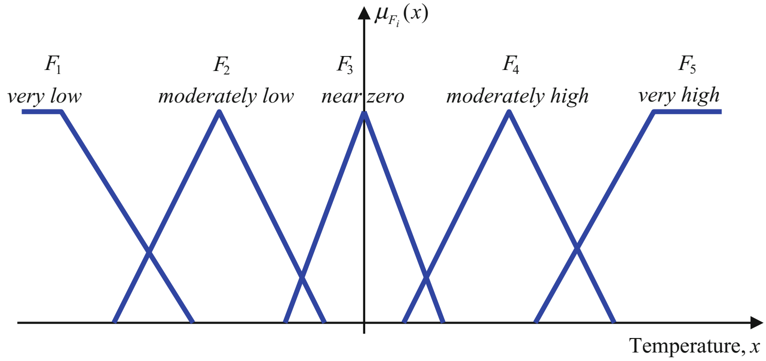

Fig. 6.1 Type-1 MFs

For illustrative purposes, triangular and piecewise linear MFs have been used. Many decisions were made to draw Fig. 6.1 MFs, including where to center the triangles, where to locate the shoulder points of the piecewise linear MFs, where to locate the basepoints of the triangles on the x-axis, and how much overlap there should be between neighboring MFs. All of these decisions translate into MF uncertainties, which, as shall be demonstrated in this chapter, can be handled by type-2 fuzzy sets and their MFs.

Example 6.2 This is a continuation of Example 2.2, so it would be a good time for the reader to reread that example. Now a model is sought that lets uncertainties about the overlap of the fuzzy sets be incorporated into it. A type-2 fuzzy set lets us do that and in different ways. In Fig. 6.2b, c, each of the type-1 overlap points [b, c, d, and e in (a)] is allowed to have an interval of uncertainty. For the purposes of this example, imagine a multitude of straight lines each with the same slope as the straight line type-1 MF, sweeping from the left end of the interval of uncertainty to the right end of that interval (other nonparallel lines, or even curves, could also be used). A couple of these lines are shown as dashed lines in Fig. 6.2b, c. In Fig. 6.2b the same weight (which leads to the uniform shading) is assigned to every point on each of these straight lines, in which case the resulting type-2 fuzzy set is one example of what is called an *interval type-2 fuzzy set*. In Fig. 6.2c the same weight is assigned to each line but only at a level u (which leads to the nonuniform shading), in which case the resulting type-2 fuzzy set is one example of what is called a *general type-2 fuzzy set*. Although the overlapping of x is the same for both kinds of type-2 fuzzy sets, the weightings (which occur in the third dimension) are different.

So, a type-2 fuzzy set lets one partition x using *overlapping partitions*, but now the partitions allow for uncertainty about the overlap, i.e., as *second-order uncertainty partitions* (Definition 1.3), something that cannot be done by a type-1 fuzzy set. Such overlapping partitions lead to even smoother transitions from one set to

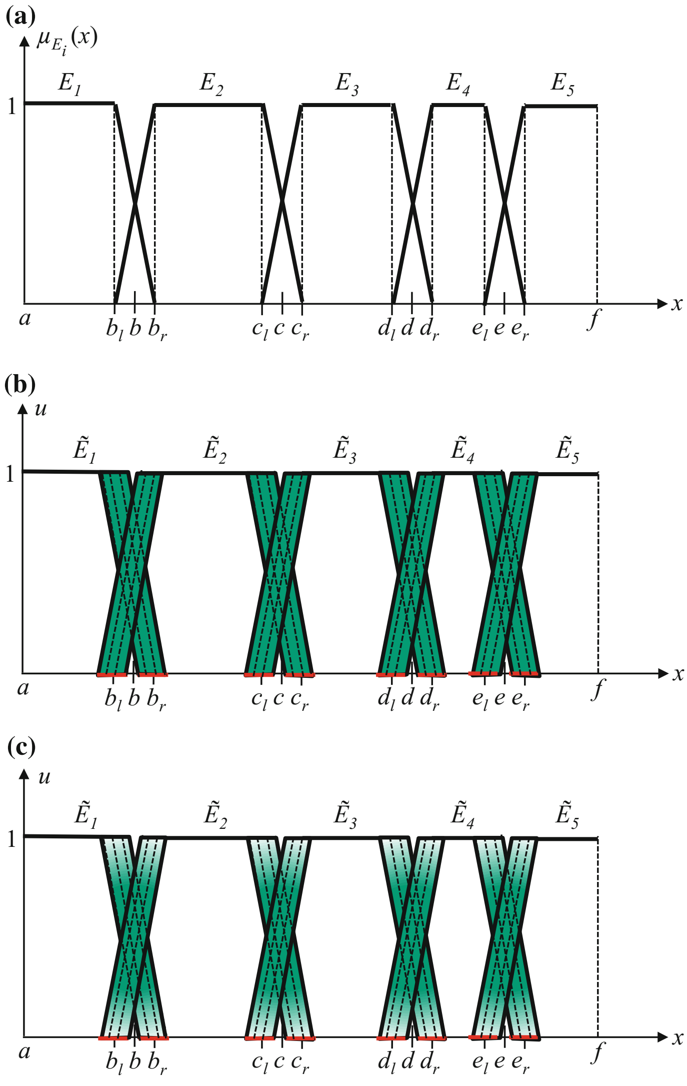

Fig. 6.2 Transitions from overlapping partitions for type-1 fuzzy sets to type-2 fuzzy sets: **a** interpreting type-1 fuzzy sets as overlapping partitions, **b** interval type-2 fuzzy set overlapping partitions (uniform shading), and **c** general type-2 fuzzy set overlapping partitions (nonuniform shading). In (**b**) and (**c**) the type-1 overlapping partition points (e.g., b_l and b_r) have expanded to intervals, $b_l \to [b_{ll}, b_{lr}]$ and $b_r \to [b_{rl}, b_{rr}]$, respectively. The endpoints of these intervals are not marked in (**b**) or (**c**)

another than the type-1 fuzzy set transitions, and so they have the potential to serve us well where it is important to allow for uncertainty about the overlap.

Additional benefits for using general type-2 fuzzy sets are described in Sect. 11.1.

Finally, the just-used phrase "second-order uncertainty partitions" may sound to some as diminishing the importance of a type-2 fuzzy set. It will be seen that a type-2 fuzzy system is analogous to a probabilistic system through first- and second-order moments, whereas a type-1 fuzzy system is analogous to a probabilistic system only through the first moment. And, of course, type-2 fuzzy sets are needed to implement a type-2 fuzzy system. Exactly how this is done is the subject of Chaps. 9 and 11.

Zadeh (1975, Definition 3.1) defined a type-2 fuzzy set as one whose MF ranges over type-1 fuzzy sets, but he left open how to mathematically model such fuzzy sets. The rest of this chapter explains some ways for doing this.

6.2 Definitions of a General Type-2 Fuzzy Set and Associated Concepts

Imagine blurring the type-1 MF depicted in Fig. 6.3a by shifting the points on the triangle either to the left or to the right and not necessarily by the same amounts, as in Fig. 6.3b. Then, at a specific value of x, say x', there no longer is a single value for the MF; instead, the MF takes on values wherever the vertical line intersects the blur. Those values need not all be weighted the same; hence, an amplitude distribution can be assigned to all of those points. Doing this for $x \in X$, a three-dimensional MF—a type-2 MF—is created that characterizes a type-2 fuzzy set. This construction can be easily remembered, as *type-1-blur-weight*.

John and Coupland (2012) make the following important cautionary statement about "blurring":

> A careful rereading of [the above paragraph] will demonstrate that the notion of blurring is used to illustrate and communicate the form for a general type-2 fuzzy MF and less so an advocated methodology for constructing a type-2 MF. In some applications blurring may work well (Coupland and John 2007), however, there is no guarantee of success and in certain application areas the blurring approach will fail.

Since the publication of the first edition of this book (Mendel 2001) some of the notations and definitions about type-2 fuzzy sets have changed for the better. A history of those changes, as well as recommended notation and definition changes appears in Mendel et al. (2016). This book abides by all of those recommended changes.

Definition 6.1 A *type-2 fuzzy set* (also called a *general type-2 fuzzy set*), denoted $\tilde{A}$, is the graph of a bivariate function (Aisbett et al. 2010)—called the MF of $\tilde{A}$—on the Cartesian product $X \times [0,1]$ into $[0,1]$, where X is the universe for the *primary variable* of $\tilde{A}$, x. The MF of $\tilde{A}$ is denoted $\mu_{\tilde{A}}(x,u)$, (or $\mu_{\tilde{A}}$ for short) and is called a *type-2 MF*, i.e.,

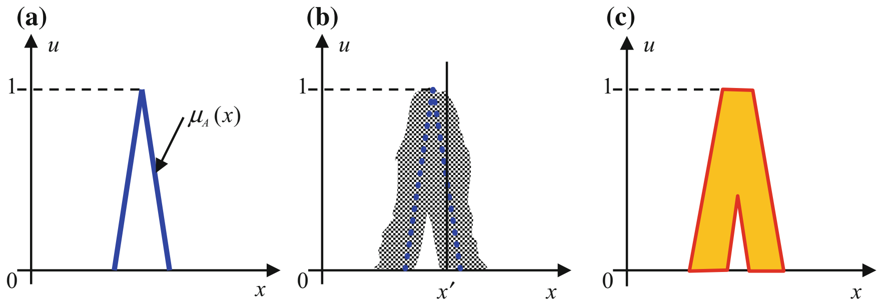

Fig. 6.3 **a** Type-1 MF, **b** blurred type-1 MF (artistic rendition), and **c** footprint of uncertainty

$$\tilde{A} = \left\{ \left((x,u), \mu_{\tilde{A}}(x,u) \right) | x \in X,\ u \in U \equiv [0,1] \right\} \tag{6.1}$$

in which $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$. U is the universe for the *secondary variable* u, and in this book U is always assumed to be[2] $[0,1]$. $\tilde{A}$ can also be expressed in fuzzy set notation as

$$\tilde{A} = \int_{x \in X} \int_{u \in [0,1]} \mu_{\tilde{A}}(x,u)/(x,u), \tag{6.2}$$

where $\iint$ denotes union over all admissible x and u.[3]

In (6.1), the restriction $u \in U \equiv [0,1]$ is consistent with the type-1 constraint that $0 \leq \mu_A(x) \leq 1$, that is, when MF uncertainties disappear, a type-2 MF must reduce to a type-1 MF, in which case the variable u equals $\mu_A(x)$ and $0 \leq \mu_A(x) \leq 1$. The restriction $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$ is consistent with the fact that the amplitude of a MF should lie between or be equal to 0 and 1.

Unless otherwise stated, in this book:

1. The elements of X and U are assumed to be continuous real numbers. Where needed, their discrete versions are denoted X_d and U_d, respectively.
2. If one begins with continuous X and U and discretizes (samples) them for computing purposes, then X and U will continue to be viewed as continuous, because the original type-2 fuzzy sets were defined on continuous X and U.

Note that $\mu_{\tilde{A}}(x,u)$, whose domain is $X \times U$, is three-dimensional [just as $\mu_A(x)$ for a type-1 fuzzy set is two-dimensional]. Consequently, regardless of what kind of a type-2 fuzzy set one may be describing, for it to be called a type-2 fuzzy set its MF must be three-dimensional.

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[2]If U differs from $[0,1]$, then use $u \in U$ in (6.1), and do the same elsewhere, e.g. in (6.4a) and (6.4c).

[3]For discrete universes of discourse, in (6.2) $\int$ is replaced by $\sum$, X is replaced by X_d and $[0,1]$ is replaced by $\{0, u_1, u_2, \ldots, u_{n-1}, 1\}$.

Example 6.3 Figure 6.4 depicts a 3-D MF for a general type-2 fuzzy set. Observe that, for this example, primary variable $x \in [0,1]$, but in general X is application dependent, whereas U is always $[0,1]$.

Definition 6.2 For every $x \in X$ and $u \in [0,1]$, the value of $\mu_{\tilde{A}}(x,u), f_x(u)$, is called the *secondary grade* of x; hence, if $x' \in X$ and $u' \in [0,1]$, then $f_{x'}(u') \equiv \mu_{\tilde{A}}(x',u')$ where $0 \le f_{x'}(u') \le 1$.

Definition 6.3 A *secondary membership function*, $\mu_{\tilde{A}(x)}(u)$, is (Aisbett et al. 2010) a restriction of function $\mu_{\tilde{A}} : X \times [0,1] \to [0,1]$ to $x \in X$, i.e., $\mu_{\tilde{A}(x)} : [0,1] \to [0,1]$, or in fuzzy set notation:

$$\mu_{\tilde{A}(x)}(u) = \int_{u \in [0,1]} \mu_{\tilde{A}}(x,u)/u = \int_{u \in [0,1]} f_x(u)/u \tag{6.3}$$

Note, importantly, that $\tilde{A}(x)$ is a type-1 fuzzy set,[4] which is also referred to as a *secondary set*, and as such it can be represented by its $\alpha-$ cut decomposition (see Sect. 2.12). $\{\mu_{\tilde{A}(x)}(u) | u \in U\}$ is also called a *vertical-slice* of $\mu_{\tilde{A}}(x,u)$.

A viable alternative to symbol $\tilde{A}(x)$ is $\tilde{A}_x$. The latter is occasionally used in later chapters where parenthetical arguments are needed for other things.

If all of the secondary MFs have the same geometric shape (e.g., triangle, trapezoidal, Gaussian, etc.), then the name of that geometric shape is used as the name for the entire type-2 fuzzy set.

Because a secondary MF is a type-1 fuzzy set it can be convex or non-convex (see Definition 2.4). Almost all published results for type-2 fuzzy sets are for convex secondary MFs; however, beginning in the year 2010 (Tahayori et al. 2010), interest in non-convex secondary MFs awakened. As of the writing of this book, there are too many unanswered theoretical and computational issues associated with using non-convex secondary MFs (more will be said about this in Chap. 7); hence, unless otherwise stated, *in this book secondary MFs will always be convex functions*.

Example 6.4 Figure 6.5 depicts triangle secondary MFs at five nonzero values of the primary variable $(x = x_1, \ldots, x_5)$ and spike secondary MFs at $x = 0$ and $x = x_6$. Triangle secondary MFs are shown only because they are easy to draw. All of the secondary MFs have one maximum membership grade equal to 1, and so they are *normal* type-1 fuzzy sets. Why there are spike secondary MFs at $x = 0$ and $x = x_6$ will be explained in Example 6.8. Finally, because the secondary MFs are triangles (spikes can be interpreted as triangles) this type-2 fuzzy set is called a *triangle type-2 fuzzy set*, whereas the fuzzy set in Fig. 6.4 is called a *Gaussian type-2 fuzzy set*, because all of its secondary MFs are Gaussians.

[4] Notation $\mu_{\tilde{A}(x)}(u)$ [which is different from the notation used in Mendel (2001)] is consistent with the usual labeling of a MF for a type-1 fuzzy set, where $\tilde{A}(x)$ is the name of that set.

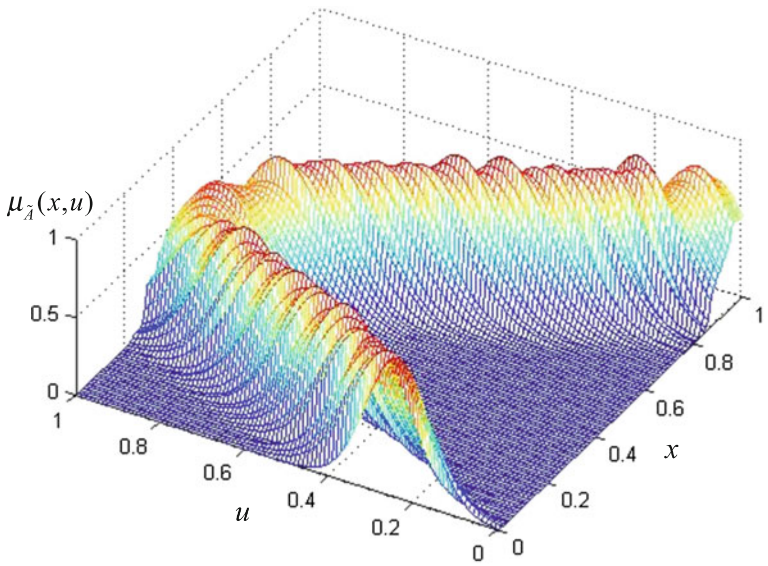

Fig. 6.4 A 3-D MF for a general type-2 fuzzy set (courtesy of Prof. Frank Rhee)

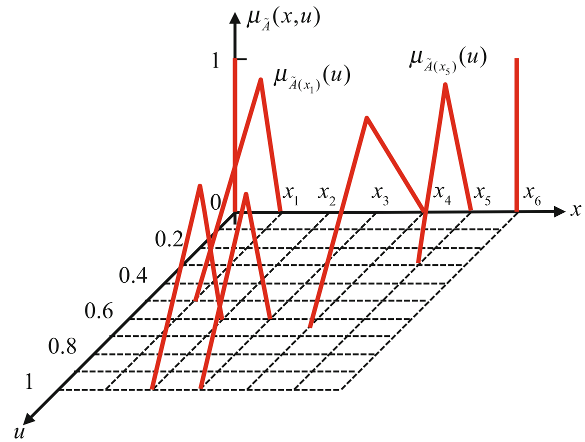

Fig. 6.5 Secondary MFs (in *red*) at seven values of the primary variable

Example 6.5 This is a continuation of Example 6.4. Shown in Fig. 6.6 are the nonzero secondary grades at the Fig. 6.5 grid points of $X \times U$. A few of the nonzero secondary grades are labeled, e.g. $f_{x_1}(0.1), f_{x_4}(0.3)$ and $f_{x_6}(0)$.

Secondary MFs can be chosen in different ways. In Examples 6.4 and 6.5 they were chosen a priori as triangles. Triangle, trapezoid, truncated Gaussian, etc., secondary MFs are very commonly used during designs of type-2 fuzzy systems when all MF parameters are optimized.

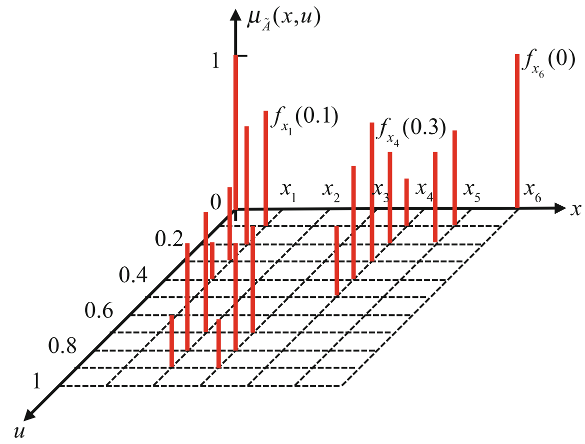

Fig. 6.6 Nonzero secondary grades at the grid points of $X \times U$ for the seven secondary MFs that are depicted in Fig. 6.5

Secondary MFs can also be obtained directly from data, e.g., Moharrer et al. (2013) collect data about words that are then processed using statistics-based techniques that include confidence intervals that vary between 0 and 1, which then play the role of secondary grades; and, Bilgin et al. (2012a, b, c, 2013a, b) begin with a linguistic term that is modeled as either a left or right shoulder, apply n concentration hedges to it [the kind that make the membership of the hedged word contained within the membership of its less-hedged word (Sect. 2.9)], and then layer the hedged FOUs one on top of another at secondary grades equal to integer multiples of $1/n$. This *linear adjective* model is so far limited to shoulder models. Rakshit et al. (2013, 2016) solve an optimization problem to determine secondary MFs that characterize the reliability in the assignment of the primary membership using subject data.

Definition 6.4 J_x, the *primary membership* of x, is[5]:

$$J_x = \{(x,u)|u \in [0,1],\ \mu_{\tilde{A}}(x,u) > 0\} \tag{6.4a}$$

It can also be expressed as a subset of $\{x\} \times I_x$ (Mendel et al. 2016), i.e.,

$$J_x = \{x\} \times I_x, \tag{6.4b}$$

where

$$I_x = \{u \in [0,1] | \mu_{\tilde{A}}(x,u) > 0\} \tag{6.4c}$$

[5]In Mendel (2001) and much of the type-2 literature it is stated that $J_x \subseteq [0,1]$ and J_x is left undefined, both of which have been very problematic. In this book, the statement $J_x \subseteq [0,1]$ has been abandoned and J_x is defined. For additional discussions about this, see Sect. 6.6 and Mendel et al. (2016).

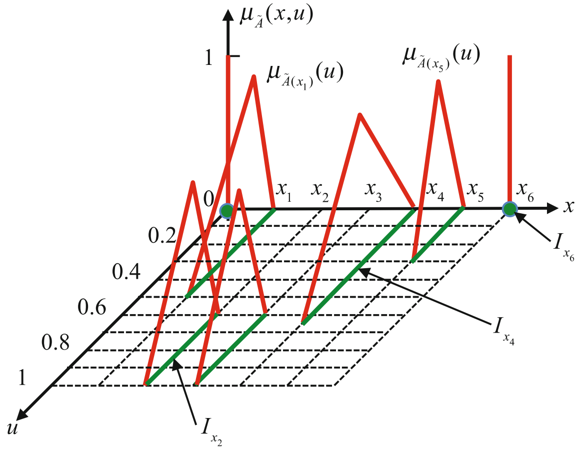

Fig. 6.7 Supports of the secondary MFs are shown in *green* for the seven secondary MFs that are depicted in Fig. 6.5

I_x is the *support of the secondary MF*, and can be *connected*[6] or *disconnected*.

Example 6.6 This is a further continuation of Example 6.4. Shown in Fig. 6.7 are the supports of the secondary MFs at the seven values of the primary variable that are depicted in Fig. 6.5. A few of the supports are labeled (e.g., I_{x_2}, I_{x_4}, and I_{x_6}). Observe that at $x = 0$ and $x = x_6$, I_0 and I_{x_6} are single points, whereas I_{x_1}, I_{x_2}, I_{x_3}, I_{x_4} and I_{x_5} are intervals. In this example each I_x is *connected*. An example where this is not so is given in Example 6.7.

Definition 6.5 (Mendel et al. 2016) The *support* of $\tilde{A}$ (Aisbett et al. 2010) comprises all $(x, u) \in X \times [0, 1]$ such that $\mu_{\tilde{A}}(x, u) > 0$, and is also called the *domain of uncertainty* of $\tilde{A}$, $\mathrm{DOU}(\tilde{A})$, i.e.,

$$\mathrm{DOU}(\tilde{A}) = \{(x, u) \in X \times [0, 1] | \mu_{\tilde{A}}(x, u) > 0\} = \bigcup_{x \in X} J_x. \tag{6.5}$$

It is worth noting (Mendel et al. 2016) that the word "uncertainty" in "DOU" refers to the uncertainty that is expressed by the secondary MF; therefore, one might argue that $\mathrm{DOU}(\tilde{A})$ should be even more restricted than above, i.e., it should be defined by removing isolated points in the support that have unity secondary membership grades. This is not done because "DOU" refers to a collective entity, and so even if there is only one element in it that has an uncertain membership grade it can be called a "DOU."

Example 6.7 Shown in Fig. 6.8 are the secondary MFs for a general type-2 fuzzy set at five values of the primary variable. Observe that each of the supports of the secondary MFs, shown in green, is *disconnected*. $\mathrm{DOU}(\tilde{A})$ is comprised of all the green supports of the secondary MFs, which also corresponds collectively to the union of all of the primary memberships, as in (6.5).

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[6]A set $A \subseteq \mathbb{R}$ is *connected* if and only if A is an interval (closed, open, or neither).

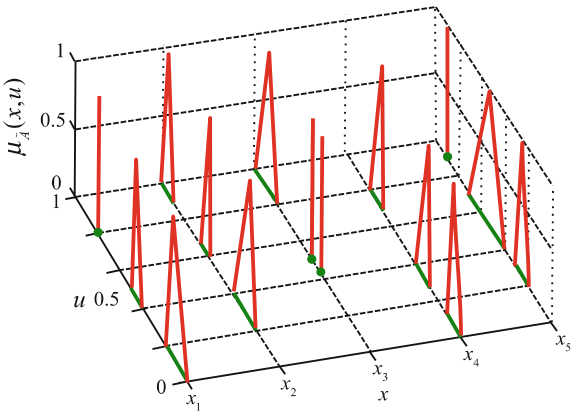

Fig. 6.8 The domain of uncertainty is shown in *green*

Definition 6.6 When the support of the secondary MF, I_x, is *closed* (so that it is connected; see footnote 6) for $x \in X$, i.e.,

$$I_x = \{u \in [0,1] | \mu_{\tilde{A}}(x,u) > 0\} = [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)], \tag{6.6}$$

where (Aisbett et al. 2010)[7]

$$\overline{\mu}_{\tilde{A}}(x) = \sup\{u | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\} \tag{6.7}$$

$$\underline{\mu}_{\tilde{A}}(x) = \inf\{u | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\}, \tag{6.8}$$

then the domain of uncertainty of $\tilde{A}$ is called the *footprint of uncertainty*[8] of $\tilde{A}$, $\mathrm{FOU}(\tilde{A})$, i.e.,

$$\mathrm{DOU}(\tilde{A}) = \mathrm{FOU}(\tilde{A}) = \{(x,u) | x \in X \text{ and } u \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]\}. \tag{6.9}$$

Note that $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ are called the *lower* and *upper MFs*of $\mathrm{FOU}(\tilde{A})$ (Mendel and Liang 1999), and are the lower and upper (type-1 fuzzy set) bounding functions

[7]Aisbett et al. (2010) were the first to express $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ as in (6.7) and (6.8) and to point out that $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ are the infimum and supremum of the support of $\mu_{\tilde{A}}$ expressed as functions on X. See Mendel et al. (2016) for additional discussions about this.

[8]The term *footprint of uncertainty* is not mentioned in Karnik and Mendel (2001b), and is defined only using words in Karnik and Mendel (2001a, Example 1), Karnik et al. (1999, Example 3.1), Liang and Mendel (2000, Definition 1), and Mendel (2007, Box 1). It was coined by Mendel as a simple way to verbalize and describe the two-dimensional domain of support for a type-2 fuzzy set's MF, and appears for the first time in Karnik et al. (1999) and Karnik and Mendel (2001a). Because the phrase *footprint of uncertainty* is not general enough to accommodate all kinds of type-2 fuzzy sets, the phrase *domain of uncertainty* was introduced in Mendel and John (2002). The concepts of *lower* and *upper MFs* were first described in Liang and Mendel (2000).

of the FOU, respectively. Commonly used abbreviations for the lower and upper MFs of $\tilde{A}$ are LMF$(\tilde{A})$ and UMF$(\tilde{A})$.

The definition of FOU$(\tilde{A})$ in (6.9) guarantees that it is connected if X is connected and if LMF$(\tilde{A})$ and UMF$(\tilde{A})$ are continuous functions $X \to [0,1] \subseteq \mathbb{R}$.

Definition 6.7 The *support* of LMF$(\tilde{A})$ [UMF$(\tilde{A})$] is the crisp set of all points $x \in X$ such that $\underline{\mu}_{\tilde{A}}(x) > 0$ [$\overline{\mu}_{\tilde{A}}(x) > 0$].

In general, the supports of LMF$(\tilde{A})$ and UMF$(\tilde{A})$ are different, and the support of LMF$(\tilde{A})$ is contained within the support of UMF$(\tilde{A})$.

Example 6.8 This is a further continuation of Example 6.4. Shown in Fig. 6.9 is the shaded-in FOU associated with the secondary MFs that are depicted in Fig. 6.5. In fact, Fig. 6.5 was actually obtained after Fig. 6.9 was constructed, by deleting the FOU in Fig. 6.9.

It should now be clear why there are spike secondary MFs at $x = 0$ and $x = x_6$; the secondary MFs at $x = 0$ and $x = x_6$ are single points.

Observe in Fig. 6.9 that both LMF$(\tilde{A})$ and UMF$(\tilde{A})$ are trapezoids, and that UMF$(\tilde{A})$ is a normal type-1 fuzzy set, whereas LMF$(\tilde{A})$ is a subnormal type-1 fuzzy set. In general, the LMF or the UMF of an FOU does not have to be a normal type-1 fuzzy set. In fact, in a type-2 fuzzy system in which fired rules are first combined by using the union operation the resulting FOU will not have normal lower and upper MFs. This will be demonstrated in Chap. 9.

Observe, also, that the support of LMF$(\tilde{A})$ is $[x_1, x_4]$, the support of UMF$(\tilde{A})$ is $[0, x_6]$, and $[x_1, x_4] \subset [0, x_6]$.

Definition 6.8 For continuous universes of discourse X and $U = [0,1]$, an[9] *embedded type-2 fuzzy set*, denoted $\tilde{A}_e$, is (Aisbett et al. 2010) $\mu_{\tilde{A}}$ restricted to $\{(x, u(x)) | x \in X\}$ for some $u : X \to [0,1]$, i.e., $\mu_{\tilde{A}_e} : X \to [0,1]$ when $\mu_{\tilde{A}_e}(x) = \mu_{\tilde{A}}(x, u(x))$, or in fuzzy set notation (Karnik and Mendel 1998)[10]:

$$\tilde{A}_e = \int_{x \in X} [f_x(u)/u]/x \quad u \in [0,1] \tag{6.10}$$

Set $\tilde{A}_e$ is embedded in set $\tilde{A}$, and there are an uncountable number of embedded type-2 fuzzy sets.

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[9] *Embedded type-2* and *type-1 fuzzy sets* were first described in Karnik and Mendel (1998).

[10] In Mendel (2001) and many later references, (6.10) is stated for $u \in J_x$ instead of $u \in [0,1]$. While this is okay for connected primary memberships, it is not okay for disconnected primary memberships, because J_x is undefined when $\mu_{\tilde{A}}(x, u) = 0$ (see Definition 6.4). Using $u \in [0,1]$ in (6.10) remedies this. The same is true for (6.13).

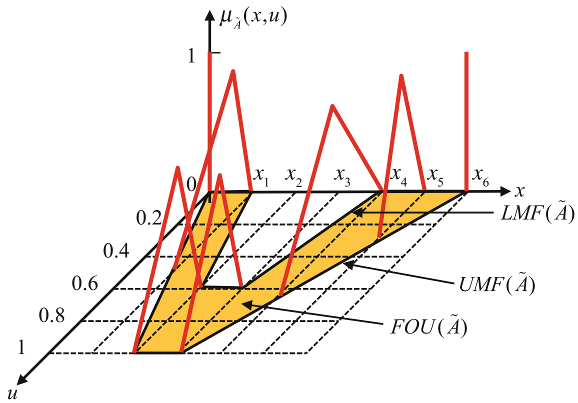


Fig. 6.9 The FOU for the secondary MFs that are depicted in Fig. 6.5

Definition 6.9 Given the closed I_x in (6.6), when $x = x_i$ and u is *discretized*, $[\underline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{A}}(x_i)]$ becomes a set with $M_i + 1$ elements, $\{u_j^i\}_{j=1}^{M_i+1}$, defined herein as:

$$u_j^i \equiv \underline{\mu}_{\tilde{A}}(x_i) + (j-1) \times \left[\frac{\overline{\mu}_{\tilde{A}}(x_i) - \underline{\mu}_{\tilde{A}}(x_i)}{M_i - 1}\right], \quad j = 1, \ldots, M_i \tag{6.11}$$

This I_x is denoted I_x^D to signify it is for discretized u.

Definition 6.10 When X and U are discretized and I_x are closed, an *embedded type-2 fuzzy set* $\tilde{A}_e^j$ has N elements, one each from $I_{x_1}^D, I_{x_2}^D, \ldots, I_{x_N}^D$, namely $u_j^1, u_j^2, \ldots, u_j^N$, where u_j^i is defined in (6.11); and, each element has an associated secondary grade, namely $f_{x_1}(u_j^1), f_{x_2}(u_j^2), \ldots, f_{x_N}(u_j^N)$, i.e., in fuzzy set notation:

$$\tilde{A}_e^j = \sum_{i=1}^{N} \left[f_{x_i}(u_j^i)/u_j^i\right]/x_i \tag{6.12}$$

Set $\tilde{A}_e^j$ is embedded in $\tilde{A}$, and, there are a total of $\prod_{i=1}^{N} M_i \; \tilde{A}_e^j$.

Of course, u can be discretized in ways that are different from (6.11),[11] in which case the formula for u_j^i would change but the conceptual idea of an embedded type-2 set would not change.

Definition 6.11 For continuous universes of discourse X and U, an *embedded type-1 fuzzy set*, denoted A_e, is (Aisbett et al. 2010) a function whose range is a subset of $[0, 1]$ determined by $\mu_{\tilde{A}}$, i.e., $u : X \to [0, 1]$ satisfying $\mu_{\tilde{A}}(x, u(x)) > 0$ for $x \in X$, or in fuzzy set notation (Karnik and Mendel 1998):

[11]Equation (6.11) leads to uniform sampling at each x_i. A *grid method* of discretization is given in Greenfield and John (2007), in which x and u are "evenly divided into a rectangular grid, as determined by the degree of discretization of the x and u axes".

$$A_e = \int_{x \in X} u/x \quad u \in [0,1] \tag{6.13}$$

Set A_e acts as the domain for $\tilde{A}_e$ in (6.10), and there are an uncountable number of embedded type-1 sets.

Definition 6.12 When X and U are discretized and I_x are closed, an *embedded type-1 fuzzy set* A_e^j has N elements, one each from $I_{x_1}^D, I_{x_2}^D, \ldots, I_{x_N}^D$, namely $u_j^1, u_j^2, \ldots, u_j^N$, where u_j^i is defined in (6.11), i.e., in fuzzy set notation:

$$A_e^j = \sum_{i=1}^{N} u_j^i / x_i \tag{6.14}$$

Set A_e^j acts as the domain for $\tilde{A}_e^j$ in (6.12), and there are a total of $\prod_{i=1}^{N} M_i \, A_e^j$.

Example 6.9 This is a further continuation of Example 6.5, in which the grid on $X \times U$ defines $X_d \times U_d$. Observe from Fig. 6.6 that there are $\prod_{i=0}^{6} M_i = 1 \times 4 \times 3 \times 3 \times 5 \times 2 \times 1 = 360$ embedded type-2 and type-1 fuzzy sets. Figure 6.10 depicts two of the embedded type-2 and type-1 fuzzy sets. The former are shown in red and the latter are shown in green. For Fig. 6.10a:

$$\begin{cases} A_e = 0/0 + 0.4/x_1 + 0.8/x_2 + 0.9/x_3 + 0.2/x_4 + 0.1/x_5 + 0/x_6 \\ \tilde{A}_e = 1/0/0 + 0.25/0.4/x_1 + 0.6/0.8/x_2 + 0.3/0.9/x_3 + 0.6/0.2/x_4 + 0.6/0.1/x_5 + 1/0/x_6 \end{cases} \tag{6.15a}$$

For Fig. 6.10b:

$$\begin{cases} A_e = 0/0 + 0.1/x_1 + 0.9/x_2 + 0.7/x_3 + 0.5/x_4 + 0.2/x_5 + 0/x_6 \\ \tilde{A}_e = 1/0/0 + 0.75/0.1/x_1 + 0.3/0.9/x_2 + 0.75/0.7/x_3 + 0.45/0.2/x_4 + 0.55/0.2/x_5 + 1/0/x_6 \end{cases} \tag{6.15b}$$

Definition 6.13 Assume that each of the secondary MFs of a type-2 fuzzy set has only one secondary grade that equals 1. A *principal MF*is the union of all such points at which this occurs, i.e.,

$$\mu_{\text{principal}}(x) = \int_{x \in X} u/x \quad \text{where } f_x(u) = 1 \tag{6.16}$$

and[12] is associated with a type-1 fuzzy set.

[12]Note that $f_x(u) = 1$ can be solved for u (at $x \in X$), so that u can be expressed as $u = f_x^{-1}(1)$.

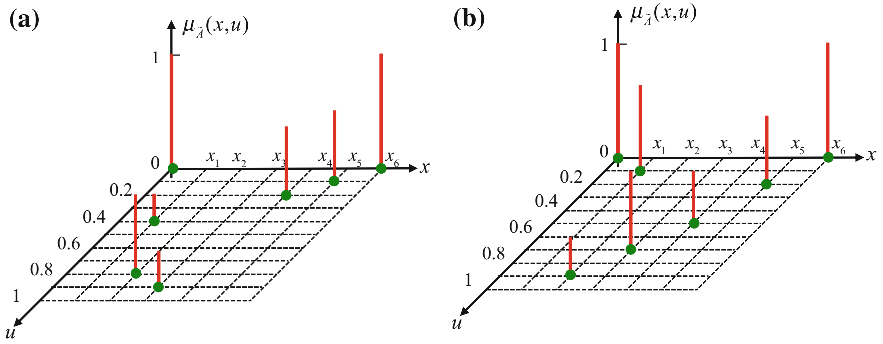
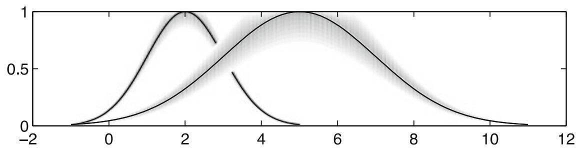


Fig. 6.10 Two examples of (*red*) embedded type-2 and (*green*) type-1 fuzzy sets, each associated with the sampled type-2 fuzzy set that is depicted in Fig. 6.6

Fig. 6.11 Pictorial representation of a Gaussian type-2 fuzzy set



Example 6.10 The principal MFs for the two Gaussian type-2 fuzzy sets whose MFs are depicted in Fig. 6.11 are the solid Gaussian curves.

Definition 6.14 The MF of a type-2 fuzzy set is said to be *parsimonious* when it is described by a small number of parameters.

Parsimonious models have a very long history in the field of mathematical modeling, e.g., in system identification [e.g., Ljung (1999)] one always tries to use a model with the fewest number of parameters to fit data. Because a type-2 fuzzy set is a mathematical model, parsimony should be adhered to for such models. Justification for preferring a parsimonious model seems to go all the way back to Occam.[13]

[13]William of Ockham (also Occam) (c. 1288–1348) was an English Franciscan friar and scholastic philosopher, from Ockham, a small village in Surrey, near East Horsley. One important contribution that he made to modern science and modern intellectual culture was through the principle of parsimony in explanation and theory building what came to be known as Ockham's razor. This maxim, as interpreted by Bertrand Russell, states that if one can explain a phenomenon without assuming this or that hypothetical entity, there is no ground for assuming it, i.e., one should always opt for an explanation in terms of the fewest possible number of causes, factors, or variables. The most useful statement of the Ockham’s razor principle is “when you have two competing theories which make exactly the same predictions, the one that is simpler is the better.” This principle is sometimes misstated as “keep it as simple as possible.” One can have two (or more) competing theories that lead to different predictions. Occam’s Razor does not apply in that case, because the results that are obtained from the competing theories are different.

Although parsimony is advocated in this book, this does not mean there can be no flexibility about it, e.g., if a type-2 fuzzy set that is described by n_1 parameters does not lead to improved performance for an application, a type-2 fuzzy set model that is described by $n_1 + 1$ parameters should then be tried, etc.

Definition 6.15 A *totally symmetrical type-2 fuzzy set* is one whose lower and upper MFs, as well as its secondary MFs, are symmetrical.

Definition 6.16 (Pedrycz 2015) An *information granule* is a collection of elements drawn together by their closeness (resemblance, proximity, functionality, etc.), articulated in terms of some useful spatial, temporal, or functional relationship.

Clearly, a type-2 fuzzy set is an information granule (as is a type-1 fuzzy set). So, for example, when a word is modeled by a type-2 fuzzy set, this can be called a *granular model of a word*.

In the rest of this chapter and book, type-2 fuzzy set, interval type-2 fuzzy set and type-1 fuzzy set are often abbreviated to T2 FS, IT2 FS and T1 FS, respectively.

6.3 Definitions of an IT2 FS and Associated Concepts

The most widely studied T2 FS has all of its secondary grades equal to 1 and is called an IT2 FS.

Definition 6.17 When $u \in [0, 1]$ and $\mu_{\tilde{A}}(x, u) = 1$ for $x \in X$, then $\tilde{A}$ is called an *IT2 FS*. An IT2 FS is completely described by its DOU, so that:

$$\tilde{A} = 1/\text{DOU}(\tilde{A}) \tag{6.17}$$

(6.17) is an expressive equation that means $\mu_{\tilde{A}}(x, u) = 1$ for $(x, u) \in \text{DOU}(\tilde{A})$, where $\text{DOU}(\tilde{A})$ is given in (6.5) in which $\mu_{\tilde{A}}(x, u) > 0$ is replaced by $\mu_{\tilde{A}}(x, u) = 1$.

Equation (6.17) does not mean that one should ignore the secondary grades, because doing so would reduce the IT2 FS from a 3-D entity to a 2-D entity and so it would no longer be a T2 FS. All of the earlier definitions in this chapter can be used for an IT2 FS simply by setting the secondary grades equal to 1, and so they are not repeated here.

An IT2 FS is said to be *maximally uncertain* because all of its secondary membership grades are the same value. A general T2 FS is said to be *less uncertain than an IT2 FS* because its secondary grades are not all the same. How to provide a quantitative measure of uncertainty for a T2 FS is discussed in Chap. 8.

Definition 6.18 (Mendel et al. 2016) An IT2 FS is called a *closed IT2 FS* (CIT2 FS) when I_x is closed for $x \in X$ (see Definition 6.6). In this case, $\mathrm{DOU}(\tilde{A}) = \mathrm{FOU}(\tilde{A})$; hence, for a CIT2 FS, (6.17) can be expressed as:

$$\tilde{A} = 1/\mathrm{FOU}(\tilde{A}) \tag{6.18}$$

where $\mathrm{FOU}(\tilde{A})$ is defined in (6.9). Alternate ways to express $\mathrm{FOU}(\tilde{A})$ for a CIT2 FS are [see, also, (6.5)]:

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{x \in X} \{x\} \times I_x = \bigcup_{x \in X} J_x \tag{6.19}$$

The vertical-slice of a CIT2 FS is a type-1 interval fuzzy number on U (Definition 2.6).

The CIT2 FS has received the most attention to-date, and is the one that is emphasized in this book (more about this in Sect. 6.6). Such an IT2 FS is related to an *interval-valued fuzzy set* (IVFS).

Definition 6.19 (Bustince et al. 2015) An *IVFS* A on the universe $X \neq \varnothing$ is a mapping[14]

$$A : U \to L([0,1]) = \left\{ \mathbf{x} = [\underline{x}, \bar{x}] | (\underline{x}, \bar{x}) \in [0,1]^2 \text{ and } \underline{x} \leq \bar{x} \right\} \tag{6.20}$$

such that the membership degree of $x \in X$ is given by $A(x) = [\underline{A}(x), \overline{A}(x)] \in L([0,1])$, where $\underline{A} : X \to [0,1]$ and $\overline{A} : X \to [0,1]$ are mappings defining the lower and upper bound of the membership interval $A(x)$, respectively.

Observe that no secondary grade is assigned to an IVFS and so it is not the same as an IT2 FS. *It is* $\mathrm{FOU}(\tilde{A})$ *that is an IVFS.*[15] More will be said about IT2 FSs and IVFSs in Sect. 6.6.

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[14]The statement of this definition uses the notation in Bustince et al. (2015). To connect that notation with the notation that is used in this book, set $A(x) = I_x$, $\underline{A}(x) = \underline{\mu}_{\tilde{A}}(x)$ and $\bar{A}(x) = \bar{\mu}_{\tilde{A}}(x)$. There are also other notations that are used for an IVFS, e.g. (1) Goralczany (1987) calls them an i–v fuzzy set, where $\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ ($\bar{\mu}_A$ does not denote an UMF); (2) Bustince (2000) denotes the degree of membership of an element x of an IVFS A as $M_A(x) = [M_{AL}(x), M_{AU}(x)]$; and, (3) Klir and Yuan (1995) denote the degree of membership of an element x of an IVFS A as $A(x) = [L_A(x), U_A(x)]$.

[15]Mendel et al. (2016, p. 341) states: "… every closed IT2 FS is an IVFS." This is not quite correct. An IT2 FS has always had a secondary grade associated with it, whereas an IVFS has not. Put a different way, the MF of a closed IT2 FS is three-dimensional whereas the MF of an IVFS is two-dimensional. An analogy can be made between these two kinds of fuzzy sets and a rectangular skyscraper. The IVFS is analogous to the foundation of this building, whereas the closed IT2 FS is analogous to the entire rectangular building. Clearly, the foundation cannot claim to be the entire building. The distinction between an IVFS and a closed IT2 FS is not very important when one is only interested in studying (closed) interval type-2 fuzzy systems because the unity secondary grades of the closed IT2 FSs convey no useful information; it is the FOU of the closed IT2 FS that plays the important role, and, as just stated, the FOU is the same as an IVFS; it is in this sense that the above quoted phrase should be

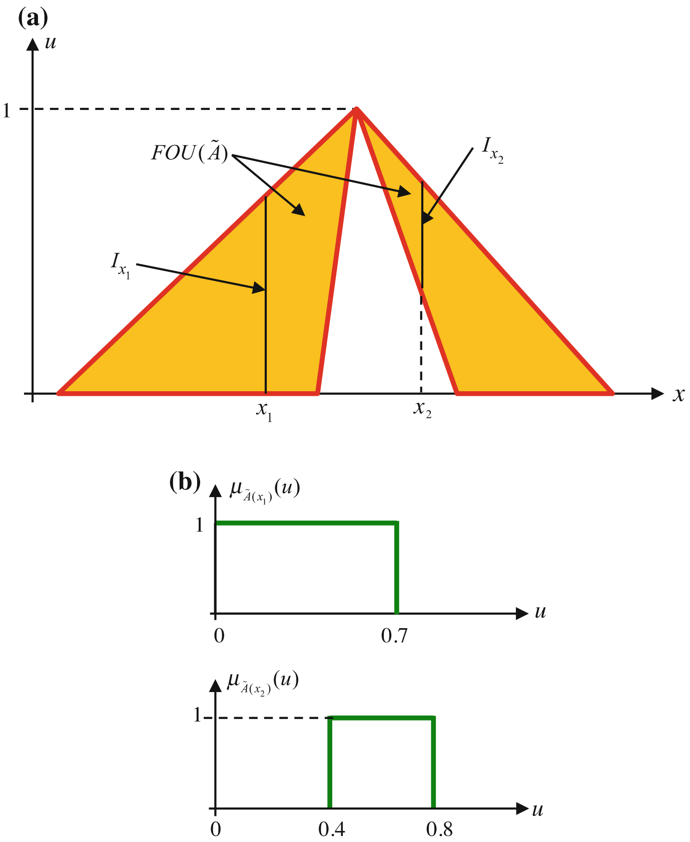


Fig. 6.12 The shaded region in **a** is the FOU of a CIT2 FS. The supports of the secondary MFs, I_{x_1} and I_{x_2}, are shown at x_1 and x_2 in (**a**), and their associated secondary MFs, $\mu_{\tilde{A}(x_1)}(u)$ and $\mu_{\tilde{A}(x_2)}(u)$, are shown in (**b**)

FOUs can be chosen in different ways. For CWW applications, data can be collected from a group of subjects about a word (as already explained in Sects. 4.4 and 5.2) after which that data is mapped into an FOU (more about this is explained in Sect. 10.4). For non-CWW applications, a type-1 MF may be blurred to obtain an FOU, or (more commonly) parsimonious parametric UMFs and LMFs are chosen a priori to define an FOU (e.g., triangle, trapezoid, Gaussian, etc.); their parameters are either prespecified or optimized during a design procedure (more about this is explained in Sect. 10.2).

(Footnote 15 continued)

interpreted. However, the distinction between an IVFS and a closed IT2 FS is very important when one studies general T2 FSs, for which the secondary grades are no longer all unity. The horizontal-slice representation of a general T2 FS (Sect. 6.7.3) uses slices in the third dimension, which exist for closed IT2 FSs but not for IVFSs.

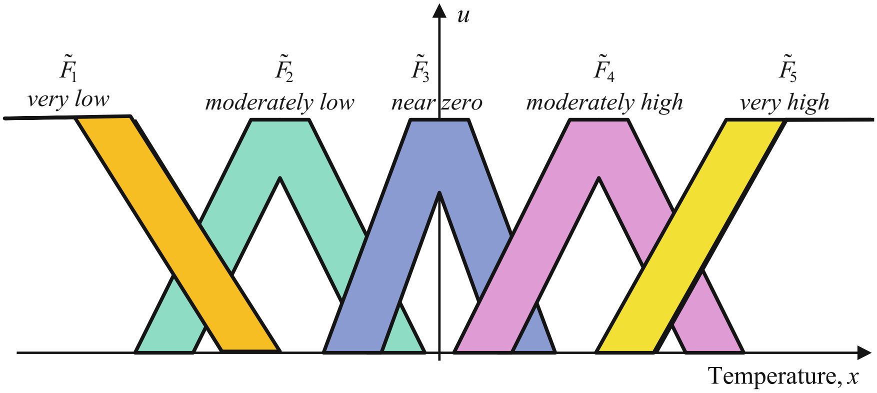
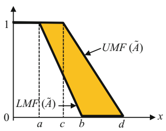
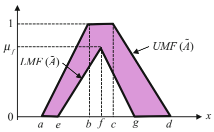
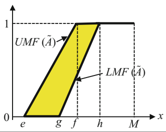


Fig. 6.13 FOUs for Fig. 6.1 MFs

Example 6.11 An example of an FOU for a CIT2 FS is depicted in Fig. 6.12a by using shaded regions. The uniform shading indicates that the FOU is for a CIT2 FS. Shown also on Fig. 6.12a are the supports of two secondary MFs and in Fig. 6.12b the two secondary MFs at x_1 and x_2.

Example 6.12 Figure 6.13 is an "uncertain" version of Fig. 6.1, in which uncertainty has been assumed about the knowledge of where to locate the triangle apex and basepoints, and where to locate the shoulder points of the two end MFs. As in Fig. 6.11a, the uniformly shaded FOUs denote the fact that it is for a CIT2 FS.

Table 6.1 Formulas for upper and lower MFs for piecewise linear left shoulder, interior and right shoulder FOUs

FOU	$\overline{\mu}_{\tilde{A}}(x)$	$\underline{\mu}_{\tilde{A}}(x)$
Left shoulder 	$\overline{\mu}_{\tilde{A}}(x)=\begin{cases}1 & 0\le x\le c\\ \frac{d-x}{d-c} & c<x\le d\\ 0 & x>d\end{cases}$	$\underline{\mu}_{\tilde{A}}(x)=\begin{cases}1 & 0\le x\le a\\ \frac{b-x}{b-a} & a<x\le b\\ 0 & x>b\end{cases}$
Interior 	$\overline{\mu}_{\tilde{A}}(x)=\begin{cases}0 & x\le a\\ \frac{x-a}{b-a} & a<x\le b\\ 1 & b<x\le c\\ \frac{d-x}{d-c} & c<x\le d\\ 0 & x>d\end{cases}$	$\underline{\mu}_{\tilde{A}}(x)=\begin{cases}0 & x\le e\\ \frac{x-e}{f-e}\mu_f & e<x\le f\\ \frac{g-x}{g-f}\mu_f & f<x\le g\\ 0 & x>g\end{cases}$
Right shoulder 	$\overline{\mu}_{\tilde{A}}(x)=\begin{cases}0 & x\le e\\ \frac{x-e}{f-e} & e<x\le f\\ 1 & f<x\le M\end{cases}$	$\underline{\mu}_{\tilde{A}}(x)=\begin{cases}0 & x\le g\\ \frac{x-g}{h-g} & g<x\le h\\ 1 & h<x\le M\end{cases}$

Formulas for the upper and lower MFs for piecewise linear left shoulder, interior and right shoulder FOUs are given in Table 6.1.

Example 6.13 As another example of an FOU, the word survey described in Sect. 4.4.2 is returned to. Figure 6.14 indicates how the uncertainties associated with the two endpoints of an interval can be translated into an FOU for pre-chosen triangular type-1 MFs. Let $x = m_a$ denote the average value for the left-hand point of subject data intervals and $x = m_b$ denote the average value for the right-hand point of subject data intervals. The standard deviation for the location of the left-hand point is denoted σ_a, and the standard deviation for the location of the right-hand point is denoted σ_b. m_a and m_b are shown as solid circles in Fig. 6.14.

Uncertainty intervals are defined for m_a and m_b, as $[m_a - \sigma_a, m_a + \sigma_a]$ and $[m_b - \sigma_b, m_b + \sigma_b]$, respectively. There are two cases that must be considered: (1) $m_b - \sigma_b > m_a + \sigma_a$ and (2) $m_b - \sigma_b < m_a + \sigma_a$. In the first case (Fig. 6.14), the uncertainty interval for m_a does not overlap with the uncertainty interval for m_b, but in the second case it does. A construction for the FOU is provided next for the first case, and the construction of the FOU for the second case is left as an exercise (Exercise 6.2).

When $m_b - \sigma_b > m_a + \sigma_a$, as in Fig. 6.14:

1. Let $m_b - m_a = l$.
2. Locate the apex of the triangle at $l/2$, and assign it unity height.
3. The left-hand vertex of the triangle, on the horizontal axis, can range from $m_a - \sigma_a$ to $m_a + \sigma_a$; the region of uncertainty for the left-hand leg is the shaded left-hand triangle whose vertices are at: $(m_a - \sigma_a, 0)$, $(m_a + \sigma_a, 0)$ and $(l/2, 1)$.
4. The right-hand vertex of the triangle, on the horizontal axis, can range from $m_b - \sigma_b$ to $m_b + \sigma_b$; the region of uncertainty for the right-hand leg is the shaded right-hand triangle whose vertices are at: $(l/2, 1)$, $(m_b - \sigma_b, 0)$ and $(m_b + \sigma_b, 0)$.
5. The FOU is the union of all points in the two shaded triangles. This construction also gives some meaning to the FOU shown in Fig. 6.12a.

This is by no means the only way in which an FOU can be associated with the two endpoints of an interval for a linguistic term. Section 10.4 describes another (better) choice.

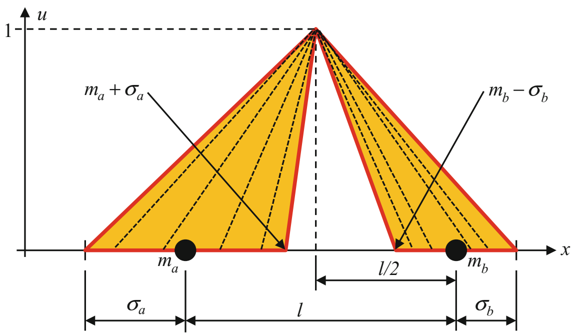


Fig. 6.14 FOU when interval end-point information is requested. Uncertainty interval for m_a does not overlap with the uncertainty interval for m_b. The *dashed black lines* are representative type-1 fuzzy set MFs for each subject

Example 6.14 Ulu et al. (2013) define a *rectangular type-2 fuzzy granule* as a rectangular prism whose base area represents the FOU and height $\alpha \in [0, 1]$ represents the secondary MFs. When $\alpha = 1$ this fuzzy granule becomes a *rectangular interval type-2 fuzzy granule*. A T2 FS whose FOU is formed by using rectangular interval type-2 fuzzy granules is called a *Granular type-2 fuzzy set*, and a CIT2 FS whose FOU is formed by using rectangular interval type-2 fuzzy granules is called a *Granular CIT2 FS*.

Each rectangular interval type-2 fuzzy granule is characterized by four parameters (Fig. 6.15a) x_L, x_R, u_L and u_R; however, because adjacent granules touch, a Granular CIT2 FS comprised of K granules is characterized by $4 + 3(K - 1)$ parameters. Ulu et al. (2013) note that Granular type-2 MFs provide an "opportunity to express the [MF] uncertainties without any dependence on the shape of a specific function." Unfortunately, this shape independence comes at the expense of many more parameters than are needed to describe a nongranular CIT2 FS, e.g., the FOU in Fig. 6.15b can be approximated by using trapezoidal lower and upper MFs, with a total of 9 parameters (this assumes that the UMF is a normal type-1 fuzzy set, so it can be defined by 4 parameters, and that the LMF is subnormal so it can be defined by 5 parameters, one of which includes a parameter for its height), whereas the four granules in that figure need 13 parameters to describe them. Additionally, because of the discontinuous nature of adjacent granules, K must also be considered as a design parameter.

Because of the non-parsimonious nature of Granular CIT2 FS they are not used in the rest of this book.

Definition 6.20 A *primary MF* (Aisbett et al. 2010) is a member of a family of functions into the unit interval which are parameterized by Ω, i.e., $\mu_{\varphi} : X \to [0, 1]$, $\varphi \in \Omega$. For short, $\mu_A(x)$ is used to denote a primary MF. It will be subject to some restrictions on its parameters. The family of all primary MFs creates an FOU for a CIT2 FS.

Example 6.15 An example of a primary MF is the triangle, depicted in Fig. 6.3a whose vertices (parameters) have been assumed to vary over some interval of values. The FOU associated with this primary MF is shown in Fig. 6.3c.

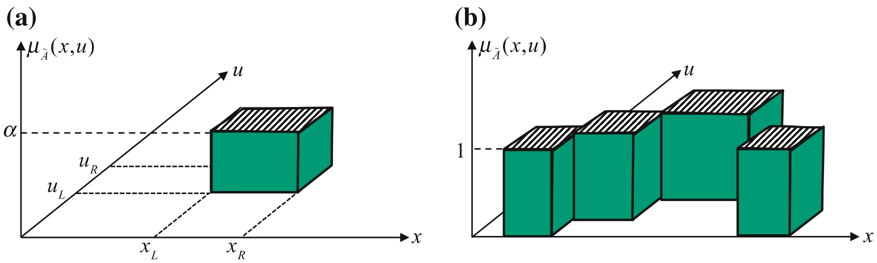


Fig. 6.15 **a** A rectangular type-2 fuzzy granule, and **b** an example of a Granular IT2 FS. In **b** when the *top shaded rectangles* are moved down to the $x - u$ plane, the result is the FOU of $\tilde{A}$

Definition 6.21 (Garibaldi et al. 2005) A *nonstationary fuzzy set* $\dot{A}$ is characterized by a MF $\mu_{\dot{A}}(x,t)$, where $x \in X$, $\mu_{\dot{A}}(x,t) \in [0,1]$ and t is a free variable, *time*—the time at which the fuzzy set is instantiated, i.e., in fuzzy set notation:

$$\dot{A} = \int_{x \in X} \mu_{\dot{A}}(x,t)/x \tag{6.21}$$

Beginning with a type-1 MF, its three main alternative kinds of non-stationarity are: variation in location, variation in slope and noise variation. For example, let c denote the center value of a type-1 MF and $c(t) \equiv c + kf(t)$ where $f(t)$ is called a "perturbation function" which may be random, hence the terminology "nonstationary fuzzy set".

When $f(t)$ is a known deterministic function, then $\mu_{\dot{A}}(x,t)$ can be lower and upper bounded, in which case there is a direct connection between $\dot{A}$ and a CIT2 FS. On the other hand, when $f(t)$ is random (a random process), then $\dot{A}$ is a *random fuzzy set* (Lushu 1995). Such fuzzy sets are very different from fuzzy random variables (e.g., Buckley 2003; Moller and Beere 2004), and can be treated as nonlinear transformations of random processes. If the distribution function can be computed for the now random $\mu_{\dot{A}}(x,t)$, then perhaps lower and upper probability bounds can be established for each value of a primary variable. These bounds might then be somehow related to the lower and upper MFs of a CIT2 FS. How to carry out such calculations remains to be explored.

Nonstationary fuzzy sets are not used in this book.

6.4 Examples of Two Popular FOUs

Two very popular FOUs are for a Gaussian primary MF with uncertain standard deviation and a Gaussian primary MF with uncertain mean. This section provides lots of information about both of these FOUs. Exercise 6.6 asks the reader to provide the comparable information for Gaussian primary MF with uncertain mean and standard deviation. The popularity of these FOUs is due to their parsimony and differentiability. The latter is important when a derivative-based optimization algorithm is used to optimize MF parameters during the design of an interval type-2 fuzzy system (see Sect. 10.2.3).

Example 6.16 **Gaussian primary MF with uncertain standard deviation**

Consider the case of a Gaussian primary MF having a fixed mean, m, and an uncertain standard deviation that takes on values in $[\sigma_1, \sigma_2]$, i.e.,

$$\mu_A(x) = \exp\left[-\frac{1}{2}((x-m)/\sigma)^2\right] \quad \sigma \in [\sigma_1, \sigma_2] \tag{6.22}$$

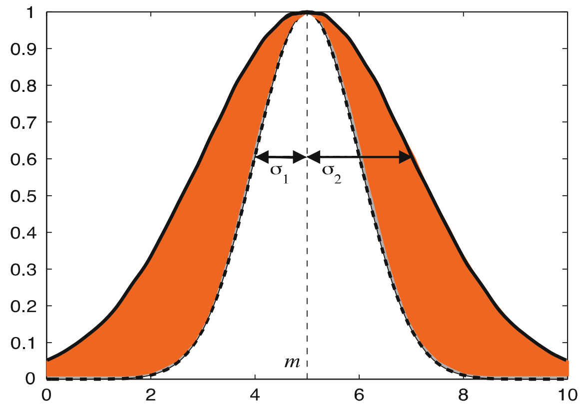


Fig. 6.16 FOU for Gaussian primary MF with uncertain standard deviation

Corresponding to each value of σ, a different membership curve is obtained. The uniform shading for the FOU (Fig. 6.16) again indicates that it is for a CIT2 FS. This primary MF and its associated interval type-2 MFs are used in Sect. 10.3.2 to model measurements that have been corrupted by non-stationary additive noise, and in Sect. 10.6 for rule-based classification of video traffic.

The thick solid curve in Fig. 6.16 denotes the UMF, and the thick dashed curve denotes the LMF; they can be expressed as:

$$\overline{\mu}_{\tilde{A}}(x) = N(x; m, \sigma_2) \tag{6.23}$$

$$\underline{\mu}_{\tilde{A}}(x) = N(x; m, \sigma_1), \tag{6.24}$$

where for example, $N(x; m, \sigma_1) \equiv \exp[-\frac{1}{2}((x-m)/\sigma_1)^2]$. Note that both the upper and lower MFs do not change formulas over $x \in X$ and they are differentiable over $x \in X$, i.e.,

$$\frac{\partial \overline{\mu}_{\tilde{A}}(x)}{\partial m} = \partial N(x; m, \sigma_2)/\partial m = (x-m)N(x; m, \sigma_2)/\sigma_2^2 \tag{6.25}$$

$$\frac{\partial \overline{\mu}_{\tilde{A}}(x)}{\partial \sigma_2} = \partial N(x; m, \sigma_2)/\partial \sigma_2 = (x-m)^2 N(x; m, \sigma_2)/\sigma_2^3 \tag{6.26}$$

$$\frac{\partial \underline{\mu}_{\tilde{A}}(x)}{\partial m} = \partial N(x; m, \sigma_1)/\partial m = (x-m)N(x; m, \sigma_1)/\sigma_1^2 \tag{6.27}$$

$$\frac{\partial \underline{\mu}_{\tilde{A}}(x)}{\partial \sigma_1} = \partial N(x; m, \sigma_1)/\partial \sigma_1 = (x-m)^2 N(x; m, \sigma_1)/\sigma_1^3 \tag{6.28}$$

As will be seen in Example 6.17, the derivatives in (6.25)–(6.28) are much simpler to compute than those for a Gaussian primary MF with uncertain mean; however, this fact should not be the deciding point as to which kind of an FOU to

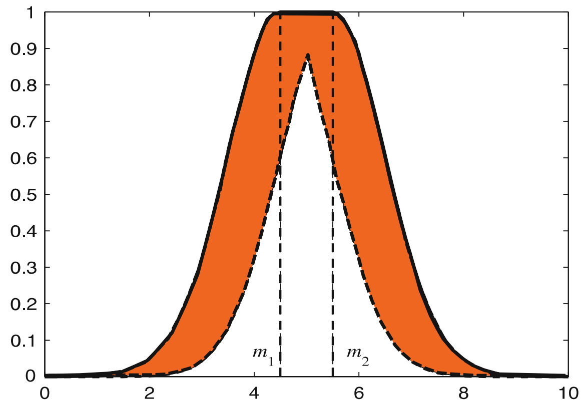


Fig. 6.17 FOU for Gaussian primary MF with uncertain mean

use in a specific application. An understanding about the nature of the application's uncertainties should always be the driving force behind this decision.

Example 6.17 **Gaussian primary MF with uncertain mean**

Consider the case of a Gaussian primary MF having a fixed standard deviation, σ, and an uncertain mean that takes on values in $[m_1, m_2]$, i.e.,

$$\mu_A(x) = \exp\left[-\frac{1}{2}((x-m)/\sigma)^2\right] \quad m \in [m_1, m_2] \tag{6.29}$$

Corresponding to each value of m a different membership curve is obtained. The uniform shading for the FOU (Fig. 6.17) again indicates that it is for a CIT2 FS. This primary MF and its associated interval type-2 MF are used in Chap. 10 for forecasting of time series.

As in Fig. 6.16, the thick solid curve in Fig. 6.17 denotes the UMF, and the thick dashed curve denotes the LMF; they can be expressed as:

$$\overline{\mu}_{\tilde{A}}(x) = \begin{cases} N(x; m_1, \sigma) & x < m_1 \\ 1 & m_1 \le x \le m_2 \\ N(x; m_2, \sigma) & x > m_2 \end{cases} \tag{6.30}$$

$$\underline{\mu}_{\tilde{A}}(x) = \begin{cases} N(x; m_2, \sigma) & x \le \frac{m_1+m_2}{2} \\ N(x; m_1, \sigma) & x > \frac{m_1+m_2}{2} \end{cases} \tag{6.31}$$

Note that both the upper and lower MFs change formulas over $x \in X$, but they are differentiable over $x \in X$, i.e.,

$$\frac{\partial \overline{\mu}_{\tilde{A}}(x)}{\partial m_1} = \begin{cases} \partial N(x; m_1, \sigma)/\partial m_1 \\ 0 \\ 0 \end{cases} = \begin{cases} (x-m_1)N(x; m_1, \sigma)/\sigma^2 & x < m_1 \\ 0 & m_1 \le x \le m_2 \\ 0 & x > m_2 \end{cases} \tag{6.32}$$

$$\frac{\partial \overline{\mu}_{\tilde{A}}(x)}{\partial m_2} = \begin{cases} 0 \\ 0 \\ \partial N(x; m_2, \sigma)/\partial m_2 \end{cases} = \begin{cases} 0 & x \le m_1 \\ 0 & m_1 \le x \le m_2 \\ (x - m_2)N(x; m_2, \sigma)/\sigma^2 & x \ge m_2 \end{cases} \tag{6.33}$$

$$\frac{\partial \overline{\mu}_{\tilde{A}}(x)}{\partial \sigma} = \begin{cases} \partial N(x; m_1, \sigma)/\partial \sigma \\ 0 \\ \partial N(x; m_2, \sigma)/\partial \sigma \end{cases} = \begin{cases} (x - m_1)^2 N(x; m_1, \sigma)/\sigma^3 & x < m_1 \\ 0 & m_1 \le x \le m_2 \\ (x - m_2)^2 N(x; m_2, \sigma)/\sigma^3 & x > m_2 \end{cases} \tag{6.34}$$

$$\frac{\partial \underline{\mu}_{\tilde{A}}(x)}{\partial m_1} = \begin{cases} 0 \\ \partial N(x; m_1, \sigma)/\partial m_1 \end{cases} = \begin{cases} 0 & x \le \dfrac{m_1 + m_2}{2} \\ (x - m_1)N(x; m_1, \sigma)/\sigma^2 & x > \dfrac{m_1 + m_2}{2} \end{cases} \tag{6.35}$$

$$\frac{\partial \underline{\mu}_{\tilde{A}}(x)}{\partial m_2} = \begin{cases} \partial N(x; m_2, \sigma)/\partial m_2 \\ 0 \end{cases} = \begin{cases} (x - m_2)N(x; m_2, \sigma)/\sigma^2 & x \le \dfrac{m_1 + m_2}{2} \\ 0 & x > \dfrac{m_1 + m_2}{2} \end{cases} \tag{6.36}$$

$$\frac{\partial \underline{\mu}_{\tilde{A}}(x)}{\partial \sigma} = \begin{cases} \partial N(x; m_2, \sigma)/\partial \sigma \\ \partial N(x; m_1, \sigma)/\partial \sigma \end{cases} = \begin{cases} (x - m_2)^2 N(x; m_2, \sigma)/\sigma^3 & x \le \dfrac{m_1 + m_2}{2} \\ (x - m_1)^2 N(x; m_1, \sigma)/\sigma^3 & x > \dfrac{m_1 + m_2}{2} \end{cases} \tag{6.37}$$

The important points to remember from these calculations are that, for a Gaussian primary MF with uncertain mean, one must be very careful to use the correct values for the derivatives of the upper and lower MFs with respect to their parameters, and these derivatives depend on where the independent variable x is in relation to the means of the left- and right-hand Gaussians, i.e., they depend on which branch of a lower or upper MF is *active.*[16]

6.5 Interval Type-2 Fuzzy Numbers

What exactly is an interval type-2 fuzzy number? There does not seem to be general agreement about the answer to this question, unlike the general agreement that exists about what a type-1 fuzzy number is (see Definition 2.5). This is because an IT2 FS is described by two T1 FSs, its lower and upper MFs, and so different possibilities avail themselves for an IT2 FS to be called an interval type-2 fuzzy number. Hamrawi and Coupland (2009) define two kinds of interval type-2 fuzzy numbers[17]:

[16]The concept of an *active branch* was first described in Liang and Mendel (2000).

[17]Interestingly, Hamrawi and Coupland (2009) also define different kinds of general type-2 fuzzy numbers. See Sect. 11.3.

- A *perfectly normal interval type-2 fuzzy number* is an IT2 FS both of whose lower and upper MFs of its FOU are type-1 fuzzy numbers.
- A *normal interval type-2 fuzzy number* is an IT2 FS only whose upper MF of its FOU is a type-1 fuzzy number.

I will now argue why, to me, only a "perfectly normal interval type-2 fuzzy number" deserves the designation of an interval type-2 fuzzy number. To begin, anything that is called a "fuzzy number" should be a model for a linguistic term that is related to the number, e.g., if n is a real number, then a fuzzy number would be for terms like *close to n*, *very close to n*, *around n*, *about n*, etc. So, at n there is no uncertainty (n is n), which means that at n the membership grade for the fuzzy number should be 1 (as occurs for a perfectly normal interval type-2 fuzzy number) and not an interval of values (as occurs for a normal interval type-2 fuzzy number).

Continuing, suppose that a group of S subjects is asked: "On a scale of l to r where would you locate the endpoints (a and b) of an interval of real numbers for a linguistic term related to the real number n (of course, $l \le n \le r$), e.g., *around n*. The result will be a set of S intervals, $[a^{(i)}, b^{(i)}]_{i=1}^{S}$. After doing some preprocessing of these S intervals (e.g., remove outliers, keep only the intervals that contain n, keep only the intervals that overlap,[18] etc.), one will be left with S' intervals, $[a'^{(i)}, b'^{(i)}]_{i=1}^{S'}$, where $S' \le S$. Overlaying these S' intervals, one will find a common interval, $[c, d]$, where $[c, d] \subset [a'^{(i)}, b'^{(i)}]_{i=1}^{S'}$. The phrase "common interval" means all S' subjects are in agreement about $[c, d]$; hence, there is no uncertainty about $[c, d]$. Consequently, the membership grade should be 1 for $[c, d]$. When an IT2 FS is used to model this fuzzy number, it is only a perfectly normal interval type-2 fuzzy number that can achieve this.

By these arguments, only a "perfectly normal interval type-2 fuzzy number" deserves the designation of an interval type-2 fuzzy number. Although, not everyone may agree with this, then, at the very least, one should use the above Hamrawi–Coupland designations. In this book the following will be abided by:

Definition 6.22 An *interval type-2 fuzzy number* is an IT2 FS whose lower and upper MFs of its FOU are type-1 fuzzy numbers.

Consequently, the Example 6.16 Gaussian primary MF with uncertain standard deviation is an interval type-2 fuzzy number, but the Example 6.17 Gaussian primary MF with uncertain mean is not.

Interval type-2 fuzzy numbers will play an important role in one kind of non-singleton fuzzification in an IT2 fuzzy system (see Sect. 9.4.2.3).

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[18]Not only is it true that "Words mean different things to different people," but, in addition, it is true that "Words must mean similar things to different people," or else people are not communicating effectively (Liu and Mendel 2008). The former adage is the motivation for modeling linguistic terms (words) using type-2 fuzzy sets, whereas the latter adage is the motivation for keeping only overlapping intervals.

6.6 Different Kinds of T2 FSs: Hierarchy

As noted in Mendel (2014, footnote 2): "In the early days of type-2 fuzzy sets and systems, the phrases 'type-2 fuzzy set' or 'type-2 fuzzy system' were used in an all-inclusive way, meaning any kind of type-2 fuzzy set or system. During the past 15 years [now more than 17 years] most of the attention has been given to [closed] interval type-2 fuzzy sets and systems. It is only within the past 5 years or so that there has been a return to more general type-2 fuzzy sets and systems, and, to distinguish them from the more specialized [closed] interval type-2 fuzzy sets and systems, the term 'general' is being used. In essence, type-2 fuzzy sets and systems now consist of the union of interval and general type-2 fuzzy sets and systems." This is captured in the Venn diagram of Fig. 6.18.

As mentioned just before Definition 6.1, T2 FSs have been beset with some problematic notations (see Mendel et al. 2016) for a comprehensive historical explanation of them, as well as recommended fixes, all of which have been adopted in this book). The most problematic notation was the use of $J_x \subseteq [0, 1]$ in the definition of $\tilde{A}$, i.e., instead of (6.1) $\tilde{A}$ was defined as:

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}.$$

Additionally, nowhere in Mendel (2001) was J_x defined formulaically, as has been done in (6.4a).

This book emphasizes the use of I_x instead of J_x, where $J_x = \{x\} \times I_x$. All of the early works on IT2 FSs were for when I_x is connected; however, by only stating that $J_x \subseteq [0, 1]$, and not defining J_x, this left $J_x \subseteq [0, 1]$ open to the richer interpretation provided in Bustince et al. (2015) for which I_x does not have to be connected. It is in that interpretation that one gains an appreciation for distinguishing between connected and disconnected I_x.

Example 6.18 In Bustince et al. (2015) one finds examples of different kinds of IT2 FSs, one of which is closed, and the others of which are not. Regarding the latter:

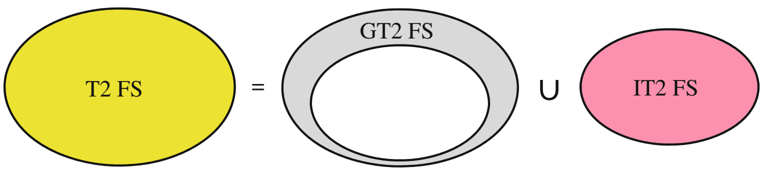

Fig. 6.18 T2 FSs and systems now consist of the union of interval and general T2 FSs and systems

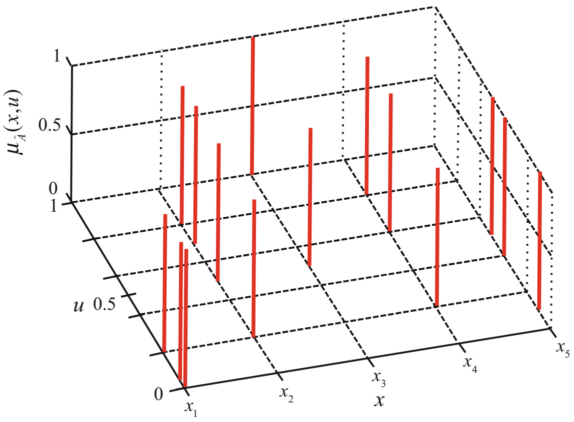

Fig. 6.19 Example of a *multiset* from an IT2 FS (adapted from Bustince et al. 2015)

- Figure 6.19 is an example of a *multiset* from an IT2 FS for which I_{x_i} is a set of points each with a secondary grade of unity. In this example,

$$\mu_{\tilde{A}}(x_1,u) = \begin{cases} 1 & u \in I_{x_1} = \{0, 0.05, 0.2\} \\ 0 & \text{otherwise} \end{cases}, \ \mu_{\tilde{A}}(x_2,u) = \begin{cases} 1 & u \in I_{x_2} = \{0.2, 0.5, 0.7, 0.8\} \\ 0 & \text{otherwise} \end{cases},$$

$$\mu_{\tilde{A}}(x_3,u) = \begin{cases} 1 & u \in I_{x_3} = \{0.5, 1\} \\ 0 & \text{otherwise} \end{cases}, \ \mu_{\tilde{A}}(x_4,u) = \begin{cases} 1 & u \in I_{x_4} = \{0.2, 0.6, 0.8\} \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$

$$\mu_{\tilde{A}}(x_5,u) = \begin{cases} 1 & u \in I_{x_5} = \{0.1, 0.4, 0.5\} \\ 0 & \text{otherwise} \end{cases}.$$

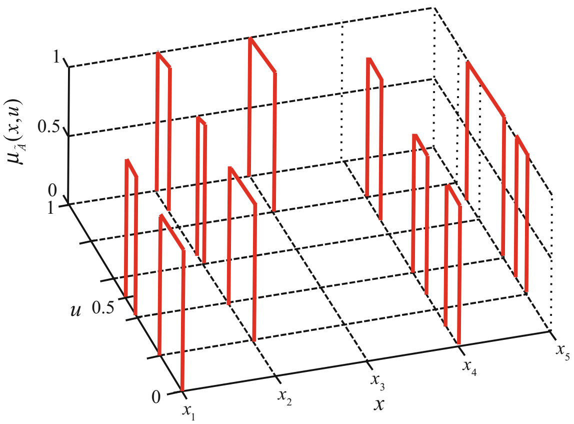

Fig. 6.20 Example of a *multi interval-valued* FS as an IT2 FS (adapted from Bustince et al. 2015)

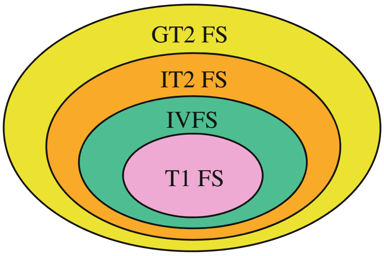

Fig. 6.21 A way to summarize fuzzy sets, in which the ellipses denote "encompass," as is explained in the text (adapted from Bustince et al. 2015)

- Figure 6.20 is an example of a *multi interval-valued* FS as an IT2 FS for which I_{x_i} is a set of nonoverlapping intervals, each with a secondary grade of unity for all of their elements. In this example:

$$\mu_{\tilde{A}}(x_1,u) = \begin{cases} 1 & u \in I_{x_1} = [0,0.2] \cup [0.4,0.5] \\ 0 & \text{otherwise} \end{cases}, \quad \mu_{\tilde{A}}(x_2,u) = \begin{cases} 1 & u \in I_{x_2} = [0.2,0.4] \cup [0.6,0.66] \cup [0.9,1] \\ 0 & \text{otherwise} \end{cases}$$
$$\mu_{\tilde{A}}(x_3,u) = \begin{cases} 1 & u \in I_{x_3} = [0.8,1] \\ 0 & \text{otherwise} \end{cases}, \quad \mu_{\tilde{A}}(x_4,u) = \begin{cases} 1 & u \in I_{x_4} = [0,0.1] \cup [0.3,0.4] \cup [0.7,0.8] \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$
$$\mu_{\tilde{A}}(x_5,u) = \begin{cases} 1 & u \in I_{x_5} = [0.2,0.3] \cup [0.4,0.7] \\ 0 & \text{otherwise} \end{cases}.$$

Of course, one can combine these two examples so that I_{x_i} is a set of points and nonoverlapping intervals, each with a secondary grade of unity for all of their elements. An example of doing this can also be found in Bustince et al. (2015), and is also left as an exercise for the reader (Exercise 6.10).

Figure 6.21 is adapted from Fig. 8 in Bustince et al. (2015). It should be interpreted in the following way: When a T1 FS and an IVFS are expressed as T2 FSs (see Sect. 6.8 for how to do this), then a GT2 FS encompasses an IT2 FS which encompasses an IVFS[19] which encompasses a T1 FS. This seems to be a very nice way to summarize the different kinds of fuzzy sets.

The rest of this book focuses for the most part only on closed general[20] *and closed IT2 FSs,* because, as of the year 2016, it is only for such T2 FSs that one knows how to perform all of the operations that are needed to implement either an interval or general type-2 fuzzy system, and it is only those fuzzy systems that have been applied to real world applications. Consequently, *in the rest of this book when the phrase "interval (or general) T2 FS (or system)" is used it is always to be understood (unless indicated otherwise) that this means a "closed interval (or general) T2 FS (or system)."* The abbreviations *IT2 FS* and *GT2 FS* that are used in the rest of this book are always for such type-2 fuzzy sets.

[19]The elevated IVFS will then be the same as a CIT2 FS.

[20]A *closed GT2 FS* is defined in Definition 6.25.

6.7 Mathematical Representations for T2 FSs[21]

There are four ways to mathematically represent a T2 FS[22] (Mendel 2014): (a) collection of points as in (6.1); (b) union of vertical slices (over $x \in X$), where each vertical slice (see Fig. 6.5) is a T1 FS (a secondary MF); (c) union of wavy slices, where each wavy slice is an embedded T2 FS (see Fig. 6.10); and, (4) fuzzy union of horizontal slices (over $\alpha \in [0, 1]$), where each horizontal slice resembles an IT2 FS raised to level α.

The *collection of points representation* is the starting point for all of the other representations, and often corresponds to the way that data about a GT2 FS can be stored; but, it does not seem to be good for much else, so nothing else will be said about it.

Just as it is possible to rigorously prove that a type-1 MF can be represented as the fuzzy union of all of its α-cuts (each raised to level α), it is also possible to rigorously prove that a type-2 MF can be represented as the union of all of its vertical slices, or horizontal slices or wavy slices. In retrospect, all of these decompositions are visually obvious.

6.7.1 Vertical Slice Representation

Theorem 6.1 *The* vertical-slice representation *of GT2 FS $\tilde{A}$ focuses on each value of the primary variable x, and expresses* (6.1) *as the union of all of its secondary T1 FSs, i.e.,*

$$\tilde{A} = \int_{x \in X} \mu_{\tilde{A}(x)}(u)/x \tag{6.38}$$

where

$$\mu_{\tilde{A}(x)}(u) = \int_{u \in [0,1]} f_x(u)/u \tag{6.39}$$

so that[23]

[21]Some of the material in this section is adapted from Mendel (2014) modulo some notational changes.

[22]Be reminded that in this book all secondary MFs are convex T1 FSs.

[23]The first statement of the vertical-slice representation in terms of its α-cuts is Mendel (2014).

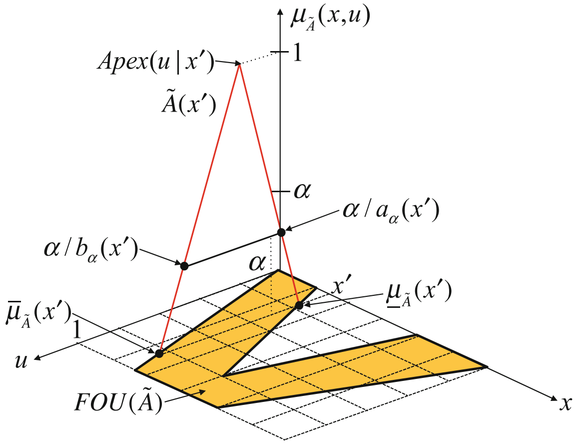

Fig. 6.22 A secondary set $\tilde{A}(x')$, its triangular MF, and an α-cut of $\tilde{A}(x')$ raised to level α. Observe that $\tilde{A}(x')$ is anchored on FOU$(\tilde{A})$ at $\underline{\mu}_{\tilde{A}}(x')$ and $\overline{\mu}_{\tilde{A}}(x')$

$$\tilde{A} = \int_{x \in X} \left[\bigcup_{\alpha \in [0,1]} \left[\alpha / \tilde{A}(x)_{\alpha} \right] \right] / x = \int_{x \in X} \sup_{\alpha \in [0,1]} \left[\alpha / \tilde{A}(x)_{\alpha} \right] \tag{6.40}$$

in which the α*-cut of the T1 FS* $\tilde{A}(x)$, $\tilde{A}(x)_{\alpha}$, *is given by (see* Definition 2.9, Example 2.27 *and* Fig. 6.22*)*

$$\tilde{A}(x)_{\alpha} = \{ u | \mu_{\tilde{A}(x)}(u) \geq \alpha \} \equiv [a_{\alpha}(x), b_{\alpha}(x)] \tag{6.41}$$

Proof Equation (6.39) is already stated in (6.3), and (6.38) collects all of the vertical slices. At each $x \in X$, $\tilde{A}(x)$ has the following $\alpha-$cut decomposition for convex secondary MFs (Theorem 2.3):

$$\begin{aligned} \tilde{A}(x) &= \bigcup_{\alpha \in [0,1]} \left[\alpha / \tilde{A}(x)_{\alpha} \right] = \sup_{\alpha \in [0,1]} \left[\alpha / \tilde{A}(x)_{\alpha} \right] \\ &= \sup_{\alpha \in [0,1]} \left[\alpha / [a_{\alpha}(x), b_{\alpha}(x)] \right] \end{aligned} \tag{6.42}$$

The first line of (6.42) is the right-hand portion of (6.40), and the second line of (6.42) uses $\tilde{A}(x)_{\alpha}$ in (6.41), which completes the proof of Theorem 6.1.

The vertical-slice representation of $\tilde{A}$ in (6.38) and (6.39) was used originally in Mizumoto and Tanaka (1976, 1981) to derive formulas for set-theoretic operations of GT2 FSs. It is excellent for computations involving GT2 FSs and can also be used for theoretical studies. In fact, as is explained in Sects. 6.7.4 and 11.14, it is one of the preferred representations for solving real-world general type-2 fuzzy system optimal design problems. The vertical-slice representation of $\tilde{A}$ in (6.40) strongly suggests that α-cuts will play a useful and central role for GT2 FSs. That this is so will be made very clear in Chaps. 7 and 11.

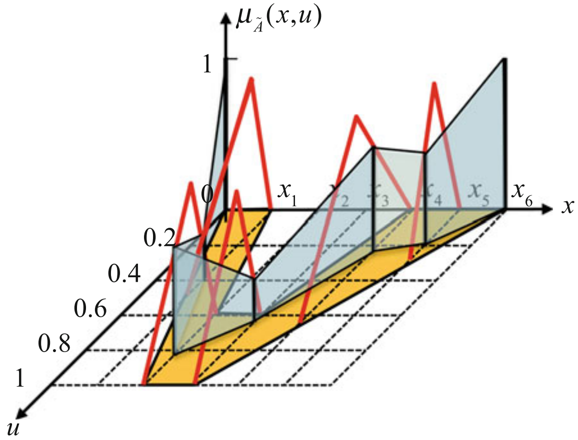

Fig. 6.23 A wavy slice for the Fig. 6.9 GT2 FS. Each secondary MF contributes one element to the wavy slice. The shading of the wavy slice is only for artistic purposes; however, for continuous X and U the shaded region becomes a continuous foil

6.7.2 Wavy Slice Representation

Theorem 6.2 [Wavy Slice Representation Theorem[24] *for a GT2 FS* $\tilde{A}$ (Mendel and John 2002)] *GT2 FS* $\tilde{A}$ *is the set-theoretic union of all of its embedded T2 FSs, i.e.*

$$\tilde{A} = \bigcup_{\forall j} \tilde{A}_e^j \tag{6.43}$$

Proof Equation (6.43) is obvious by using the following simple informal geometric argument. Create all of the possible embedded T2 FSs for $\tilde{A}$ (see Fig. 6.23 for one such set) and take their union to reconstruct $\mu_{\tilde{A}}(x,u)$. Same points, which occur in different embedded T2 FSs only appear once in the set-theoretic union.

Although impractical for most computations, because (6.43) requires the enumeration of all of the embedded T2 FSs, and their number can be enormously large depending upon the discretization levels of the primary and secondary variables, this wavy slice representation of a GT2 FS has proved to be of great value for developing new theoretical results. Arguably, its most important use has been for IT2 FSs, as is encapsulated in the following:

Theorem 6.3 [Wavy Slice Representation Theorem *for an IT2 FS* (Mendel and Wu 2010, Chap. 2)] *The FOU of an IT2 FS is (covered by) the union of all of its embedded T1 FSs, i.e.*[25]

[24]This representation theorem, which was originally stated and proved in Mendel and John (2002) for $u \in I_x\,[J_x]$ instead of for $u \in [0,1]$, has also been referred to as the *Mendel-John Representation Theorem*. I prefer "wavy slice" because it fits in very nicely with the description of the other "slice" representations of a GT2 FS. Its extension to non-closed (disconnected) GT2 FSs remains to be explored.

[25]The extension of Theorem 6.3 to non-closed (disconnected) IT2 FSs remains to be explored.

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{\forall j} A_e^j \tag{6.44}$$

Proof Create all of the possible embedded T1 FSs in $\mathrm{FOU}(\tilde{A})$ and take their union to reconstruct $\mathrm{FOU}(\tilde{A})$. Same points, which occur in different embedded T1 FSs, only appear once in the set-theoretic union.

Note that Theorem 6.3 can be interpreted as a *covering representation*, because the union of all embedded T1 FSs covers the entire FOU.

Definition 6.23 A *maximal covering* of $\mathrm{FOU}(\tilde{A})$ is one in which every possible embedded T1 FS is used to cover the FOU, and so same points will occur in (many) different embedded T1 FSs. A *minimal covering* of $\mathrm{FOU}(\tilde{A})$ is one that uses a minimal number of embedded T1 FSs to cover the FOU.

Example 6.19 A *minimal covering* of $\mathrm{FOU}(\tilde{A})$ can be achieved by choosing embedded T1 FSs as a linear combination of the lower and upper MFs of $\tilde{A}$ as follows:

$$A_e^j = (1-j)\mathrm{LMF}(\tilde{A}) + j\mathrm{UMF}(\tilde{A}), \quad j \in [0,1] \tag{6.45}$$

This is also called a *homotopy*.[26]

When, for example, (6.45) is applied to the FOU in Fig. 6.16, then that FOU will be covered by Gaussian type-1 fuzzy sets each of which passes through the common point $(m, 1)$. See, also Exercises 6.13 and 6.14.

By using Theorem 6.3 it is possible to apply everything that has already been developed for T1 FSs to each of the embedded T1 FSs, after which the union in (6.44) is performed. This has been done in Mendel et al. (2006) and will be demonstrated in Chap. 7 (Examples 7.15–7.17). Additionally, Mendel (2009) has demonstrated that Theorem 6.3 is a very good place to begin when trying to solve a new problem that involves IT2 FSs.

6.7.3 *Horizontal Slice Representation*[27]

Definition 6.24 An *α-plane* (Liu 2008; Mendel et al. 2009) for a GT2 FS $\tilde{A}$, denoted $\tilde{A}_\alpha$, is the union of all primary memberships of $\tilde{A}$ whose secondary grades are greater than or equal to $\alpha \in [0,1]$, i.e.

[26] A *homotopy* between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

[27] If a reader is interested only in IT2 FSs and fuzzy systems and not GT2 FSs and systems, this section as well as Sect. 6.7.4 can be omitted.

$$\begin{aligned}\tilde{A}_\alpha &= \left\{(x,u), \mu_{\tilde{A}}(x,u) \geq \alpha | x \in X, u \in [0,1]\right\} \\ &= \int_{x\in X} \int_{u\in[0,1]} \{(x,u)|f_x(u) \geq \alpha\}\end{aligned} \tag{6.46}$$

Alternatively, $\tilde{A}_\alpha$ can be expressed by means of (6.41), as:

$$\tilde{A}_\alpha = \int_{x\in X} \tilde{A}(x)_\alpha/x = \int_{x\in X} [a_\alpha(x), b_\alpha(x)]/x \tag{6.47}$$

$\tilde{A}_\alpha$ has a LMF and an UMF, where $(x \in X)$:

$$\begin{cases} \mathrm{LMF}(\tilde{A}_\alpha) = a_\alpha(x) \\ \mathrm{UMF}(\tilde{A}_\alpha) = b_\alpha(x) \end{cases} \tag{6.48}$$

When $\tilde{A}_\alpha$ is raised to level- α, it is a plane (horizontal slice) at that level, and can be obtained by connecting respective α-cuts of all type-1 vertical slice secondary MFs for $x \in X$.

Theorem 6.4 *The* horizontal-slice representation *of GT2 FS* $\tilde{A}$ *is*:

$$\tilde{A} = \sup_{\alpha\in[0,1]} \alpha/\left[\int_{x\in X} [a_\alpha(x), b_\alpha(x)]/x\right] = \sup_{\alpha\in[0,1]} \alpha/\tilde{A}_\alpha = \bigcup_{\alpha\in[0,1]} \alpha/\tilde{A}_\alpha \tag{6.49}$$

Proof This follows directly from substituting (6.41) into the right-hand side of (6.40), and then focusing first on x and then on α, after which (6.47) is used.

An α-plane can be obtained directly from the α-cuts of the secondary MFs. Just as an α-cut of a T1 FS resides on its 1-D domain X, $\tilde{A}_\alpha$ resides on its 2-D domain $X \times U$. Consequently, for a GT2 FS whose secondary MFs are convex T1 FSs each α-plane is an IVFS, and

$$\mathrm{FOU}(\tilde{A}) = \tilde{A}_0. \tag{6.50}$$

Recall (Sect. 2.12) that when an α-cut is raised to level α one obtains an α-level. Analogously, when an α-plane is raised to level α one obtains a *horizontal slice at level* α, $R_{\tilde{A}_\alpha}$ (Mendel 2010), i.e.

$$R_{\tilde{A}_\alpha} = \alpha/\tilde{A}_\alpha \tag{6.51}$$

Table 6.2 Comparisons of α-plane and *z*Slice descriptions (adapted from Mendel 2014)

Item	α-plane description	*z*Slice description
Coordinates	(x, u, μ)	(x, y, z)
$\alpha-$plane	$\tilde{A}_\alpha = \int_{x\in X} \int_{u\in[0,1]} \{(x,u)\vert f_x(u) \geq \alpha\}$	$\tilde{Z}_\alpha$ projected onto $X \times Y$
*z*Slice	$\alpha/\tilde{A}_\alpha = R_{\tilde{A}_\alpha}$	$\tilde{Z}_z = \int_{x\in X} \int_{y\in[0,1]} z/(x,y)$
FOU$(\tilde{A})$	$\tilde{A}_0$	$\tilde{Z}_0$
Vertical slice $(x_j \in X)$	$\int_{\alpha\in[0,1]} \int_{u\in[0,1]} \alpha/(x_j, u)$	$\int_{z\in[0,1]} \int_{y\in[0,1]} z/(x_j, y)$
Representation of $\tilde{A}$	$\tilde{A} = \bigcup_{\alpha\in[0,1]} R_{\tilde{A}_\alpha} = \sup_{\alpha\in[0,1]} [\alpha/\tilde{A}_\alpha]$	$\tilde{A} = \bigcup_{z\in[0,1]} \tilde{Z}_z = \sup_{z\in[0,1]} [\tilde{Z}_z]$

This has been called an "α-plane raised to Level α"or a[28] "zSlice" (Wagner and Hagras 2008, 2010, 2013). Of course, one can also shorten the expression "α-plane raised to Level α" to the much simpler expression "α-level plane." Note, also, $R_{\tilde{A}_\alpha}$ is an IT2 FS all of whose secondary grades equal α (rather than 1 as would be the case for the usual IT2 FS), and that

$$\text{FOU}(R_{\tilde{A}_\alpha}) = \tilde{A}_\alpha \tag{6.52}$$

Equation (6.52) provides an α-plane with an important interpretation of and connection to an IT2 FS of height α.

Example 6.20 Figure 6.24 depicts some α-planes raised to level α for the Fig. 6.9 GT2 FS. It was constructed by fixing a value of α (e.g., $\alpha = 1/3$), drawing the α-cut at level α for each of the secondary MFs, and then connecting the points where those horizontal lines intersect the MFs.[29] Observe that, because all of the secondary MFs

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[28]Zadeh (1975) does not use the term α-plane, nor does he have an α-plane decomposition for a T2 FS. The term α-plane was introduced first in Liu (2008), and the term *zSlice* was introduced first in Wagner and Hagras (2008). In the *z*Slice literature coordinates are called (x, y, z) instead of (x, u, μ); hence, z is also the same as α. The connections between these two representations are summarized in Table 6.2.

Chen and Kawase (2000) also use α-cuts, and while they do not have a formal representation theorem for a T2 FS, their Definition 3.2 expresses a secondary MF in terms of α-cuts. Tahayori et al. (2006) state a "Representation Principle" without proof; it is the same as Liu's α-plane Representation Theorem, but they do not use the term α-plane.

Starczewski (2009a, b) has done some related work, but he does not have an α-plane Representation Theorem.

Hamrawi et al. (2010) use α-cuts of both the lower and upper MFs of the FOU to represent an IVFS. It also reexamines α-planes and zSlices, and, among other things, shows that a GT2 FS can be represented as the union of all its α-cuts of all its α-planes.

[29]Drawing or sketching 3-D MFs can be quite challenging. See Mendel (2012) for step-by-step instructions for drawing or sketching 2-1/2D MFs, which are 2-D figures that give the appearance of being 3-D.

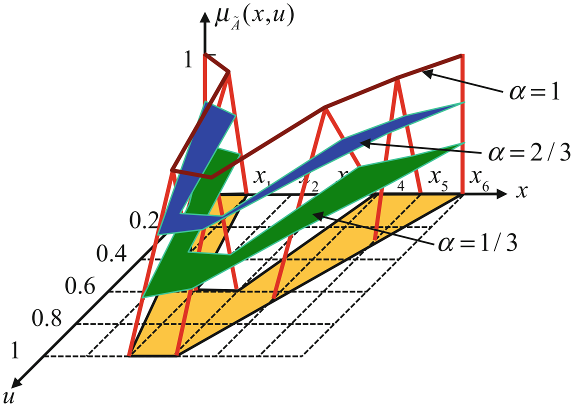


Fig. 6.24 Some α-planes raised to level α for the Fig. 6.9 GT2 FS

are normal triangles, the $\alpha = 1$ α-plane raised to level α is just a T1 FS, i.e., a function and not a plane.

One can give compelling reasons for using both of the terms "*α-plane*" or "*zSlice*." A reason for using "*α-plane*" is: each type-1 secondary MF can be expressed in terms of alpha-cuts and an alpha-plane is the union of all such alpha-cuts at a fixed value of alpha, so the term "*α-plane*" connects perfectly with the long-used term "α-cut." Two reasons for using "*zSlice*" are: (1) naming coordinates (x, y, z) has a long mathematical heritage, whereas calling them (x, u, μ) does not, and (2) z Slices are already al level z, which is exactly where we want them to be, whereas an α-plane has to be raised to level α for it to be where we want it to be.

It is a bit confusing to have two different terminologies for the same things, so in this book [as in Mendel (2014)] this kind of a representation is called a *horizontal-slice representation* since "horizontal" compliments "vertical" rather nicely. Because my own preference is to connect type-1 and type-2 results smoothly, the *α-plane* terminology is used in the rest of this book.[30]

The great value of the horizontal-slice representation is that it will let everything that is developed for IT2 FSs, in Chaps. 7 and 8, be applied to each of the horizontal slices.

Definition 6.25 A *closed GT2 FS* $\tilde{A}$ is one whose horizontal slices are closed for $\alpha \in [0, 1]$.

This follows from when Definition 6.18 is applied to (6.51) and (6.49) expressed as $\tilde{A} = \bigcup_{\alpha \in [0,1]} R_{\tilde{A}_\alpha}$. The GT2 FS in Fig. 6.24 is closed.

For a GT2 FS to be closed all of its secondary MFs must be convex.

[30] If the universes of discourse for primary or secondary variables are not continuous, then it would not be correct to use the word "plane." One could then revert to *horizontal slice at level α*; such a slice would only contain a finite set of discrete x and u, or discrete x and intervals for u. These situations are not considered in this book.

6.7.4 Which Representations Are Most Useful for Optimal Design[31] Applications?

Recall, from Chap. 4, that during the optimal design of a type-1 fuzzy system a mathematical objective function, $J(\boldsymbol{\phi})$, was established that depends upon the design parameters, $\boldsymbol{\phi}$. The elements of $\boldsymbol{\phi}$ include all of the antecedent and consequent MF parameters as well as any defuzzification parameters (if there are any). $J(\boldsymbol{\phi})$ is a nonlinear function of $\boldsymbol{\phi}$ and so some sort of mathematical programming approach had to be used to optimize it.

Chapter 10 will demonstrate that a mathematical objective function $J(\boldsymbol{\phi})$ is also established during the optimal designs of an IT2 fuzzy system, and that the design parameters in $\boldsymbol{\phi}$ not only include all of the antecedent and consequent FOU parameters, but also any parameters that are associated with mapping IT2 fuzzy sets into a crisp number.

Chapter 11 will demonstrate that a mathematical objective function $J(\boldsymbol{\phi})$ is also established during the optimal designs of a GT2 fuzzy system, and that the design parameters in $\boldsymbol{\phi}$ not only include all of the antecedent and consequent FOU parameters, but also the secondary MF parameters, and any parameters that are associated with mapping GT2 fuzzy sets into a crisp number.

The basic premise of this book is that *one should use a parsimonious parametric representation of a T2 FS during an optimal design of a T2 fuzzy system, regardless of whether it is uses IT2 FSs or GT2 FSs*. Such a representation is one that is described by as few parameters as possible (Definition 6.14).

Clearly, *a parsimonious representation of an IT2 FS is obtained when its lower and upper MFs are described parsimoniously*, e.g., as in Examples 6.16 and 6.17. Things are not as clear for a GT2 FS.

It is obvious that the point representation of a GT2 FS is not parsimonious, because it is not even a parametric representation. Similarly, the wavy slice representation, in which the wavy slices must be enumerated from all of the points of the GT2 FS, is also a nonparametric representation. Neither of these representations is useful for an optimal design of a GT2 fuzzy system.

The horizontal-slice representation needs to be made parametric for it to be useful for an optimal design of a GT2 fuzzy system. This representation requires choosing the number of horizontal slices and then parameterizing each of them. One way to do this is to assume that all of the horizontal slices have the same shape, and a higher horizontal slice is a *squished* version of the one just below it. A way to squish a horizontal slice at level α_1 into a horizontal-slice at level α_2 $(\alpha_2 > \alpha_1)$ is to begin with the FOU (at level $\alpha_1 = 0$) and then let the UMF (LMF) of the horizontal slice at level α_2 $(\alpha_2 > \alpha_1)$ be a scaled version of the UMF (LMF) of the FOU. The scale factor for the UMF (LMF) has to be less (greater) than unity, and as α increases in values towards its final value of 1, the scale factor has to become

[31]If a reader is interested only in IT2 FSs and fuzzy systems, this section can be omitted.

smaller and smaller (larger and larger), but not so large that the LMF is above the UMF (see Exercise 6.18). By this approach, if there are k horizontal slices and the two scale factors are different, then the number of parameters to represent this kind of GT2 FS is $2k + n_0$, where n_0 denotes the number of parameters that describe the FOU when $\alpha = 0$. If the scale factors are constrained to be the same (see Exercise 6.19), then there will be $k + n_0$ parameters. For a relatively small value of k (2–6) this can be a parsimonious representation.

Yet another way of squishing a horizontal slice at level α_1 into a horizontal-slice at level α_2 $(\alpha_2 > \alpha_1)$ is to begin with the FOU (at level $\alpha_1 = 0$) and use the same UMF of the FOU at all α levels, but let the LMF of the horizontal-slice at level $\alpha_2(\alpha_2 > \alpha_1)$ be a scaled version of the LMF of the FOU (see Exercise 6.20) (Kumbasar and Hagras 2015). By this approach, if there are k horizontal slices, then the number of parameters to represent this kind of GT2 FS is also $k + n_0$.

Our conclusion from this is that *when more than a few horizontal slices are used, the horizontal-slice representation of a GT2 FS may not be a useful representation for the optimal design of a GT2 fuzzy system, but if only a small number of horizontal slices are used, and any of the above squishing techniques are used, then the horizontal-slice representation of a GT2 FS is a useful representation for the optimal design of a GT2 fuzzy system.*[32]

It will now be demonstrated that the vertical-slice representation of a GT2 FS is a very flexible and parsimonious representation of such a fuzzy set.

My suggestion for parameterizing the vertical-slice representation of a GT2 FS is to: (1) Parameterize its FOU exactly as one presently parameterizes the FOU of an IT2 FS; (2) Parameterize the secondary MFs by choosing a fairly simple function that introduces only one new parameter; (3) If performance is not acceptable then use secondary MFs that can be described by using two new parameters; etc. Because the secondary MFs are vertical slices, they are always anchored on the already parameterized FOU. Examples of such secondary MFs are given next.

Example 6.21 **(triangle secondary MFs)**: The vertices of the base of each triangle are located at $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ (see Fig. 6.22), and the location of the apex of each triangle, $\text{Apex}(u|x)$, is parameterized (Liu 2008; Mendel et al. 2009) as $(w \in [0, 1])$:

$$\text{Apex}(u|x) = \underline{\mu}_{\tilde{A}}(x) + w[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)] \tag{6.53}$$

When[33] $w = 0$ the secondary MF is a right triangle whose right angle is perpendicular to $\underline{\mu}_{\tilde{A}}(x)$; when $w = 1/2$ the secondary MF is an isosceles triangle; and, when $\overline{\mu}_{\tilde{A}}(x)\, w = 1$ the secondary MF is a right triangle whose right angle is

[32]This conclusion is different from the one that is stated in Mendel (2014) because when that article was written it was unknown to the author how to squish an FOU. Thanks to others, it is now known how to do this.

[33]Almaraashi et al. (2016) refer to w as an *apex factor* and even let it be a function of x. When the latter is done, then there will be one apex factor at each discretized value of the primary variable, and this is no longer a parsimonious representation.

perpendicular to $\overline{\mu}_{\tilde{A}}(x)$. w is treated as a design parameter, but it is the *same* for all of the vertical slices.

During computations, one will need to use α-cuts of the triangle secondary MF. They are easily found (Exercise 6.15), as $(w \in [0,1])$:

$$\begin{cases} \tilde{A}(x)_\alpha = [a_\alpha(x), b_\alpha(x)] \\ a_\alpha(x) = \underline{\mu}_{\tilde{A}}(x) + w[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]\alpha \\ b_\alpha(x) = \overline{\mu}_{\tilde{A}}(x) - (1-w)[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]\alpha \end{cases} \tag{6.54}$$

Example 6.22 **(symmetrical trapezoid secondary MFs)**: The vertices of the base of each trapezoid are located at $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$, and its top is defined by left and right endpoints (see Fig. 6.25), $EP_l(u|x)$ and $EP_r(u|x)$, that are parameterized (Liu 2008; Mendel et al. 2009) as $(w \in [0,1])$:

$$EP_l(u|x) = \underline{\mu}_{\tilde{A}}(x) + \frac{1}{2}w[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)] \tag{6.55}$$

$$EP_r(u|x) = \overline{\mu}_{\tilde{A}}(x) - \frac{1}{2}w[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)] \tag{6.56}$$

When $w = 0$, the trapezoid reduces to a square well, and *the GT2 FS reduces to an IT2 FS*; and, when $w = 1$, $EP_l(u|x) = EP_r(u|x) = [\underline{\mu}_{\tilde{A}}(x) + \overline{\mu}_{\tilde{A}}(x)]/2$, so that the trapezoid reduces to an isosceles triangle. w is treated as a design parameter, but it is the *same* for all of the vertical slices.

During computations, one will need to use α-cuts of the symmetrical trapezoid secondary MF. They are also easily found (Exercise 6.16), as $(w \in [0,1])$:

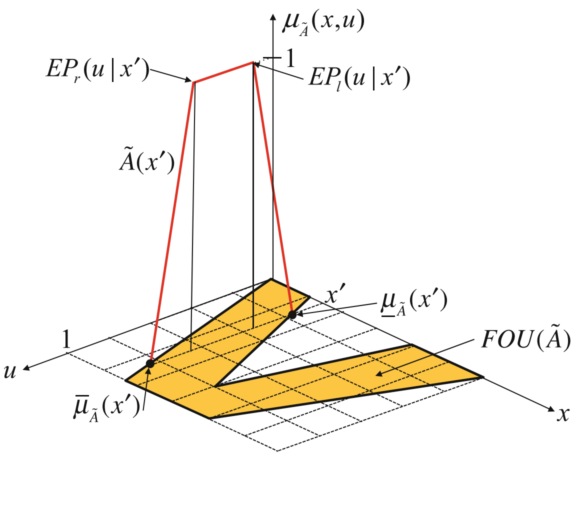


Fig. 6.25 Symmetrical trapezoid secondary MF for Example 6.22

$$\begin{cases} \tilde{A}(x)_\alpha = [a_\alpha(x), b_\alpha(x)] \\ a_\alpha(x) = \underline{\mu}_{\tilde{A}}(x) + \frac{1}{2}w[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]\alpha \\ b_\alpha(x) = \overline{\mu}_{\tilde{A}}(x) - \frac{1}{2}w[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]\alpha \end{cases} \tag{6.57}$$

Example 6.23 **(nonsymmetrical trapezoid secondary MFs)**: Another choice for a secondary MF is a nonsymmetrical trapezoid; however, it requires two parameters to define it. The vertices of the base of each trapezoid are located at $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$, and its top is defined by left and right endpoints (see Fig. 6.25), but $EP_l(u|x)$ and $EP_r(u|x)$ are now parameterized as:

$$EP_l(u|x) = \underline{\mu}_{\tilde{A}}(x) + w_1[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)] \tag{6.58}$$

$$EP_r(u|x) = \overline{\mu}_{\tilde{A}}(x) - w_2[\overline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)] \tag{6.59}$$

$$\begin{cases} w_2 \neq w_1 \\ w_1 + w_2 < 1 \end{cases} \tag{6.60}$$

Equation (6.60) guarantees non-symmetry and $EP_r(u|x) > EP_l(u|x)$ (Exercise 6.17). When $w_1 = w_2 = 0$, the nonsymmetrical trapezoid reduces to a square well, and *the GT2 FS reduces to an IT2 FS*; when $w_1 = 0$, $EP_l(u|x) = \underline{\mu}_{\tilde{A}}(x)$ so that the non-symmetrical trapezoid has a vertical left leg; and, when $w_2 = 0$, $EP_r(u|x) = \overline{\mu}_{\tilde{A}}(x)$ so that the nonsymmetrical trapezoid has a vertical right leg. w_1 and w_2 are treated as design parameters, but they are the *same* for all of the vertical slices.

Exercise 6.17 also asks the reader to obtain formulas for the α-cuts of the nonsymmetrical trapezoid MF.

Our conclusion is that the *vertical-slice representation of a GT2 FS is a very parsimonious representation for the optimal design of a GT2 fuzzy system.*

6.8 Representing Non T2 FSs as T2 FSs

It may happen that a calculation involves a mixture of T1, IT2 and GT2 FSs, e.g., the union or intersection of such a mixture may be needed. This is easily accomplished by elevating a T1 FS to an IT2 FS. Nothing has to be done to an IT2 FS because it already is a special case of a GT2 FS.

Elevating a T1 FS, A, to an IT2 FS, $\tilde{A}$, is accomplished by assigning a secondary grade of 1 to A at $x \in X$, i.e.,

$$A \to \tilde{A} = \left\{\left((x,u), \mu_{\tilde{A}}(x,u) = 1\right) | x \in X,\ u = \mu_A(x)\right\} \tag{6.61}$$

This can also be summarized by means of the following expressive equation:

$$A \to \tilde{A} = 1/A \tag{6.62}$$

This expressive notation has already been used in (6.17) and (6.18).

When one wants to elevate an IVFS to an IT2 FS this is done by means of (6.18), i.e.,

$$\text{IVFS} \to \tilde{A} = 1/\text{IVFS} \tag{6.63}$$

Note that $1/\text{IVFS} = \text{CIT2 FS}$.

Definition 6.26 An element $x = x'$ is said (Mizumoto and Tanaka 1976) to have a *zero membership in a T2 FS* if it has a secondary grade equal to 1 corresponding to the primary membership of 0, and if it has all other secondary grades equal to 0. Such a 0 secondary membership at x' is denoted 1/0.

An element $x = x'$ is said to have a membership of one in a T2 FS—a *type-2 fuzzy singleton*—if it has a secondary grade equal to 1 corresponding to the primary membership of 1, and, if it has all other secondary grades equal to 0. Such a secondary membership at x' is denoted 1/1.

6.9 Returning to Linguistic Labels for T2 FSs

As explained in Sect. 2.2.5, sometimes it is necessary to go from MF numerical values for a variable to a linguistic description of that variable. This section examines how to do this for T2 FSs.

Consider the situation depicted in Fig. 6.26 at $x = x'$. This value of x only generates a nonzero membership in the T2 FS $\tilde{F}_4$ = *moderately high*; hence, $x = x'$ can be described linguistically, without any ambiguity, as “moderately high.” The situation at $x = x''$ is quite different, because this value of x generates a range of

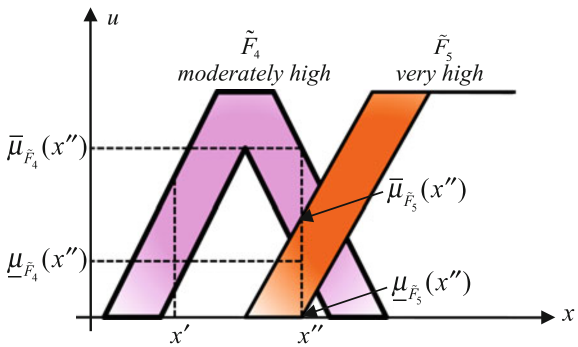


Fig. 6.26 Returning to a linguistic label for T2 FSs. Secondary MFs are not shown

nonzero secondary MF values in the two T2 FSs $\tilde{F}_4$ = *moderately high* and $\tilde{F}_5$ = *very high*. It would be extraordinarily difficult to communicate this linguistically. One approach, described next, is to first convert the intersection of the vertical line at $x = x''$ with the FOUs into a collection of numbers, after which the linguistic label at $x = x''$ can be chosen using an algorithm similar to the one for T1 FSs in (2.10).

The type-2 MFs $\mu_{\tilde{F}_4}(x,u)$ and $\mu_{\tilde{F}_5}(x,u)$ are described by $\mathrm{FOU}(\tilde{F}_4)$ and $\mathrm{FOU}(\tilde{F}_5)$, respectively, as well as by their secondary MFs. The upper and lower MFs for $\mathrm{FOU}(\tilde{F}_4)$ are $\overline{\mu}_{\tilde{F}_4}(x)$ and $\underline{\mu}_{\tilde{F}_4}(x)$, whereas the comparable quantities for $\mathrm{FOU}(\tilde{F}_5)$ are $\overline{\mu}_{\tilde{F}_5}(x)$ and $\underline{\mu}_{\tilde{F}_5}(x)$. Consider, for example, the vertical line at $x = x''$, and its intersections with $\mathrm{FOU}(\tilde{F}_4)$. Associated with the interval $[\underline{\mu}_{\tilde{F}_4}(x''), \overline{\mu}_{\tilde{F}_4}(x'')]$ is the secondary MF, $\mu_{\tilde{F}_4(x'')}(u)$, $u \in [\underline{\mu}_{\tilde{F}_4}(x''), \overline{\mu}_{\tilde{F}_4}(x'')]$. Let the center of gravity of the T1 FS $\mu_{\tilde{F}_4(x'')}(u)$ be denoted $u^{\mathrm{cg}}_{\tilde{F}_4(x'')}$. In a similar manner, $u^{\mathrm{cg}}_{\tilde{F}_5(x'')}$ can be computed so that $u^{\mathrm{cg}}_{\tilde{F}_4(x'')}$ and $u^{\mathrm{cg}}_{\tilde{F}_5(x'')}$ can then be compared. If $u^{\mathrm{cg}}_{\tilde{F}_4(x'')} > u^{\mathrm{cg}}_{\tilde{F}_5(x'')}$, then one would speak of x'' as "being moderately high"; otherwise one would speak of x'' as "being very high."

What has just been explained can be formalized as follows. Given p T2 FSs $\tilde{F}_i$ with MFs $\mu_{\tilde{F}_i}(x,u)$ that are characterized by $\mathrm{FOU}(\tilde{F}_i)$ $(i = 1,\ldots,p)$, whose upper and lower MFs, are $\overline{\mu}_{\tilde{F}_i}(x)$ and $\underline{\mu}_{\tilde{F}_i}(x)$, respectively, and by their secondary MFs. Consider an arbitrary value of x, say $x = x'$, and compute $\max[u^{\mathrm{cg}}_{\tilde{F}_1(x')}, u^{\mathrm{cg}}_{\tilde{F}_2(x')}, \ldots, u^{\mathrm{cg}}_{\tilde{F}_p(x')}] \equiv u^{\mathrm{cg}}_{\tilde{F}_m(x')}$, where $u^{\mathrm{cg}}_{\tilde{F}_i(x')}$ is the center of gravity of the secondary MF $\mu_{\tilde{F}_i(x')}(u)$, $u \in [\underline{\mu}_{\tilde{F}_i}(x'), \overline{\mu}_{\tilde{F}_i}(x')]$. Let $L(x')$ denote the linguistic label associated with x'. Then, $L(x') \equiv \tilde{F}_m$, i.e.,

$$L(x') = \arg\max_{\forall \tilde{F}_i} [u^{\mathrm{cg}}_{\tilde{F}_1(x')}, u^{\mathrm{cg}}_{\tilde{F}_2(x')}, \ldots, u^{\mathrm{cg}}_{\tilde{F}_p(x')}] \tag{6.64}$$

Example 6.24 For interval secondary MFs, it is easy to compute $u^{\mathrm{cg}}_{\tilde{F}_i(x')}$, as

$$u^{\mathrm{cg}}_{\tilde{F}_i(x')} = \frac{1}{2}\left[\overline{\mu}_{\tilde{F}_i}(x') + \underline{\mu}_{\tilde{F}_i}(x')\right] \tag{6.65}$$

For non-interval secondary MFs, the computations of the $u^{cg}_{\tilde{F}_i(x')}$ will be more complicated, and will, most likely, have to be done numerically.

Note that when T2 FSs reduce to T1 FSs, i.e., when $(x' \in X)$ $\overline{\mu}_{\tilde{F}_i}(x') = \underline{\mu}_{\tilde{F}_i}(x') = \mu_{F_i}(x')$, (6.64) reduces to (2.10). This is consistent with our basic design requirement that, when all MF uncertainties disappear, type-2 results must reduce to their well-established type-1 results.

Sometimes it is necessary to go from a complete FOU (for an IT2 FS) to a linguistic description of that variable. The approach that has just been described cannot be used to do this because it focuses on mapping only a single numerical value for a variable into a linguistic description of that variable. Similarity of IT2 FSs can be used to accomplish the former [e.g., see Mendel and Wu (2010, Chap. 4)].

6.10 Multivariable Membership Functions

All of the discussions in this chapter, so far, have been for T2 FSs that depend on only one variable. This section describes how to characterize T2 FSs that depend on up to p variables, $x_1, x_2, \ldots, x_p$.

To begin, our focus is on a T2 FS, $\tilde{A}$, that depends on only two variables, x_1 and x_2. When $X_1 \times X_2$ is continuous, then, analogous to (6.38) and (6.39), $\tilde{A}$ can be expressed as

$$\tilde{A} = \int_{x_1 \in X_1} \int_{x_2 \in X_2} \mu_{\tilde{A}(x_1,x_2)}(u)/(x_1, x_2) \tag{6.66}$$

where

$$\mu_{\tilde{A}(x_1,x_2)}(u) = \int_{u \in [0,1]} f_{(x_1,x_2)}(u)/u \tag{6.67}$$

When the multivariable secondary MF $\mu_{\tilde{A}(x_1,x_2)}(u)$, which is a T1 FS at each (x_1, x_2) pair, is[34] *separable* (see Sect. 2.14) then it can be expressed in terms of $\mu_{\tilde{A}(x_1)}(u)$ and $\mu_{\tilde{A}(x_2)}(u)$, as

$$\mu_{\tilde{A}(x_1,x_2)}(u) = \mu_{\tilde{A}(x_1)} \sqcap \mu_{\tilde{A}(x_2)} \tag{6.68}$$

where $\sqcap$ denotes the meet operation which is associated with computing the intersection of T2 FSs, and is discussed in great detail in Chap. 7. (6.68) is analogous to (2.115).

The extensions of these two-variable results to more than two variables is straightforward, e.g., for p variables, for which $\mu_{\tilde{A}(x_1,\ldots,x_p)}(u)$ is separable,

$$\mu_{\tilde{A}(x_1,\ldots,x_p)}(u) = \mu_{\tilde{A}(x_1)} \sqcap \mu_{\tilde{A}(x_2)} \sqcap \cdots \sqcap \mu_{\tilde{A}(x_p)} \tag{6.69}$$

Equation (6.69) is analogous to (2.116).

Note that (6.68) and (6.69) are widely used in Chaps. 7, 9 and 11, and how to actually compute them is explained in Chap. 7.

[34]To-date, there is no literature about non-separable multivariable secondary MFs.

Exercises

6.1 Provide an example that illustrates the difference between a DOU and an FOU.

6.2 As in Example 6.13, determine the FOU for a triangular MF (see Fig. 6.14), but for when $m_b - \sigma_b < m_a + \sigma_a$.

6.3 A possible alternative to asking a subject to provide interval end-point values for a linguistic term is to ask the subject to provide a *center location* (c) and *interval length* (l) for the interval.

(a) Formulate the question you would ask a subject so as to obtain these values.

(b) Determine the FOU when c and l information are obtained from a group of subjects. Assume that c has uncertainty $\pm\sigma_c$, l is centered about c, and that its uncertainty, $\pm\sigma_l$, is the same about its two endpoints $c - l/2$ and $c + l/2$. Consider two cases, when: (i) uncertainty interval for c, $m_c \pm \sigma_c$, does not overlap with the uncertainty intervals for the interval endpoints; and (ii) uncertainty interval for c, $m_c \pm \sigma_c$, overlaps with the uncertainty intervals for the interval endpoints.

6.4 Show three different FOUs that have lower and upper trapezoidal MFs.

6.5 Sketch FOU($\tilde{A}$) when, $\text{UMF}(\tilde{A}) = \exp\left\{-\frac{1}{2}[(x-m)/\sigma]^2\right\}$, $\text{LMF}(\tilde{A}) = s\exp\left\{-\frac{1}{2}[(x-m)/\sigma]^2\right\}$ and $0 < s < 1$.

6.6 Consider the combined case of Examples 6.16 and 6.17, i.e., a Gaussian primary MF with both an uncertain mean and standard deviation.

(a) Sketch its FOU.

(b) Determine formulas for its upper and lower MFs.

6.7 For Example 6.17, derive:

$$\partial N(x; m_1, \sigma)/\partial m_1 = (x - m_1)N(x; m_1, \sigma)/\sigma^2 \ (x < m_1)$$

$$\partial N(x; m_1, \sigma)/\partial \sigma = (x - m_1)^2 N(x; m_1, \sigma)/\sigma^3 \ (x < m_1)$$

6.8 Write formulas for:

(a) The LMF and the UMF of the FOU that is depicted in Fig. 6.9.

(b) The seven vertical slices that are also depicted on that figure.

6.9 Sketch four embedded T1 FSs for the type-2 MF that is depicted in:

(a) Fig. 6.16.

(b) Fig. 6.17.

6.10 Create an example that uses a mixture of a multiset from an IT2 FS and a multi interval-valued FS as an IT2 FS with the same level of detail as in Example 6.18 (figure and equations). Explain why such an IT2 FS can also be called a *multi interval-valued FS from an IT2 FS.*

6.11 Using the multiset from an IT2 FS that is depicted in Fig. 6.19:

 (a) At each x_i, sample u so that only the spikes are picked up. How many embedded T2 FSs will there be?
 (b) Enumerate all of the embedded T2 FSs.
 (c) Verify the truth of Theorem 6.2.
 (d) Verify the truth of Theorem 6.3.

6.12 Describe the minimal covering of $\mathrm{FOU}(\tilde{A})$ that is depicted in Fig. 6.27.

6.13 Describe the minimal covering of $\mathrm{FOU}(\tilde{B})$ in Fig. 6.28.

6.14 It has been cleverly suggested (Wu 2011) that an FOU can be covered by embedded T1 FSs that are both normal and convex, in which case there could be a "Constrained Representation Theorem" which would represent any IT2 FS as the union of such embedded T1 FSs. However, Wu (2011) provides a counterexample to this representation, so that such a theorem does not exist. In the FOU that is depicted in Fig. 6.29, shade in the region that cannot be covered by embedded T1 FSs that are both normal and convex.

Fig. 6.27 $\mathrm{FOU}(\tilde{A})$ for Exercise 6.12 (Wu and Mendel 2007)

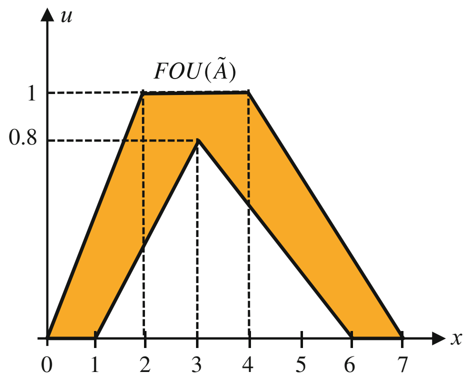
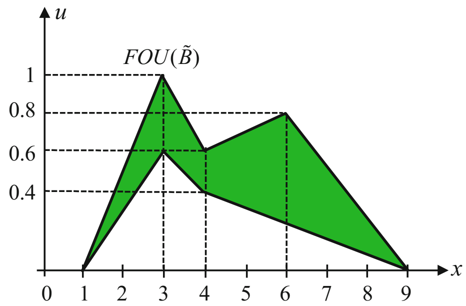


Fig. 6.28 $\mathrm{FOU}(\tilde{B})$ for Exercise 6.13



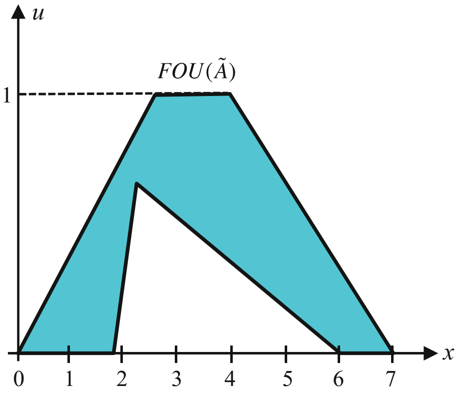


Fig. 6.29 FOU($\tilde{A}$) for Exercise 6.14

6.15 Derive the α-cut formulas that are in (6.54).

6.16 Derive the α-cut formulas that are in (6.57).

6.17 (a) Explain why the second line of (6.60) guarantees $EP_r(u|x) > EP_l(u|x)$. (b) Derive the α-cut formulas for the nonsymmetrical trapezoid in (6.58)–(6.60).

6.18 Explain how a squished FOU can be obtained by using two homotopies (see Example 6.19).

6.19 In this exercise (McCulloch and Wagner 2016), the lower and upper MFs of FOU($\tilde{A}$) are normal triangles, where a triangle MF for a T1 FS A is denoted $\mu_A(x) = \text{trimf}(x; [a, b, c; w])$ and

$$\mu_A(x) = \begin{cases} w(x-a)/(b-a) & a \le x \le b \\ w(c-x)/(c-b) & b < x \le c \\ 0 & \text{otherwise} \end{cases}$$

w is a weight that is used to adjust the height of the MF. In this exercise, $w = 1$, $\text{UMF}(\tilde{A}) = \text{trimf}(x; [\bar{a}, b, \bar{c}; 1])$, $\text{LMF}(\tilde{A}) = \text{trimf}(x; [\underline{a}, b, \underline{c}; 1])$, where $\bar{a} < \underline{a}$ and $\bar{c} > \underline{c}$, and $\alpha_k = k/k_{\max}$, $k = 0, 1, \ldots, k_{\max}$.

(a) Sketch FOU($\tilde{A}$).

(b) What are the formulas for $\bar{a}_k$, b_k, $\bar{c}_k$, $\underline{a}_k$ and $\underline{c}_k$ such that FOU($\tilde{A}_k$) is a squished version of FOU($\tilde{A}_{k-1}$), where the squishing is uniform as one goes from one level to the next?

(c) What is the squishing parameter?

(d) Is there a connection between the way in which FOU($\tilde{A}$) is squished in this exercise and the way in which it is squished in Exercise 6.18?

6.20 Another way to squish a horizontal slice at level α_1 into a horizontal-slice at level α_2 ($\alpha_2 > \alpha_1$) is to begin with the FOU (at level $\alpha_1 = 0$) and use the same UMF of the FOU at all α levels, but let the LMF of the horizontal slice at level $\alpha_2 (\alpha_2 > \alpha_1)$ be a scaled version of the UMF of the FOU. Let γ_k denote the squishing parameter at level α_k.

(a) What is the formula for the $\alpha-$cut of the secondary MF for this GT2 FS?
(b) Sketch the secondary MF.

6.21 When $X \rightarrow X_d$ and $U \rightarrow U_d$, explain which definitions in this chapter can be easily modified by replacing X by X_d, U by U_d, and $\int$ by $\sum$.

6.22 When $X \rightarrow X_d$ but U remains a continuous universe of discourse, explain which definitions in this chapter can be easily modified, and how.

References

Aisbett, J., J.T. Rickard, and D.G. Morgenthaler. 2010. Type-2 fuzzy sets as functions on spaces. *IEEE Transactions on Fuzzy Systems* 18: 841–844.

Almaraashi, M., R. John, A. Hopgood, and S. Ahmadi. 2016. Learning of interval and general type-2 fuzzy logic systems using simulated annealing: Theory and practice. *Information Sciences* 360: 21–42.

Bilgin, A., H. Hagras, A. Malibari, M. Alhaddad, and D. Alghazzawi. 2012a. Towards a general type-2 fuzzy logic approach for computing with words using linear adjectives. In *Proceedings FUZZ-IEEE 2012*, pp. 1130–1137, Brisbane, AU.

Bilgin, A., J.Dooley, L. Whittington, H. Hagras, M. Henson, C. Wagner, A. Malibari, A. Al-Ghamdi, M. Alhaddad, and D. Alghazzawi. 2012b. Dynamic profile-selection for zslices based type-2 fuzzy agents controlling multi-user ambient intelligent environments. In *Proceedings FUZZ-IEEE 2012*, pp. 1392–1399, Brisbane, AU.

Bilgin, A., H. Hagras, A. Malibari, M. J. Alhaddad, and D. Alghazzawi. 2012c. A general type-2 fuzzy logic approach for adaptive modeling of perceptions for computing with words. In *Proceedings 2012 12th UK workshop on computational intelligence (UKCI)*, pp. 1–8.

Bilgin, A., H. Hagras, A. Malibari, M. J. Alhaddad, and D. Alghazzawi. 2013a. An experience based linear general type-2 fuzzy logic approach for computing with words. In *Proceedings of IEEE international conference on fuzzy systems*, Paper #1139, Hyderabad, India.

Bilgin, A., H. Hagras, A. Malibari, M. J. Alhaddad, and D. Alghazzawi. 2013b. Towards a linear general type-2 fuzzy logic based approach for computing with words. *International Journal of Soft Computing* 17:2203–2222.

Buckley, J.J. 2003. *Fuzzy probabilities: New approaches and new applications*. New York: Physica-Verlag.

Bustince, H. 2000. Indicator of inclusion grade for interval-valued fuzzy sets: Applications to approximate reasoning based on interval-valued fuzzy sets. *International Journal of Approximate Reasoning* 23: 137–209.

Bustince, H., J. Fernandez, H. Hagras, F. Herrera, M. Pagola, and E. Barrenechea. 2015. Interval type-2 fuzzy sets are generalization of interval-valued fuzzy sets: towards a wider view on their relationship. *IEEE Transactions on Fuzzy Systems* 23: 1876–1882.

Chen, Q., and S. Kawase. 2000. On fuzzy-valued fuzzy reasoning. *Fuzzy Sets and Systems* 113: 237–251.

Coupland, S., and R.I. John. 2007. Geometric type-1 and type-2 fuzzy logic systems. *IEEE Transactions on Fuzzy Systems* 15: 3–15.

Garibaldi, J. M., S. Musikasuwan, and T. Ozen. 2005. The association between non-stationary and interval type-2 fuzzy sets: A case study. In *Proceedings IEEE FUZZ conference*, pp. 224–229, Reno, NV.

Gorzalczany, M.B. 1987. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems* 21: 1–17.

Greenfield, S., and R. John. 2007. Optimized generalized type-2 join and meet operations. In *Proceedings FUZZ-IEEE 2007,* pp. 141–146, London, UK.

Hamrawi, H., and S. Coupland. 2009. Type-2 fuzzy arithmetic using alpha-planes. In *Proceedings IFSA/EUSFLAT*, pp. 606–611, Portugal.

Hamrawi, H., S. Coupland, and R. John. 2010. A novel alpha-cut representation for type-2 fuzzy sets. In *Proceedings FUZZ-IEEE 2010, IEEE world congress on computational intelligence*, pp. 351–358, Barcelona, Spain.

John, R., and S. Coupland. 2012. type-2 fuzzy logic: Challenges and misconceptions. *IEEE Computational Intelligence Magazine* 7 (3): 48–52.

Karnik, N.N., and J.M. Mendel. 1998. *An introduction to type-2 fuzzy logic systems*. USC-SIPI Report #418, Univ. of Southern Calif., Los Angeles, CA, June 1998. This can be accessed at: http://sipi.usc.edu/research; then choose "sipi technical reports/418.".

Karnik, N.N., and J.M. Mendel. 2001a. Operations on type-2 fuzzy sets. *Fuzzy Sets and Systems* 122: 327–348.

Karnik, N.N., and J.M. Mendel. 2001b. Centroid of a type-2 fuzzy set. *Information Sciences* 132: 195–220.

Karnik, N.N., J.M. Mendel, and Q. Liang. 1999. Type-2 fuzzy logic systems. *IEEE Transactions on Fuzzy Systems* 7: 643–658.

Klir, G.J., and B. Yuan. 1995. *Fuzzy sets and fuzzy logic: Theory and applications*. Upper Saddle River, NJ: Prentice Hall.

Kumbasar, T., and H. Hagras. 2015. A self-tuning zSlices based general type-2 fuzzy PI controller. *IEEE Transactions on Fuzzy Systems* 23: 991–1013.

Liang, Q., and J.M. Mendel. 2000. Interval type-2 fuzzy logic systems. In *Proceedings FUZZ-IEEE '00*, San Antonio, TX.

Liu, F. 2008. An efficient centroid type-reduction strategy for general type-2 fuzzy logic system. *Information Sciences* 178: 2224–2236.

Liu, F., and J.M. Mendel. 2008. Encoding words into interval type-2 fuzzy sets using an interval approach. *IEEE Transactions on Fuzzy Systems* 16: 1503–1521.

Ljung, L. 1999. *System identification: Theory for the user*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall.

Lushu, L. 1995. Random fuzzy sets and fuzzy martingales. *Fuzzy Sets and Systems* 69: 181–192.

McCulloch, J., and C. Wagner. 2016. Measuring the similarity between zSlices general type-2 fuzzy sets with non-normal secondary membership functions. In *Proceedings of FUZZ-IEEE 2016*, pp. 461–468, Vancouver, Canada.

Mendel, J.M. 2001. *Introduction to rule-based fuzzy logic systems*. Upper Saddle River, NJ: Prentice-Hall.

Mendel, J.M. 2007. Type-2 fuzzy sets and systems: An overview. *IEEE Computational Intelligence Magazine* 2: 20–29.

Mendel, J.M. 2009. On answering the question 'Where do I start in order to solve a new problem involving interval type-2 fuzzy sets?'. *Information Sciences* 179: 3418–3431.

Mendel, J.M. 2010. Comments on 'α-plane representation for type-2 fuzzy sets: Theory and applications'. *IEEE Transactions on Fuzzy Systems* 18: 229–230.

Mendel, J.M. 2012. Plotting 2-1/2 D figures for general type-2 fuzzy sets by hand or by powerpoint. In *Proceedings FUZZ-IEEE 2012*, pp. 1490-1497, Brisbane, AU.

Mendel, J.M. 2014. General type-2 fuzzy logic systems made simple: A tutorial. *IEEE Transactions on Fuzzy Systems* 22: 1162–1182.

Mendel, J.M., and R.I. John. 2002. Type-2 fuzzy sets made simple. *IEEE Transactions on Fuzzy Systems* 10: 117–127.

Mendel, J.M., and Q. Liang. 1999. Pictorial comparisons of type-1 and type-2 fuzzy logic systems. In *Proceedings IASTED international conference on intelligent systems & control*, Santa Barbara, CA.

Mendel, J.M., and D. Wu. 2010. *Perceptual computing: Aiding people in making subjective judgments*. Hoboken, NJ: Wiley and IEEE Press.

Mendel, J.M., R.I. John, and F. Liu. 2006. Interval type-2 fuzzy logic systems made simple. *IEEE Transactions on Fuzzy Systems* 14: 808–821.

Mendel, J.M., F. Liu, and D. Zhai. 2009. Alpha-plane representation for type-2 fuzzy sets: Theory and applications. *IEEE Transactions an Fuzzy Systems* 17: 1189–1207.

Mendel, J.M., M.R. Rajati, and P. Sussner. 2016. On clarifying some notations used for type-2 fuzzy sets as well as some recommended notational changes. *Information Sciences* 340–341: 337–345.

Mizumoto, M., and K. Tanaka. 1976. Some properties of fuzzy sets of type-2. *Information and Control* 31: 312–340.

Mizumoto, M., and K. Tanaka. 1981. Fuzzy sets of type-2 under algebraic product and algebraic sum. *Fuzzy Sets and Systems* 5: 277–290.

Moharrer, M., H. Tahayori, H., and A. Sadeghian. 2013. Modeling complex concepts with type-2 fuzzy sets: The case of user satisfaction of online services. In *Advances in type-2 fuzzy sets and systems: theory and applications*, ed. Sadeghian, et al., 133–146. NY: Springer.

Moller, B., and M. Beere. 2004. *Fuzzy randomness: Uncertainty in civil engineering and computational mechanics*. New York: Springer.

Pedrycz, W. 2015. Concepts and design aspects of granular models of type-1 and type-2. *International Journal of Fuzzy Logic and Intelligent Systems* 15: 87–95.

Rakshit, P., A. Chakraborty, A. Konar, and A.K. Nagar. 2013. Secondary membership evaluation in generalized type-2 fuzzy sets by evolutionary optimization algorithm. In *Proceedings FUZZ-IEEE 2013*, Paper #1334, Hyderabad, India.

Rakshit, P., A. Saha, A. Konar, and S. Saha. 2016. A type-2 fuzzy classifier for gesture recognition induced pathological disorder recognition. *Fuzzy Sets and Systems* 305: 95–130.

Starczewski, J.T. 2009a. Efficient triangular type-2 fuzzy logic systems. *International Journal of Approximate Reasoning* 50: 799–811.

Starczewski, J.T. 2009b. Extended triangular norms. *Information Sciences* 179: 742–757.

Tahayori, H., A.G.B. Tettamanzi, and G.D. Antoni. 2006. Approximated type-2 fuzzy set operations. In *Proceedings FUZZ-IEEE 2006*, Vancouver, B. C. Canada, pp. 9042–9049.

Tahayori, H., A.G.B. Tettamanzi, G.D. Antoni, A. Visconti, and M. Moharrer. 2010. Concave type-2 fuzzy sets: Properties and operations. *Soft Computing Journal* 14(7): 749–756.

Ulu, C., M. Güzellkaya, and I. Eksin. 2013. Granular type-2 membership functions: A new approach to formation of footprint of uncertainty in type-2 fuzzy sets. *Applied Soft Computing* 13: 3713–3728.

Wagner, C., and H. Hagras. 2008. z Slices–towards bridging the gap between interval and general type-2 fuzzy logic. In *Proceedings IEEE FUZZ conference*, Paper # FS0126, Hong Kong, China.

Wagner, C., and H. Hagras. 2010. Towards general type-2 fuzzy logic systems based on zSlices. *IEEE Transactions on Fuzzy Systems* 18: 637–660.

Wagner, C., and H. Hagras. 2013. zSlices based general type-2 fuzzy sets and systems. In *Advances in type-2 fuzzy sets and systems: Theory and applications*, eds. Sadeghian, A., J.M. Mendel, and H. Tahayori. New York: Springer.

Wu, D. 2011. A constrained representation theorem for interval type-2 fuzzy sets using convex and normal embedded type-1 fuzzy sets and its application to centroid computation. In *Proceedings of world conference on soft computing*, Paper #200, San Francisco, CA.

Wu, D., and J.M. Mendel. 2007. Uncertainty measures for interval type-2 fuzzy sets. *Information Sciences* 177: 5378–5393.

Zadeh, L.A. 1975. The concept of a linguistic variable and its application to approximate reasoning–1. *Information Sciences* 8: 199–249.