What do Thompson's group V and the Collatz conjecture have in common?

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\begin{aligned}
V \;=\; \langle \kappa, \lambda, \mu, \nu \mid\ & \kappa^2 = \lambda^2 = \mu^2 = \nu^2 = \\
& \lambda\kappa\mu\kappa\lambda\nu\kappa\nu\mu\kappa\lambda\kappa\mu = \kappa\nu\lambda\kappa\mu\nu\kappa\lambda\nu\mu\nu\lambda\nu\mu = \\
& (\lambda\kappa\mu\kappa\lambda\nu)^3 = (\mu\kappa\lambda\kappa\mu\nu)^3 = \\
& (\lambda\nu\mu)^2\kappa(\mu\nu\lambda)^2\kappa = (\lambda\nu\mu\nu)^5 = \\
& (\lambda\kappa\nu\kappa\lambda\nu)^3\kappa\nu\kappa(\mu\kappa\nu\kappa\mu\nu)^3\kappa\nu\kappa\nu = \\
& ((\lambda\kappa\mu\nu)^2(\mu\kappa\lambda\nu)^2)^3 = (\lambda\nu\lambda\kappa\mu\kappa\mu\nu\lambda\nu\mu\kappa\mu\kappa)^4 = \\
& (\mu\nu\mu\kappa\lambda\kappa\lambda\nu\mu\nu\lambda\kappa\lambda\kappa)^4 = (\lambda\mu\kappa\lambda\kappa\mu\lambda\kappa\nu\kappa)^2 = \\
& (\mu\lambda\kappa\mu\kappa\lambda\mu\kappa\nu\kappa)^2 = 1\rangle.
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The Collatz Conjecture

The Collatz conjecture asserts that iterated application of the mapping

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How is it possible to put these into a common framework?

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Example: We have

$$\tau_{0(2),1(4)} \in \mathrm{Sym}(\mathbb{Z}) : \ n \mapsto \begin{cases} 2n+1 & \text{if } n \in 0(2), \\ (n-1)/2 & \text{if } n \in 1(4). \end{cases}$$

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It is easy to check with RCWA [3] that these generators satisfy indeed Higman's relations, which verifies the isomorphism.

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