Particle Markov chain Monte Carlo methods

Vortrag am X.09.2010

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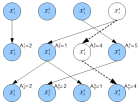
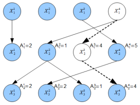


Outline

- paper by Andrieu, Doucet, Holenstein: **Particle Markov chain Monte Carlo methods**
- goal: inference in State Space Models
- how: combination of sequential Monte Carlo (SMC) and Markov chain Monte Carlo (MCMC) methods



state space models: Hidden Markov Models

state assumptions

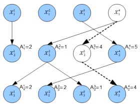
- states: $\{X_n; n \geq 1\} \subset \mathcal{X}^{\mathbb{N}}$
- $X_1 \sim \mu_\theta(\cdot)$
- $X_{n+1} | (X_n = x) \sim f_\theta(\cdot | x)$
- parameters: $\theta \in \Theta$

observation assumptions

- observations: $\{Y_n; n \geq 1\} \subset \mathcal{Y}^{\mathbb{N}}$
- $Y_{n+1} | (X_1, ..., X_n = x, ..., X_m) \sim g_\theta(\cdot | x)$

interested in:

- $p_\theta(x_{1:n} | y_{1:n})$
- even more: $p(\theta, x_{1:n} | y_{1:n})$



Bayesian inference conditional on observations $y_{1:T}$

Bayesian inference

- $p_\theta(x_{1:n}|y_{1:n}) \propto p_\theta(x_{1:n}, y_{1:n})$
- with

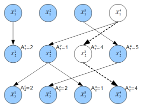
$$p_\theta(x_{1:T}, y_{1:T}) = \mu_\theta(x_1) \prod_{n=2}^{T} f_\theta(x_n|x_{n-1}) \prod_{n=1}^{T} g_\theta(y_n|x_n)$$

- also: $p_\theta(x_{1:T}|y_{1:T}) \propto p_\theta(x_{1:T-1}, y_{1:T-1}) f_\theta(x_T|x_{T-1}) g_\theta(y_T|x_T)$

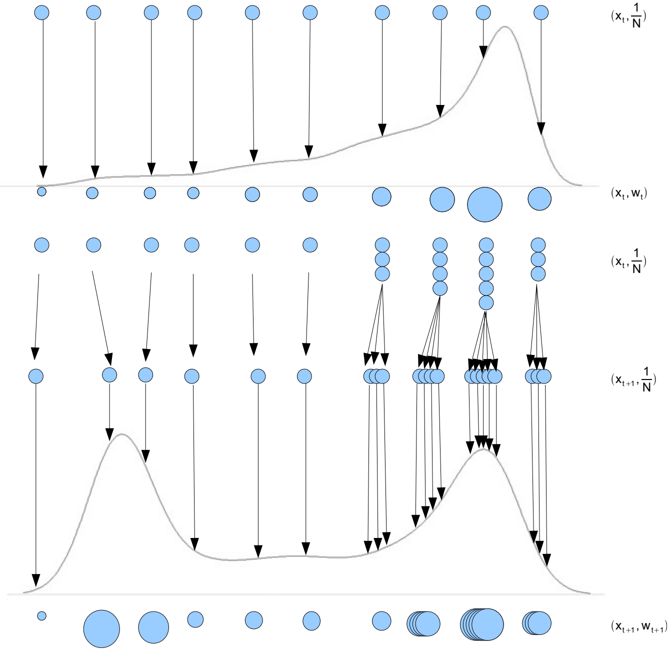
particle filter

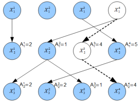
- weights and particles: $(w_n^{(i)}, x_{1:n}^{(i)})_{i=1}^{N} \sim p_\theta(x_{1:n}|y_{1:n})$
- new particles: $x_{n+1}^{(i)} \sim q(\cdot|x_{1:n}^{(i)}, y_{n+1})$
- remember Importance Sampling: correct for "wrong distribution"
-

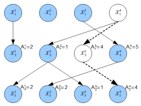
$$w_{n+1}^{(i)} = w_n^{(i)} \frac{f_\theta(x_{n+1}^{(i)}|x_n^{(i)}) g_\theta(y_{n+1}|x_{n+1}^{(i)})}{q(x_{n+1}|x_{1:n}^{(i)}, y_{n+1})}$$



bootstrap filter illustration







from Particle Filter to Markov chain Monte Carlo

- we get particles and weight $\sim p_\theta(x_{1:T}|y_{1:T})$
- we also get marginal likelihood:

$$\prod_{n=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N} w_n^i\right) \xrightarrow{N\to\infty} p_\theta(y_{1:T}) := \hat{Z}$$

- Particle MCMC idea: use SMC as proposal in Metropolis-Hastings algorithms
- use SMC distributions as conditional distributions in Gibbs algorithm

Particle Independent Metropolis Hastings algorithm

- run SMC, sample

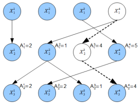
$$x^*_{1:T} \sim \hat{p}_\theta(dx_{1:T}|y_{1:T})$$

- compute simulated likelihood

$$\hat{Z}^* = p_\theta(\hat{y}_{1:T})$$

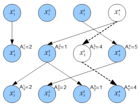
- accept with

$$1 \wedge \frac{\hat{Z}^*}{\hat{Z}^i}$$



nonlinear toy example

- $$X_n = \frac{X_{n-1}}{2} + 25\frac{X_{n-1}}{1+X_{n-1}^2} + 8cos(1.2n) + V_n$$
- $$Y_n = \frac{X_n^2}{20} + W_n$$
- $X_1 \sim^{IID} \mathcal{N}(0,5)$
- $V_n \sim^{IID} \mathcal{N}(0,\sigma_v^2)$
- $W_n \sim^{IID} \mathcal{N}(0,\sigma_w^2)$



single boostrap filter run

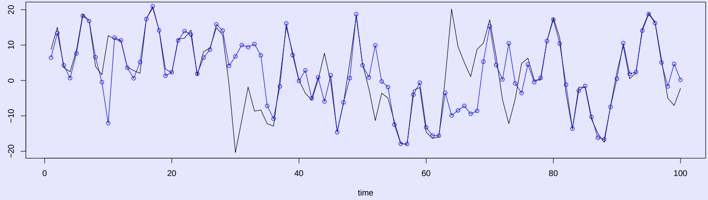
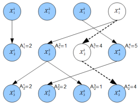


Figure: true states x and Bootstrap median estimates marked with points



Outline

SCM methods

Particle Independent Metropolis Hastings

Particle Marginal Metropolis Hastings

Particle Gibbs

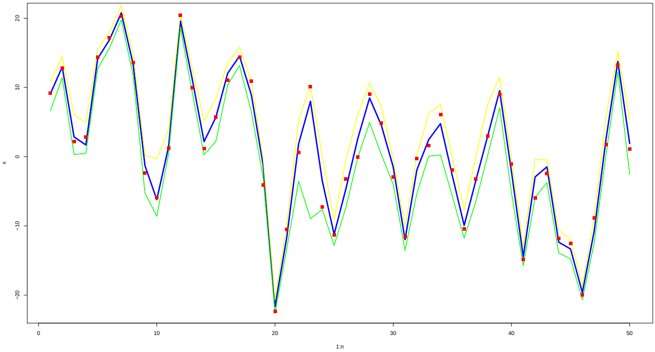
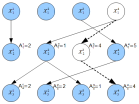
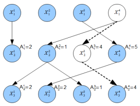


Figure: PIMH N=50 particles, n=50 observations, 10000 iterations





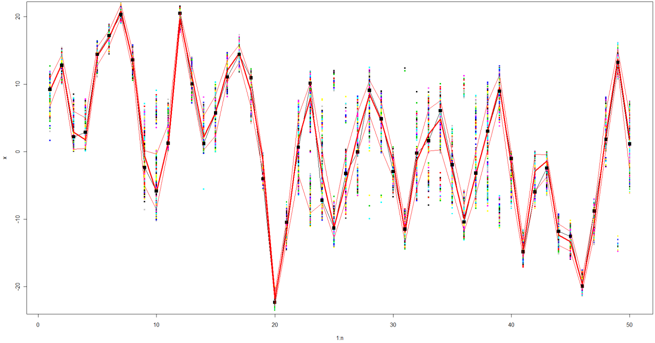
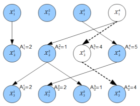


Figure: PIMH N=6000 particles, n=50 observations, 10000 iterations



PMMH

- interested in θ and $x_{1:T}$ from $p_\theta(\theta, x_{1:T}|y_{1:T}) = p(\theta|y_{1:T})p_\theta(x_{1:T}|y_{1:T})$
- Andrieu, Doucet, Holenstein, proved to be the correct MH ratio:

$$1 \wedge \frac{\hat{p}_{\theta*}(y_{1:T})p(\theta^*)q\{\theta(i-1)|\theta^*\}}{\hat{p}_{\theta(i-1)}(y_{1:T})p(\theta)q\{\theta^*|\theta(i-1)\}}$$

- applied to a basic stochastic volatility (SV) model:

$$y_t = e^{h_t/2}\varepsilon_t$$
$$h_t = \mu + \phi h_{t-1} + \tau\eta_t$$

where $\varepsilon_t, \eta_t \sim \mathcal{N}(0,1), \mathbb{E}(\varepsilon_t\eta_{t+h}) = 0$ for all h and $\mathbb{E}(\varepsilon_t \varepsilon t + l = \mathbb{E}(\eta_t\eta_{t+l}) = 0$ for all $l \neq 0$

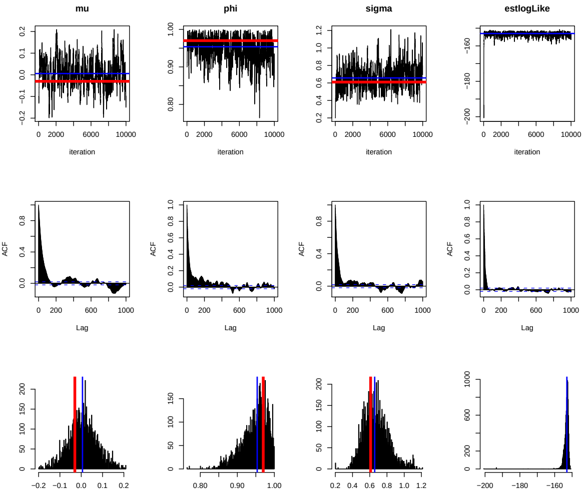
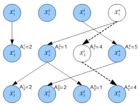


Figure: SV model N=1000 Particles n=100 observations 10000 iterations



Outline

SCM methods

Particle Independent Metropolis Hastings

Particle Marginal Metropolis Hastings

Particle Gibbs

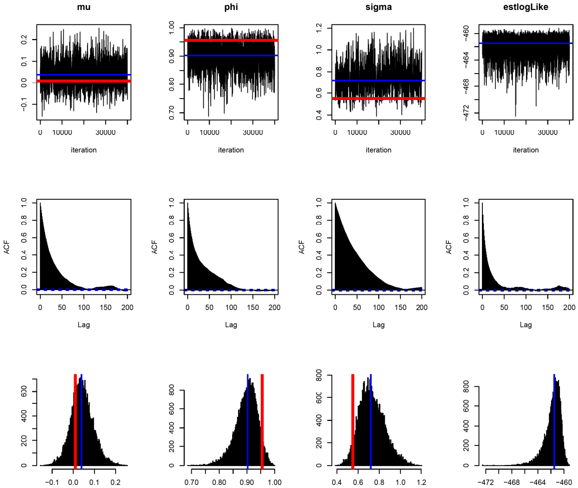
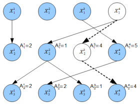


Figure: SV model N=1500 particles 250 observations 40000 mcmc iterations



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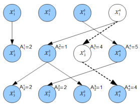
SCM methods
Particle Independent Metropolis Hastings
Particle Marginal Metropolis Hastings
Particle Gibbs



Particle Gibbs

- iteratively sample from $p(\theta | x_{1:T}, y_{1:T})$ and with SMC from $p_\theta(x_{1:T} | y_{1:T})$
- important in the Particle Gibbs: conditioning on last particle-path

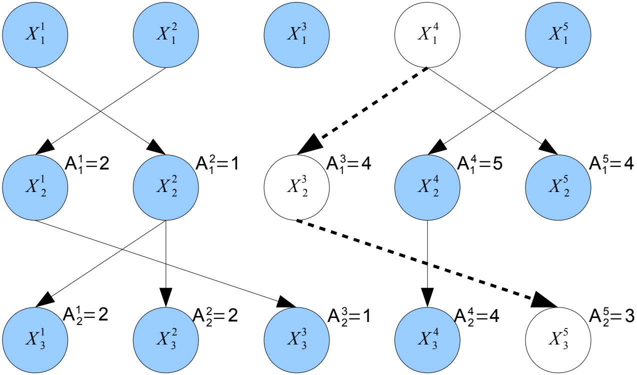
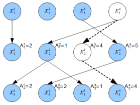


Figure: conditionalSMC starting with dashed lineage



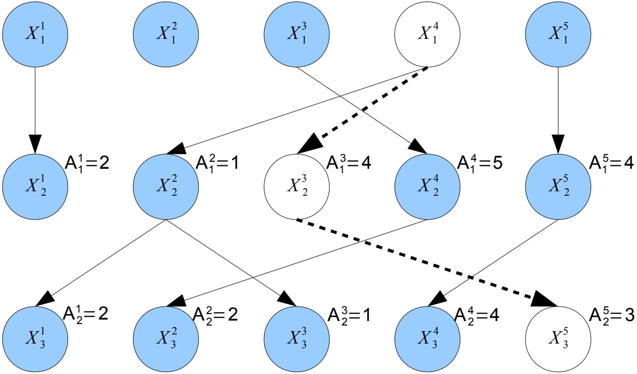
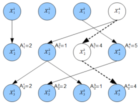


Figure: conditionalSMC N-1 new path conditioned on dashed path

Proofs



Proofs

- formalize SMC on extended space including all random variables
- for Particle MH show: acceptance ratios lead to target distribution
- for Particle Gibbs show full conditional densities and "collapsed Gibbs"-step

PG

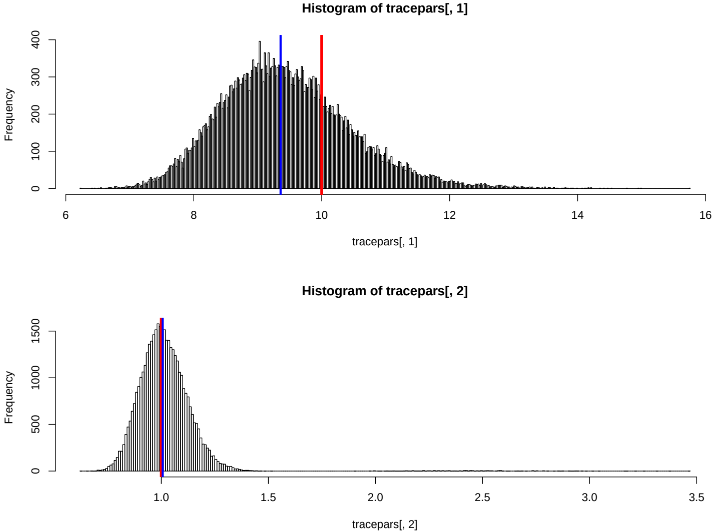
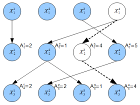


Figure: Particle Gibbs with MH step for nonlinear model, 40.000 iterations



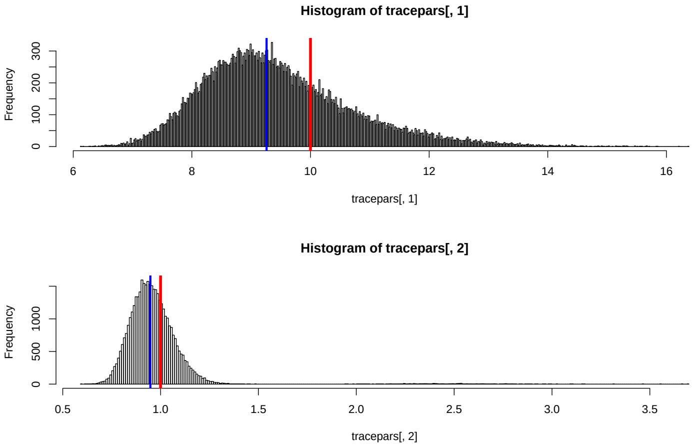
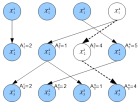


Figure: Particle Gibbs for nonlinear model, 40.000 iterations



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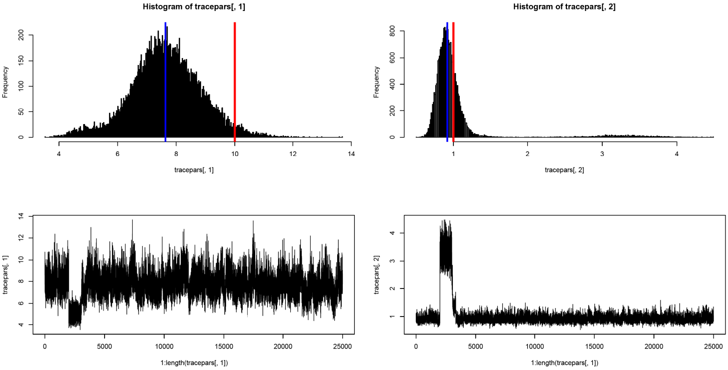
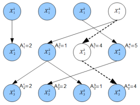


Figure: toy model N250 particles n250 observations



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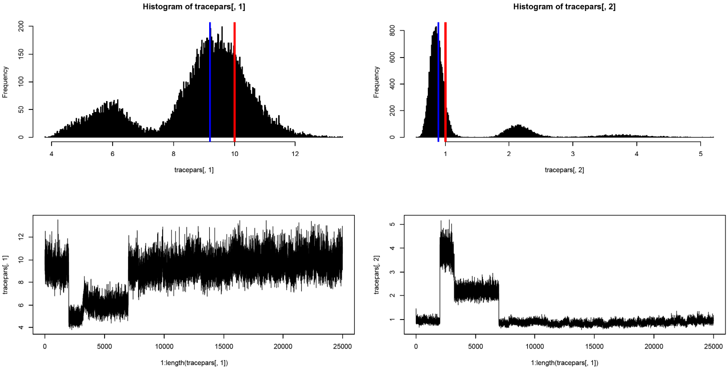
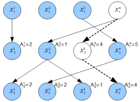


Figure: toy model N250 particles n300 observations



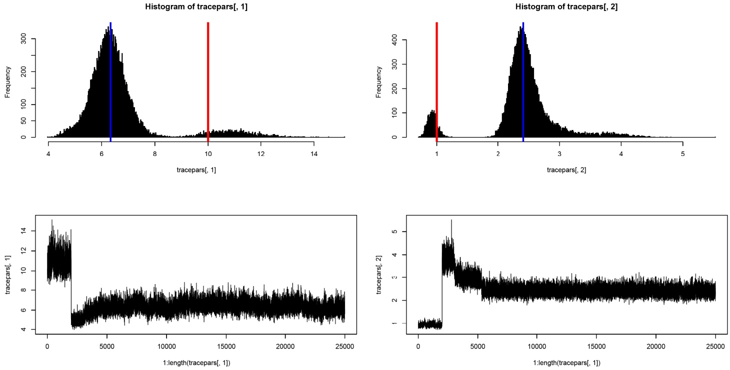
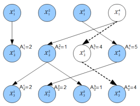


Figure: toy model N250 particles n350 observations



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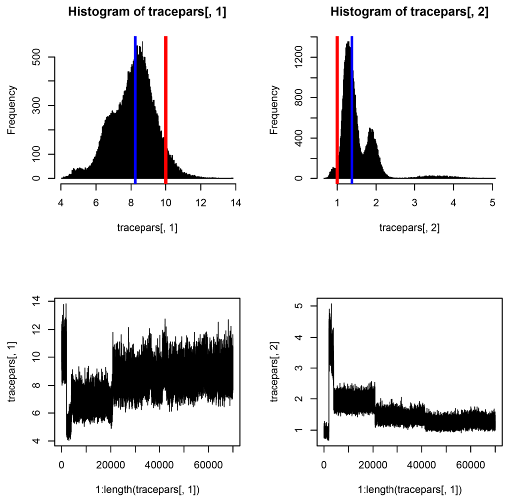
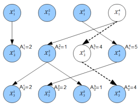


Figure: toy model N250 particles n350 observations 70000 mcm iterations



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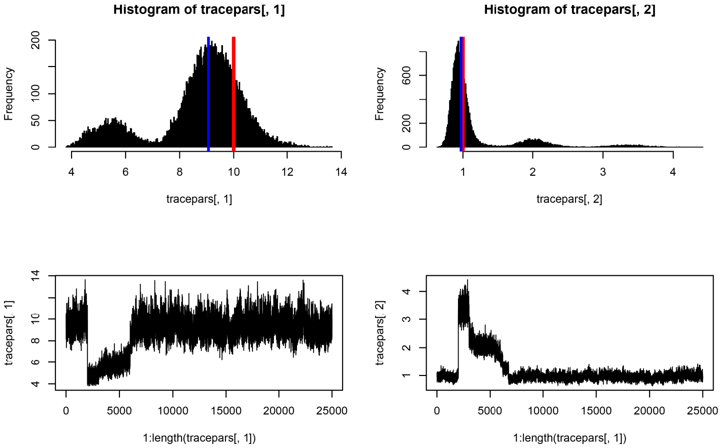
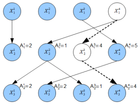


Figure: toy model N500 particles n350 observations



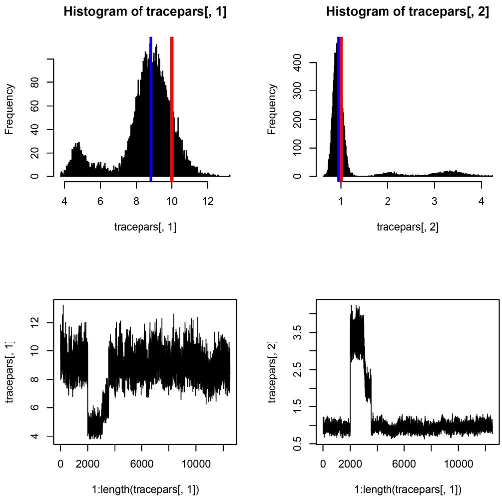
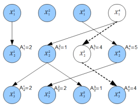


Figure: PH with MH toy model N500 particles n350 observations



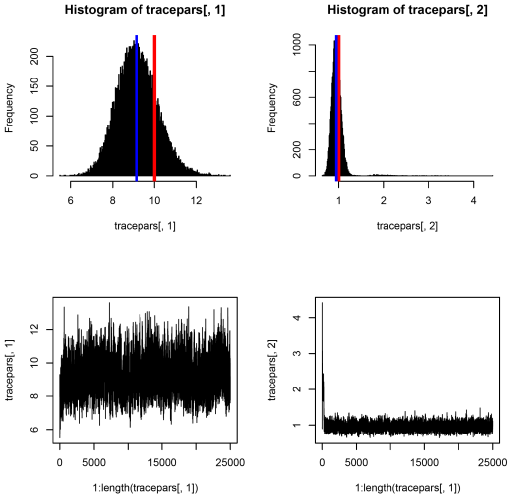
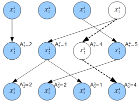


Figure: PG toy model N700 particles n350 observations



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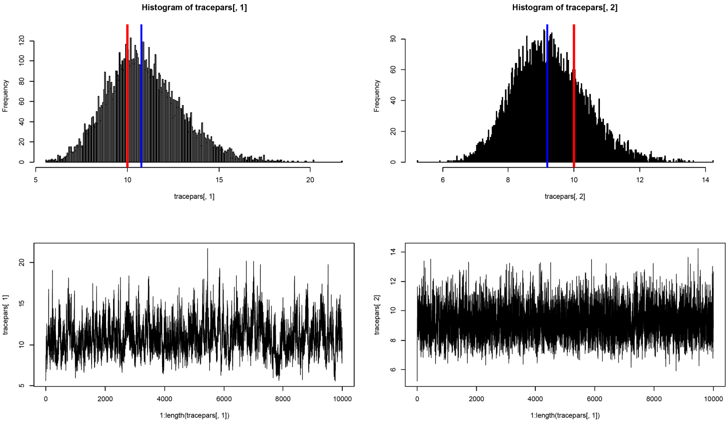
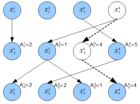


Figure: PG for toy model (10,10) 10000 iterations



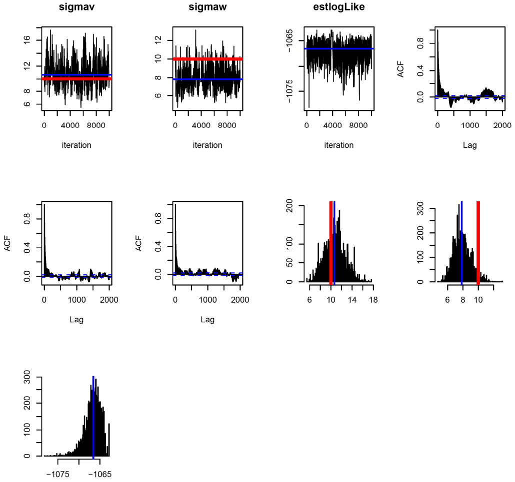


Figure: PMMH for toy model (10,10) 10000 iterations