Curs 1

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• Adunarea matricelor - exemplu

Fie $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 5 & 7 \\ -3 & 3 \\ 2 & 0 \end{pmatrix}$; $A, B \in \mathcal{M}_{3,2}(\mathbb{R})$

$$A+B = \begin{pmatrix} 2+5 & -1+7 \\ 0-3 & 2+3 \\ 3+2 & 1+0 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ -3 & 5 \\ 5 & 1 \end{pmatrix}$$

• Înmulțirea matricelor cu scalari - exemplu

$$\alpha \in \mathbb{R} \ ; \ A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix} . \quad \alpha \cdot A = \begin{pmatrix} \alpha \cdot 2 & \alpha \cdot (-1) \\ \alpha \cdot 0 & \alpha \cdot 2 \\ \alpha \cdot 3 & \alpha \cdot 1 \end{pmatrix} = \begin{pmatrix} 2\alpha & -\alpha \\ 0 & 2\alpha \\ 3\alpha & \alpha \end{pmatrix}$$

• Înmulțirea a două matrice

$$A \in \mathcal{M}_{m,n}(\mathbb{R}) \ ; \ B \in \mathcal{M}_{n,p}(\mathbb{R}) \ ; \ A \cdot B \in \mathcal{M}_{m,p}(\mathbb{R})$$

$(A \cdot B)_{i,j} = L_i(A) \cdot C_j(B)$ unde $L_i(A)$ este linia i a matricii A și $C_j(B)$ este coloana j a matricii B.

$1 \le i \le m$

$1 \le j \le p$ $\quad L_i(A) \in \mathcal{M}_{1,n}(\mathbb{R})$; $C_j(B) \in \mathcal{M}_{n,1}(\mathbb{R})$.

$$L_i(A) \cdot C_j(B) = \sum_{k=1}^{n} a_{ik} b_{kj}$$

<u>Exemplu</u>: • Fie $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ $A \cdot B \in \mathcal{M}_{3,1}(\mathbb{R})$.

$$A \cdot B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 7 + (-1) \cdot (-3) \\ 0 \cdot 7 + 2 \cdot (-3) \\ 3 \cdot 7 + 1 \cdot (-3) \end{pmatrix} = \begin{pmatrix} 17 \\ -6 \\ 18 \end{pmatrix}$$

• Ptr. $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & 0 \end{pmatrix}$ $A \cdot B \in \mathcal{M}_{3,3}(\mathbb{R})$; $B \cdot A \in \mathcal{M}_{2,2}(\mathbb{R})$

$$A \cdot B = \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 5 & -4 \\ 4 & -2 & 0 \\ 5 & 5 & -6 \end{pmatrix}$$

Cele două produse nu au nici măcar aceeași formă

$$B \cdot A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}$$ deci nu pot fie egale.

Proprietățile determinanților (vom demonstra numai o parte dintre proprietățile enunțate.

2) dc $L_i(A) = (0\ 0\ 0 \dots 0) = (a_{i1}\ a_{i2}\ a_{i3} \dots a_{in}) \in \mathcal{M}_{1,n}(\mathbb{R})$ atunci $\det(A) = 0$

$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{i\sigma(i)} \dots a_{n\sigma(n)}$. În fiecare produs din expresia $\det(A)$ avem un element de pe linia i, anume $a_{i\sigma(i)} = 0$. Deci fiecare produs este $\operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots 0 \dots a_{n\sigma(n)} = 0$

Deci $\det(A) = \sum_{\sigma \in S_n} 0 = 0$.

(3) $L_i(B) = (\lambda a_{i1}\ \lambda a_{i2} \dots \lambda a_{in}) = \lambda L_i(A)$; $L_j(B) = L_j(A)$ ptr. $(\forall)$ $j \neq i$ (toate celelalte linii sunt identice în matricele A și B).

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \dots b_{i\sigma(i)} \dots b_{n\sigma(n)} =$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots (\lambda a_{i\sigma(i)}) \dots a_{n\sigma(n)} =$$

$$= \sum_{\sigma \in S_n} \lambda \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{i\sigma(i)} \dots a_{n\sigma(n)} =$$

$$= \lambda \cdot \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{i\sigma(i)} \dots a_{n\sigma(n)}.$$

(4) Avem $L_i(A) = (a_{i1}\ a_{i2} \dots a_{in}) = (b_{i1} + c_{i1}\ \ b_{i2} + c_{i2} \dots b_{in} + c_{in})$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{i\sigma(i)} \dots a_{n\sigma(n)} =$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots (b_{i\sigma(i)} + c_{i\sigma(i)}) \dots a_{n\sigma(n)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots b_{i\sigma(i)} \dots a_{n\sigma(n)} +$$

$$+ \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots c_{i\sigma(i)} \dots a_{n\sigma(n)} = \det(B) + \det(C),$$ unde:

$$B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ b_{i1} & \cdots & b_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \; ; \quad C = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ c_{i1} & \cdots & c_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

5) Avem $L_j(A) = \lambda L_i(A)$, unde $i<j$. Demonstrăm afirmația pentru $\lambda = 1$ și apoi aplicăm proprietatea 3).

Considerăm A_n subgrupul altern (al permutărilor pare) al lui S_n.

Din cursul de algebră se știe că $S_n = A_n \cup A_n \underset{\substack{\nearrow \\ \text{transpoziția } (ij)}}{(ij)} =$

$= \{\sigma \mid \sigma \in A_n\} \cup \{\sigma\cdot(ij) \mid \sigma \in A_n\}$.

$$\det(A) = \sum_{\sigma \in A_n \cup A_n(ij)} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \sum_{\sigma \in A_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} -$$

$\operatorname{sgn}(\sigma) = 1$
$(\forall)\, \sigma \in A_n$

$$- \sum_{\sigma \in A_n} a_{1\sigma(ij)(1)} \cdots a_{i\sigma(ij)(i)} \cdots a_{j\sigma(ij)(j)} \cdots a_{n\sigma(ij)(n)} =$$

$$\sigma\cdot(ij)(k) = \begin{cases} \sigma(k) & k \neq i,j \\ \sigma(j) & k = i \\ \sigma(i) & k = j \end{cases}$$

$$= \sum_{\sigma \in A_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} - \sum a_{1\sigma(1)} \cdots \underset{\substack{\parallel \\ a_{j\sigma(j)}}}{a_{i\sigma(j)}} \cdots \underset{\substack{\parallel \\ a_{i\sigma(i)}}}{a_{j\sigma(i)}} \cdots a_{n\sigma(n)} =$$

pentru că $L_j(A) = L_i(A)$

$$= \sum_{\sigma \in A_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} - \sum_{\sigma \in A_n} a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{j\sigma(j)} a_{n\sigma(n)} = 0.$$

6) Fie $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ și $B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

Considerăm

$$0 \underset{5)}{=} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1}+a_{j1} & \cdots & a_{i1}+a_{jn} \\ a_{i1}+a_{j1} & \cdots & a_{i1}+a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \underset{4)}{=}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{i1}+a_{j1} & \cdots & a_{in}+a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{j1} & \cdots & a_{jn} \\ a_{i1}+a_{j1} & \cdots & a_{in}+a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \underset{4)}{=}$$

$$\overset{4)}{=}\underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{i1} & \cdots & a_{in} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{\overset{5)}{=}0} + \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{\det(A)} + \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{j1} & \cdots & a_{jn} \\ a_{i1} & \cdots & a_{in} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{\det(B)} + \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{j1} & \cdots & a_{jn} \\ a_{j1} & \cdots & a_{jn} \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{\overset{5)}{=}0}$$

Deci $0 = \det(A) + \det(B) \Leftrightarrow \det(B) = -\det(A)$.

7) Rezultă din 4) și 5).

<u>Exemplu pentru formula Laplace</u> Fie $A \in \mathcal{M}_4(\mathbb{R})$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Voi scrie formula Laplace pentru mulțimea de indici $I = \{1,2\}$.

Conform formulei Laplace

$$\det(A) = \sum_{\substack{M \text{ minor} \\ \text{format pe liniile } 1,2 \text{ și} \\ \text{cu două coloane}}} M \cdot M'$$

unde M' este complementul algebric al minorului M.

Mulțimile J de două coloane sunt $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$ în număr de $C_4^2 = 6$.

$$\text{Deci } \det(A) = \underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}_{J=\{1,2\}} \cdot (-1)^{1+2+1+2} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + \underbrace{\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}}_{J=\{1,3\}} \cdot (-1)^{1+2+1+3} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ \underbrace{\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}}_{J=\{1,4\}} \cdot (-1)^{1+2+1+4} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + \underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}_{J=\{2,3\}} \cdot (-1)^{1+2+2+3} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} + \underbrace{\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}}_{J=\{2,4\}} \cdot (-1)^{1+2+2+4} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix}$$

$$+ \underbrace{\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}}_{J=\{3,4\}} \cdot (-1)^{1+2+3+4} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}.$$