THE COLLATZ CONJECTURE
IN A GROUP THEORETIC CONTEXT

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ABSTRACT. In this paper we exhibit a permutation group which acts transitively on $\mathbb{N}_0$ if and only if the Collatz conjecture holds. We also give an infinite series of finitely generated simple groups many of which contain this group as a subgroup, and whose intersection is isomorphic to Thompson's group V.

1. INTRODUCTION

By $r(m)$ we denote the residue class $r + m\mathbb{Z}$, where we assume that $0 \leqslant r < m$. The Collatz conjecture asserts that iterated application of the mapping

$$C: \mathbb{Z} \to \mathbb{Z}, \;\; n \;\mapsto\; \begin{cases} \frac{n}{2} & \text{if } n \in 0(2), \\ 3n+1 & \text{if } n \in 1(2) \end{cases}$$

to any positive integer yields 1 after a finite number of steps (cf. Lagarias [7], [8]).

The mapping C is surjective, but not injective. It is affine on residue classes, and it maps negative to negative and nonnegative to nonnegative integers. The most basic *bijective* mappings which share the latter properties are those which interchange two disjoint residue classes:

Definition 1.1. Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ of $\mathbb{Z}$, let the *class transposition* $\tau_{r_1(m_1),r_2(m_2)}$ be the permutation which interchanges $r_1 + km_1$ and $r_2 + km_2$ for each integer k and which fixes all other points.

The set of all class transpositions generates a countable simple group $\mathrm{CT}(\mathbb{Z}) < \mathrm{Sym}(\mathbb{Z})$ which has a rich class of subgroups, cf. [5]. In this paper we exhibit subgroups of $\mathrm{CT}(\mathbb{Z})$ which act transitively on the set of nonnegative integers in their support if and only if the Collatz conjecture holds:

Proposition 1.2. *The following hold:*

a) The group $G_C := \langle \tau_{1(2),4(6)}, \tau_{1(3),2(6)}, \tau_{2(3),4(6)} \rangle$ *acts transitively on* $\mathbb{N} \setminus 0(6)$ *if and only if the Collatz conjecture holds.*

b) The group $G_T := \langle \tau_{0(2),1(2)}, \tau_{1(2),2(4)}, \tau_{1(4),2(6)} \rangle$ *acts transitively on* $\mathbb{N}_0$ *if and only if the Collatz conjecture holds.*

By Corollary 3.7 in [5], the following subgroups of $\mathrm{CT}(\mathbb{Z})$ are simple as long as $2 \in \mathbb{P}$:

Definition 1.3. Given a set $\mathbb{P}$ of prime numbers, let $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z}) \leqslant \mathrm{CT}(\mathbb{Z})$ denote the subgroup which is generated by all class transpositions $\tau_{r_1(m_1),r_2(m_2)}$ for which all prime factors of m_1 and m_2 lie in $\mathbb{P}$.

Both G_C and G_T are subgroups of $\mathrm{CT}_{\{2,3\}}(\mathbb{Z})$.

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Remark 1.4. The group $\mathrm{CT}_{\{2\}}(\mathbb{Z})$ is isomorphic to Higman's group $G_{2,1}$ defined in [3]. This finitely presented infinite simple group is usually treated in the literature under the name *Thompson's group V*.

The isomorphism between $\mathrm{CT}_{\{2\}}(\mathbb{Z})$ and Thompson's group V has been pointed out by John P. McDermott in response to the question of the author which known simple group the former group would be isomorphic to.

If $|\mathbb{P}| > 1$, the group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ has no underlying tree structure. This makes the situation notably more complicated. Anyway if $\mathbb{P}$ is finite, then $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is still finitely generated – cf. Theorem 3.2.

2. A PERMUTATION GROUP EQUIVALENT OF THE COLLATZ CONJECTURE

In this section we prove Proposition 1.2.

Proposition 2.1. *Let* $a := \tau_{1(2),4(6)}$, $b := \tau_{1(3),2(6)}$ *and* $c := \tau_{2(3),4(6)}$. *Then the group* $G_C := \langle a, b, c\rangle < \mathrm{CT}(\mathbb{Z})$ *acts transitively on* $\mathbb{N} \setminus 0(6)$ *if and only if the Collatz conjecture holds.*

Proof. We observe that $C^{-1}(0(3)) = 0(6) \subset 0(3)$, that the restrictions of C and a to $3(6)$ are the same and map this residue class to $10(18) \subset \mathbb{Z} \setminus 0(3)$, that $10(18)^a = 3(6)$, and that no trajectory of C contains only multiples of 3. Therefore it suffices to show that for any $n \in \mathbb{N} \setminus 0(3)$ we have $\{n, n^a, n^b, n^c\} = \{n\} \cup \{n^C\} \cup C^{-1}(n)$. We treat four cases:

n mod 6	n	n^a	n^b	n^c	n	n^C	$C^{-1}(n)$
1	n	$3n+1$	$2n$	n	n	$3n+1$	$\{2n\}$
2	n	n	$\frac{n}{2}$	$2n$	n	$\frac{n}{2}$	$\{2n\}$
4	n	$\frac{n-1}{3}$	$2n$	$\frac{n}{2}$	n	$\frac{n}{2}$	$\{\frac{n-1}{3}, 2n\}$
5	n	$3n+1$	n	$2n$	n	$3n+1$	$\{2n\}$

□

With a little more effort, we can get rid of the set $0(6)$ of fixed points:

Proposition 2.2. *Let* $a := \tau_{0(2),1(2)}$, $b := \tau_{1(2),2(4)}$ *and* $c := \tau_{1(4),2(6)}$. *Then the group* $G_T := \langle a, b, c\rangle < \mathrm{CT}(\mathbb{Z})$ *acts transitively on* $\mathbb{N}_0$ *if and only if the Collatz conjecture holds.*

Proof. Let

$$T : \mathbb{Z} \to \mathbb{Z}, \ n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \in 0(2), \\ \frac{3n+1}{2} & \text{if } n \in 1(2) \end{cases}$$

be the Collatz mapping, and put

$$f : \mathbb{Z} \to \mathbb{Z}, \ n \mapsto \begin{cases} n^{ac} &= \frac{3n+4}{2} & \text{if } n \in 0(4), \\ n^{c} &= \frac{3n+1}{2} & \text{if } n \in 1(4), \\ n^{b} &= \frac{n}{2} & \text{if } n \in 2(4), \\ n^{aba} &= \frac{n-3}{2} & \text{if } n \in 3(4) \end{cases}$$

and

$$r : \mathbb{Z} \to \mathbb{Z}, \ n \mapsto \begin{cases} 2n-2 & \text{if } n \in 0(3) \cup 2(3), \\ 2n-1 & \text{if } n \in 1(3). \end{cases}$$

Then rf and Tr coincide on $\mathbb{Z} \setminus 0(6)$, and we have $rf^2 = T^2r$. Further, a interchanges the image of r with its complement in $\mathbb{Z}$. Therefore if the Collatz conjecture holds, then the group G_T acts transitively on $\mathbb{N}_0$. It remains to show the other direction. Put

$$s: \ \mathbb{Z} \to \mathbb{Z}, \ \ n \ \mapsto \begin{cases} \frac{n+2}{2} & \text{if } n \in 0(2), \\ \frac{n+1}{2} & \text{if } n \in 1(2). \end{cases}$$

The mapping s is a right inverse of r, and for all integers n we have $n^s = n^{as}$. It suffices to check that for all $n \in \mathbb{N}_0$ we have $\{n^{bs}, n^{cs}\} \subseteq \{n^s, n^{sT}\} \cup T^{-1}(n^s)$. Indeed we have

- $n^{bs} = n^s$ if $n \in 0(4)$,
- $n^{bs} = n^{sT}$ if $n \in 2(4)$,
- $n^{bsT} = n^s$ if $n \in 1(2)$,
- $n^{cs} = n^s$ if $n \in 3(4) \cup 0(6) \cup 4(6)$,
- $n^{cs} = n^{sT}$ if $n \in 1(4)$, and
- $n^{csT} = n^s$ if $n \in 2(6)$,

which shows that if G_T acts transitively on $\mathbb{N}_0$, then the Collatz conjecture holds. □

Note however that for *some* groups generated by 3 class transpositions it is easy to find out that they act transitively on $\mathbb{N}_0$:

Remark 2.3. With the GAP [2] package RCWA [6], using Method 10.4 in [4] one can check that the group $G_5 := \langle \tau_{0(2),1(2)}, \tau_{1(2),2(4)}, \tau_{0(3),2(3)} \rangle$ acts at least 5-transitively on $\mathbb{N}_0$. The group G_5 can be obtained from G_T by replacing the generator $\tau_{1(4),2(6)}$ by $\tau_{0(3),2(3)}$. The important difference between G_5 and G_T is as follows: while there is a finite set S of elements of G_5 such that for every integer $n > 0$ there is some $g \in S$ such that $n^g < n$, the group G_T does not have a finite subset with this property.

3. Thompson's group V and further subgroups of $\mathrm{CT}(\mathbb{Z})$

By Theorem 2.3 in [5], the group $\mathrm{CT}(\mathbb{Z})$ is not finitely generated. By the arguments used in the proof of that theorem, it follows also that $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is not finitely generated if $\mathbb{P}$ is infinite. However we will see that $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely generated if $\mathbb{P}$ is finite.

Definition 3.1. Given a positive integer m, let $\mathcal{C}_m$ be the set of all class transpositions which interchange residue classes whose moduli divide m.

Theorem 3.2. *Let $\mathbb{P}$ be a finite set of primes. Then the group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely generated. More precisely, $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is generated by $\mathcal{C}_m$, where $m := \prod_{p \in \mathbb{P}} p^2$ if $2 \notin \mathbb{P}$ and $m := 2 \cdot \prod_{p \in \mathbb{P}} p^2$ otherwise.*

Proof. Let m be as above, and let $\tau = \tau_{r_1(m_1),r_2(m_2)} \in \mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ be a class transposition. We need to show that τ can be written as a product of elements of $\mathcal{C}_m$.

Let $p \in \mathbb{P}$, and let k_1 and k_2 be the exponents of the highest powers of p which divide m_1 or m_2, respectively. Without loss of generality, we can assume $k_2 \geqslant k_1$ and $k_2 > 2$.

We put $m_3 := \gcd(m, m_2)$ and $m_4 := m_3/p$. Since $r_1(m_1)$ and $r_2(m_2)$ are disjoint residue classes and $m_4 \geqslant 3$, we can choose a residue class $r_4(m_4)$ which intersects trivially with the support of τ. Putting $\sigma := \tau_{r_2(m_3),r_4(m_4)} \in \mathcal{C}_m$, we have

$$\tau^\sigma \ = \ \tau_{r_1(m_1),r_4(m_2/p)}.$$

Now we can conclude by induction on $k_i, i = 1, 2$, carried out for all primes $p \in \mathbb{P}$, that there is a product π of elements of $\mathcal{C}_m$ such that $\tau^\pi \in \mathcal{C}_m$. The assertion follows. □

Small generating sets for the groups $\mathrm{CT}_{\{2\}}(\mathbb{Z}) \cong G_{2,1}$ and $\mathrm{CT}_{\{3\}}(\mathbb{Z})$ are immediate, and from Theorem 3.2, by means of computation with the GAP [2] package RCWA [6] we can also derive one for $\mathrm{CT}_{\{2,3\}}(\mathbb{Z})$:

Proposition 3.3. *We have*

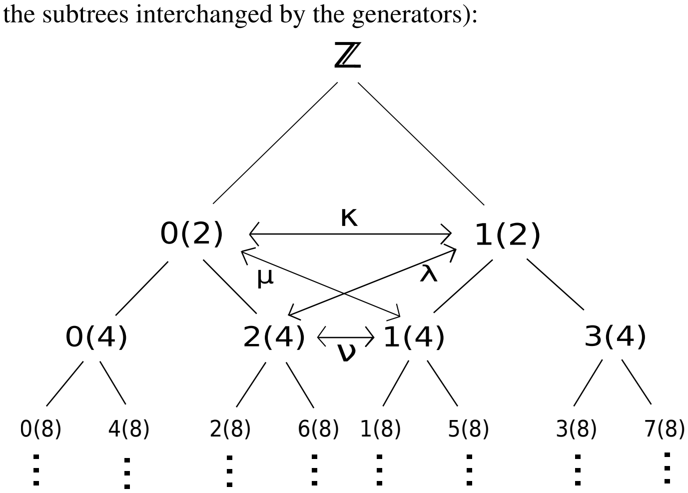
$$\begin{aligned}
\mathrm{CT}_{\{2\}}(\mathbb{Z}) &= \langle \tau_{0(2),1(2)}, \tau_{1(2),2(4)}, \tau_{0(2),1(4)}, \tau_{1(4),2(4)} \rangle, \\
\mathrm{CT}_{\{3\}}(\mathbb{Z}) &= \langle \tau_{0(3),1(3)}, \tau_{1(3),2(3)}, \tau_{2(9),3(9)}, \tau_{5(9),6(9)}, \tau_{2(3),3(9)} \rangle, \\
\mathrm{CT}_{\{2,3\}}(\mathbb{Z}) &= \langle \tau_{0(2),1(2)}, \tau_{0(3),1(3)}, \tau_{1(3),2(3)}, \tau_{0(2),1(4)}, \tau_{0(2),5(6)}, \tau_{0(3),1(6)} \rangle.
\end{aligned}$$

The generators for $\mathrm{CT}_{\{2\}}(\mathbb{Z})$ given in Proposition 3.3 correspond directly to Higman's generators for $G_{2,1}$:

Remark 3.4. As one can check by straightforward calculation, the generators $\kappa := \tau_{0(2),1(2)}$, $\lambda := \tau_{1(2),2(4)}$, $\mu := \tau_{0(2),1(4)}$ and $\nu := \tau_{1(4),2(4)}$ for $\mathrm{CT}_{\{2\}}(\mathbb{Z})$ given in Proposition 3.3 satisfy the defining relations

(1) $\kappa^2 = \lambda^2 = \mu^2 = \nu^2 = 1$,
(2) $\lambda\kappa\mu\kappa\lambda\nu\kappa\nu\mu\kappa\lambda\kappa\mu = 1$,
(3) $\kappa\nu\lambda\kappa\mu\nu\kappa\lambda\nu\mu\nu\lambda\nu\mu = 1$,
(4) $(\lambda\kappa\mu\kappa\lambda\nu)^3 = (\mu\kappa\lambda\kappa\mu\nu)^3 = 1$,
(5) $(\lambda\nu\mu)^2\kappa(\mu\nu\lambda)^2\kappa = 1$,
(6) $(\lambda\nu\mu\nu)^5 = 1$,
(7) $(\lambda\kappa\nu\kappa\lambda\nu)^3\kappa\nu\kappa(\mu\kappa\nu\kappa\mu\nu)^3\kappa\nu\kappa\nu = 1$,
(8) $((\lambda\kappa\mu\nu)^2(\mu\kappa\lambda\nu)^2)^3 = 1$,
(9) $(\lambda\nu\lambda\kappa\mu\kappa\mu\nu\lambda\nu\mu\kappa\mu\kappa)^4 = 1$,
(10) $(\mu\nu\mu\kappa\lambda\kappa\lambda\nu\mu\nu\lambda\kappa\lambda\kappa)^4 = 1$,
(11) $(\lambda\mu\kappa\lambda\kappa\mu\lambda\kappa\nu\kappa)^2 = 1$, and
(12) $(\mu\lambda\kappa\mu\kappa\lambda\mu\kappa\nu\kappa)^2 = 1$

of the group $G_{2,1}$ given on Page 50 in Higman [3]. Since $G_{2,1}$ is simple, it follows that $\mathrm{CT}_{\{2\}}(\mathbb{Z}) \cong G_{2,1}$. Another presentation for this group can be found on Page 242 in [1]. The generators A, B, C and π_0 used there can be related to κ, λ, μ and ν via $A = \lambda\kappa\mu$, $B = \mu\nu\lambda\kappa$, $C = \mu\kappa\lambda\kappa$ and $\pi_0 = \mu$, respectively, $\kappa = AC$, $\lambda = AC\pi_0A^{-1}$, $\mu = \pi_0$ and $\nu = A\pi_0B^{-1}\pi_0$. The group $\mathrm{CT}_{\{2\}}(\mathbb{Z})$ can be visualized as follows (the arrows point to the roots of the subtrees interchanged by the generators):



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