ON A SERIES OF FINITELY GENERATED INFINITE SIMPLE GROUPS

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ABSTRACT. We present a series of finitely generated infinite simple groups. One of the groups in our series is isomorphic to Higman's group $G_{2,1}$, which is also known as Thompson's group V.

1. INTRODUCTION

This paper continues the work on the countable simple group $\mathrm{CT}(\mathbb{Z}) < \mathrm{Sym}(\mathbb{Z})$ in [3]. The group $\mathrm{CT}(\mathbb{Z})$ is generated by the set of all *class transpositions*:

Definition 1.1. Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ of $\mathbb{Z}$, let the *class transposition* $(r_1(m_1), r_2(m_2))$ be the permutation which interchanges $r_1 + km_1$ and $r_2 + km_2$ for each integer k and which fixes all other points. Here we assume that $0 \leqslant r_1 < m_1$ and that $0 \leqslant r_2 < m_2$.

By Corollary 3.7 in [3], the following subgroups of $\mathrm{CT}(\mathbb{Z})$ are simple as well:

Definition 1.2. Given a set $\mathbb{P}$ of odd primes, let $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z}) \leqslant \mathrm{CT}(\mathbb{Z})$ denote the subgroup which is generated by all class transpositions $(r_1(m_1), r_2(m_2))$ for which all odd prime factors of m_1 and m_2 lie in $\mathbb{P}$.

Therefore, $\mathrm{CT}(\mathbb{Z})$ has an uncountable series of simple subgroups, which is parametrized by the sets of odd primes. In this article we are interested in the groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ for finite $\mathbb{P}$. In particular, we prove the following:

Theorem 1.3. *If $\mathbb{P}$ is a finite set of odd primes, then the group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely generated. The group $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is isomorphic to Higman's group $G_{2,1}$ defined in* [2]*, thus in particular finitely presented.*

The last-mentioned isomorphism has been found by John P. McDermott (Galway), and reported to the author in private communication.

2. THE GROUPS $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ FOR FINITE $\mathbb{P}$ ARE FINITELY GENERATED

By Theorem 2.3 in [3], the group $\mathrm{CT}(\mathbb{Z})$ is not finitely generated. By the arguments used in the proof of that theorem, it follows also that $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is not finitely generated if $\mathbb{P}$ is infinite. However we will see that $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely generated if $\mathbb{P}$ is finite.

Definition 2.1. Given a positive integer m, let $\mathcal{C}_m$ be the set of all class transpositions which interchange residue classes whose moduli divide m.

Theorem 2.2. *Let $\mathbb{P}$ be a finite set of odd primes. Then the group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely generated. More precisely, $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is generated by $\mathcal{C}_m$, where $m := 8 \cdot \prod_{p \in \mathbb{P}} p^2$.*

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Proof. Let $m := 8 \cdot \prod_{p \in \mathbb{P}} p^2$, and let $\tau = (r_1(m_1), r_2(m_2)) \in \mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ be a class transposition. We need to show that τ can be written as a product of elements of $\mathcal{C}_m$.

Let $p \in \mathbb{P} \cup \{2\}$, and let k_1 and k_2 be the exponents of the highest powers of p which divide m_1 or m_2, respectively. Without loss of generality, we can assume that $k_2 \geqslant k_1$ and $k_2 > 2$.

We put $m_3 := \gcd(m, m_2)$ and $m_4 := m_3/p$. Since $r_1(m_1)$ and $r_2(m_2)$ are disjoint residue classes and $m_4 \geqslant 3$, we can choose a residue class $r_4(m_4)$ which intersects trivially with the support of τ. Putting $\sigma := (r_2(m_3), r_4(m_4))$, we have

$$\tau^\sigma \ = \ (r_1(m_1), r_2(m_2/p)).$$

Now we can conclude by induction on k_i, $i = 1, 2$, carried out for all primes $p \in \mathbb{P} \cup \{2\}$, that there is a product π of elements of $\mathcal{C}_m$ such that $\tau^\pi \in \mathcal{C}_m$. The assertion follows. □

3. The Group $\mathrm{CT}_\emptyset(\mathbb{Z})$ is Isomorphic to the Higman-Thompson Group

Now we prove our assertion on the group $\mathrm{CT}_\emptyset(\mathbb{Z})$, which is generated by all class transpositions which interchange residue classes modulo powers of 2. We need a lemma:

Lemma 3.1. *We have*

$$\begin{aligned} \mathrm{CT}_\emptyset(\mathbb{Z}) &= \langle (0(2),1(4)), (0(4),1(4)), (1(4),2(4)), (2(4),3(4)) \rangle \\ &= \langle (0(2),1(2)), (1(2),2(4)), (0(2),1(4)), (1(4),2(4)) \rangle. \end{aligned}$$

Proof. By Theorem 2.2, the group $\mathrm{CT}_\emptyset(\mathbb{Z})$ is generated by the set $\mathcal{C}_8$ of the 71 class transpositions which interchange residue classes whose moduli divide 8.

Obviously, both sets of generators given in the lemma generate subgroups of $\mathrm{CT}_\emptyset(\mathbb{Z})$. Therefore it is sufficient to check that they generate the same group, and that this group contains $\mathcal{C}_8$.

This is done by means of computation, using the GAP [1] package RCWA [4]. It turns out that any element of one of the two sets of generators can be written as a product of 7 or fewer elements of the other, and that any element of $\mathcal{C}_8$ can be written as a product of 19 or fewer elements of the second-mentioned set of generators. □

Theorem 3.1. *It is* $\mathrm{CT}_\emptyset(\mathbb{Z}) \cong G_{2,1}$*, where* $G_{2,1}$ *is the finitely-presented infinite simple group found by Higman* [2].

Proof. Let $\kappa := (0(2), 1(2))$, $\lambda := (1(2), 2(4))$, $\mu := (0(2), 1(4))$ and $\nu := (1(4), 2(4))$. By Lemma 3.1, we have $\mathrm{CT}_\emptyset(\mathbb{Z}) = \langle \kappa, \lambda, \mu, \nu \rangle$. We use the GAP [1] package RCWA [4] to verify by means of computation that the generators κ, λ, μ and ν satisfy the defining relations

(1) $\kappa^2 = \lambda^2 = \mu^2 = \nu^2 = 1$,
(2) $\lambda\kappa\mu\kappa\lambda\nu\kappa\nu\mu\kappa\lambda\kappa\mu = 1$,
(3) $\kappa\nu\lambda\kappa\mu\nu\kappa\lambda\nu\mu\nu\lambda\nu\mu = 1$,
(4) $(\lambda\kappa\mu\kappa\lambda\nu)^3 = (\mu\kappa\lambda\kappa\mu\nu)^3 = 1$,
(5) $(\lambda\nu\mu)^2\kappa(\mu\nu\lambda)^2\kappa = 1$,
(6) $(\lambda\nu\mu\nu)^5 = 1$,
(7) $(\lambda\kappa\nu\kappa\lambda\nu)^3\kappa\nu\kappa(\mu\kappa\nu\kappa\mu\nu)^3\kappa\nu\kappa\nu = 1$,
(8) $((\lambda\kappa\mu\nu)^2(\mu\kappa\lambda\nu)^2)^3 = 1$,
(9) $(\lambda\nu\lambda\kappa\mu\kappa\mu\nu\lambda\nu\mu\kappa\mu\kappa)^4 = 1$,
(10) $(\mu\nu\mu\kappa\lambda\kappa\lambda\nu\mu\nu\lambda\kappa\lambda\kappa)^4 = 1$,
(11) $(\lambda\mu\kappa\lambda\kappa\mu\lambda\kappa\nu\kappa)^2 = 1$, and
(12) $(\mu\lambda\kappa\mu\kappa\lambda\mu\kappa\nu\kappa)^2 = 1$

of the group $G_{2,1}$ given on page 50 in Higman [2]. This establishes that $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is an homomorphic image of $G_{2,1}$. However since Higman's group is simple and $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is not the trivial group, it is in fact isomorphic. □

Remark 3.2. From the proof of Theorem 4.1 in [3], we see that the free group of rank 2 embeds into $\mathrm{CT}_{\emptyset}(\mathbb{Z})$. In particular we have $\langle (\kappa\mu)^2, (\lambda\kappa)^2 \rangle \cong \mathrm{F}_2$.

Remark 3.3. As a consequence of Theorem 2.6 in [3], the Prüfer p-groups embed into $\mathrm{CT}(\mathbb{Z})$. Given a set $\mathbb{P}$ of odd primes and $p \in \mathbb{P}$, the construction of p-th roots of torsion elements of $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ in the proof of that theorem is done within the group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$. Therefore the Prüfer p-group embeds into $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$. Likewise, the construction of square roots of elements of $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is done within $\mathrm{CT}_{\emptyset}(\mathbb{Z})$. Hence by Theorem 3.1, the Prüfer 2-group embeds into Higman's group $G_{2,1}$.

4. A Small Generating Set for $\mathrm{CT}_{\{3\}}(\mathbb{Z})$

By means of computation with RCWA [4], from Theorem 2.2 we can also derive a small generating set for the group $\mathrm{CT}_{\{3\}}(\mathbb{Z})$:

Lemma 4.1. *We have*

$$\begin{aligned}\mathrm{CT}_{\{3\}}(\mathbb{Z}) \ = \ \langle & (0(2),1(2)),(0(2),1(4)),(1(2),2(4)),(1(4),2(4)),\\ & (0(3),1(3)),(1(3),2(3)),\\ & (0(3),1(9)),(0(3),4(9)),(0(3),7(9)),\\ & (0(2),1(6)),(0(2),5(6)),\\ & (0(3),1(6)),(0(4),1(6)),(0(6),1(8))\rangle.\end{aligned}$$

5. Open Problems

By Theorem 3.1, the group $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is finitely presented. However it is not known so far whether any of the other groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely presented.

Question 5.1. *Is the group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ finitely presented, provided that $\mathbb{P}$ is finite?*

Given sets $\mathbb{P}_1$ and $\mathbb{P}_2$ of odd primes, we have $\mathrm{CT}_{\mathbb{P}_1 \cap \mathbb{P}_2}(\mathbb{Z}) \leqslant \mathrm{CT}_{\mathbb{P}_1}(\mathbb{Z}) \cap \mathrm{CT}_{\mathbb{P}_2}(\mathbb{Z})$, since all generators of $\mathrm{CT}_{\mathbb{P}_1 \cap \mathbb{P}_2}(\mathbb{Z})$ lie in both $\mathrm{CT}_{\mathbb{P}_1}(\mathbb{Z})$ and $\mathrm{CT}_{\mathbb{P}_2}(\mathbb{Z})$. We even have equality if and only if every element of $\mathrm{CT}_{\mathbb{P}_1}(\mathbb{Z}) \cap \mathrm{CT}_{\mathbb{P}_2}(\mathbb{Z})$ can be factored into class transpositions $(r_1(m_1), r_2(m_2))$ where all odd prime factors of m_1 and m_2 lie in $\mathbb{P}_1 \cap \mathbb{P}_2$:

Conjecture 5.2. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be sets of odd primes. Then, $\mathrm{CT}_{\mathbb{P}_1}(\mathbb{Z}) \cap \mathrm{CT}_{\mathbb{P}_2}(\mathbb{Z}) = \mathrm{CT}_{\mathbb{P}_1 \cap \mathbb{P}_2}(\mathbb{Z})$. In particular, $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is the intersection of all groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$.*

So far, RCWA [4] provides a heuristic method for factoring elements of $\mathrm{CT}(\mathbb{Z})$ into class transpositions, which works sometimes quite well (see e.g. [3], Section 5), but which is not an algorithm in the sense that termination is guaranteed. It seems likely that Conjecture 5.2 may be proved together with Conjecture 6.8 in [3], as a result of developing a proper algorithm for factoring elements of $\mathrm{CT}(\mathbb{Z})$ into class transpositions.

References

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