Counting the Orbits in Finite Groups under the Action of the Automorphism Group - Suzuki Groups vs. Linear Groups

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- Abstract -

Let G be a finite group and let $\omega(G)$ denote the number of orbits in G under the action of its automorphism group. One interesting question arising here is what could be stated about the group G when $\omega(G)$ is prescribed, or if it does not exceed a given upper bound. Limiting $\omega(G)$ means demanding that G has to fulfill a certain 'homogeneity condition'. For example, $\omega(G) = 2$ implies that $G \cong \mathrm{C}_p^k$ for a prime p and a positive integer k. The case $\omega(G) = 3$ is also treatable, but increasing the bound for $\omega(G)$ some further leads to a fast growth of the complexity of the problem - so it seems sensible to slightly 'simplify' the problem by prescribing additional properties of the group G. For example, it is possible to ask for simple groups G with limited $\omega(G)$ - this in fact increases the 'treatable' upper bound on $\omega(G)$ significantly. On the other hand it might be of enormous value for gaining further progress here to explicitly determine the value of $\omega(G)$ for certain 'interesting' types of groups. In my diploma thesis, I have done this for all of the minimal simple groups as well as all of the simple Zassenhaus groups. The method I used is very roughly the following : at first, I looked at the conjugacy classes, and then I determined which of them are fused by outer automorphisms. This leads to complicated recursion formulas. Perhaps the most remarkable result of my thesis is that

$$\omega(\mathrm{Sz}(q)) = \omega(\mathrm{PSL}(2,q)) + 2$$

for all admissible values of q, where $\mathrm{Sz}(q)$ denotes the Suzuki group over the field with q elements. This reflects also similarities in the structure of the two types of groups; the summand '2' arises just from the fact that this equation holds for $q = 2$, when you extend the definition of the Suzuki groups to this case, and that the sets of elements of $\mathrm{Sz}(q)$ resp. $\mathrm{PSL}(2,q)$ which are not conjugated to elements of $\mathrm{Sz}(2)$ resp. $\mathrm{PSL}(2,2)$ are partitioned under the action of the respective automorphism group in some sense in an 'equal' manner. The explicit formula for $\omega(\mathrm{Sz}(q))$ is

$$\omega(\mathrm{Sz}(q)) = \frac{q+3}{n} + \sum_{\varnothing \neq M \subseteq \pi(n)} (-1)^{|M|} \left(\frac{2^{T_{n,M}} + 3}{n} - \omega(\mathrm{Sz}(2^{T_{n,M}})) \right),$$

where $q = 2^n$, $T_{n,M} = \gcd_{t \in M} \frac{n}{t}$ and $\pi(n)$ denotes the set of prime divisors of n.

Further questions are, among many others :

- Let $G \leq H$ be two simple groups. Does this imply that $\omega(G) \leq \omega(H)$? (For 'general' groups this implication is invalid, for example it holds that $\omega(\mathrm{C}_2 \times \mathrm{C}_4) = 4$, but $\omega(\mathrm{C}_4^2) = 3$).
- How it is possible to extend these results to groups which are not simple ?