MULTIMI MASURABILE JORDAN

1) INEL DE MULTIMI, MASURA POZITIVA PE UN INEL DE MULTIMI

Definitia 1. Fie $X \neq \varnothing$. O multime nevida de parti $\mathcal{F} \subseteq \wp(X)$ se numeste inel de multimi daca are urmatoarele proprietati:

a) $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$

b) $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \setminus A_2 \in \mathcal{F}$.

Observatie. Daca $\mathcal{F} \subseteq \wp(X)$ este inel de multimi, atunci sunt adevarate urmatoarele afirmatii:

a) $\varnothing \in \mathcal{F}$

b) $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$.

Definitia 2. Fie $\mathcal{F} \subseteq \wp(X)$ un inel de multimi. Se numeste masura pozitiva pe $\mathcal{F}$ o functie $\mu : \mathcal{F} \to [0, +\infty]$ care are urmatoarele proprietati:

a) $\mu(\varnothing) = 0$

b) $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)\ \ \forall A_1, A_2 \in \mathcal{F}\ cu\ A_1 \cap A_2 = \varnothing$.

Observatie. Fie $\mu : \mathcal{F} \to [0, +\infty]$ o masura pozitiva. Atunci:

a) $A_1, A_2 \in \mathcal{F}, A_1 \subseteq A_2 \Rightarrow \mu(A_1) \leq \mu(A_2)$

b) $\mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2)\ \ \forall A_1, A_2 \in \mathcal{F}$

c) $A_1, A_2 \in \mathcal{F}, A_1 \subseteq A_2, \mu(A_1) < +\infty \Rightarrow \mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1)$

d) $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)\ \ \forall A_1, A_2 \in \mathcal{F}$.

2) INTERVALE MARGINITE n-DIMENSIONALE, MULTIMI ELEMENTARE in $\mathbb{R}^n$

Pentru orice interval marginit $J \subseteq \mathbb{R}$ notam $l(J) \in \mathbb{R}$ lungimea acestuia.

Definitia 3. Multimea $I \subseteq \mathbb{R}^n$ se numeste interval marginit n-dimensional daca exista o familie finita $\{I_1, I_2, ..., I_n\}$ de intervale marginite din $\mathbb{R}$ astfel incat $I = I_1 \times I_2 \times .... \times I_n$.

Definitia 4. Fie $I = I_1 \times I_2 \times .... \times I_n$ un interval marginit n-dimensional. Numarul real $v(I) = l(I_1) \cdot l(I_2) \cdot \cdot \cdot l(I_n)$ se numeste volumul lui I.

Definitia 5. O multime $E \subseteq \mathbb{R}^n$ se numeste elementara daca exista o familie finita $\{I_1, I_2, ..., I_p\}$ de intervale marginite n-dimensionale astfel incat $E = I_1 \cup I_2 \cup ... \cup I_p$.

Notatie. $\mathcal{L} \overset{not}{=} \{E \subseteq \mathbb{R}^n |\ E\ multime\ elementara\} \subseteq \wp(\mathbb{R}^n)$

Propozitia 1. Fie $E \in \mathcal{L}$. Atunci $\exists I_1, I_2, ..., I_k \subseteq \mathbb{R}^n$ intervale marginite n-dimensionale astfel incat $E = I_1 \cup I_2 \cup ... \cup I_k$ si $\overset{0}{I_i} \cap \overset{0}{I_j} = \varnothing\ \forall i \neq j \in \{1, 2, ..., k\}$.

Definitia 6. Fie $E \in \mathcal{L}$ astfel ca $E = I_1 \cup I_2 \cup ... \cup I_k$ cu $I_j \subseteq \mathbb{R}^n$ interval marginit n-dimensional $\forall j \in \{1, 2, ..., k\}$ si $\overset{0}{I_i} \cap \overset{0}{I_j} = \varnothing\ \forall i \neq j \in \{1, 2, ..., k\}$. Numarul real $v(E) = \sum\limits_{j=1}^{k} v(I_j)$ se numeste volumul multimii elementare E.

Observatie. a) Orice interval marginit n-dimnesional $I \subset \mathbb{R}^n$ este multime elementara in $\mathbb{R}^n$.

b) $\varnothing \in \mathcal{L}$ si $v(\varnothing) = 0$.

c) $E \in \mathcal{L} \Rightarrow \overset{0}{E}, \overline{E} \in \mathcal{L}$ si $v(E) = v\left(\overset{0}{E}\right) = v(\overline{E})$.

3) MASURA EXTERIOARA JORDAN, MASURA INTERIOARA JORDAN

Definitia 6. Fie $A \subset \mathbb{R}^n$ o multime marginita. Numarul real $\mu^*(A) = \inf_{E \in \mathcal{L}, A \subset E} v(E)$ se numeste masura exterioara Jordan a multimii A.

Observatie. a) $\mu^*(E) = v(E)\ \forall E \in \mathcal{L}$.

b) $\mu^*(\varnothing) = 0$.

Propozitia 2. Fie $A_1, A_2 \subset \mathbb{R}^n$ doua multimi marginite. Sunt adevarate urmatoarele afirmatii:

i) $A_1 \subset A_2 \Rightarrow \mu^*(A_1) \leq \mu^*(A_2)$.

ii) $\mu^*(A_1 \cup A_2) \leq \mu^*(A_1) + \mu^*(A_2)$.

Definitia 7. O multime marginita $A \subset \mathbb{R}^n$ este multime de masura Jordan nula daca $\mu^*(A) = 0$.

Notatie. $J_0 \overset{not}{=} \{A \subset \mathbb{R}^n |\, A\ m \arg inita, \mu^*(A) = 0\}$

Observatie. $A \in J_0 \Leftrightarrow \forall \varepsilon > 0 \exists E_\varepsilon \in \mathbb{R}^n$ multime elementara astfel incat $A \subset E_\varepsilon$ si $v(E_\varepsilon) < \varepsilon$.

Exemple. a) Orice multime finita din $\mathbb{R}^n$ este multime de masura Jordan nula.

b) Fie $f : [a,b] \to \mathbb{R}$ o functie integrabila Riemann. Atunci $G_f = \{(x, f(x)) |\, x \in [a,b]\} \subset \mathbb{R}^2$ este multime de masura Jordan nula.

Propozitia 3. O multime marginita $A \subset \mathbb{R}^n$ este de masura Jordan nula daca si numai daca $\forall \varepsilon > 0, \exists I_1.I_2, ..., I_p \subset \mathbb{R}^n$ o familie finita de intervale marginite n-dimensionale astfel incat $A \subset I_1 \cup I_2 \cup ... \cup I_p$ si $\sum_{i=1}^{p} v(I_i) < \varepsilon$.

Definitia 8. Fie $A \subset \mathbb{R}^n$ o multime marginita. Numarul real $\mu_*(A) = \sup_{E \in \mathcal{L}, E \subset A} v(E)$ se numeste masura interioara Jordan a multimii A.

Observatie. a) $\mu_*(E) = v(E)\ \forall E \in \mathcal{L}$.

b) $\mu_*(\varnothing) = 0$.

Propozitia 4. Sunt adevarate urmatoarele afirmatii:

i) $\mu_*(A) \leq \mu^*(A)$ oricare ar fi $A \subset \mathbb{R}^n$ multime marginita.

ii) Daca $A_1 \subset A_2$ sunt doua multimi marginite din $\mathbb{R}^n$, atunci $\mu_*(A_1) \leq \mu_*(A_2)$.

Definitia 9. O multime marginita $A \subset \mathbb{R}^n$ se numeste multime masurabila Jordan daca $\mu_*(A) = \mu^*(A)$.

Notatii. 1) $J(\mathbb{R}^n) \overset{not}{=} \{A \subset \mathbb{R}^n |\, A\ \textit{multime masurabila Jordan}\}$

2) $\mu(A) \overset{not}{=} \mu_*(A) = \mu^*(A) \in \mathbb{R}$ masura Jordan a multimii A

Teorema 1. a) $J(\mathbb{R}^n) \subset \wp(\mathbb{R}^n)$ este inel de multimi.

b) Functia $\mu : J(\mathbb{R}^n) \to \mathbb{R}$ definita prin $\mu(A) = \mu_*(A) = \mu^*(A)\ \forall A \in J(\mathbb{R}^n)$ este masura pozitiva pe $J(\mathbb{R}^n)$.

Definitia 10. Masura $\mu : J(\mathbb{R}^n) \to \mathbb{R}_+$ se numeste masura Jordan in $\mathbb{R}^n$.

Teorema 2. a) $\mathcal{L} \subset J(\mathbb{R}^n)$ si $\mu(E) = v(E)\ \forall E \in \mathcal{L}$.

b) $J_0 \subset J(\mathbb{R}^n)$.

c) O multime marginita $A \subset \mathbb{R}^n$ este masurabila Jordan daca si numai daca FrA este masurabila Jordan si $\mu(FrA) = 0$.

d) O multime marginita $A \subset \mathbb{R}^n$ este masurabila Jordan daca si numai daca $\forall \varepsilon > 0 \ \exists E, F \in \mathcal{L}$ astfel incat $E \subset A \subset F$ si $v(F) - v(E) < \varepsilon$.

e) Daca $A \in J(\mathbb{R}^n)$, atunci $\overline{A}, \overset{0}{A} \in J(\mathbb{R}^n)$ si $\mu(A) = \mu(\overline{A}) = \mu\left(\overset{0}{A}\right)$.

Definitia 11. a) O multime $E \subset \mathbb{R}^n$ se numeste convexa daca $\forall x, y \in E, \forall t \in [0,1]$ avem ca $(1-t) \cdot x + t \cdot y \in E$.

b) O multime $E \subset \mathbb{R}^n$ se numeste simpla in raport cu axa $Ox_j, 1 \leq j \leq n$ daca $\exists K \subset \mathbb{R}^{n-1}$ multime compacta, masurabila Jordan si doua functii continue $\varphi_1, \varphi_2 : K \to \mathbb{R}$ astfel incat

$$E = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | \left(x_1, x_2, .., \overset{\wedge}{x_j}, .., x_n\right) \in K, \varphi_1\left(x_1, x_2, .., \overset{\wedge}{x_j}, .., x_n\right) \leq x_j \leq \varphi_2\left(x_1, x_2, .., \overset{\wedge}{x_j}, .., x_n\right)\right\}.$$

Teorema 3 (Proprietatile multimilor masurabile Jordan)

a) $\forall E \in J(\mathbb{R}^n), \forall x \in \mathbb{R}^n$ *si* $\forall \alpha \in \mathbb{R}^*$ avem ca $E+x, \alpha E \in J(\mathbb{R}^n)$ si $\mu(E+x) = \mu(E), \mu(\alpha E) = \alpha^n \mu(E)$.

b) Orice multime $E \subset \mathbb{R}^n$ marginita si convexa este masurabila Jordan.

c) Orice multime $E \subset \mathbb{R}^n$ simpla in raport cu axa $Ox_j, 1 \leq j \leq n$ este masurabila Jordan.

Teorema de transport a multimilor masurabile Jordan. Fie $D = \overset{0}{D} \subset \mathbb{R}^n$ o multime deschisa, marginita si $\varphi : D \to \mathbb{R}^n$ o functie injectiva, de clasa C^1 cu $\det J_\varphi(x) \neq 0 \ \forall x \in D$. Daca $E \subset \mathbb{R}^n$ este o multime masurabila Jordan cu $\overline{E} \subset D$, atunci $\varphi(E) \subset \mathbb{R}^n$ este multime masurabila Jordan.