What do Thompson's group V and the Collatz conjecture have in common?

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The Collatz conjecture asserts that iterated application of the mapping

$$T: \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

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How is it possible to put these into a common framework?

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Example: We have

$$\tau_{0(2),1(4)} \in \text{Sym}(\mathbb{Z}): n \mapsto \begin{cases} 2n+1 & \text{if } n \in 0(2), \\ (n-1)/2 & \text{if } n \in 1(4). \end{cases}$$

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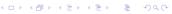
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It is easy to check with RCWA [3] that these generators satisfy indeed Higman's relations, which verifies the isomorphism.

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 - $|\langle \tau_{1(5),4(5)}, \tau_{0(3),1(6)}, \tau_{3(4),0(6)} \rangle| = 2^{200} \cdot 3^{103} \cdot 5^{48} \cdot 7^{28} \cdot 11^{16} \cdot 13^{13} \cdot 17^{8} \cdot 19^{6} \cdot 23^{6} \cdot 29.$

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