25.02.2022

CURS 2 - GAL

Spații vectoriale
Sisteme liniar independente/dependente
Sisteme de generatori. Baze

Def $(\mathbb{K},+,\cdot)$ corp comutativ. Fie V mulțime nevidă.
V are o structură de spațiu vectorial peste corpul $\mathbb{K}$
$\Leftrightarrow \exists\ + : V\times V \longrightarrow V$ (lege internă)
$\cdot : \mathbb{K}\times V \longrightarrow V$ (lege externă)
care verifică axiomele:

1) $(V,+)$ grup abelian
2) $a\cdot(b\cdot x) = (a\cdot b)\cdot x$
3) $(a+b)\cdot x = a\cdot x + b\cdot x$
4) $a\cdot(x+y) = a\cdot x + a\cdot y$
5) $1_{\mathbb{K}}\cdot x = x$, $\forall\, a,b\in\mathbb{K}$ (scalari)
$\forall\, x,y\in V$ (vectori)

Notăm $(V,+,\cdot)/_{\mathbb{K}}$

OBS
a) $0_{\mathbb{K}}\cdot x = 0_V$
b) $a\cdot 0_V = 0_V$
c) $(a-b)\cdot x = a\cdot x - b\cdot x$
d) $a\cdot(x-y) = a\cdot x - a\cdot y$, $\forall\, a,b\in\mathbb{K}$, $\forall\, x,y\in V$.

Exemple
1) $(\mathcal{V},+,\cdot)/\mathbb{R}$ spațiul vectorilor liberi
2) $(\mathbb{K},+,\cdot)$ corp $\Rightarrow (\mathbb{K},+,\cdot)/_{\mathbb{K}}$ sp. vectorial.
Cazuri particulare: $(\mathbb{R},+,\cdot)/\mathbb{R}$; $(\mathbb{C},+,\cdot)/\mathbb{C}$, $(\mathbb{Z}_p,+,\cdot)/\mathbb{Z}_p$.

3) $(\mathbb{K},+,\cdot)$ corp
$\mathbb{K}' \subset \mathbb{K}$ subcorp. $\Rightarrow (\mathbb{K},+,\cdot)/_{\mathbb{K}'}$ sp. vect.

Cazuri particulare: $(\mathbb{R},+,\cdot)/_{\mathbb{Q}}$; $(\mathbb{C},+,\cdot)/_{\mathbb{R}}$; $(\mathbb{C},+,\cdot)/_{\mathbb{Q}}$.

4) $(V_1,\oplus,\odot)/_{\mathbb{K}}$, $(V_2,\boxplus,\boxdot)/_{\mathbb{K}}$ sp. vect $\rightarrow (V_1\times V_2,+,\cdot)/_{\mathbb{K}}$

$+ : (V_1\times V_2)\times(V_1\times V_2) \longrightarrow V_1\times V_2$.

$(x_1,y_1)+(x_2,y_2)=(x_1\oplus x_2,\ y_1\boxplus y_2)$

$\cdot\ : \mathbb{K}\times(V_1\times V_2) \longrightarrow V_1\times V_2$

$a\cdot(x,y)=(a\odot x,\ a\boxdot y)$, $\forall (x_1,y_1),(x_2,y_2),(x,y)\in V_1\times V_2$
$\forall a\in\mathbb{K}$.

Caz particular

$(\mathbb{R},+,\cdot)/_{\mathbb{R}} \Rightarrow (\mathbb{R}^n,+,\cdot)/_{\mathbb{R}}$.

$(x_1,..,x_n)+(y_1,..,y_n)=(x_1+y_1,..,x_n+y_n)$

$a(x_1,..,x_n)=(ax_1,..,ax_n)$

$\forall (x_1,..,x_n),(y_1,..,y_n)\in\mathbb{R}^n$, $\forall a\in\mathbb{R}$.

5) $(\mathcal{M}_{m,n}(\mathbb{K}),+,\cdot)/_{\mathbb{K}}$ sp. vect

$(a_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \longrightarrow (a_{11},..,a_{1n},a_{21},..,a_{2n},..,a_{m1},..,a_{mn}) \in \mathbb{R}^{m\cdot n}$.

6) $(\mathbb{K}[X],+,\cdot)/_{\mathbb{K}}$ sp. vect al polinoamelor.

$P=a_0+a_1X+\ldots+a_nX^n \Longleftrightarrow (a_0,a_1,..,a_n)\in\mathbb{K}^{n+1}$

7) $I=[a,b]$, $a<b$.

$(\mathcal{C}(I)=\{f:I\rightarrow\mathbb{R} \mid f \text{ cont}\},+,\cdot)/_{\mathbb{R}}$ sp. vect

$\mathcal{D}(I)=\{f:I\rightarrow\mathbb{R} \mid f \text{ derivab}\},+,\cdot)/_{\mathbb{R}}$ sp vect.

$\mathcal{P}(I)=\{f:I\rightarrow\mathbb{R} \mid f \text{ primitivabile}\},+,\cdot)/_{\mathbb{R}}$ sp vect

$\mathcal{I}(I)=\{f:I\rightarrow\mathbb{R} \mid f \text{ integrabila}\},+,\cdot)/_{\mathbb{R}}$ sp vect.

Def $(V,+,\cdot)/_{\mathbb{K}}$ sp vect, $V' \subseteq V$ subm. nevidă.
V' s.n. <u>subspațiu vectorial</u> $\Leftrightarrow$ subm. este închisă la adunarea vect. și la „$\cdot$" cu scalari
i.e. $\forall x, y \in V' \Rightarrow x+y \in V'$
$\forall x \in V', \forall a \in \mathbb{K} \Rightarrow a \cdot x \in V'$

OBS $V' \subset V$ subsp vect $\Rightarrow (V',+,\cdot)/_{\mathbb{K}}$ sp. vect. (cu op. induse).

Prop $(V,+,\cdot)/_{\mathbb{K}}$ sp vect.
$V' \subset V$ subsp. vect $\Leftrightarrow [\forall a, b \in \mathbb{K}, \forall x, y \in V' \Rightarrow ax+by \in V']$
$\Leftrightarrow [\forall a_i \in \mathbb{K}, x_i \in V', i=\overline{1,n} \Rightarrow \sum_{i=1}^{n} a_i x_i \in V']$

Dem
„$\Rightarrow$" Ip: V' subsp. vect.
Fie $a \in \mathbb{K}, x \in V' \Rightarrow ax \in V'$
Fie $b \in \mathbb{K}, y \in V' \Rightarrow by \in V'$ $\Rightarrow ax+by \in V'$
„$\Leftarrow$" $\forall a, b \in \mathbb{K}, \forall x, y \in V' : ax+by \in V'$
Fie $a = b = 1_{\mathbb{K}}$
$1_{\mathbb{K}} x + 1_{\mathbb{K}} \cdot y = x+y \in V'$
Fie $b = 0_{\mathbb{K}}$
$ax + 0_{\mathbb{K}} \cdot y = a \cdot x + 0_V = a \cdot x \in V' \Rightarrow V'$ subsp. vect.

Exemple
1) $(V,+,\cdot)/_{\mathbb{K}}$, $\{0_V\}$, V subsp. vect.
2) $n < m$, $m \geq 2$ $\mathbb{R}^n \subset \mathbb{R}^m$ subsp vect.
3) $(\mathcal{M}_n(\mathbb{R}),+,\cdot)/_{\mathbb{R}}$.

a) $V' = \{ A = \mathrm{diag}(a_1, .., a_n) \in \mathcal{M}_n(\mathbb{R}) \}$.

b) $V'' = \{ A \in \mathcal{M}_n(\mathbb{R}) \mid Tr(A) = 0 \}$.

c) $V''' = \{ A \in \mathcal{M}_n(\mathbb{R}) \mid A = A^T \} = \mathcal{M}_n^s(\mathbb{R})$

d) $V''' = \{ A \in \mathcal{M}_n(\mathbb{R}) \mid A = -A^T \} = \mathcal{M}_n^a(\mathbb{R})$

OBS

$GL(n, \mathbb{R})$
$O(n)$ $\subset \mathcal{M}_n(\mathbb{R})$ nu sunt subsp. vect.
$SO(n)$

4) $W = \{ (x, y) \in \mathbb{R}^2 \mid ax + by = 0, a^2 + b^2 > 0 \} \subset \mathbb{R}^2$ ssp vect

$W' = \{ (x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0, a^2 + b^2 + c^2 > 0 \} \subset \mathbb{R}^3$ ssp vect

$W'' = \{ (x_1, .., x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i x_i = 0, \sum_{i=1}^{n} a_i^2 > 0 \} \subset \mathbb{R}^n$.

(hiperplan care trece prin 0)

$U = S(A) = \{ (x_1, .., x_n) \in \mathbb{R}^n \mid \underset{(m,n)}{A}\,\underset{(n,1)}{X} = \underset{(m,1)}{0} \} \subset \mathbb{R}^n$ ssp vect.

(mulț. sol. unui SLO)

($\cap$ a m hiperplane)

Subspațiul vectorial generat de o mulțime

Def $(V, +, \cdot)/_{\mathbb{K}}$ sp. vect, $S \subset V$ subm $\neq \phi$

$$\langle S \rangle = \{ x \in V \mid x = \sum_{i=1}^{n} a_i x_i, \ a_1, .., a_n \in \mathbb{K}, \ x_1, .., x_n \in S \}$$

(combinații liniare finite de vectori din S cu scalari din $\mathbb{K}$).

OBS Dc $V = \langle S \rangle$, atunci S s.n. sistem de generatori. (SG)

V s.n. spatiu vect. finit generat, dacă
$\exists$ S finită aî $V = \langle S \rangle$

OBS

a) $S \subset \langle S \rangle$

b) $\langle S \rangle$ = cel mai mic subsp. vect al lui V, care conține S

c) Convenție $\langle \emptyset \rangle = \{0_V\}$.

Def $(V, +, \cdot)/\mathbb{K}$ sp vect, $S \subset V$ subm. nevidă.

1) S s.n. sistem liniar independent (SLI) $\Leftrightarrow$
$\forall a_1, .., a_n \in \mathbb{K}$
$\left[\forall x_1, .., x_n \in S\right.$ aî $\sum_{i=1}^{n} a_i x_i = 0_V \Rightarrow a_1 = \ldots = a_n = 0_{\mathbb{K}}\left.\right]$
($\forall$ combinație liniară nulă este trivială).

2) S s.n. sistem liniar dependent (SLD) $\Leftrightarrow$
$\exists x_1, .., x_n \in S$
$\exists a_1, .., a_n \in \mathbb{K}$, nu toți nuli aî $\sum_{i=1}^{n} a_i x_i = 0_V$

Prop $(V, +, \cdot)/\mathbb{K}$, $x \in V$, $x \neq 0_V$ $\Rightarrow \{x\}$ este SLI

Dem
Fie $a \in \mathbb{K}$ aî $a \cdot x = 0_V$.
Pp. prin absurd că $a \neq 0_{\mathbb{K}} \Rightarrow \exists a^{-1} \in \mathbb{K}$

$\underbrace{a^{-1} \cdot a}_{1_{\mathbb{K}}} \cdot x = a^{-1} \cdot 0_V \Rightarrow x = 0_V$ Contrad.

Pp. este falsă $\Rightarrow a = 0_{\mathbb{K}} \Rightarrow \{x\}$ SLI

Def $(V,+,\cdot)_{/\mathbb{K}}$ sp vect, $S \subseteq V$ subm nevidă

S sn. bază $\Leftrightarrow$ $\begin{cases} 1) \text{ S este SLI} \\ 2) \text{ S este SG} \end{cases}$

Exemple

1) $(\mathbb{R}^n,+,\cdot)_{/\mathbb{R}}$, $B_0 = \{e_1 = (1,0,..,0), e_2 = (0,1,0,..,0),.., e_n = (0,..,0,1)\}$

B_0 este bază canonică

a) B_0 este SLI

Fie $a_1,..,a_n \in \mathbb{R}$ a.î. $a_1 e_1 + ... + a_n e_n = (0,..,0)$

$a_1(1,0,..0) + a_2(0,1,0,.., 0) + ... + a_n(0,..,0,1) = (0,..,0)$

$(a_1,..,a_n) = (0,..,0) \Rightarrow a_i = 0, \forall i = \overline{1,n} \Rightarrow B_0$ este SLI

b) B_0 este SG.

$$\forall \underset{\mathbb{R}^n}{\underset{\cap}{x}} = (x_1,..,x_n) = (x_1,0,..,0) + ... + (0,..,0,x_n)$$
$$= x_1(1,0,..,0) + ... + x_n(0,..,0,1)$$
$$= x_1 e_1 + .. + x_n e_n.$$

$x \in \langle B_0 \rangle \Rightarrow B_0$ este SG $\quad x_1,..,x_n \in \mathbb{R}$

Deci B_0 bază

2) $(\mathbb{K}[X],+,\cdot)_{/\mathbb{K}}$, $B_0 = \{1, X, X^2, ...\}$ bază

nu este sp. vect finit generat

$(\mathbb{K}_n[X],+,\cdot)_{/\mathbb{K}}$, $\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] \mid \text{grad } P \leq n\}$

$B_0 = \{1, X, X^2, .., X^n\}$ bază canonică

- SLI $\quad a_0 + a_1 X + ... + a_n X^n = 0 \Leftrightarrow a_0 = ... = a_n = 0_{\mathbb{K}}$
- SG $\quad \forall \underset{\mathbb{K}_n[X]}{\underset{\cap}{P}} = a_0 + a_1 X + .. + a_n X^n \in \langle B_0 \rangle$

Deci B_0 este bază

3) $(\mathcal{M}_{m,n}(\mathbb{K}), +, \cdot)/\mathbb{K}$.

$$B_0 = \left\{ E_{ij} = i\begin{pmatrix} 0 & \overset{j}{\vdots} & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \right\}_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$$ baza canonică

<u>OBS</u>

a) $\forall$ subm $\neq \emptyset$ a unui SLI este un SLI

b) $\forall$ supramulțime a unui SLD este un SLD.

c) $\forall$ supramulțime a unui SG este un SG.

<u>Teorema schimbului</u>

Fie $(V, +, \cdot)/\mathbb{K}$ sp. vect finit generat

Fie $\{x_1, .., x_n\}$ SG
$\{y_1, .., y_n\}$ SLI $\Rightarrow \{y_1, .., y_n\}$ este SG.

<u>Dem</u>.

$V = \langle \{x_1, .., x_n\} \rangle \Rightarrow \exists a_1, .., a_n \in \mathbb{K}$ a.î. $y_1 = a_1 x_1 + ... + a_n x_n$

$\ni$
y_1 Pp. prin abs. $a_1 = .. = a_n = 0_{\mathbb{K}} \Rightarrow y_1 = 0_V$

$\underset{\overset{\|}{y_1}}{1_{\mathbb{K}} \cdot 0_V} + 0_{\mathbb{K}} \cdot y_2 + ... + 0_{\mathbb{K}} y_n = 0_V \Rightarrow \{y_1, .., y_n\}$ SLD ✗.

Pp. $a_1 \neq 0_{\mathbb{K}}$ (eventual renumerotăm)

$y_1 = a_1 x_1 + ... + a_n x_n. \Rightarrow x_1 = a_1^{-1}(y_1 - a_2 x_2 - ... - a_n x_n)$

$V = \langle \{x_1, .., x_n\} \rangle = \langle \{y_1, x_2, x_3, .., x_n\} \rangle$

$\ni$
y_2 $\exists\, b_1, a_2, .., a_n \in \mathbb{K}$ a.î.

$y_2 = b_1 y_1 + \underline{a_2 x_2 + ... + a_n x_n}$

Pp. prin abs $a_2 = .. = a_n = 0_{\mathbb{K}} \Rightarrow y_2 = b_1 y_1$

$$b_1 y_1 - 1_{\mathbb{K}} \cdot y_2 + 0_{\mathbb{K}} y_3 + \dots + 0_{\mathbb{K}} y_n = 0_V \Rightarrow \{y_1, \dots, y_n\} \text{ SLD abs}$$

$$1_{\mathbb{K}} \neq 0_{\mathbb{K}}$$

Considerăm $a_2 \neq 0_{\mathbb{K}}$ (eventual renumerotăm)

$y_2 = b_1 y_1 + a_2 \underline{x_2} + a_3 x_3 + \dots + a_n x_n$.

$x_2 = a_2^{-1} (y_2 - b_1 y_1 - a_3 x_3 - \dots - a_n x_n)$

$V = \langle \{x_1, \dots, x_n\} \rangle = \langle \{y_1, x_2, \dots, x_n\} \rangle = \langle \{y_1, y_2, x_3, \dots, x_n\} \rangle$

Analog, după un nr finit de pași $\Rightarrow$

$V = \langle \{y_1, \dots, y_n\} \rangle \Rightarrow \{y_1, \dots, y_n\}$ este SG.

<u>Prop</u>

Card $\forall$ SG (finit) $\geq$ card $\forall$ SLI (finit)

<u>Dem</u>

Fie $\{x_1, \dots, x_n\}$ SG.

Fie $\{y_1, \dots, y_{n+1}\}$. Dem că $\{y_1, \dots, y_{n+1}\}$ este SLD.

1) $\{y_1, \dots, y_n\}$ SLI $\xRightarrow{\text{Th. Sch}}$ $\{y_1, \dots, y_n\}$ SG.

$\underset{y_{n+1}}{\overset{\cup\!\!\!\!\;\;}{V}} = \langle \{y_1, \dots, y_n\} \rangle \Rightarrow \exists a_1, \dots, a_n \in \mathbb{K}$ a.î. $y_{n+1} = a_1 y_1 + \dots + a_n y_n$.

$a_1 y_1 + \dots + a_n y_n - 1_{\mathbb{K}} \cdot y_{n+1} = 0_V \Rightarrow$

$\{y_1, \dots, y_n, y_{n+1}\}$ SLD.

2) $\{y_1, \dots, y_n\}$ SLD. $\Rightarrow \{y_1, \dots, y_n, y_{n+1}\}$ SLD

($\forall$ supram a unui SLD este SLD)

Teoremă $(V,+,\cdot)_{/\mathbb{K}}$ sp. vect finit generat.

Dacă B_1, B_2 sunt baze, atunci $card\, B_1 = card\, B_2 = n$

$n = \dim_{\mathbb{K}} V$ (dimensiunea sp. vect)

Dem

1) B_1 este SG
B_2 este SLI $\xRightarrow{Prop} |B_1| \geq |B_2|$

$\Rightarrow |B_1| = |B_2| = n$.

2) B_2 este SG
B_1 este SLI $\xRightarrow{Prop} |B_2| \geq |B_1|$

OBS $(V,+,\cdot)_{/\mathbb{K}}$, $\dim_{\mathbb{K}} V = n$.

$B = \{v_1, \dots, v_n\} \subset V$ sistem de vectori

UAE
1) B bază.
2) B este SLI
3) B este SG.

OBS $n = \dim_{\mathbb{K}} V$ = nr. max de vectori care formează SLI
= nr minim de vectori —"— SG

OBS
a) $\forall$ SLI (finit) se poate completa la o bază

b) Din $\forall$ SG (finit) ~~se poate~~, care conține cel puțin un vector nenul, se poate extrage o bază.