THE MONSTER IS THE 28917828424943086TH SMALLEST NON-ABELIAN SIMPLE GROUP

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Abstract. Using the CFSG, we determine the positions of the 26 sporadic simple groups in a list of non-abelian simple groups sorted by ascending order. In particular, we find that the largest of these groups, the Fischer-Griess monster, occurs at position 28917828424943086.

1. Introduction

Based on [1] (cf. in particular Table 6 on Page xvi), using GAP [2] and primecount [3] we work out the following:

Theorem 1.1. The positions of the 26 sporadic simple groups in a list of non-abelian simple groups sorted by ascending order are as follows:

Group	Position	Group	Position
M_{11}	15	ON	1334
M_{12}	31	Co_3	1364
J_1	36	Co_2	4771
M_{22}	46	Fi_{22}	5396
J_2	50	HN	8240
M_{23}	98	Ly	39569
HS	138	Th	46944
J_3	143	Fi_{23}	150537
M_{24}	203	Co_1	151306
McL	269	J_4	386553
He	388	Fi_{24}'	7696143
Ru	976	B	8105003887
Suz	1325	M	28917828424943086

The positions of the groups M_{11}, M_{12}, J_1, M_{22} and J_2 can be read off directly from [1], Page 238. The positions of these and the other

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groups of order less than 10^{18} are also readily available from GAP [2]. This further covers the groups M_{23}, HS, J_3, M_{24}, McL, He, Ru, Suz, ON, Co_3, Co_2, Fi_{22}, HN, Ly and Th. Therefore we can restrict our considerations to the remaining 6 largest groups. Since the counting process in all of these cases is completely analogous, we discuss here only the case of the monster group M.

Also in a completely analogous manner, we obtain the following:

Theorem 1.2. The positions of the smallest members of 15 of the 16 series of simple groups of Lie type in a list of non-abelian simple groups sorted by ascending order are as follows (in the cases where a group belongs to more than one series, we count it as a member of the series which is mentioned earlier in the table):

Series	Smallest member	Position
$\mathrm{A}_n(q)$	$\mathrm{A}_1(5)$	1
${}^2\mathrm{A}_n(q)$	${}^2\mathrm{A}_2(3)$	12
$\mathrm{B}_n(q)$	$\mathrm{B}_2(4)$	56
${}^2\mathrm{B}_2(q)$	${}^2\mathrm{B}_2(8)$	23
$\mathrm{C}_n(q)$	$\mathrm{C}_3(3)$	403
$\mathrm{D}_n(q)$	$\mathrm{D}_4(2)$	185
${}^2\mathrm{D}_n(q)$	${}^2\mathrm{D}_4(2)$	191
${}^3\mathrm{D}_4(q)$	${}^3\mathrm{D}_4(2)$	194
$\mathrm{G}_2(q)$	$\mathrm{G}_2(3)$	79
${}^2\mathrm{G}_2(q)$	${}^2\mathrm{G}_2(27)$	495
$\mathrm{F}_4(q)$	$\mathrm{F}_4(2)$	17283
${}^2\mathrm{F}_4(q)$	${}^2\mathrm{F}_4(8)$	4721721
$\mathrm{E}_6(q)$	$\mathrm{E}_6(2)$	4421593
${}^2\mathrm{E}_6(q)$	${}^2\mathrm{E}_6(2)$	3199322
$\mathrm{E}_7(q)$	$\mathrm{E}_7(2)$	844883561753
$\mathrm{E}_8(q)$	$\mathrm{E}_8(2)$	?

The determination of the position of $\mathrm{E}_8(2)$ would be possible as well, but take more resources. – The key step of this would be counting the prime numbers less than or equal to 877469272728536815850 3849.

2. Counting the simple groups smaller than the monster

We count the groups by series, taking care to avoid counting any group twice. –

(1) First of all, we determine the number of simple groups of type $\mathrm{A}(1, q) = \mathrm{PSL}(2, q)$ of order less than $|\mathrm{M}|$. These make up by

far the majority of all non-abelian simple groups whose order lies in that range. We need to distinguish two cases:

(a) q is odd. The largest group of type $\mathrm{PSL}(2,q)$ of order less than $|\mathrm{M}|$ is the one for $q_{max} = 1173501297218634619$. Letting $\pi(n)$ denote the number of primes $\leq n$, the number of odd prime powers $q \leq q_{max}$ is equal to

$$\sum_{r=1}^{\log_3(q_{max})} \left(\pi(\sqrt[r]{q_{max}}) - 1\right).$$

We evaluate the first two terms with primecount [3], and obtain the values 28917828369160216 and 54858748, respectively. The remaining terms for $r = 3, \dots, 37$ are easily evaluated as 82463, 3526, 564, 171, 74, 41, 25, 17, 13, 10, 8, 7, 5, 5, 4, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, and 1, respectively. The sum of these 37 numbers is 28917828424105929. However we need to subtract $|\{3\}| = 1$, since $\mathrm{PSL}(2,3)$ is not simple. All in all, we get 28917828424105928 groups to be counted.

(b) q is even. Here, the 57 groups $\mathrm{PSL}(2,2^k)$ for $3 \leq k \leq 59$ have order less than $|\mathrm{M}|$. – Note that we have already counted the group $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ above.

(2) The series A of projective special linear groups $\mathrm{PSL}(n+1,q)$ contributes another 406149 groups of Lie rank ≥ 2. –

(a) A total of 405436 groups of Lie rank 2. –
   (i) 14 groups of type $\mathrm{PSL}(3,3^k)$ for $1 \leq k \leq 14$,
   (ii) 215733 groups of type $\mathrm{PSL}(3,q)$ for the prime powers $q \equiv 1(\bmod 3)$ in the range $4 \leq q \leq 6281491$, and
   (iii) 189689 groups of type $\mathrm{PSL}(3,q)$ for the prime powers $q \equiv 2(\bmod 3)$ in the range $5 \leq q \leq 5475539$.

Note that the group $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ has already been counted above.

(b) A total of 618 groups of Lie rank 3. –
   (i) 11 groups of type $\mathrm{PSL}(4,2^k)$ for $1 \leq k \leq 11$,
   (ii) 314 groups of type $\mathrm{PSL}(4,q)$ for the prime powers $q \equiv 1(\bmod 4)$ in the range $5 \leq q \leq 4297$, and
   (iii) 293 groups of type $\mathrm{PSL}(4,q)$ for the prime powers $q \equiv 3(\bmod 4)$ in the range $3 \leq q \leq 4099$.

(c) A total of 55 groups of Lie rank 4. –
   (i) 3 groups of type $\mathrm{PSL}(5,5^k)$ for $1 \leq k \leq 3$,
   (ii) 12 groups of type $\mathrm{PSL}(5,q)$ for the prime powers $q \equiv 1(\bmod 5)$ in the range $11 \leq q \leq 181$, and

   (iii) 40 groups of type $\mathrm{PSL}(5, q)$ for the prime powers $q \equiv 2, 3, 4(\mod 5)$ in the range $2 \leq q \leq 173$.
 (d) 18 groups of Lie rank 5, namely the groups $\mathrm{PSL}(6, q)$ for the prime powers q in the range $2 \leq q \leq 32$.
 (e) 9 groups of Lie rank 6, namely the groups $\mathrm{PSL}(7, q)$ for the prime powers q in the range $2 \leq q \leq 13$.
 (f) 5 groups of Lie rank 7, namely the groups $\mathrm{PSL}(8, q)$ for $q \in \{2, 3, 4, 5, 7\}$.
 (g) 8 groups of Lie rank ≥ 8, namely the groups $\mathrm{PSL}(9, 2)$, $\mathrm{PSL}(9, 3)$, $\mathrm{PSL}(9, 4)$, $\mathrm{PSL}(10, 2)$, $\mathrm{PSL}(10, 3)$, $\mathrm{PSL}(11, 2)$, $\mathrm{PSL}(12, 2)$, and $\mathrm{PSL}(13, 2)$.

(3) The series ${}^2\mathrm{A}$ of projective special unitary groups $\mathrm{PSU}(n+1, q)$ contributes a total of 406270 groups. –
 (a) A total of 405560 groups of Lie rank 2. –
   (i) 14 groups of type $\mathrm{PSU}(3, 3^k)$ for $1 \leq k \leq 14$,
   (ii) 189964 groups of type $\mathrm{PSU}(3, q)$ for the prime powers $q \equiv 1(\mod 3)$ in the range $4 \leq q \leq 5475529$, and
   (iii) 215582 groups of type $\mathrm{PSU}(3, q)$ for the prime powers $q \equiv 2(\mod 3)$ in the range $5 \leq q \leq 6281537$. –
 (b) A total of 615 groups of Lie rank 3. –
   (i) 11 groups of type $\mathrm{PSU}(4, 2^k)$ for $1 \leq k \leq 11$,
   (ii) 300 groups of type $\mathrm{PSU}(4, q)$ for the prime powers $q \equiv 1(\mod 4)$ in the range $5 \leq q \leq 4093$, and
   (iii) 304 groups of type $\mathrm{PSL}(4, q)$ for the prime powers $q \equiv 3(\mod 4)$ in the range $3 \leq q \leq 4283$.
 (c) A total of 55 groups of Lie rank 4. –
   (i) 3 groups of type $\mathrm{PSU}(5, 5^k)$ for $1 \leq k \leq 3$,
   (ii) 14 groups of type $\mathrm{PSU}(5, q)$ for the prime powers $q \equiv 4(\mod 5)$ in the range $4 \leq q \leq 179$, and
   (iii) 38 groups of type $\mathrm{PSL}(5, q)$ for the prime powers $q \equiv 1, 2, 3(\mod 5)$ in the range $2 \leq q \leq 173$.
 (d) 18 groups of Lie rank 5, namely the groups $\mathrm{PSU}(6, q)$ for the prime powers q in the range $2 \leq q \leq 32$.
 (e) 9 groups of Lie rank 6, namely the groups $\mathrm{PSU}(7, q)$ for the prime powers q in the range $2 \leq q \leq 13$.
 (f) 5 groups of Lie rank 7, namely the groups $\mathrm{PSU}(8, q)$ for $q \in \{2, 3, 4, 5, 7\}$.
 (g) 8 groups of Lie rank ≥ 8, namely the groups $\mathrm{PSU}(9, 2)$, $\mathrm{PSU}(9, 3)$, $\mathrm{PSU}(9, 4)$, $\mathrm{PSU}(10, 2)$, $\mathrm{PSU}(10, 3)$, $\mathrm{PSU}(11, 2)$, $\mathrm{PSU}(12, 2)$, and $\mathrm{PSU}(13, 2)$.

(4) The series B of orthogonal groups of type $\mathrm{O}(2n + 1, q)$ contributes a total of 23385 groups. –

   (a) 23260 groups of type $\mathrm{O}(5,q)$ for the prime powers q in the range $4 \leq q \leq 263537$ and $q \neq 2^{18}$,
   (b) 94 groups of type $\mathrm{O}(7,q)$ for the prime powers q in the range $2 \leq q \leq 379$,
   (c) 17 groups of type $\mathrm{O}(9,q)$ for the prime powers q in the range $2 \leq q \leq 31$,
   (d) 7 groups $\mathrm{O}(11,q)$ for $q \in \{2,3,4,5,7,8,9\}$, and the
   (e) 7 groups $\mathrm{O}(13,2)$, $\mathrm{O}(13,3)$, $\mathrm{O}(13,4)$, $\mathrm{O}(15,2)$, $\mathrm{O}(15,3)$, $\mathrm{O}(17,2)$ and $\mathrm{O}(19,2)$.
(5) The series ${}^2\mathrm{B}_2$ of Suzuki groups $\mathrm{Sz}(2^{2m+1})$ contributes the 17 groups $\mathrm{Sz}(2^3), \mathrm{Sz}(2^5), \ldots, \mathrm{Sz}(2^{35})$.
(6) The series C of projective symplectic groups $\mathrm{PSp}(2n,q)$ contributes a total of 105 groups. –
   (a) 86 groups of type $\mathrm{PSp}(6,q)$ for the odd prime powers q in the range $3 \leq q \leq 379$,
   (b) 13 groups of type $\mathrm{PSp}(8,q)$ for the odd prime powers q in the range $3 \leq q \leq 31$,
   (c) 4 groups $\mathrm{PSp}(10,q)$ for $q \in \{3,5,7,9\}$, and the
   (d) 2 groups $\mathrm{PSp}(12,3)$ and $\mathrm{PSp}(14,3)$.
(7) The series D of orthogonal groups of type $\mathrm{O}^+(2n,q)$ contributes a total of 50 groups. –
   (a) 33 groups of type $\mathrm{O}^+(8,q)$ for the prime powers q in the range $2 \leq q \leq 83$,
   (b) 9 groups of type $\mathrm{O}^+(10,q)$ for the prime powers q in the range $2 \leq q \leq 13$,
   (c) 4 groups $\mathrm{O}^+(12,q)$ for $q \in \{2,3,4,5\}$, and the
   (d) 4 groups $\mathrm{O}^+(14,2)$, $\mathrm{O}^+(14,3)$, $\mathrm{O}^+(16,2)$ and $\mathrm{O}^+(18,2)$.
(8) The series ${}^2\mathrm{D}$ of orthogonal groups of type $\mathrm{O}^-(2n,q)$ contributes a total of 50 groups. –
   (a) 33 groups of type $\mathrm{O}^-(8,q)$ for the prime powers q in the range $2 \leq q \leq 83$,
   (b) 9 groups of type $\mathrm{O}^-(10,q)$ for the prime powers q in the range $2 \leq q \leq 13$,
   (c) 4 groups $\mathrm{O}^-(12,q)$ for $q \in \{2,3,4,5\}$, and the
   (d) 4 groups $\mathrm{O}^-(14,2)$, $\mathrm{O}^-(14,3)$, $\mathrm{O}^-(16,2)$ and $\mathrm{O}^-(18,2)$.
(9) The series ${}^3\mathrm{D}_4$ of Steinberg triality groups contributes the 33 groups ${}^3\mathrm{D}_4(q)$ for prime powers q satisfying $2 \leq q \leq 83$.
(10) The series G_2 contributes the 956 groups $\mathrm{G}_2(q)$ for prime powers q satisfying $3 \leq q \leq 7079$.
(11) The series ${}^2\mathrm{G}_2$ of Ree groups of type $\mathrm{Ree}(3^{2m+1})$ contributes the 7 groups $\mathrm{Ree}(3^3), \mathrm{Ree}(3^5), \ldots, \mathrm{Ree}(3^{15})$.

(12) The series F_4 contributes the 7 groups $\mathrm{F}_4(2)$, $\mathrm{F}_4(3)$, $\mathrm{F}_4(4)$, $\mathrm{F}_4(5)$, $\mathrm{F}_4(7)$, $\mathrm{F}_4(8)$ and $\mathrm{F}_4(9)$.
(13) The series ${}^2\mathrm{F}_4$ of Ree groups of type $\mathrm{Ree}(2^{2m+1})$ contributes 3 groups – the Tits group $\mathrm{Ree}(2)'$, $\mathrm{Ree}(2^3)$ and $\mathrm{Ree}(2^5)$.
(14) The series E_6 contributes the 3 groups $\mathrm{E}_6(2)$, $\mathrm{E}_6(3)$ and $\mathrm{E}_6(4)$.
(15) The series ${}^2\mathrm{E}_6$ contributes the 3 groups ${}^2\mathrm{E}_6(2)$, ${}^2\mathrm{E}_6(3)$ and ${}^2\mathrm{E}_6(4)$.
(16) The series E_7 contributes only one group, namely $\mathrm{E}_7(2)$.
(17) The series E_8 contributes no group, since already $\mathrm{E}_8(2)$ is larger than M.
(18) The largest alternating group whose order is less than $|\mathrm{M}|$ is A_{43}. Now we have already counted $\mathrm{A}_5 \cong \mathrm{PSL}(2,5)$, $\mathrm{A}_6 \cong \mathrm{PSL}(2,9)$ and $\mathrm{A}_8 \cong \mathrm{PSL}(4,2)$, so the series contributes 36 new groups.
(19) Finally, we need to count the 25 sporadic simple groups other than the monster.

Adding all these numbers up, we find that there are 28917828424943085 non-abelian simple groups smaller than the monster – or in other words, that the monster is at position 28917828424943086 in the list, as claimed.

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