

Classifying Finite Simple Groups with Respect to the Number of Orbits Under the Action of the Automorphism Group

– Supplementary Tables, Updated 2020-04-28 –

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The values $\omega(G)$ in Table 1 have mostly been computed using **GAP** [4], all other data has been taken from the *Atlas of finite groups* [3]. Tables 2 and 4 have in parts been computed using **GAP**, and in parts using **MAGMA** [2] by Eamonn O’Brien in December 2019 – cf. Table 3. For Table 4, among various other information, also the bounds from [1] have been taken into account. The value $\omega(\text{Ree}(8)) = 57$ has been determined independently by Frank Lübeck and Rob Wilson from the character table of the group and some insight on fusion of classes, and the value $\omega(^2\text{E}_6(2)) = 91$ has been determined in a similar way by Rob Wilson.

1 Orbit Numbers for Small Simple Groups

Table 1: Values $\omega(G)$ for simple groups G (sorted by group order, the enumeration of groups $G = \text{PSL}(2, q)$ was stopped at $|G| = 10^6$).

G	$\omega(G)$	$ G $	Prime factorization of $ G $	$\text{Out}(G)$
$A_5 \cong \text{PSL}(2, 4)$				
$\cong \text{PSL}(2, 5)$	4	60	$2^2 \cdot 3 \cdot 5$	C_2
$\text{PSL}(3, 2) \cong \text{PSL}(2, 7)$	5	168	$2^3 \cdot 3 \cdot 7$	C_2
$A_6 \cong \text{PSL}(2, 9)$	5	360	$2^3 \cdot 3^2 \cdot 5$	C_2^2
$\text{PSL}(2, 8)$	5	504	$2^3 \cdot 3^2 \cdot 7$	C_3
$\text{PSL}(2, 11)$	7	660	$2^2 \cdot 3 \cdot 5 \cdot 11$	C_2
$\text{PSL}(2, 13)$	8	1092	$2^2 \cdot 3 \cdot 7 \cdot 13$	C_2
$\text{PSL}(2, 17)$	10	2448	$2^4 \cdot 3^2 \cdot 17$	C_2
A_7	8	2520	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	C_2
$\text{PSL}(2, 19)$	11	3420	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	C_2
$\text{PSL}(2, 16)$	7	4080	$2^4 \cdot 3 \cdot 5 \cdot 17$	C_4
$\text{PSL}(3, 3)$	9	5616	$2^4 \cdot 3^3 \cdot 13$	C_2
$\text{PSU}(3, 3) \cong G_2(2)'$	10	6048	$2^5 \cdot 3^3 \cdot 7$	C_2
$\text{PSL}(2, 23)$	13	6072	$2^3 \cdot 3 \cdot 11 \cdot 23$	C_2
$\text{PSL}(2, 25)$	10	7800	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	C_2^2
M_{11}	10	7920	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
$\text{PSL}(2, 27)$	7	9828	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	C_6
To be continued.				

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G	$\omega(G)$	$ G $	Prime factorization of $ G $	$\text{Out}(G)$
PSL(2, 29)	16	12180	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	C_2
PSL(2, 31)	17	14880	$2^5 \cdot 3 \cdot 5 \cdot 31$	C_2
$A_8 \cong \text{PSL}(4, 2)$	12	20160	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	C_2
PSL(3, 4)	6	20160	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	D_6
PSL(2, 37)	20	25308	$2^2 \cdot 3^2 \cdot 19 \cdot 37$	C_2
$\text{PSU}(4, 2) \cong \text{O}(5, 3)$	15	25920	$2^6 \cdot 3^4 \cdot 5$	C_2
Sz(8)	7	29120	$2^6 \cdot 5 \cdot 7 \cdot 13$	C_3
PSL(2, 32)	9	32736	$2^5 \cdot 3 \cdot 11 \cdot 31$	C_5
PSL(2, 41)	22	34440	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$	C_2
PSL(2, 43)	23	39732	$2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$	C_2
PSL(2, 47)	25	51888	$2^4 \cdot 3 \cdot 23 \cdot 47$	C_2
PSL(2, 49)	17	58800	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	C_2^2
$\text{PSU}(3, 4)$	9	62400	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	C_4
PSL(2, 53)	28	74412	$2^2 \cdot 3^3 \cdot 13 \cdot 53$	C_2
M_{12}	12	95040	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	C_2
PSL(2, 59)	31	102660	$2^2 \cdot 3 \cdot 5 \cdot 29 \cdot 59$	C_2
PSL(2, 61)	32	113460	$2^2 \cdot 3 \cdot 5 \cdot 31 \cdot 61$	C_2
$\text{PSU}(3, 5)$	10	126000	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	S_3
PSL(2, 67)	35	150348	$2^2 \cdot 3 \cdot 11 \cdot 17 \cdot 67$	C_2
J_1	15	175560	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1
PSL(2, 71)	37	178920	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$	C_2
A_9	16	181440	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	C_2
PSL(2, 73)	38	194472	$2^3 \cdot 3^2 \cdot 37 \cdot 73$	C_2
PSL(2, 79)	41	246480	$2^4 \cdot 3 \cdot 5 \cdot 13 \cdot 79$	C_2
PSL(2, 64)	15	262080	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	C_6
PSL(2, 81)	15	265680	$2^4 \cdot 3^4 \cdot 5 \cdot 41$	$C_2 \times C_4$
PSL(2, 83)	43	285852	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$	C_2
PSL(2, 89)	46	352440	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 89$	C_2
PSL(3, 5)	19	372000	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	C_2
M_{22}	11	443520	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	C_2
PSL(2, 97)	50	456288	$2^5 \cdot 3 \cdot 7^2 \cdot 97$	C_2
PSL(2, 101)	52	515100	$2^2 \cdot 3 \cdot 5^2 \cdot 17 \cdot 101$	C_2
PSL(2, 103)	53	546312	$2^3 \cdot 3 \cdot 13 \cdot 17 \cdot 103$	C_2
J_2	16	604800	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	C_2
PSL(2, 107)	55	612468	$2^2 \cdot 3^3 \cdot 53 \cdot 107$	C_2
PSL(2, 109)	56	647460	$2^2 \cdot 3^3 \cdot 5 \cdot 11 \cdot 109$	C_2
PSL(2, 113)	58	721392	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 113$	C_2
PSL(2, 121)	37	885720	$2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$	C_2^2
PSL(2, 125)	24	976500	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	C_6
$\text{O}(5, 4) \cong \text{PSp}(4, 4)$	12	979200	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	C_4
$\text{O}(7, 2) \cong \text{PSp}(6, 2)$	30	1451520	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_{10}	22	1814400	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	C_2
PSL(3, 7)	16	1876896	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	S_3
$\text{PSU}(4, 3)$	14	3265920	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	D_4
$G_2(3)$	17	4245696	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	C_2
$\text{O}(5, 5) \cong \text{PSp}(4, 5)$	27	4680000	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	C_2
<i>To be continued.</i>				

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G	$\omega(G)$	$ G $	Prime factorization of $ G $	$\text{Out}(G)$
PSU(3, 8)	10	5515776	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	$C_3 \times S_3$
PSU(3, 7)	34	5663616	$2^7 \cdot 3 \cdot 7^3 \cdot 43$	C_2
PSL(4, 3)	26	6065280	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	C_2^2
PSL(5, 2)	20	9999360	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	C_2
M_{23}	17	10200960	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
PSU(5, 2)	30	13685760	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	C_2
PSL(3, 8)	17	16482816	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$	C_6
${}^2F_4(2)'$ (Tits-G.)	17	17971200	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	C_2
A_{11}	29	19958400	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	C_2
Sz(32)	11	32537600	$2^{10} \cdot 5^2 \cdot 31 \cdot 41$	C_5
PSL(3, 9)	32	42456960	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	C_2^2
PSU(3, 9)	29	42573600	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$	C_4
HS	21	44352000	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	C_2
J_3	17	50232960	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	C_2
PSU(3, 11)	30	70915680	$2^5 \cdot 3^2 \cdot 5 \cdot 11^3 \cdot 37$	S_3
$O(5, 7) \cong \text{PSp}(4, 7)$	43	138297600	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	C_2
$O^+(8, 2)$	27	174182400	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	S_3
$O^-(8, 2)$	33	197406720	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	C_2
${}^3D_4(2)$	21	211341312	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	C_3
PSL(3, 11)	73	212427600	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	C_2
A_{12}	40	239500800	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	C_2
M_{24}	26	244823040	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
$G_2(4)$	24	251596800	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	C_2
PSL(3, 13)	39	270178272	$2^5 \cdot 3^2 \cdot 7 \cdot 13^3 \cdot 61$	S_3
PSU(3, 13)	100	811273008	$2^4 \cdot 3 \cdot 7^2 \cdot 13^3 \cdot 157$	C_2
McL	19	898128000	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	C_2
PSL(4, 4)	36	987033600	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	C_2^2
PSU(4, 4)	35	1018368000	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	C_4
$O(5, 8) \cong \text{PSp}(4, 8)$	21	1056706560	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	C_6
PSL(3, 16)	20	1425715200	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	$C_4 \times S_3$
$O(5, 9) \cong \text{PSp}(4, 9)$	41	1721606400	$2^8 \cdot 3^8 \cdot 5^2 \cdot 41$	C_2^2
PSU(3, 17)	62	2317678272	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	S_3
A_{13}	52	3113510400	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	C_2
He	26	4030387200	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	C_2
PSU(3, 16)	40	4279234560	$2^{12} \cdot 3 \cdot 5 \cdot 17^2 \cdot 241$	C_8
$\text{PSp}(6, 3)$	50	4585351680	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	C_2
$O(7, 3)$	52	4585351680	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	C_2
PSL(3, 19)	75	5644682640	$2^4 \cdot 3^4 \cdot 5 \cdot 19^3 \cdot 127$	S_3
$G_2(5)$	44	5859000000	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
PSL(3, 17)	163	6950204928	$2^9 \cdot 3^2 \cdot 17^3 \cdot 307$	C_2
PSL(4, 5)	34	7254000000	$2^7 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$	D_4
PSU(6, 2)	34	9196830720	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	S_3

2 Simple Groups by Orbit Number

Table 2: Simple groups G for given $\omega(G)$; if several groups are generically isomorphic, only one of them is mentioned. The table is complete for $\omega(G) \leq 63$.

n	Simple groups G satisfying $\omega(G) = n$
4	$\text{PSL}(2, 4) \cong \text{PSL}(2, 5) \cong A_5$
5	$\text{PSL}(2, 7) \cong \text{PSL}(3, 2)$, $\text{PSL}(2, 9) \cong A_6$, $\text{PSL}(2, 8)$
6	$\text{PSL}(3, 4)$
7	$\text{PSL}(2, 11)$, $\text{PSL}(2, 16)$, $\text{PSL}(2, 27)$, $\text{Sz}(8)$
8	$\text{PSL}(2, 13)$, A_7
9	$\text{PSL}(3, 3)$, $\text{PSL}(2, 32)$, $\text{PSU}(3, 4)$
10	$\text{PSL}(2, 17)$, $\text{PSU}(3, 3)$, $\text{PSL}(2, 25)$, M_{11} , $\text{PSU}(3, 5)$, $\text{PSU}(3, 8)$
11	$\text{PSL}(2, 19)$, M_{22} , $\text{Sz}(32)$
12	$\text{PSL}(4, 2) \cong A_8$, M_{12} , $O(5, 4)$
13	$\text{PSL}(2, 23)$
14	$\text{PSU}(4, 3)$
15	$\text{PSU}(4, 2) \cong O(5, 3)$, J_1 , $\text{PSL}(2, 64)$, $\text{PSL}(2, 81)$
16	$\text{PSL}(2, 29)$, A_9 , J_2 , $\text{PSL}(3, 7)$
17	$\text{PSL}(2, 31)$, $\text{PSL}(2, 49)$, $G_2(3)$, M_{23} , $\text{PSL}(3, 8)$, ${}^2F_4(2)'$, J_3
18	
19	$\text{PSL}(3, 5)$, McL , $\text{Ree}(27)$
20	$\text{PSL}(2, 37)$, $\text{PSL}(5, 2)$, $\text{PSL}(3, 16)$
21	$\text{PSL}(2, 128)$, HS , ${}^3D_4(2)$, $O(5, 8)$
22	$\text{PSL}(2, 41)$, A_{10}
23	$\text{PSL}(2, 43)$, $\text{Sz}(128)$
24	$\text{PSL}(2, 125)$, $G_2(4)$
25	$\text{PSL}(2, 47)$, $O'N$
26	$\text{PSL}(4, 3)$, M_{24} , He
27	$O(5, 5)$, $\text{PSL}(2, 243)$, $O^+(8, 2)$
28	$\text{PSL}(2, 53)$
29	A_{11} , $\text{PSU}(3, 9)$
30	$O(7, 2)$, $\text{PSU}(5, 2)$, $\text{PSU}(3, 11)$
31	$\text{PSL}(2, 59)$
32	$\text{PSL}(2, 61)$, $\text{PSL}(3, 9)$
33	$O^-(8, 2)$
34	$\text{PSU}(3, 7)$, $\text{PSL}(4, 5)$, $\text{PSU}(5, 4)$, $\text{PSU}(6, 2)$
35	$\text{PSL}(2, 67)$, $\text{PSU}(4, 4)$
36	$\text{PSL}(4, 4)$, Ru
37	$\text{PSL}(2, 71)$, $\text{PSL}(2, 121)$, $\text{PSL}(2, 256)$, Suz
38	$\text{PSL}(2, 73)$, $O^+(8, 3)$
39	$\text{PSL}(3, 13)$
40	A_{12} , $\text{PSU}(3, 16)$
41	$\text{PSL}(2, 79)$, $O(5, 9)$
42	$\text{PSU}(3, 32)$, Co_3
43	$\text{PSL}(2, 83)$, $O(5, 7)$
<i>To be continued.</i>	

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n	Simple groups G satisfying $\omega(G) = n$
44	$G_2(5)$, $PSL(6, 2)$, HN
45	$O(5, 16)$
46	$PSL(2, 89)$
47	
48	Th
49	
50	$PSL(2, 97)$, $PSL(2, 169)$, $PSp(6, 3)$
51	
52	$PSL(2, 101)$, A_{13} , $O(7, 3)$
53	$PSL(2, 103)$, Ly
54	
55	$PSL(2, 107)$
56	$PSL(2, 109)$, ${}^3D_4(3)$
57	Ree(8)
58	$PSL(2, 113)$
59	Fi_{22}
60	Co_2
61	$PSL(2, 343)$, $PSL(2, 512)$
62	$PSU(3, 17)$, $F_4(2)$, J_4
63	$Sz(512)$
64	$PSU(4, 5)$
65	$PSL(2, 127)$
66	
67	$PSL(2, 131)$
68	
69	$PSL(2, 729)$, A_{14}
70	$PSL(2, 137)$
71	$PSL(2, 139)$
72	$PSL(3, 25)$, $PSL(5, 3)$, $G_2(7)$
73	$PSL(3, 11)$
74	
75	$PSL(3, 19)$, $O(7, 4)$
76	$PSL(2, 149)$, $PSU(4, 7)$
77	$PSL(2, 151)$, $O^-(8, 3)$, $PSL(7, 2)$
78	${}^3D_4(4)$
79	
80	$PSL(2, 157)$
81	$O(9, 2)$
82	$PSL(2, 289)$
83	$PSL(2, 163)$
84	$O^+(10, 2)$, $O^+(8, 4)$
85	$PSL(2, 167)$, $PSL(4, 9)$
86	
87	$O(5, 11)$
88	$PSL(2, 173)$, $PSL(2, 625)$
89	$PSU(5, 3)$
<i>To be continued.</i>	

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n	Simple groups G satisfying $\omega(G) = n$
90	A_{15}
91	$\text{PSL}(2, 179), {}^2E_6(2)$
92	$\text{PSL}(2, 181)$
93	$O^-(10, 2)$
94	
95	
96	
97	$\text{PSL}(2, 191), \text{Fi}'_{24}$
98	$\text{PSL}(2, 193), \text{Fi}_{23}$
99	
100	$\text{PSL}(2, 197), \text{PSU}(3, 13)$

Table 3: Values $\omega(G)$ computed by Eamonn O'Brien with MAGMA in December 2019.

G	$\omega(G)$
$O(5, 8)$	21
$\text{PSU}(6, 2)$	34
$\text{PSU}(5, 4)$	34
$\text{PSU}(4, 4)$	35
$O^+(8, 3)$	38
$\text{PSU}(3, 16)$	40
$O(5, 9)$	41
$\text{PSU}(3, 32)$	42
$O(5, 16)$	45
${}^3D_4(3)$	56
$\text{PSU}(3, 17)$	62
$F_4(2)$	62
$\text{PSU}(4, 5)$	64
$\text{PSL}(3, 25)$	72
$\text{PSL}(3, 11)$	73
$\text{PSL}(3, 19)$	75
$O(7, 4)$	75
$\text{PSU}(4, 7)$	76
$\text{PSL}(7, 2)$	77
$O^-(8, 3)$	77
${}^3D_4(4)$	78
$O^+(8, 4)$	84
$O^+(10, 2)$	84
$\text{PSL}(4, 9)$	85
$O(5, 11)$	87
$\text{PSU}(5, 3)$	89
$O^-(10, 2)$	93
$\text{PSU}(3, 13)$	100
$\text{PSU}(3, 23)$	106
$\text{PSL}(5, 4)$	110
<i>To be continued.</i>	

<i>Continued.</i>	
G	$\omega(G)$
$O(5, 13)$	115
$O^+(8, 5)$	116
$PSL(4, 8)$	119
$PSL(6, 3)$	122
$E_6(2)$	132
$PSp(6, 5)$	133
$O^-(8, 4)$	133
$O(7, 5)$	136
$PSL(4, 7)$	137
$PSU(4, 9)$	142
$O^-(10, 3)$	151
$O(5, 27)$	151
$PSU(6, 3)$	156
$PSU(3, 29)$	162
$PSL(6, 4)$	169
$O(5, 25)$	203
$PSU(4, 11)$	232
$PSU(9, 2)$	240
$O^+(10, 3)$	268
$O(7, 9)$	307
$PSU(3, 41)$	310

3 Remaining ‘Candidates’

Table 4: Bounds on orbit numbers for all remaining simple groups G which possibly satisfy $\omega(G) \leq 100$. We give the best lower bound computed so far.

n	Simple groups G satisfying $\omega(G) \geq n$
64	$PSU(6, 5)$
77	$PSU(5, 9)$

References

- [1] Alexander Bors, Michael Giudici, and Cheryl E. Praeger. *Documentation for the GAP code file OrbOrd.txt*, 2019. (<https://arxiv.org/abs/1910.12570>).
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