CURS 3 - GAL

Repere. Coordonate în raport cu un reper. Subspații vectoriale.

Teoremă $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect. finit generat

$\forall B_1, B_2$ baze $\Rightarrow |B_1| = |B_2| = n = \dim_{\mathbb{K}} V$.

Def Fie $(V,+,\cdot)/_{\mathbb{K}}$ un sp vect finit generat

$\mathcal{R} = \{e_1, .., e_n\}$ bază

$\mathcal{R}$ s.n. reper $\Leftrightarrow$ este o bază ordonată

Prop $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect n-dim, $\mathcal{R} = \{e_1, .., e_n\}$ reper.

$\Rightarrow \forall x \in V$, $\exists! (x_1, .., x_n) \in \mathbb{K}^n$ (coordonatele sau componentele lui x în raport cu $\mathcal{R}$) a.î.

$x = x_1 e_1 + \ldots + x_n e_n$.

Dem $\mathcal{R}$ reper $\Rightarrow$ SG $\Rightarrow V = \langle \mathcal{R} \rangle$

$\Rightarrow \forall x \in V$, $\exists\ x_1, .., x_n \in \mathbb{K}$ a.î. $x = x_1 e_1 + \ldots + x_n e_n$.

Pp. prin absurd $\exists\ x'_1, .., x'_n \in \mathbb{K}$ a.î. $x = x'_1 e_1 + .. + x'_n e_n$.

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x'_i e_i \Rightarrow \sum_{i=1}^{n} (x_i - x'_i) e_i = 0_V.$$

$$\xRightarrow{\mathcal{R}\ SLI} x_i - x'_i = 0, \forall i = \overline{1,n}$$

$\Rightarrow$ scrierea $x = \sum_{i=1}^{n} x_i e_i$ este unică.

Modificarea coordonatelor la schimbarea reperului

$$\mathcal{R}=\{e_1,\dots,e_n\} \xrightarrow{A} \mathcal{R}'=\{e_1',\dots,e_n'\} \quad , A=(a_{ij})_{i,j=\overline{1,n}}$$

$$e_i'=\sum_{j=1}^{n} a_{ji}e_j \;,\; \forall i=\overline{1,n}$$

$$x=\sum_{i=1}^{n} x_i' e_i' = \underline{\sum_{j=1}^{n} x_j e_j} \quad (1)$$

$$\sum_{i=1}^{n} x_i' e_i' = \sum_{i=1}^{n} x_i' \left(\sum_{j=1}^{n} a_{ji}e_j\right) = \underline{\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji}x_i'\right)e_j} \quad (2)$$

Din (1), (2) $\overset{\mathcal{R}\,SLI}{\Longrightarrow} x_j=\sum_{i=1}^{n} a_{ji}x_i' \;,\; \forall j=\overline{1,n}$

$$\boxed{X=AX'} \quad X=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \;,\; X'=\begin{pmatrix}x_1'\\ \vdots\\ x_n'\end{pmatrix}$$

Prop $A\in GL(n,\mathbb{K})$

Dem

$$\begin{tikzcd} \mathcal{R}=\{e_1,\dots,e_n\} \arrow[r, "A"] \arrow[rr, bend right, "C"'] & \mathcal{R}'=\{e_1',\dots,e_n'\} \arrow[r, "B"] & \mathcal{R}''=\{e_1'',\dots,e_n''\} \end{tikzcd}$$

$C=AB$

$$e_i''=\underline{\sum_{k=1}^{n} c_{ki}e_k} \quad (*)$$

$$e_i''=\sum_{j=1}^{n} b_{ji}e_j' = \sum_{j=1}^{n} b_{ji}\left(\sum_{k=1}^{n} a_{kj}e_k\right)=$$

$$=\sum_{k=1}^{n}\underline{\sum_{j=1}^{n}(a_{kj}b_{ji})e_k} \quad (**)$$

$$(*),(**) \quad c_{ki}=\sum_{j=1}^{n} a_{kj}b_{ji} \Rightarrow C=AB$$

$$\mathcal{R}=\{e_1,..,e_n\} \xrightarrow{A} \mathcal{R}'=\{e_1',..,e_n'\} \xrightarrow{B} \mathcal{R}=\{e_1,..,e_n\}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{I_n}$$

$$\mathcal{R}'=\{e_1',..,e_n'\} \xrightarrow{B} \mathcal{R}=\{e_1,..,e_n\} \xrightarrow{A} \mathcal{R}'=\{e_1',..,e_n'\}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{I_n.}$$

$AB = BA = I_n \Rightarrow B = A^{-1}$

$A \in GL(n, \mathbb{K})$.

<u>Teoremă (criteriu de LI)</u>

$(V,+,\cdot)/\mathbb{K}$ sp vect n-dim.

Fie $S=\{v_1,..,v_m\}$ sistem de vectori din V, $m \le n$.

S este SLI $\iff$ rangul matricei componentelor vectorilor din S în raport cu $\forall$ reper din V este maxim i.e. m.

<u>Dem</u>: $\mathcal{R}=\{e_1,..,e_n\}$ reper în V, $V=\langle \mathcal{R}\rangle$.

$v_i = \sum_{j=1}^{n} v_{ji} e_j$, $\forall i=\overline{1,m}$ $\quad C=(v_{ji})_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}}$

S este SLI $\iff$

$[\forall a_1,..,a_m \in \mathbb{K}: \ a_1v_1+...+a_mv_m = 0_V \Rightarrow a_1=...=a_m=0_{\mathbb{K}}]$

$$\sum_{i=1}^{m} a_i v_i = 0_V \Rightarrow \sum_{i=1}^{m} a_i\left(\sum_{j=1}^{n} v_{ji} e_j\right) = 0_V \Rightarrow$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} v_{ji} a_i\right) e_j = 0_V \xrightarrow{\mathcal{R}\ SLI} \sum_{i=1}^{m} v_{ji} a_i = 0, \forall j=\overline{1,n}$$

$$(*)\begin{cases} v_{11}a_1 + v_{12}a_2 + ... + v_{1m}a_m = 0 \\ \vdots \\ v_{n1}a_1 + v_{n2}a_2 + ... + v_{nm}a_m = 0 \end{cases}$$

$(*)$ SLO de n ecuații cu m necunoscute: $a_1,..,a_m$.

(*) are sol unică nulă (SCD) $\Leftrightarrow rg\, C = n$ (maxim).

Dem că rangul nu depinde de reperul ales.

$$\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{A} \mathcal{R}' = \{e_1', .., e_n'\}.$$

A = matricea de trecere de la $\mathcal{R}$ la $\mathcal{R}'$.

$$e_i' = \sum_{j=1}^{n} a_{ji} e_j, \ \forall i = \overline{1,n}$$

$$i=\overline{1,m} \quad v_i = \sum_{k=1}^{n} v_{ki}' e_k' = \sum_{k=1}^{n} v_{ki}' \left(\sum_{j=1}^{n} a_{jk} e_j \right)$$

$$= \sum_{j=1}^{n} \underline{\left(\sum_{k=1}^{n} a_{jk} v_{ki}' \right)} e_j = \sum_{j=1}^{n} \underline{v_{ji}} e_j$$

$$v_{ji} = \sum_{k=1}^{n} a_{jk} v_{ki}' \Rightarrow C = AC'$$

$$C' = (v_{ki}')_{\substack{k=\overline{1,n} \\ i=\overline{1,m}}}$$

$rg(C) = rg(AC') = rg\, C' = n$ (este un invariant).

<u>Exemplu</u> $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $R_0 = \{e_1 = (1,0), e_2 = (0,1)\}$ reperul canonic.

Fie $\mathcal{R}' = \{(2,1) = e_1', (3,0) = e_2'\}$.

a) $\mathcal{R}'$ este un reper în $\mathbb{R}^2$

b) $R_0 \xrightarrow{A} \mathcal{R}'$, $\mathcal{R}' \xrightarrow{B} R_0$, $A, B = ?$

c) $x = (1,2)$

Să se afle coordonatele lui x în raport cu $R_0, \mathcal{R}'$

<u>SOL</u>

a), b) $e_1' = (2,1) = (2,0) + (0,1) = 2(1,0) + 1 \cdot (0,1) = \underline{2}e_1 + \underline{1} \cdot e_2$

$e_2' = (3,0) = 3e_1 + 0e_2 \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}, \ B = A^{-1}$

$$\left.\begin{array}{l} rg\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = 2 \underset{LI}{\overset{crit}{\Longrightarrow}} \mathcal{R}' \text{ este SLI} \\ \text{dar } \dim \mathbb{R}^2 = 2 = |\mathcal{R}'| \end{array}\right\} \Rightarrow \text{reper.}$$

c) $x = (1,2) = 1e_1 + 2e_2$

$(1,2)$ coord. în rap. cu R_0.

$$X = AX' \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$\begin{cases} 2x_1' + 3x_2' = 1 \Rightarrow x_2' = \frac{1-4}{3} = -1 \\ x_1' = 2 \end{cases}$$

$(x_1', x_2') = (2, -1)$. coord. în rap cu R_1'

<u>SAU</u>

$x = (1,2) = x_1'(2,1) + x_2'(3,0) = (2x_1' + 3x_2', x_1')$

$$\begin{cases} 2x_1' + 3x_2' = 1 \\ x_1' = 2 \end{cases}$$

Operații cu subspații vectoriale

$(V, +, \cdot)_{/\mathbb{K}}$ sp. vect, $V' \subset V$ subm. nevidă.

$V' \subset V$ <u>subspațiu vectorial</u> $\Leftrightarrow$ $\forall a, b \in \mathbb{K}$, $\forall x, y \in V'$: $ax + by \in V'$

<u>Ex</u> $(\mathbb{R}^2, +, \cdot)_{/\mathbb{R}}$

$V' = \{ (x, y) \in \mathbb{R}^2 \mid x + y = 1 \}$. Este subsp. vect? NU

$(0_{\mathbb{R}^2} \notin V')$

<u>Prop</u> $(V, +, \cdot)_{/\mathbb{K}}$ sp. vect

Dacă $V_1, V_2 \subset V$ subsp. vect, at $V_1 \cap V_2 \subset V$ subsp. vect

<u>Dem</u>.

$\forall x, y \in V_1 \cap V_2 \Rightarrow$ $x, y \in V_1$ și $x, y \in V_2$ $\Rightarrow$ $ax + by \in V_1$ și $ax + by \in V_2$ $\Rightarrow$

$\forall a, b \in \mathbb{K}$

$\Rightarrow ax + by \in V_1 \cap V_2 \Rightarrow V_1 \cap V_2$ subspațiu vect.

<u>OBS</u> În general, dacă $V_1, V_2 \subset V$ subsp. vect, atunci $V_1 \cup V_2 \subset V$ <u>NU</u> este subsp. vect.

<u>Def</u> $$\underset{\| not}{\langle V_1 \cup V_2 \rangle} = \left\{ \sum_{i=1}^{n} a_i x_i \,,\ x_i \in V_1 \cup V_2,\ a_i \in \mathbb{K},\ i = \overline{1,n},\ n \in \mathbb{N}^* \right\}.$$

$V_1 + V_2$ = sp. vect generat de $V_1 \cup V_2$.

<u>Prop</u> $(V, +, \cdot)/\mathbb{K}$ sp vect, $V_1, V_2 \subset V$ subsp. vect

$$V_1 + V_2 = \{ v_1 + v_2,\ v_1 \in V_1,\ v_2 \in V_2 \}.$$

<u>Dem</u>

"$\subseteq$" $V_1 + V_2 = \langle V_1 \cup V_2 \rangle \subseteq \{ v_1 + v_2,\ v_1 \in V_1,\ v_2 \in V_2 \}$.

Fie $x \in V_1 + V_2 \Rightarrow x = \sum_{i=1}^{n} a_i x_i = \underbrace{\sum_{i=1}^{m} a_i x_i}_{\substack{\| \\ x_1 \\ \cap \\ V_1}} + \underbrace{\sum_{j=m+1}^{n} a_j x_j}_{\substack{\| \\ x_2 \\ \cap \\ V_2}}$

Convenție $x_1, \dots, x_m \in V_1$

$x_{m+1}, \dots, x_n \in V_2$.

$x = \underset{\in V_1}{x_1} + \underset{\in V_2}{x_2}$

"$\supseteq$" Fie $x = x_1 + x_2 \in \langle V_1 \cup V_2 \rangle$

(caz particular de comb. liniară finită).

<u>Teorema Grassmann</u>

$(V, +, \cdot)/\mathbb{K}$ sp. vect finit generat, $V_1, V_2 \subset V$ subsp. vect.

$$\Rightarrow \dim_{\mathbb{K}} (V_1 + V_2) = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}} (V_1 \cap V_2)$$

<u>Dem</u>

$\dim_{\mathbb{K}} V = n$, $\dim_{\mathbb{K}} V_1 = n_1$, $\dim_{\mathbb{K}} V_2 = n_2$, $\dim_{\mathbb{K}} (V_1 \cap V_2) =$

p

Fie $\mathcal{B}_0 = \{e_1, \dots, e_p\}$ bază în $V_1 \cap V_2$.

Extindem la $\mathcal{B}_1 = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}\}$ bază în V_1

— // — $\mathcal{B}_2 = \{e_1, \dots, e_p, g_{p+1}, \dots, g_{n_2}\}$ bază în V_2.

Considerăm $\mathcal{B} = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}, g_{p+1}, \dots, g_{n_2}\}$.

Dem că $\mathcal{B}$ este bază în $V_1 + V_2$

1) $\mathcal{B}$ este SLI.

Fie $a_1, \dots, a_p, b_{p+1}, \dots, b_{n_1}, c_{p+1}, \dots, c_{n_2} \in \mathbb{K}$ a.î.

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k = 0_V.$$

$$\underbrace{\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j}_{\in V_1} = \underbrace{- \sum_{k=p+1}^{n_2} c_k g_k}_{\in V_2} \in V_1 \cap V_2 = \langle \mathcal{B}_0 \rangle$$

$$= \sum_{i=1}^{p} a_i' e_i$$

- $$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j = \sum_{i=1}^{p} a_i' e_i \Rightarrow$$

$$\sum_{i=1}^{p} (a_i - a_i') e_i + \sum_{j=p+1}^{n_1} b_j f_j = 0_V \xRightarrow{\mathcal{B}_1 \text{ SLI}} \begin{cases} a_i - a_i' = 0_{\mathbb{K}}, & i = \overline{1,p} \\ b_j = 0_{\mathbb{K}}, & j = \overline{p+1, n_1} \end{cases}$$

- $$- \sum_{k=p+1}^{n_2} c_k g_k = \sum_{i=1}^{p} a_i' e_i \Rightarrow \sum_{i=1}^{p} a_i' e_i + \sum_{k=p+1}^{n_2} c_k g_k = 0_V$$

$$\xRightarrow{\mathcal{B}_2 \text{ SLI}} \begin{cases} a_i' = 0_{\mathbb{K}}, & i = \overline{1,p} \\ c_k = 0_{\mathbb{K}}, & k = \overline{p+1, n_2} \end{cases}$$

Deci $\begin{cases} a_i = 0_{\mathbb{K}}, & \forall i = \overline{1,p} \\ b_j = 0_{\mathbb{K}}, & \forall j = \overline{p+1, n_1} \\ c_k = 0_{\mathbb{K}}, & \forall k = \overline{p+1, n_2} \end{cases} \Rightarrow \mathcal{B}$ este SLI

2) $\mathcal{B}$ este SG pt V_1+V_2 i.e $V_1+V_2=\langle \mathcal{B}\rangle$.

$\forall x \in V_1+V_2 \Rightarrow x = \underset{V_1}{\underset{\cap}{x_1}} + \underset{V_2}{\underset{\cap}{x_2}} =$

$= \sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{i=1}^{p} a'_i e_i + \sum_{k=p+1}^{n_2} c_k g_k$

$= \sum_{i=1}^{p} (a_i+a'_i) e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k$

Deci $\mathcal{B}$ este bază

$\dim_K (V_1+V_2) = |\mathcal{B}| = \cancel{p} + n_1 - \cancel{p} + n_2 - p$

$= n_1+n_2-p = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}} (V_1 \cap V_2)$

<u>Def</u> V_1+V_2 s.n. <u>sumă directă</u> și se notează $V_1 \oplus V_2$
$\Leftrightarrow V_1 \cap V_2 = \{0_V\}$

<u>Convenție</u> $\dim_{\mathbb{K}} \{0_V\} = 0$

<u>Caz particular T. Grassmann</u>

$\dim (V_1 \oplus V_2) = \dim V_1 + \dim V_2$.

<u>OBS</u> 1) $V_1 \oplus V_2$ $\quad \mathcal{B}_i$ bază în V_i, $i=\overline{1,2} \Rightarrow$
$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ bază în $V_1 \oplus V_2$.

2) $(V, +, \cdot)/\mathbb{K}$ sp. vect finit generat și $\mathcal{B}$ bază în V. Partiționăm $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$.
Dacă considerăm $V_1 = \langle \mathcal{B}_1 \rangle$, $V_2 \langle \mathcal{B}_2 \rangle$, atunci $V = V_1 \oplus V_2$.

Prop $V_1 + V_2$ este sumă directă $\Leftrightarrow$.
$\forall v \in V_1 + V_2$, $\exists!\ v_1 \in V_1, v_2 \in V_2$ aî $v = v_1 + v_2$.

Dem

"$\Rightarrow$" Ipoteză $V_1 \oplus V_2$ i.e $V_1 \cap V_2 = \{0_V\}$.

Pp. prin absurd că scrierea nu este unică.

$$\underset{\in V_1}{v_1} + \underset{\in V_2}{v_2} = \underset{\in V_1}{v_1'} + \underset{\in V_2}{v_2'} \Rightarrow \underset{\in V_1}{v_1 - v_1'} = \underset{\in V_2}{v_2' - v_2} \in V_1 \cap V_2 = \{0_V\}$$

$v_1 = v_1'$, $v_2 = v_2'$

Deci scrierea este unica

"$\Leftarrow$" v se scrie în mod unic $v = \underset{\in V_1}{v_1} + \underset{\in V_2}{v_2}$

Pp. prin abs $\exists\ x \in V_1 \cap V_2 \Rightarrow v = \underbrace{v_1 + x}_{\in V_1} + \underbrace{v_2 - x}_{\in V_2}$, $x \neq 0_V$

Contrad. $\Rightarrow x = 0_V$.

Def $(V, +, \cdot)/\mathbb{K}$ finit generat, $V_1, V_2 \subset V$ subsp. vect.

Dacă $V = V_1 \oplus V_2$, at

V_2 s.n. subspațiu complementar lui V_1

V_1 s.n. —||— V_2.

OBS Subspațiul complementar nu este unic.

Fie $V = (\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V_1 = \{(x, y, 0),\ x, y \in \mathbb{R}\}$

$V_2 = \{(0, 0, z),\ z \in \mathbb{R}\}$

$V_2' = \{(t, t, t),\ t \in \mathbb{R}\}$.

$\mathbb{R}^3 = \boxed{V_1} \oplus \underline{V_2} = V_1 \oplus \underline{V_2'}$

- $V_1 = \langle\{e_1 = (1,0,0), e_2 = (0,1,0)\}\rangle$ 2-dim.
  $V_2 = \langle\{e_3 = (0,0,1)\}\rangle$; $V_2' = \langle\{(1,1,1)\}\rangle$.

$\mathcal{B}_1 = \{e_1, e_2\}$ bază în V_1.

completăm $\mathcal{B}_1$ la o bază în $\mathbb{R}^3$.

$\mathcal{B}_1 \cup \{e_3\}$

$\mathcal{B}_1 \cup \{(1,1,1)\}$.

Ex $(\mathbb{R}^4, +, \cdot)/\mathbb{R}$, $V' = \{(a,b,c,0) \mid a,b,c \in \mathbb{R}\}$

$V'' = \{(0,0,d,e) \mid d,e \in \mathbb{R}\}$.

Este V'' subsp. complementar al lui V'?

$(0,0,1,0) \in V' \cap V'' \Rightarrow V''$ nu e subsp. complem. pt V'

$\dim(V'+V'') = \dim V' + \dim V'' - \dim(V' \cap V'') =$

$= 3+2-1 = 4$.

$V' = \langle\{e_1, e_2, e_3\}\rangle$

$V'' = \langle\{e_3, e_4\}\rangle$ $\qquad V'+V'' = \mathbb{R}^4$

$V' \cap V'' = \langle\{e_3\}\rangle$ $\qquad \oplus$ nu este directă.

Ex $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

$V' = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{cases} x+y-z=0 \\ x-y+z=0 \end{cases}\right\} = \{(0,y,y), y \in \mathbb{R}\}$

$= \{y(0,1,1), y \in \mathbb{R}\}$

$\mathbb{R}^3 = V' \oplus V''$

$V'' = ?$

$V' = \langle\{(0,1,1)\}\rangle$

$rg\begin{pmatrix} 0 & \boxed{1} & \boxed{0} \\ 1 & \boxed{0} & \boxed{0} \\ 1 & \boxed{0} & \boxed{1} \end{pmatrix} = 3$ $\qquad V'' = \langle\{e_1, e_3\}\rangle$

$\mathbb{R}^3 = V' \oplus V''$.