CURS 9 - GAL

Transformări ortogonale.

• $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r , $f \in End(V)$

f transformare ortogonală $\Leftrightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle$ $\forall x, y \in E$.

$f \in O(E)$

Prop $f \in End(V)$

$f \in O(E) \Leftrightarrow \| f(x) \| = \| x \|, \forall x \in E$.

Prop $f \in O(E) \Rightarrow f$ inj

Prop $f \in O(E)$

$[f]_{R,R} \in O(n), \forall R =$ reper ortonormat

OBS

$f \in O(E) \Leftrightarrow$ Schimbare de repere ortonormate.

OBS $n = 2$

$A \in O(2) \Rightarrow \exists \varphi \in (-\pi, \pi]$

1) $\det A = 1 \Rightarrow A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

2) $\det A = -1 \Rightarrow A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$

Prop $(E, \langle \cdot, \cdot \rangle)$ sver, $f \in O(E)$, $U \subseteq E$ subsp vect. invariant în raport cu f i.e $f(U) \subseteq U$. Atunci

a) $f(U) = U$

b) $U^{\perp}$ este subsp. invariant al lui f

c) $f|_{U^{\perp}} : U^{\perp} \to U^{\perp}$ transf. ortogonală.

a) $f: U \longrightarrow f(U)$ izomorfism de sp vect.

$\left.\begin{array}{l} \dim f(U) = \dim U \\ \text{dar } f(U) \subseteq U \end{array}\right\} \Rightarrow f(U) = U.$

b) Dem $f(U^\perp) \subseteq U^\perp$ (subsp. invariant)

Fie $x \in U^\perp \Rightarrow f(x) \in U^\perp$

$\langle f(x), \underset{U}{\underset{\cap}{y}} \rangle = \langle f(x), f(z) \rangle = \langle \underset{U^\perp}{\underset{\cap}{x}}, \underset{U}{\underset{\cap}{z}} \rangle = 0 \Rightarrow$

$y \in U$, $\exists z \in U$ a.î. $f(z) = y$

$\Rightarrow f(x) \in U^\perp \Rightarrow f(U^\perp) \subseteq U^\perp \overset{a)}{\Longrightarrow} f(U^\perp) = U^\perp$

c) $f|_{U^\perp} : U^\perp \longrightarrow U^\perp$ transf. ortogonală.

$\boxed{\text{OBS}}$ $p, s \in End(E)$, $p^2 = p$ (proiecție), $s^2 = id_E$ (~~sime~~ simetrie.

$E' = Ker\, p$, $E'' = Im\, p$.

$\dim E' = k$, $\dim E'' = n - k$

$R' = \{e_1, .., e_k\}$, $R'' = \{e_{k+1}, .., e_n\}$ repere orton. în E', resp E''.

$p(x') = 0$ $\quad \forall x \in E$, $\exists! x', x''$ a.î. $x = x' + x''$

$p(x'') = x''$ $\quad E = E' \oplus E''$, $E'' = E'^\perp$

p = proiecția ortogonală pe E''

$\begin{cases} s(x') = -x' \\ s(x'') = x'' \end{cases}$ $\quad s = 2p - id_E$. simetria ortogonală față de E''

$A_p = [p]_{R,R} = \left(\begin{array}{c|c} O_k & O \\ \hline O & I_{n-k} \end{array}\right)$; $A_s = [s]_{R,R} = \left(\begin{array}{c|c} -I_k & O \\ \hline O & I_{n-k} \end{array}\right)$

$\quad A \notin O(n)$ $\quad s \in O(E)$ $\quad \in O(n)$

Prop $f\in O(E) \Rightarrow$ valorile proprii $\in\{-1,1\}$.

Dem

λ = valoare proprie $\Leftrightarrow \exists\, x\in E$, $x \neq 0_E$ aî $f(x)=\lambda x$

$$\underbrace{\|f(x)\|}_{=|\lambda|\,\|x\|} = \|x\| \Rightarrow |\lambda| = 1 \Rightarrow \lambda = \pm 1.$$

Clasificare transf. ortogonale.

① $n=1$. $f: E\to E$, $f\in O(E)$

$R=\{e\}$ e = versor propriu

$f(e)=\pm e$ $\qquad f(x) = f(\alpha e) = \pm x$

$f\in\{id_E, -id_E\}$.

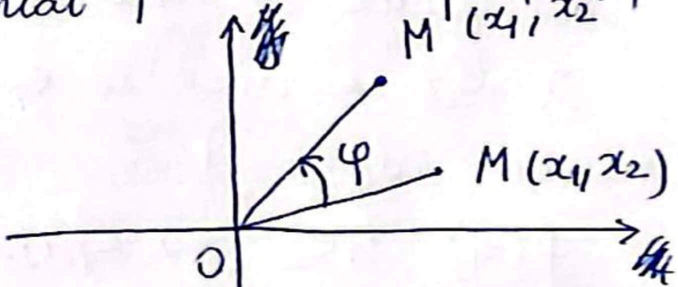
② $n=2$. $A\in O(2)$

a) $\det A = 1$, $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$f: \mathbb{R}^2\to\mathbb{R}^2$

$$f(x_1,x_2) = (\overbrace{x_1\cos\varphi - x_2\sin\varphi}^{x_1'}, \overbrace{x_1\sin\varphi + x_2\cos\varphi}^{x_2'})$$

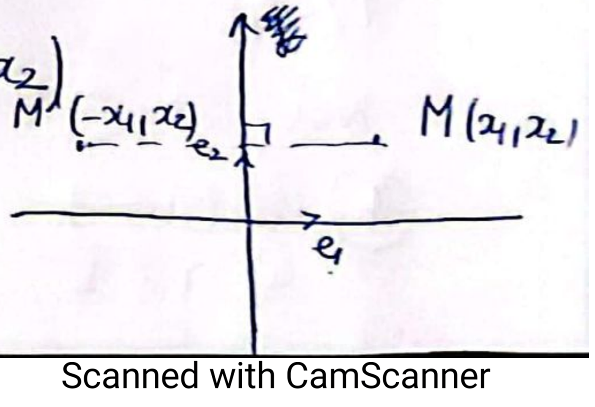
rotație de unghi orientat φ



b) $\det A = -1$, $\exists$ un reper $R=\{e_1,e_2\}$ ortonormat aî $[f]_{R,R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x_1,x_2) = (-x_1, x_2)$

f = simetrie ortogonală față de $\langle\{e_1\}\rangle^{\perp} = \langle\{e_2\}\rangle$



Teoremă $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $\dim E = n = 2$

Dacă $f \in O(E)$, $f \neq id_E$, atunci f se scrie ca o compunere de cel mult 2 simetrii ortogonale (față de drepte)

Dem.

1) $f \in O(E)$, de speță 1 i.e $[f]_{R,R} = A_f \in SO(2)$

$\det(A_f) = 1$.

Fie $s' \in O(E)$ simetrie ortogonală i.e $\det A_{s'} = -1$.

$s' \circ f \in O(E)$ $\det(A_{s' \circ f}) = -1$. $\Rightarrow$

$s' \circ f = s \Rightarrow f = s' \circ s$ $(s' \circ s' = id_E)$

2) $f \in O(E)$, de speță 2 i.e. $\det(A_f) = -1$

$\Rightarrow f = s$ simetrie ortogonală.

3) $\boxed{n = 3}$, $f \in O(E)$, $A \in O(3)$

$P(\lambda) = \det(A - \lambda I_3)$

polinom de grd al 3-lea cu coef reali $\Rightarrow$ cel puțin o răd $\lambda \in \mathbb{R}$ $(\lambda = \pm 1)$.

λ val. proprie și e_1 = versor propriu coresp. valorii proprii λ.

$f(e_1) = \lambda e_1 \Rightarrow \langle \{e_1\} \rangle$ subsp. invar. al lui f.

$\Rightarrow \langle \{e_1\} \rangle^{\perp}$ este subsp. invariant.

$f/_{\langle \{e_1\} \rangle^{\perp}} : \langle \{e_1\} \rangle^{\perp} \longrightarrow \langle \{e_1\} \rangle^{\perp}$ transf. ortog în dim 2

Not $\tilde{A}$ matricea asociată

A) $\det A = 1$

a) $\lambda = 1$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{A}$}} \\ 0 & & \end{pmatrix}$$

$\det \tilde{A} = 1$ $\quad \tilde{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

b) $\lambda = -1$ $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ 0 & & \tilde{A} \end{pmatrix}$

$\det \tilde{A} = -1$

$\exists$ un reper $\{e_2, e_3\}$ a.î. $\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathcal{R} = \{e_1, e_2, e_3\}$: $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathcal{R}' = \{e_3, e_1, e_2\}$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$

<u>Teoremă</u> $\dim E = 3$

$f \in O(E)$ de speța 1

$\Rightarrow \exists$ un reper ortonormat $\mathcal{R} = \{e_1, e_2, e_3\}$ a.î.

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} = A$$

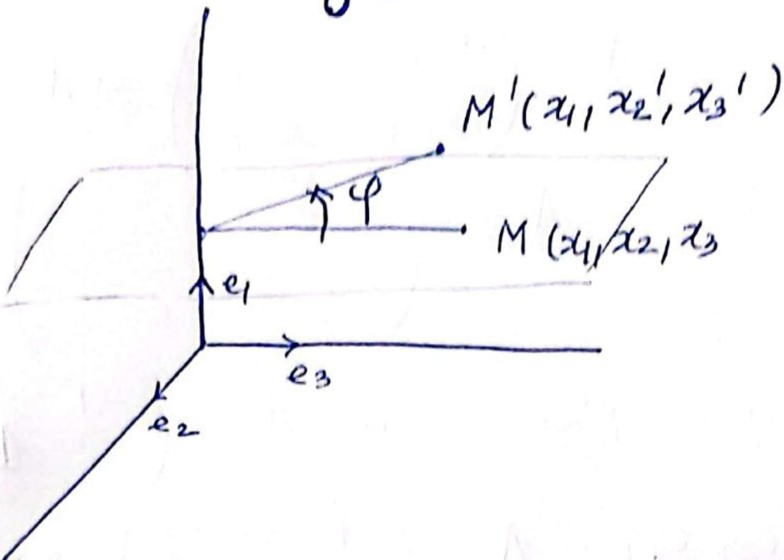
$f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = (x_1, \underbrace{x_2\cos\varphi - x_3\sin\varphi}_{x_1'}, \underbrace{x_2\sin\varphi + x_3\cos\varphi}_{x_2'})$$

<u>OBS</u> a) $\operatorname{Tr} A = 1 + 2\cos\varphi$. invariant la sch. de repere ortonormate.

b) $f \overset{R\varphi}{=}$ este o rotație de unghi orientat φ în planul $\langle\{e_1\}\rangle^{\perp}$ și axă $\langle\{e_1\}\rangle$.

<u>AXA</u>: $x \in \langle\{e_1\}\rangle$ $\quad f(x) \overset{?}{=} x$



(B) $\det A = -1$.

a) $\lambda = 1$ $\qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix}$

$\det \tilde{A} = -1$ $\quad \exists \{e_2, e_3\}$ ai $\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\{e_1, e_2, e_3\}$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\{e_2, e_1, e_3\}$: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

b) $\lambda = -1$ $\qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$\det \tilde{A} = 1$

$\tilde{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

<u>Teoremă</u> $\dim E = 3$, $f \in O(E)$ de speță 2 $\Rightarrow$
$\exists$ un reper $R = \{e_1, e_2, e_3\}$ reper ortonormat ai

$$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = (-x_1, \overbrace{x_2\cos\varphi - x_3\sin\varphi}^{x_2'}, \overbrace{x_2\sin\varphi + x_3\cos\varphi}^{x_3'})$$

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<u>OBS</u> a) $\mathrm{Tr}\, A = -1 + 2\cos\varphi$ invariant la sch. de repere ortonormate.

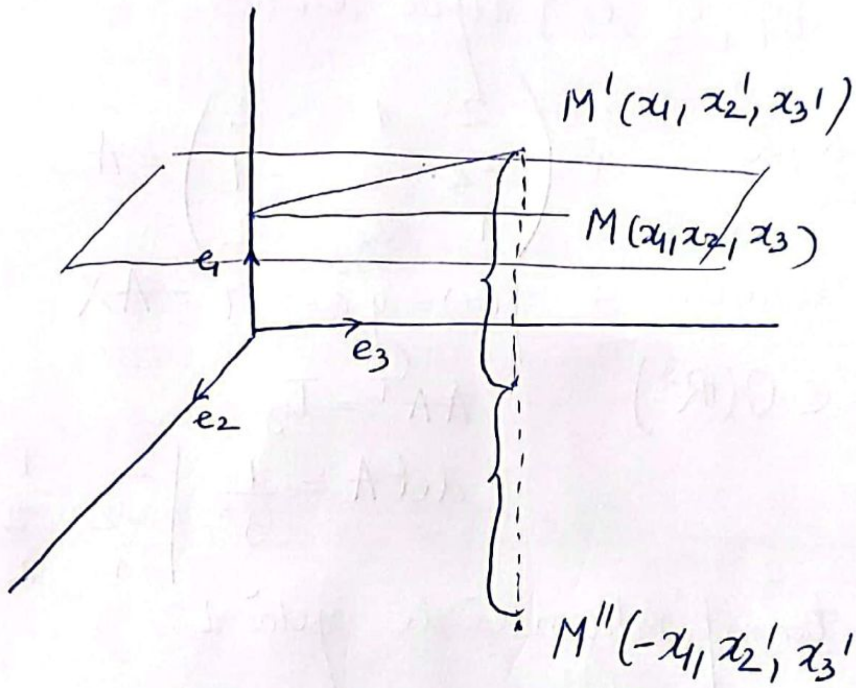
b) $f = s \circ R_\varphi$.

R_φ = rotatie de unghi orientat φ în $<\{e_1\}>^\perp$

s = simetrie ortogonală față de $<\{e_1\}>^\perp$

AXA : $f(x) = -x$

$\overset{\shortparallel}{<\{e_1\}>}$



④ $\dim E \geq 4$

$\exists$ un reper ortonormat R a.î.

$$[f]_{R,R} = A = \begin{pmatrix} \overbrace{1 \cdots 1}^{r} & & & & & \\ & \overbrace{-1 \cdots -1}^{k-r} & & & O & \\ & & A_1 & \cdots & & \\ & O & & & A_j & \end{pmatrix}$$

$k + 2j = n$

$$A_K = \begin{pmatrix} \cos\varphi_K & -\sin\varphi_K \\ \sin\varphi_K & \cos\varphi_K \end{pmatrix}$$

$K = \overline{1,j}$

<u>T. Cartan</u> $n \geq 2$

$f \in O(E)$, $f \neq id_E$

f se scrie ca o compunere de cel mult n simetrii ortogonale (față de hiperplane).

Aplicatie 2 $(\mathbb{R}^3, g_0)$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = \frac{1}{3}(2x_1 + x_2 - 2x_3,\ -2x_1 + 2x_2 - x_3,\ x_1 + 2x_2 + 2x_3)$

a) $f \in O(E)$ $E = \mathbb{R}^3$

b) $\exists$ un reper ortonormat $R = \{e_1', e_2', e_3'\}$ aî

$$[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

SOL

a) $R_0 = \{e_1^0, e_2^0, e_3^0\}$ reper canonic în $\mathbb{R}^3$

$$[f]_{R_0,R_0} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix} = A$$

f liniară : $f(x) = y \iff Y = AX$

$f \in O(\mathbb{R}^3)$: $AA^T = I_3$

$$\det A = \frac{1}{3^3}\begin{vmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 1.$$

f transf. ortogonală de speță 1.

b) Axa $f(X) = X$.

$$\begin{cases} \frac{1}{3}(2x_1 + x_2 - 2x_3) = x_1 \\ \frac{1}{3}(-2x_1 + 2x_2 - x_3) = x_2 \\ \frac{1}{3}(x_1 + 2x_2 + 2x_3) = x_3 \end{cases} \Rightarrow \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ -2x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0. \end{cases}$$

$$B = \left(\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ -2 & -1 & -1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right)$$

$\det B = 0.$

$rg\, B = 2$

$$\begin{cases} -x_1 + x_2 = 2x_3 \\ -2x_1 - x_2 = x_3 \end{cases}$$

$-3x_1 \quad / = 3x_3$

$x_1 = -x_3$

$x_2 = 2x_3 - x_3 = x_3$

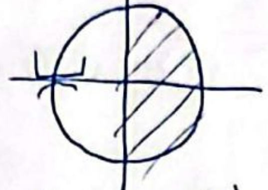
$(x_1, x_2, x_3) = (-x_3, x_3, x_3) =$

$= x_3(-1, 1, 1)$

$e_1' = \frac{1}{\sqrt{3}}(-1,1,1)$ versorul axei

$Tr A = \frac{1}{3}\cdot 6 = 2 = 1+2\cos\varphi \Rightarrow 2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2}$.

$\varphi \in (-\pi,\pi]$



$\varphi \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$.

$\langle\{e_1\}\rangle^{\perp} = \{x\in\mathbb{R}^3 \mid \underbrace{g_0(x,(-1,1,1))}_{-x_1+x_2+x_3} = 0\}$

$= \{(\underbrace{x_2+x_3, x_2, x_3}_{x_2(1,1,0)+x_3(1,0,1)}) \mid x_2,x_3\in\mathbb{R}\} = \langle\{\underbrace{(1,1,0)}_{f_2}, \underbrace{(1,0,1)}_{f_3}\}\rangle$.

$\dim \langle\{e_1\}\rangle^{\perp} = 2$

$\{f_2, f_3\}$ reper arbitrar în $\langle\{e_1\}\rangle^{\perp}$

Aplicăm Gram-Schmidt.

$e_2 = f_2 = (1,1,0)$

$e_3 = f_3 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2 = (1,0,1) - \frac{1}{2}(1,1,0)$

$= \left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \frac{1}{2}(1,-1,2)$

$e_2' = \frac{1}{\sqrt{2}}(1,1,0)$, $e_3' = \frac{1}{\sqrt{6}}(1,-1,2)$

$\left\{\frac{1}{\sqrt{3}}(-1,1,1), \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{6}}(1,-1,2)\right\}$ reperul ortonormat cerut.

<u>Prop</u> $(E, \langle,\rangle)$ s.v.e.r, $U \subseteq E$ ssp vect $\Rightarrow$

$E = U \oplus U^{\perp}$ (scriere unică).

<u>Dem</u>.

$U + U^{\perp} \subseteq E$ subsp. vect

Fie $x \in U \cap U^{\perp} \Rightarrow \overset{Q(x)}{g(x,x)} = 0 \to x = 0_E \Rightarrow U \oplus U^{\perp} \subseteq E$

$\dim U = k$. Fie $\mathcal{R}' = \{e_1, \dots, e_k\}$ reper în U.

Dem că $E \subseteq U \oplus U^\perp$.

Fie $v \in E$.

$$v' = v - \sum_{i=1}^{k} \langle v, e_i \rangle e_i$$. Arătăm că $v' \in U^\perp$.

$$\langle v', e_j \rangle = \langle v, e_j \rangle - \underbrace{\sum_{i=1}^{k} \langle v, e_i \rangle \underset{\delta_{ij}}{\langle e_i, e_j \rangle}}_{\langle v, e_j \rangle} = 0, \ \forall j = \overline{1,k}$$

$\forall x \in U$, $x = x_1 e_1 + \dots + x_k e_k$

$$\langle v', x \rangle = \sum_{j=1}^{k} x_j \langle v', e_j \rangle = 0 \Rightarrow v' \in U^\perp$$

$$v = \underset{U^\perp}{v'} + \underset{U}{v''}, \quad v'' = \sum_{i=1}^{k} \langle v, e_i \rangle e_i$$

$$E = U \oplus U^\perp$$

<u>Aplicație</u>

$$U = \{x \in \mathbb{R}^4 \mid \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases}$$

$$\left(\left. \begin{array}{cccc} \boxed{\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}} & \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \end{array} \right) \right| \begin{array}{c} 0 \\ 0 \end{array}$$

a) $U^\perp$; b) Să se afle $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper ortonormat în $\mathbb{R}^4$ aî $\mathcal{R}_1$ reper ortonormat în U

$\mathcal{R}_2$ —"— $U^\perp$

<u>SOL</u> $\dim U = 4 - 2 = 2$

$$U = \{ \underset{x_1(1,0,-1,1) + x_2(0,1,1,1)}{(x_1, x_2, -x_1 + x_2, x_1 + x_2)}, \ x_1, x_2 \in \mathbb{R} \} = \langle \{f_1, f_2\} \rangle$$

$f_1 = (1,0,-1,1)$, $f_2 = (0,1,1,1)$

$$U^{\perp} = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} g_0(x, f_1) = 0 \\ g_0(x, f_2) = 0 \end{cases} \right\} = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x_1 - x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{matrix} \right\}$$

$$= \left\{ (x_3 - x_4, -x_3 - x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{R} \right\}$$

$$\underbrace{x_3 (1,-1,1,0) + x_4 (-1,-1,0,1)}$$

$$(1,-1,1,0) = f_3, \quad (-1,-1,0,1) = f_4.$$

R: $\{f_1, f_2\}$ reper ortog în U

$\{f_3, f_4\}$ —''— $U^{\perp}$

$$\mathcal{R} = \left\{ \frac{1}{\sqrt{3}}(1,0,-1,1), \frac{1}{\sqrt{3}}(0,1,1,1), \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{\sqrt{3}}(-1,-1,0,1) \right\}$$

reperul cerut.