$\boxed{C_8 - GAL}$

Spații vectoriale euclidiene reale
Procedeul Gram - Schmidt

Def $(V,+,\cdot)/\mathbb{R}$, $g: V\times V \to \mathbb{R}$ s.n. produs scalar

$\Leftrightarrow$ 1) $g \in L^s(V,V;\mathbb{R})$

2) g poz. definită.

OBS. A da un produs scalar $\Leftrightarrow$ a declara un reper $R=\{e_1, e_n\}$ ortonormat

"$\Rightarrow$" g produs scalar $\Rightarrow R=\{e_1,..,e_n\}$ ortonormat

$g(e_i, e_j) = \delta_{ij}$

(versori și mutual ortogonali)

$\|x\| = \sqrt{g(x,x)}$.

"$\Leftarrow$" $R=\{e_1,..,e_n\}$ reper ortonormat

Fie $g: V\times V \to \mathbb{R}$ produs scalar a.î. $g(e_i, e_j) = \delta_{ij}$

Prelungim g prin liniaritate în fiecare argument

$$g(x,y) = g\left(\sum_1^n x_i e_i, \sum_{j=1}^n y_j e_j\right) = \sum_{i,j=1}^n \underbrace{g(e_i,e_j)}_{\delta_{ij}} x_i y_j$$
$$= \sum_{i=1}^n x_i y_i$$

Exemplu $(\mathbb{R}^n, g_0)$, $g_0: \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ produs scalar canonic

$g_0(x,y) = \sum_{i=1}^n x_i y_i$ $\quad G_0 = I_n$ matricea în rap cu reperul canonic $R_0 = \{e_1,.., e_n\}$

Def (produs vectorial)

Fie $(\mathbb{R}^3, g_0)$ sp. vect euclidian real, cu str. euclidiană canonică. $S=\{x,y\} \subset \mathbb{R}^3$

Definim $z = x \times y$ produsul vectorial astfel:

1) Dacă S este un SLD, atunci $z = 0_{\mathbb{R}^3}$; $\overset{\text{Not}}{g_0} = \langle\ \rangle$

2) Dacă S este un SLI, atunci:

a) $\|z\|^2 = \begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}$

b) $\langle z, x\rangle = \langle z, y\rangle = 0$

c) $R = \{x, y, z\}$ reper pozitiv orientat
(la fel orientat ca și reperul canonic din R_0
$R_0 \xrightarrow{A} R$, $\det A > 0$)

OBS $S = \{x, y\}$ SLI în $\mathbb{R}^3$

$z = x \times y$ este un „determinant" formal.

$$z = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$$

$$= e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1) \ (*)$$

Prop a) $x \times y = -y \times x$

b) $(x \times y) \times z = \langle x, z\rangle y - \langle y, z\rangle x$

c) $\sum\limits_{x, y, z}^{c} (x \times y) \times z = 0$ (id. Jacobi)

Dem

a) $x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = - \begin{vmatrix} e_1 & e_2 & e_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = -y \times x$

b) $(x \times y) \times z = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ z_1 & z_2 & z_3 \end{vmatrix} = (\alpha, \beta, \gamma)$

$\langle x, z\rangle y - \langle y, z\rangle x =$

$= (x_1 z_1 + x_2 z_2 + x_3 z_3)(y_1 e_1 + y_2 e_2 + y_3 e_3) - (y_1 z_1 + y_2 z_2 + y_3 z_3) \cdot (x_1 e_1 + x_2 e_2 + x_3 e_3)$

$= e_1 \overbrace{\left[y_1 (\cancel{x_1 z_1} + x_2 z_2 + x_3 z_3) - (\cancel{y_1 z_1} + y_2 z_2 + y_3 z_3) x_1 \right]}^{\alpha} +$

$+ e_2 \cdot \beta + e_3 \gamma$

$= e_1 \left[y_1 (x_2 z_2 + x_3 z_3) - x_1 (y_2 z_2 + y_3 z_3) \right] + e_2 \beta + e_3 \gamma$

c) $\sum^{c}_{x,y,z} (x \times y) \times z = (x \times y) \times z + (y \times z) \times x + (z \times x) \times y$

$= \underline{\langle x, z\rangle y} - \underline{\langle y, z\rangle x} +$

$\langle y, x\rangle z - \underline{\langle z, x\rangle y} +$

$\underline{\langle z, y\rangle x} - \langle x, y\rangle z = 0$

Def $(\mathbb{R}^3, g_0)$, $S = \{x, y\} \subset \mathbb{R}^3$, $z \in \mathbb{R}^3$

Produsul mixt: $z \wedge x \wedge y := \langle z, x \times y\rangle$

$$= \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

OBS

$$z \wedge x \wedge y = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x \wedge y \wedge z$$

Aplicatie $(\mathbb{R}^3, g_0)$

Fie $u = (1, -1, 2)$, $v = (0, 1, 3)$, $w = (1, 4, 0)$

a) $u \times v$

b) $w \wedge u \wedge v$

$rg \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} = 2 \Rightarrow S = \{u, v\}$ este SLI

SOL

a) $u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = e_1 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + e_3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$

$= (-5, -3, 1)$

b) $w \wedge u \wedge v = \langle w, u \times v \rangle = -5 - 12 + 0 = -17$

$w = (1, 4, 0)$

SAU $w \wedge u \wedge v = \begin{vmatrix} 1 & 4 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -17$

$\{u, v, u \times v\}$ reper poz. orientat.

Teoremă (procedeul Gram-Schmidt)

Fie $(E, \langle \cdot, \cdot \rangle)$ sp. v. e. real

$\mathcal{R} = \{f_1, .., f_n\}$ reper arbitrar în V

$\Rightarrow \exists \mathcal{R}' = \{e_1, .., e_n\}$ reper ortogonal în V a.î.

$Sp\{e_1, .., e_i\} = Sp\{f_1, .., f_i\}, \forall i = \overline{1,n}$

Dem Metoda este inductivă.

$f_1 \neq 0_V$

Considerăm $\boxed{e_1 = f_1}$

Fie $e_2 = f_2 + \alpha f_1 = f_2 + \alpha e_1$

$\langle e_2, e_1 \rangle = 0 \Rightarrow \langle f_2 + \alpha e_1, e_1 \rangle = 0 \Rightarrow$

$\langle f_2, e_1 \rangle + \alpha \langle e_1, e_1 \rangle = 0 \Rightarrow \alpha = -\dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle}$

$$\boxed{e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1}$$

$\begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \end{cases} \Rightarrow Sp\{f_1, f_2\} = Sp\{e_1, e_2\}.$

Pp. adev P_{k-1}.

Construim. $e_k = f_k + \sum_{i=1}^{k-1} \alpha_{ki} e_i$

Scanned with CamScanner

$$\langle e_k, e_j \rangle = 0, \forall j = \overline{1, k-1}$$

$$\langle f_k + \sum_{i=1}^{k-1} \alpha_{ki} e_i, e_j \rangle = 0, \forall j = \overline{1, k-1}$$

$$\langle f_k, e_j \rangle + \underbrace{\sum_{i=1}^{k-1} \alpha_{ki} \langle e_i, e_j \rangle}_{= \alpha_{kj} \langle e_j, e_j \rangle} = 0, \forall j = \overline{1, k-1}$$

$$\alpha_{kj} = - \frac{\langle f_k, e_j \rangle}{\langle e_j, e_j \rangle}, \forall j = \overline{1, k-1}$$

$$e_k = f_k - \sum_{j=1}^{k-1} \frac{\langle f_k, e_j \rangle}{\langle e_j, e_j \rangle} e_j$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_k = \frac{\langle f_k, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle f_k, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 + \dots + \frac{\langle f_k, e_{k-1} \rangle}{\langle e_{k-1}, e_{k-1} \rangle} e_{k-1} + e_k \end{cases}$$

$Sp\{f_1, .., f_k\} = Sp\{e_1, .., e_k\}$

Deci $Sp\{f_1, .., f_i\} = Sp\{e_1, .., e_i\}, i = \overline{1, k}$

Repetăm raționamentul și obț.

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_n = \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} + e_n . \end{cases}$$

$R' = \{e_1, .., e_n\}$ sist vect mutual ortog $\overset{\text{Prop}}{\Rightarrow} R'$ este SLI

$\dim V = n = |R|$

$\Rightarrow R'$ reper ortogonal.

$Sp\{e_1, .., e_i\} = Sp\{f_1, .., f_i\}, i = \overline{1, n}$

$\boxed{\text{OBS}}$

$$\underset{\substack{\| \\ \{f_1, .., f_n\} \\ \text{reper } \forall}}{R} \xrightarrow{A^{-1}} \underset{\substack{\| \\ \{e_1, .., e_n\} \\ \text{reper ortog}}}{R'} \xrightarrow{B} \underset{\substack{\| \\ \left\{\frac{e_1}{\|e_1\|}, .., \frac{e_n}{\|e_n\|}\right\} \\ \text{reper ortonormat}}}{R''}$$

$$A^{-1} = \begin{pmatrix} 1 & \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} & \dots & \frac{\langle f_n, e_1\rangle}{\langle e_1, e_1\rangle} \\ 0 & 1 & & \vdots \\ \vdots & 0 & & \frac{\langle f_n, e_{n-1}\rangle}{\langle e_{n-1}, e_{n-1}\rangle} \\ 0 & 0 & & 1 \end{pmatrix} ; \quad B = \begin{pmatrix} \frac{1}{\|e_1\|} & 0 & \dots & 0 \\ 0 & \frac{1}{\|e_2\|} & & \vdots \\ \vdots & 0 & & \\ 0 & 0 & & \frac{1}{\|e_n\|} \end{pmatrix}$$

$$1 = \det(A^{-1}) = \frac{1}{\det A} \Rightarrow \det A = 1 > 0$$

$$\det B = \frac{1}{\|e_1\| \cdot \ldots \cdot \|e_n\|} > 0$$

R, R', R'' sunt la fel orientate.

OBS $\left\| \frac{v}{\|v\|} \right\|^2 = \left\langle \frac{v}{\|v\|}, \frac{v}{\|v\|} \right\rangle = \frac{1}{\|v\|^2} \|v\|^2 = 1$

Def.

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r

a) $x \in E$, $\langle \{x\} \rangle^{\perp} = \{ y \in E \mid \langle x, y \rangle = 0 \}$.

b) $\underset{\text{subsp vect}}{U \subseteq E}$ $\quad U^{\perp} = \{ y \in E \mid \langle x, y \rangle = 0, \forall x \in U \} \underset{\text{subsp vect}}{\subseteq E}$.

Aplicatie

$(\mathbb{R}^3, g_0)$, $u = (1, 2, -1)$

a) $\langle \{u\} \rangle^{\perp}$; b) Să se det un reper ortonormat în $\langle \{u\} \rangle^{\perp}$ (Gram-Schmidt)

SOL

a) $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$.

$\langle \{u\} \rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid \underset{\substack{\| \\ x_1 + 2x_2 - x_3}}{g_0(x, u)} = 0 \} = \{ (x_1, x_2, x_1 + 2x_2) \mid x_1, x_2 \in \mathbb{R} \}$ (plan)

$R = \{ f_1 = (1, 0, 1), f_2 = (0, 1, 2) \}$.

$$(x_1, x_2, x_1+2x_2) = (x_1, 0, x_1) + (0, x_2, 2x_2) =$$
$$= x_1 \underbrace{(1,0,1)}_{f_1} + x_2 \underbrace{(0,1,2)}_{f_2}$$

$\left.\begin{array}{l} R \text{ este SG pt } \langle\{u\}\rangle^{\perp} = V \\ \dim V = 3-1 = 2 = |R| \end{array}\right\} \Rightarrow R$ reper arbitrar în V

b) Aplicăm procedeul Gram-Schmidt

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 = (0,1,2) - \frac{2}{2}(1,0,1) = (-1,1,1)$

$\langle f_2, e_1\rangle = \langle f_2, f_1\rangle = 2$; $\langle e_1, e_1\rangle = \langle f_1, f_1\rangle = 2$

$R' = \{e_1 = (1,0,1),\ e_2 = (-1,1,1)\}$ reper ortogonal.

$\|e_1\| = \sqrt{\langle e_1, e_1\rangle} = \sqrt{1+0+1} = \sqrt{2}$; $\|e_2\| = \sqrt{\langle e_2, e_2\rangle} = \sqrt{3}$

$R'' = \left\{e_1' = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1,0,1),\ e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{3}}(-1,1,1)\right\}$
reper ortonormat în V

Transformări ortogonale

Def $(E_i, \langle\cdot,\cdot\rangle_i)$ $i=\overline{1,2}$ s.v.e.r.

Apl. liniară $f: E_1 \to E_2$ s.n. aplicație ortogonală

$\Leftrightarrow \langle f(x), f(y)\rangle_2 = \langle x, y\rangle_1$, $\forall x, y \in E_1$.

Prop Fie $f: E_1 \to E_2$ aplicație ortogonală

a) $\|f(x)\|_2 = \|x\|_1$, $\forall x \in E_1$

b) f aplicație injectivă.

Dem:

a) f apl. ortogonală $\Rightarrow \underbrace{\langle f(x), f(x) \rangle_2}_{\|f(x)\|_2^2} = \underbrace{\langle x, x \rangle_1}_{\|x\|_1^2}, \forall x \in E_1$

$\Rightarrow \|f(x)\|_2 = \|x\|_1, \forall x \in E_1$

b) f liniară

Fie $x \in \mathrm{Ker} f \Rightarrow f(x) = 0_{E_2}$.

$\underbrace{\|f(x)\|_2}_{\|x\|_1} = 0_{\mathbb{R}} \Rightarrow x = 0_{E_1} \Rightarrow \mathrm{Ker} f = \{0_{E_1}\} \Rightarrow f$ inj

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Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in \mathrm{End}(E)$

f s.n. transformare ortogonală $\Leftrightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle$, $\forall x, y \in E$

Not $O(E) = \{ f \in \mathrm{End}(E) \mid f$ transf. ortogonală$\}$.

Prop $f \in \mathrm{End}(E)$

$f \in O(E) \Leftrightarrow \|f(x)\| = \|x\|, \forall x \in E$

Dem

"$\Rightarrow$" din prop. precedentă

"$\Leftarrow$" $\|f(x+y)\|^2 = \|x+y\|^2$

$\langle f(x+y), f(x+y) \rangle = \langle x+y, x+y \rangle$

$\langle f(x)+f(y), f(x)+f(y) \rangle$

$\langle f(x), f(x) \rangle + \langle f(y), f(y) \rangle + 2\langle f(x), f(y) \rangle = \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle$

$\cancel{\|f(x)\|^2} + \cancel{\|f(y)\|^2} + 2\langle f(x), f(y) \rangle = \cancel{\|x\|^2} + \cancel{\|y\|^2} + 2\langle x, y \rangle$.

$\Rightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle, \forall x, y \in E \Rightarrow f \in O(E)$.

Matricea asociată unei transf. ortogonale în raport cu un reper ortonormat

$\mathcal{R} = \{e_1, \dots, e_n\}$ reper ortonormat

$A = [f]_{\mathcal{R}, \mathcal{R}}$, $f(e_i) = \sum_{j=1}^{n} a_{ji} e_j$, $\forall i = \overline{1,n}$

$$\langle f(e_i), f(e_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij}$$

$$\langle \sum_{r=1}^{n} a_{ri} e_r , \sum_{s=1}^{n} a_{sj} e_s \rangle$$

$$\sum_{r,s=1}^{n} a_{ri} a_{sj} \underbrace{\langle e_r, e_s\rangle}_{\delta_{rs}}$$

$$\sum_{r=1}^{n} a_{ri} a_{rj} = \delta_{ij} \Rightarrow A^T A = I_n.$$

OBS $\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{C} \mathcal{R}' = \{e_1', .., e_n'\}$ repere ortonormate

$C \in O(n)$ i.e $C \cdot C^T = C^T C = I_n.$

$$A' = [f]_{\mathcal{R}', \mathcal{R}'} = C^{-1} A C = C^T A C.$$

$$A'^T A' = (C^T A C)^T (C^T A C) = C^T A^T (C^T)^T C^T A C$$

$$= C^T A^T \underbrace{C C^T}_{I_n} A C = C^T \underbrace{A^T A}_{I_n} C = I_n.$$

$f \in O(E) \Leftrightarrow$ matricea asociată, în $\forall$ reper ortonormat, este ortogonală.

! $\boxed{\text{OBS}}$ $f \in O(E) \Leftrightarrow$ schimbare de repere ortonormate.

- $f \in O(E)$ $\quad$ $\mathcal{R} \xrightarrow{A} \mathcal{R}'$ repere ortonormate

$\Updownarrow$ $\quad$ $\mathcal{R} = \{e_1, .., e_n\}$, $\mathcal{R}' = \{e_1', .., e_n'\}$

$A \in O(n)$, $A = [f]_{\mathcal{R}, \mathcal{R}}$.

- $\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{A} \mathcal{R}' = \{e_1', .., e_n'\}$ repere orton $\Rightarrow A \in O(n)$

$$e_i' = \sum_{j=1}^{n} a_{ji} e_j$$

$f : E \to E$, $f(e_i) = e_i'$, $\forall i = \overline{1,n}$

Prelungim prin liniaritate $f(x) = f(\sum_{i=1}^{n} x_i e_i) =$
$= \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i e_i' = x'$