AUTOMORPHISM GROUP ORBITS ON FINITE SIMPLE GROUPS

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Abstract. Let $\omega(G)$ denote the number of orbits on the elements of a group G under the action of its automorphism group. With just three possible exceptions, we determine all finite simple groups G such that $\omega(G) \leq 100$.

1. Introduction

Let G be a finite group. Let $\omega(G)$, its *orbit number*, denote the number of orbits on the elements of G under the action of its automorphism group. In a sense, the orbit number tells us how many different 'kinds' of elements G has. Various results on orbit numbers have appeared in the literature. The study was initiated by Laffey and MacHale [13] who classified groups with orbit number at most 3 (all are solvable), showed that A_5 is the only non-solvable group with 4 orbits, and gave a structure theorem for certain solvable groups with orbit number 4. Those non-solvable groups with orbit number 5 were classified in [2]. Dantas, Garonzi and Bastos [8] classified those with orbit number 6 and showed that there are infinitely many with orbit number 7. Bastos and Dantas [3] gave structure theorems for those infinite groups which have both finite conjugacy classes and finite orbit number.

Of particular interest are the finite non-abelian simple groups. Kohl [11] determined the orbit numbers for all minimal simple groups. In [12] he showed that for every positive integer n there are only *finitely* many finite non-abelian simple groups having orbit number n and classified those with orbit number at most 17. Here we extend this classification to all simple groups G satisfying $\omega(G) \leq 100$, with the possible exception of $\mathrm{PSU}(6,5)$, $\mathrm{PSU}(5,9)$ and $\mathrm{O}^+(8,9)$. We consider the alternating groups, the sporadic simple groups, and the finite simple groups of Lie type in turn. We summarise the resulting classification in Table 2.

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2. The alternating groups

The orbit number for an alternating group A_n ($n \neq 6$) equals the number of partitions of n which have an even number of even parts.

If $\omega(\mathrm{A}_n) \leq 100$ then $n \leq 15$. In summary: $\omega(\mathrm{A}_5) = 4$, $\omega(\mathrm{A}_6) = 5$, $\omega(\mathrm{A}_7) = 8$, $\omega(\mathrm{A}_8) = 12$, $\omega(\mathrm{A}_9) = 16$, $\omega(\mathrm{A}_{10}) = 22$, $\omega(\mathrm{A}_{11}) = 29$, $\omega(\mathrm{A}_{12}) = 40$, $\omega(\mathrm{A}_{13}) = 52$, $\omega(\mathrm{A}_{14}) = 69$, and $\omega(\mathrm{A}_{15}) = 90$.

3. The sporadic simple groups

The orbit number of a sporadic simple group G can be deduced from [6, 7].

If G has no outer automorphism, then its orbit number $\omega(G)$ equals its class number $h(G)$. Hence $\omega(\mathrm{M}_{11}) = 10$, $\omega(\mathrm{M}_{23}) = 17$, $\omega(\mathrm{M}_{24}) = 26$, $\omega(\mathrm{Co}_3) = 42$, $\omega(\mathrm{Co}_2) = 60$, $\omega(\mathrm{Co}_1) = 101$, $\omega(\mathrm{Fi}_{23}) = 98$, $\omega(\mathrm{Th}) = 48$, $\omega(\mathrm{J}_1) = 15$, $\omega(\mathrm{Ly}) = 53$, $\omega(\mathrm{Ru}) = 36$, $\omega(\mathrm{J}_4) = 62$, $\omega(\mathrm{B}) = 184$ and $\omega(\mathrm{M}) = 194$.

If G has index 2 in its automorphism group, then we can read off which pairs of classes are fused by outer automorphisms. In summary: $\omega(\mathrm{M}_{12}) = 12$, $\omega(\mathrm{M}_{22}) = 11$, $\omega(\mathrm{J}_2) = 16$, $\omega({}^2\mathrm{F}_4(2)') = 17$, $\omega(\mathrm{HS}) = 21$, $\omega(\mathrm{J}_3) = 17$, $\omega(\mathrm{McL}) = 19$, $\omega(\mathrm{He}) = 26$, $\omega(\mathrm{Suz}) = 37$, $\omega(\mathrm{O'N}) = 25$, $\omega(\mathrm{Fi}_{22}) = 59$, $\omega(\mathrm{HN}) = 44$, and $\omega(\mathrm{Fi}_{24}') = 97$.

4. The finite simple groups of Lie type

We record a basic observation which is surprisingly useful.

Remark 4.1. The orbit number of a group G is at least

$$\left\lceil 1 + \frac{h(G) - 1}{|\operatorname{Out}(G)|} \right\rceil,$$

where $h(G)$ is the number of conjugacy classes and $\operatorname{Out}(G)$ is the outer automorphism group of G.

Consider [12, Theorem 2.1, Part (2)]: if G is a simple group of Lie rank l over $\mathbb{F}_{p^f}$, then

$$\omega(G) \ \geq \ \frac{h(G)}{|\operatorname{Out}(G)|} \ \geq \ \frac{p^{lf}}{6l(l+1)f}.$$

For 312 triples (l, p, f) the rightmost expression is at most 100. For each finite simple group G determined by a triple, we check whether the lower bound for $\omega(G)$ given in [12, Theorem 2.1, Part (3)] is greater than 100. By also employing the bounds from [12, Theorem 2.2] for $\omega(\mathrm{PSL}(n, q))$, we can restrict to the following "candidate" simple groups of Lie type.

- The groups $\mathrm{PSL}(2, p^f)$
  - for $f = 1$ and primes $7 \leq p \leq 199$,
  - for $f = 2$ and primes $3 \leq p \leq 19$,
  - for $f = 3$ and $p \in \{2, 3, 5, 7\}$,
  - for $f = 4$ and $p \in \{2, 3, 5\}$,
  - for $f \in \{5, 6\}$ and $p \in \{2, 3\}$, and
  - for $f \in \{7, 8, 9\}$ and $p = 2$;

  and the groups $\mathrm{PSL}(n, q)$
  - for $n = 3$ and $q \in \{3, 4, 5, 7, 8, 9, 11, 13, 16, 19, 25\}$,
  - for $n = 4$ and $q \in \{3, 4, 5, 7, 8, 9\}$,
  - for $n \in \{5, 6\}$ and $q \in \{2, 3, 4\}$, and
  - for $n \in \{7, 8\}$ and $q = 2$.
- The groups $\mathrm{PSU}(n, q)$
  - for $n = 3$ and $q \in \{3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 23, 29, 32, 41\}$,
  - for $n = 4$ and $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$,
  - for $n = 5$ and $q \in \{2, 3, 4, 9\}$,
  - for $n = 6$ and $q \in \{2, 3, 5\}$, and
  - for $(n, q) \in \{(7, 2), (8, 2), (8, 3), (9, 2)\}$.
- The groups $\mathrm{O}(n, q)$
  - for $n = 5$ and $q \in \{4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 25, 27\}$,
  - for $n = 7$ and $q \in \{2, 3, 4, 5, 7, 9\}$, and
  - for $(n, q) \in \{(9, 2), (9, 3), (11, 2), (11, 3), (13, 2)\}$.
- The groups $\mathrm{Sz}(q)$ for $q \in \{8, 32, 128, 512\}$.
- The groups $\mathrm{PSp}(n, q)$ for $(n, q) \in \{(6, 3), (6, 5), (6, 7), (6, 9), (8, 3), (10, 3)\}$.
- The groups $\mathrm{O}^+(n, q)$
  - for $n = 8$ and $q \in \{2, 3, 4, 5, 7, 9\}$, and
  - for $(n, q) \in \{(10, 2), (10, 3), (10, 5), (12, 2), (12, 3), (14, 2)\}$.
- The groups $\mathrm{O}^-(n, q)$
  - for $n = 8$ and $q \in \{2, 3, 4, 5\}$, and
  - for $(n, q) \in \{(10, 2), (10, 3), (12, 2), (12, 3), (14, 2), (14, 3)\}$.
- The groups ${}^3\mathrm{D}_4(2)$, ${}^3\mathrm{D}_4(3)$ and ${}^3\mathrm{D}_4(4)$.
- The groups $\mathrm{G}_2(q)$ for $q \in \{3, 4, 5, 7, 8, 9, 16\}$.
- The group ${}^2\mathrm{G}_2(27) = \mathrm{Ree}(27)$.
- The groups $\mathrm{F}_4(2)$, $\mathrm{F}_4(3)$ and $\mathrm{F}_4(4)$.
- The group ${}^2\mathrm{F}_4(8) = \mathrm{Ree}(8)$.
- The groups $\mathrm{E}_6(2)$ and ${}^2\mathrm{E}_6(2)$.

We now consider each collection of groups in turn. Sometimes (lower bounds to) class or orbit numbers were known from existing sources, including [4], the Atlas

[7], and Lübeck's database [9]. Explicit values were computed using GAP [10] and Magma [5]. Computations with larger groups relied on the infrastructure of [1] available in Magma, and minimal-degree permutation representations for automorphism groups of simple groups provided by Derek Holt.

- From [11, Theorem 2.5, Part (2)], we know formulae for the orbit numbers of $\mathrm{PSL}(2,p^f)$ in odd characteristic:
  - $\omega(\mathrm{PSL}(2,p)) = \frac{1}{2}(p+3)$,
  - $\omega(\mathrm{PSL}(2,p^2)) = \frac{1}{4}(p^2+2p+5)$,
  - $\omega(\mathrm{PSL}(2,p^3)) = \frac{1}{6}(p^3+2p+9)$,
  - $\omega(\mathrm{PSL}(2,p^4)) = \frac{1}{8}(p^4+2p^2+4p+9)$,
  - $\omega(\mathrm{PSL}(2,p^5)) = \frac{1}{10}(p^5+4p+15)$ and
  - $\omega(\mathrm{PSL}(2,p^6)) = \frac{1}{12}(p^6+2p^3+2p^2+4p+15)$.

  We deduce that all the candidate groups $\mathrm{PSL}(2,p^f)$, apart from $\mathrm{PSL}(2,199)$ and $\mathrm{PSL}(2,361)$, have orbit numbers at most 100.

  From [11, Theorem 2.5, Part (1)], we obtain the orbit numbers for $\mathrm{PSL}(2,q)$ in characteristic 2.

  The orbit numbers for $\mathrm{PSL}(3,q)$ for $q \le 8$ and $q = 16$ were computed in [12]. Since $h(\mathrm{PSL}(8,2)) = 246$ and $|\operatorname{Out}(\mathrm{PSL}(8,2))| = 2$, by Remark 4.1 we deduce that $\omega(\mathrm{PSL}(8,2)) \ge 1 + (246-1)/2 > 100$. The remaining orbit numbers for $\mathrm{PSL}(n,q)$ were computed using GAP and Magma.
- The values $\omega(\mathrm{PSU}(3,q))$ for $q \in \{3,4,5,8\}$ and $\omega(\mathrm{PSU}(4,q))$ for $q \in \{2,3\}$ were computed in [12]. Since $\mathrm{PSU}(4,8)$ has 602 conjugacy classes and $|\operatorname{Out}(\mathrm{PSL}(4,8))| = 6$, by Remark 4.1 we deduce that $\omega(\mathrm{PSU}(4,8)) \ge 102$.

  Since $h(\mathrm{PSU}(5,9)) = 1520$ and $|\operatorname{Out}(\mathrm{PSU}(5,9))| = 20$, by Remark 4.1 we deduce that $\omega(\mathrm{PSU}(5,9)) \ge 77$. Similarly, since $h(\mathrm{PSU}(6,5)) = 752$ and $|\operatorname{Out}(\mathrm{PSU}(6,5))| = 12$, by Remark 4.1 we deduce that $\omega(\mathrm{PSU}(6,5)) \ge 64$. In neither of these cases could we deduce that the orbit number must be greater than 100.

  For $\mathrm{PSU}(7,2)$, $\mathrm{PSU}(8,2)$ and $\mathrm{PSU}(8,3)$, the same approach shows that there are more than 100 orbits under the action of the automorphism group.

  The orbit numbers for the remaining $\mathrm{PSU}(n,q)$ were computed using GAP and Magma.
- The value $\omega(\mathrm{O}(5,4))$ was computed in [12]. The orbit numbers for the remaining $\mathrm{O}(5,q)$, apart from $\mathrm{O}(5,17)$ and $\mathrm{O}(5,19)$, and for the remaining $\mathrm{O}(7,q)$, apart from $\mathrm{O}(7,7)$, were computed using GAP and Magma. The orbit and class number for $\mathrm{O}(9,2)$ coincide since its outer automorphism group is trivial. By Remark 4.1, we deduce that the remaining $\mathrm{O}(n,q)$ have more than 100 automorphism orbits.
- The orbit numbers for $\omega(\mathrm{Sz}(q))$ are deduced from Theorem 3.4 in [11] which states that $\omega(\mathrm{Sz}(q)) = \omega(\mathrm{PSL}(2,q)) + 2$.

- The values $\omega(\mathrm{PSp}(6,3))$ and $\omega(\mathrm{PSp}(6,5))$ were computed using GAP and MAGMA respectively. By Remark 4.1, we deduce that the remaining $\mathrm{PSp}(n,q)$ have more than 100 automorphism orbits.
- The values $\omega(\mathrm{O}^+(8,q))$ for $q \in \{2,3,4,5,7\}$ and $\omega(\mathrm{O}^+(10,q))$ for $q \in \{2,3\}$ were computed using GAP and MAGMA.

  Since $h(\mathrm{O}^+(8,9)) \geq 2194$ and $|\,\mathrm{Out}(\mathrm{O}^+(8,9))| = 48$, by Remark 4.1 we deduce that $\omega(\mathrm{O}^+(8,9)) \geq 47$. In particular we could not deduce that the orbit number must be greater than 100.

  By Remark 4.1, we deduce that the remaining $\mathrm{O}^+(n,q)$ have more than 100 automorphism orbits, by using sufficiently good lower bounds for the class numbers.
- The values $\omega(\mathrm{O}^-(8,q))$ for $q \in \{2,3,4\}$ and $\omega(\mathrm{O}^-(10,q))$ for $q \in \{2,3\}$ were computed using GAP and MAGMA. By Remark 4.1, we deduce that the remaining $\mathrm{O}^-(n,q)$ have more than 100 automorphism orbits, by using sufficiently good lower bounds for the class numbers.
- The values ${}^3\mathrm{D}_4(q)$ for $q \in \{2,3,4\}$ were computed using GAP and MAGMA.
- The values $\omega(\mathrm{G}_2(3))$ and $\omega(\mathrm{G}_2(4))$ were computed in [12], and the values $\omega(\mathrm{G}_2(5))$ and $\omega(\mathrm{G}_2(7))$ were computed using GAP. By Remark 4.1, we deduce that the remaining $\mathrm{G}_2(q)$ have more than 100 automorphism orbits.
- The value $\omega(\mathrm{Ree}(27))$ was computed in [12].
- The orbit number for $\mathrm{F}_4(2)$ was computed using MAGMA. By Remark 4.1, we deduce that $\mathrm{F}_4(3)$ and $\mathrm{F}_4(4)$ have more than 100 automorphism orbits.
- The value $\omega(\mathrm{Ree}(8))$ was determined independently by Frank Lübeck and Robert A. Wilson using the character table and insights on fusion of classes.
- The orbit number for $\mathrm{E}_6(2)$ was computed using MAGMA. The value $\omega({}^2\mathrm{E}_6(2))$ was determined by Wilson using an approach similar to that for $\omega(\mathrm{Ree}(8))$.

As part of this project, we determined $\omega(G)$ for some groups G omitted from our final classification because $\omega(G) > 100$; since this data may be of independent interest, we record it in Table 1.

5. The classification

We summarise the resulting classification. We observe that there is no finite simple group G such that $\omega(G) = 18$. For completeness, we include the list from [11] of those groups having orbit number at most 17; note that $\omega(\mathrm{PSL}(3,7)) = 15$, not 16 as claimed there.

Theorem 5.1. *The finite non-abelian simple groups G with $\omega(G) \leq 100$ are listed in Table 2 where each isomorphism type occurs precisely once. The classification is complete apart from three possible exceptions: $\omega(\mathrm{O}^+(8,9)) \geq 47$, $\omega(\mathrm{PSU}(6,5)) \geq 64$ and $\omega(\mathrm{PSU}(5,9)) \geq 77$.*

TABLE 1. Some finite simple groups G with $\omega(G) > 100$.

G	$\omega(G)$	G	$\omega(G)$	G	$\omega(G)$
PSL$(4,7)$	137	PSU$(4,11)$	232	PSp$(6,5)$	133
PSL$(4,8)$	119	PSU$(6,3)$	156		
PSL$(5,4)$	110	PSU$(9,2)$	240	$\mathrm{O}^+(8,5)$	116
PSL$(6,3)$	122			$\mathrm{O}^+(8,7)$	290
PSL$(6,4)$	169	O$(5,13)$	115	$\mathrm{O}^+(10,3)$	268
		O$(5,25)$	203		
PSU$(3,23)$	106	O$(5,27)$	151	$\mathrm{O}^-(8,4)$	133
PSU$(3,29)$	162	O$(7,5)$	136	$\mathrm{O}^-(10,3)$	151
PSU$(3,41)$	310	O$(7,9)$	307		
PSU$(4,9)$	142			$\mathrm{E}_6(2)$	132

Table 2: Finite simple groups G for given $\omega(G) \leq 100$.

n	Finite simple groups G satisfying $\omega(G) = n$
4	PSL$(2,4) \cong$ PSL$(2,5) \cong \mathrm{A}_5$
5	PSL$(2,7) \cong$ PSL$(3,2)$, PSL$(2,9) \cong \mathrm{A}_6$, PSL$(2,8)$
6	PSL$(3,4)$
7	PSL$(2,11)$, PSL$(2,16)$, PSL$(2,27)$, Sz(8)
8	PSL$(2,13)$, A_7
9	PSL$(3,3)$, PSL$(2,32)$, PSU$(3,4)$
10	PSL$(2,17)$, PSU$(3,3)$, PSL$(2,25)$, M_{11}, PSU$(3,5)$, PSU$(3,8)$
11	PSL$(2,19)$, M_{22}, Sz(32)
12	PSL$(4,2) \cong \mathrm{A}_8$, M_{12}, O$(5,4)$
13	PSL$(2,23)$
14	PSU$(4,3)$
15	PSU$(4,2) \cong$ O$(5,3)$, J_1, PSL$(2,64)$, PSL$(2,81)$, PSL$(3,7)$
16	PSL$(2,29)$, A_9, J_2
17	PSL$(2,31)$, PSL$(2,49)$, $\mathrm{G}_2(3)$, M_{23}, PSL$(3,8)$, ${}^2\mathrm{F}_4(2)'$, J_3
18	
19	PSL$(3,5)$, McL, Ree(27)
20	PSL$(2,37)$, PSL$(5,2)$, PSL$(3,16)$
21	PSL$(2,128)$, PSL$(4,3)$, HS, ${}^3\mathrm{D}_4(2)$, O$(5,8)$
22	PSL$(2,41)$, A_{10}
23	PSL$(2,43)$, Sz(128)
24	PSL$(2,125)$, $\mathrm{G}_2(4)$
25	PSL$(2,47)$, O′N
26	M_{24}, He
	To be continued.

Continued.	
n	Simple groups G satisfying $\omega(G) = n$
27	$\mathrm{O}(5,5)$, $\mathrm{PSL}(2,243)$, $\mathrm{O}^+(8,2)$
28	$\mathrm{PSL}(2,53)$
29	A_{11}, $\mathrm{PSU}(3,9)$
30	$\mathrm{O}(7,2)$, $\mathrm{PSU}(5,2)$, $\mathrm{PSU}(3,11)$
31	$\mathrm{PSL}(2,59)$
32	$\mathrm{PSL}(2,61)$, $\mathrm{PSL}(3,9)$
33	$\mathrm{O}^-(8,2)$
34	$\mathrm{PSU}(3,7)$, $\mathrm{PSL}(4,5)$, $\mathrm{PSU}(5,4)$, $\mathrm{PSU}(6,2)$
35	$\mathrm{PSL}(2,67)$, $\mathrm{PSU}(4,4)$
36	$\mathrm{PSL}(4,4)$, Ru
37	$\mathrm{PSL}(2,71)$, $\mathrm{PSL}(2,121)$, $\mathrm{PSL}(2,256)$, Suz
38	$\mathrm{PSL}(2,73)$, $\mathrm{O}^+(8,3)$
39	$\mathrm{PSL}(3,13)$
40	A_{12}, $\mathrm{PSU}(3,16)$
41	$\mathrm{PSL}(2,79)$, $\mathrm{O}(5,9)$
42	$\mathrm{PSU}(3,32)$, Co_3
43	$\mathrm{PSL}(2,83)$, $\mathrm{O}(5,7)$
44	$\mathrm{G}_2(5)$, $\mathrm{PSL}(6,2)$, HN
45	$\mathrm{O}(5,16)$
46	$\mathrm{PSL}(2,89)$
47	
48	Th
49	
50	$\mathrm{PSL}(2,97)$, $\mathrm{PSL}(2,169)$, $\mathrm{PSp}(6,3)$
51	
52	$\mathrm{PSL}(2,101)$, A_{13}, $\mathrm{O}(7,3)$
53	$\mathrm{PSL}(2,103)$, Ly
54	
55	$\mathrm{PSL}(2,107)$
56	$\mathrm{PSL}(2,109)$, ${}^3\mathrm{D}_4(3)$
57	$\mathrm{Ree}(8)$
58	$\mathrm{PSL}(2,113)$
59	Fi_{22}
60	Co_2
61	$\mathrm{PSL}(2,343)$, $\mathrm{PSL}(2,512)$
62	$\mathrm{PSU}(3,17)$, $\mathrm{F}_4(2)$, J_4
63	$\mathrm{Sz}(512)$
64	$\mathrm{PSU}(4,5)$
	To be continued.

Continued.	
n	Simple groups G satisfying $\omega(G) = n$
65	$\mathrm{PSL}(2,127)$
66	
67	$\mathrm{PSL}(2,131)$
68	
69	$\mathrm{PSL}(2,729)$, A_{14}
70	$\mathrm{PSL}(2,137)$
71	$\mathrm{PSL}(2,139)$
72	$\mathrm{PSL}(3,25)$, $\mathrm{PSL}(5,3)$, $\mathrm{G}_2(7)$
73	$\mathrm{PSL}(3,11)$
74	
75	$\mathrm{PSL}(3,19)$, $\mathrm{O}(7,4)$
76	$\mathrm{PSL}(2,149)$, $\mathrm{PSU}(4,7)$
77	$\mathrm{PSL}(2,151)$, $\mathrm{O}^-(8,3)$, $\mathrm{PSL}(7,2)$
78	${}^3\mathrm{D}_4(4)$
79	
80	$\mathrm{PSL}(2,157)$
81	$\mathrm{O}(9,2)$
82	$\mathrm{PSL}(2,289)$
83	$\mathrm{PSL}(2,163)$
84	$\mathrm{O}^+(10,2)$, $\mathrm{O}^+(8,4)$
85	$\mathrm{PSL}(2,167)$, $\mathrm{PSL}(4,9)$
86	
87	$\mathrm{O}(5,11)$
88	$\mathrm{PSL}(2,173)$, $\mathrm{PSL}(2,625)$
89	$\mathrm{PSU}(5,3)$
90	A_{15}
91	$\mathrm{PSL}(2,179)$, ${}^2\mathrm{E}_6(2)$
92	$\mathrm{PSL}(2,181)$
93	$\mathrm{O}^-(10,2)$
94	
95	
96	
97	$\mathrm{PSL}(2,191)$, Fi_{24}'
98	$\mathrm{PSL}(2,193)$, Fi_{23}
99	
100	$\mathrm{PSL}(2,197)$, $\mathrm{PSU}(3,13)$

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