On Conjugates of Collatz-Type Mappings

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Abstract

A mapping $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is called *residue class-wise affine* if there is a positive integer m such that it is affine on residue classes (mod m). If there is a finite set $S \subset \mathbb{Z}$ which intersects nontrivially with any trajectory of f, then f is called *almost contracting*. Assume that f is a surjective but not injective residue class-wise affine mapping, and that the preimage of any integer under f is finite. Then f is almost contracting if and only if there is a permutation σ of $\mathbb{Z}$ such that f^σ is monotonous almost everywhere. In this article it is shown that if there is no positive integer k such that applying $f^{(k)}$ decreases the absolute value of almost all integers, then σ cannot be residue class-wise affine itself. The original motivation for the investigations in this article comes from the famous $3n + 1$ Conjecture.

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1 Introduction

In the 1930s, Lothar Collatz made the following conjecture which is still open today (see [3] for a survey article and [4] for an annotated bibliography):

1.1 3n+1 Conjecture Iterated application of the mapping

$$T : \mathbb{Z} \longrightarrow \mathbb{Z}, \quad n \longmapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ even}, \\ \frac{3n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

to any positive integer yields 1 after a finite number of steps. In short this means that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}_0$ such that $T^{(k)}(n) = 1$.

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Obviously this conjecture holds if and only if there is a permutation σ of $\mathbb{Z}$ which maps positive integers to positive integers and fixes 1 such that T^σ maps any integer $n > 1$ to a smaller one. Since T is surjective but not injective, this is essentially equivalent to requiring that T^σ is monotonous almost everywhere (imagine the graph of a monotonous conjugate!).

In this article, on the one hand we generalize the question for the existence of such a monotonous conjugate to other mappings similar to the Collatz mapping T. On the other we specialize it to the case that the conjugating permutation σ itself is similar to T, i.e. *residue class-wise affine*:

1.2 Definition We call a mapping $f : \mathbb{Z} \to \mathbb{Z}$ *residue class-wise affine* if there is a positive integer m such that the restrictions of f to the residue classes $r(m) \in \mathbb{Z}/m\mathbb{Z}$ are all affine, i.e. given by

$$f|_{r(m)} : \ r(m) \to \mathbb{Z}, \quad n \ \mapsto \ \frac{a_{r(m)} \cdot n + b_{r(m)}}{c_{r(m)}}$$

for certain coefficients $a_{r(m)}, b_{r(m)}, c_{r(m)} \in \mathbb{Z}$ depending on $r(m)$. We call the smallest possible m the *modulus* of f, written $\mathrm{Mod}(f)$. For reasons of uniqueness, we assume that $\gcd(a_{r(m)}, b_{r(m)}, c_{r(m)}) = 1$ and that $c_{r(m)} > 0$. We define the *multiplier* $\mathrm{Mult}(f)$ of f by $\mathrm{lcm}_{r(m)\in\mathbb{Z}/m\mathbb{Z}} \ a_{r(m)}$ and the *divisor* $\mathrm{Div}(f)$ of f by $\mathrm{lcm}_{r(m)\in\mathbb{Z}/m\mathbb{Z}} \ c_{r(m)}$. We always assume that $\mathrm{Mult}(f) \neq 0$.

1.3 Definition Let $f : \mathbb{Z} \to \mathbb{Z}$ be a mapping. We call f *almost contracting* if there is a finite set $S \subset \mathbb{Z}$ which intersects nontrivially with any trajectory of f. We call f *monotonizable* if there is a permutation $\sigma \in \mathrm{Sym}(\mathbb{Z})$ and a finite set $S \subset \mathbb{Z}$ such that f^σ is monotonous on $\mathbb{Z} \setminus S$. Further we call f *rcwa-monotonizable* if σ can be chosen to be residue class-wise affine.

1.4 Remark It is easy to see that surjective monotonizable mappings are also almost contracting, and that almost contracting mappings are monotonizable provided that the preimage of any integer is finite.

1.5 Example We look at the residue class-wise affine mappings

$$f: \ n \mapsto \begin{cases} \frac{n+1}{2} & \text{if } n \in 1(6), \\ \frac{9n+1}{2} & \text{if } n \in 3(6), \\ \frac{9n+11}{2} & \text{if } n \in 5(6), \\ \frac{n-2}{18} & \text{if } n \in 2(54), \\ \frac{n+8}{18} & \text{if } n \in 28(54), \\ \frac{n}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma: \ n \mapsto \begin{cases} 9n+1 & \text{if } n \in 0(3), \\ \frac{n-1}{9} & \text{if } n \in 1(27), \\ n & \text{otherwise.} \end{cases}$$

The mapping f is surjective and rcwa-monotonizable. Indeed its conjugate under σ is $f^\sigma : \ n \mapsto \lfloor (n+1)/2 \rfloor$, which is monotonous on $\mathbb{Z}$. Therefore f is almost contracting. This is nontrivial, as there are trajectories like $21, 95, 433, 217, 109, 55, 28, \dots$ and $63, 284, 142, 71, 325, 163, 82, \dots$.

2 A Condition for rcwa-Monotonizability

In this article we derive a necessary condition for rcwa-monotonizability. In the proof we need the following lemmata:

2.1 Lemma *Assume that f is a non-injective residue class-wise affine mapping. Then there are a residue class $r_0(m_0)$ and two disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ of $\mathbb{Z}$ such that $r_0(m_0) = f(r_1(m_1)) = f(r_2(m_2))$.*

Proof: Let $m := \mathrm{Mod}(f)$. Since f is not injective, there are two residue classes $\tilde{r}_1(m)$ and $\tilde{r}_2(m)$ whose images under f are not disjoint. The images $f(\tilde{r}_1(m))$ and $f(\tilde{r}_2(m))$ are also residue classes. Thus their intersection $r_0(m_0)$ is a residue class, too. The preimages $r_1(m_1)$ and $r_2(m_2)$ of $r_0(m_0)$ under the affine mappings of $f|_{\tilde{r}_1(m)}$ resp. $f|_{\tilde{r}_2(m)}$ are residue classes as well. They are disjoint since they are subsets of distinct residue classes (mod m). $\square$

2.2 Lemma *Given a residue class-wise affine mapping f, there is a constant $c \in \mathbb{N}$ such that $\forall n \in \mathbb{Z} \ \ |f(n)| \leqslant \mathrm{Mult}(f) \cdot |n| + c$.*

Proof: Take upper bounds on the absolute values of the images of n under the affine partial mappings of f. $\square$

We get a quite restrictive condition for rcwa-monotonizability:

2.3 Theorem *Assume that f is a residue class-wise affine mapping which is not injective, but surjective and rcwa-monotonizable. Then there is a $k \in \mathbb{N}$ such that there are at most finitely many $n \in \mathbb{Z}$ which satisfy $|f^{(k)}(n)| \geqslant |n|$.*

Proof: By assumption, we can choose a residue class-wise affine permutation σ and a finite subset $S \subset \mathbb{Z}$ such that $\mu := f^\sigma$ is monotonous on $\mathbb{Z} \setminus S$.

Surjectivity and non-injectivity are inherited from f to μ. Consequently, by Lemma 2.1 there is a residue class $r(m)$ such that any $n \in r(m)$ has at least two distinct preimages under μ.

From the surjectivity of μ, the monotonity of μ on $\mathbb{Z} \setminus S$ and the finiteness of S we can conclude that there is a constant $c' \in \mathbb{N}$ such that we have $\forall n \in \mathbb{Z} \ |\mu(n)| < m/(m+1) \cdot |n| + c'$, and induction over $k \in \mathbb{N}$ yields

$$\forall k \in \mathbb{N} \ \forall n \in \mathbb{Z} \ \ |\mu^{(k)}(n)| \ < \ (m/(m+1))^k \cdot |n| + k \cdot c'.$$

For any $k \in \mathbb{N}$ we have $f^{(k)}(n) = \sigma^{-1}\mu^{(k)}\sigma(n)$. We choose k such that

$$(m/(m+1))^k \ < \ 1/\left(2 \cdot \mathrm{Mult}(\sigma) \cdot \mathrm{Div}(\sigma)\right).$$

Since inversion interchanges multiplier and divisor, by Lemma 2.2 for some constant c depending on σ the following holds:

$$\begin{aligned} |f^{(k)}(n)| \ &= \ |\sigma^{-1}\mu^{(k)}\sigma(n)| \ < \ \mathrm{Div}(\sigma) \cdot (m/(m+1))^k \cdot |n| \cdot \mathrm{Mult}(\sigma) + c \\ &< \ |n|/2 + c. \end{aligned}$$

Since neither k nor c depends on n, this completes our proof. □

2.4 Example Let f be as in Example 1.5. Then Theorem 2.3 asserts that there is some $k \in \mathbb{N}$ such that for almost all $n \in \mathbb{Z}$ it is $|f^{(k)}(n)| < |n|$.

An easy computation with the GAP [1] package RCWA [2] shows indeed that $k = 7$ is the smallest value which satisfies this condition. Further one can check that the integers $n \notin \{-1, 0, 1\}$ which fail to satisfy the inequality $|f^{(k)}(n)| < |n|$ for all $k \leqslant 6$ are those in the set $21(192) \cup 63(192) \cup 105(192)$.

Theorem 2.3 has consequences for the $3n + 1$ Conjecture:

2.5 Corollary *The Collatz mapping T is not rcwa-monotonizable.*

Proof: The Collatz mapping T is surjective and not injective. Further, given $n = 2^k m - 1$ for some $k, m \in \mathbb{N}$ we have $T^{(k)}(n) = (3^k n + (3^k - 2^k))/2^k > n$. Hence if there is a conjugate T^σ ($\sigma \in \mathrm{Sym}(\mathbb{Z})$) which is monotonous almost everywhere, then by Theorem 2.3, σ is not residue class-wise affine. □

2.6 Remark Corollary 2.5 shows mainly that a 'conjugating' permutation which certifies that the Collatz mapping is almost contracting cannot be 'very simple'. To the author it seems likely that it must be quite complicate.

References

[1] The GAP Group. *GAP – Groups, Algorithms, and Programming; Version 4.4.7*, 2006. http://www.gap-system.org.

[2] Stefan Kohl. *RCWA - Residue Class-Wise Affine Groups*, 2005. GAP package, http://www.gap-system.org/Packages/rcwa.html.

[3] Jeffrey C. Lagarias. The 3x+1 problem and its generalizations. *Amer. Math. Monthly*, 92:1–23, 1985.

[4] Jeffrey C. Lagarias. The 3x+1 problem: An annotated bibliography, 2006. (`http://arxiv.org/abs/math.NT/0309224`).