12 mai 202.

Varietăți liniare

Apl. 1 Cons. $L_1 = \left\{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_2 = 3 \\ x_1 - x_3 = 2 \end{cases} \right\}$

$L_1' = \left\{ x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_2 = 1 \\ x_1 - x_3 = 2 \end{cases} \right\}$

$L_2 = \left\{ x \in \mathbb{R}^3 \mid \{ 2x_1 + x_2 - x_3 = 5 \} \right\}$

Arătați că $L_1 \cap L_2 = L_1$ și $L_1' \cap L_2 = L_1 \cap L_1' = \emptyset$.
Ce reprezintă aceste varietăți?

Rezolvare $L_1 \to \begin{cases} x_1 + x_2 = 3 \\ x_1 - x_3 = 2 \end{cases}$ $(+)$

$2x_1 + x_2 - x_3 = 5 \to L_2$

$L_1' \cap L_2 \quad \begin{cases} x_1 + x_2 \quad = 1 \\ x_1 \quad - x_3 = 2 \\ 2x_1 + x_2 - x_3 = 5 \end{cases}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix} \xRightarrow{(L_1 + L_2 = L_3)} \det A = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{rang } A = 2 \ (1)$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 5 \end{pmatrix} \qquad \Delta_3 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 5 \end{vmatrix} = -2 \neq 0 \Rightarrow \text{rang } \bar{A} = 3 \ (2)$$

rang $A \neq$ rang $\bar{A} \Rightarrow S$ incompatibil $\Rightarrow \mathcal{S} = \emptyset \Rightarrow L_1' \cap L_2 = \emptyset$

$$L_1 \cap L_1' = \begin{cases} x_1 + x_2 = 3 \\ x_1 - x_3 = 2 \\ x_1 + x_2 = 1 \\ x_1 - x_3 = 2 \end{cases} \Rightarrow \nexists \Rightarrow L_1 \cap L_1' = \emptyset$$

$\operatorname{dir} L_1 = \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_3 = 0 \end{cases}$ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ x_1, x_2 nec pr.

$x_3 = \alpha \in \mathbb{R}$ nec sec

$\begin{cases} x_1 + x_2 = 0 \Rightarrow x_2 = -\alpha \\ x_1 - \alpha = 0 \Rightarrow x_1 = \alpha \end{cases}$

$\Rightarrow \begin{cases} x_1 = \alpha \\ x_2 = -\alpha, \ \alpha \in \mathbb{R} \\ x_3 = \alpha \end{cases}$

$\operatorname{dir} L_1 = \{(\alpha, -\alpha, \alpha) \mid \alpha \in \mathbb{R}\} = \{\alpha(1,-1,1) \mid \alpha \in \mathbb{R}\} = \langle \underbrace{(1,-1,1)} \rangle \Rightarrow \dim \operatorname{dir} L_1 = 1 \Rightarrow$

$\Rightarrow \dim L_1 = 1 \Rightarrow L_1$ dreaptă (var lin 1-dim)

Obs. că $\operatorname{dir} L_1' = \operatorname{dir} L_1 \Rightarrow \dim L_1' = 1$ dreaptă

$\{L_1' \parallel L_1\}$

$L_2 \to A = (2 \quad 1 \quad -1)$

$\dim L_2 = n - \operatorname{rg} A = 3 - 1 = 2 \Rightarrow L_2$ plan

2 $p = (2, -1, 0) \in \mathbb{R}^3$

$d: \begin{cases} x_1 = 1 - 2t \\ x_2 = 2 + 3t, \quad t \in \mathbb{R} \\ x_3 = 4 - t \end{cases}$ ec. param.

Să se det. dir d. Stabiliți dacă $p \in d$. Scrieți ec. implicite

Rezolvare.

(V1) $d = \{(1-2t, 2+3t, 4-t) \mid t \in \mathbb{R}\}$

$= \{\underbrace{(1,2,4)}_{P_0} + t\underbrace{(-2,3,-1)}_{v} \mid t \in \mathbb{R}\}$

$\operatorname{dir} d = \langle v \rangle = \{tv \mid t \in \mathbb{R}\}$

(V2)

$d: \dfrac{x_1 - 1}{-2} = \dfrac{x_2 - 2}{3} = \dfrac{x_3 - 4}{-1}$ $\operatorname{dir} d \langle (-2,3,-1) \rangle = \{\alpha(-2,3,-1) \mid \alpha \in \mathbb{R}\}$

$\left. \begin{array}{l} p(2,-1,0) \in d \\ \dfrac{2-1}{-2} = \dfrac{-1-2}{3} = \dfrac{0-4}{-1} \ (F) \end{array} \right\} \Rightarrow p \notin d$

$$\begin{cases} 3(x_1-1)=-2(x_2-2) \\ -(x_2-2)=3(x_3-4) \end{cases} \iff \begin{cases} 3x_1+2x_2-7=0 \\ -x_2-3x_3+14=0 \end{cases}$$

$d=\pi_1\cap\pi_2$

Ec param $\xrightarrow[\substack{\text{param.} \\ \text{rezolv} \\ \text{sist. de ec.}}]{\text{elim}}$ Ec implicite

Scrieți ec. implicite, indicați subspațiul director și precizați dim var. lin.:

a) $L_1=\{x\in\mathbb{R}^3 \mid x=(s-t+1, s-t+2, s-3), s,t\in\mathbb{R}\}$

b) $L_2=\{x\in\mathbb{R}^3 \mid x=(s+1, s-2, s-3), s\in\mathbb{R}\}$

Rez: L_1

ec param $\begin{cases} x_1=s-t+1 \\ x_2=s-t+2 \\ x_3=s-3 \end{cases}, s,t\in\mathbb{R}$

V1 $\text{dir } L_1=\{x\in\mathbb{R}^3 \mid x=\underbrace{(s-t, s-t, s)}_{\text{sol. omogenă}}, s,t\in\mathbb{R}\}$

V2 $L_1=\{(1,2,-3)+s\overbrace{(1,1,1)}^{v_1}+t\overbrace{(-1,-1,0)}^{v_2} \mid s,t\in\mathbb{R}\}=$

$=\{(1,2,-3)+sv_1+tv_2 \mid s,t\in\mathbb{R}\}=(1,2,-3)+\underbrace{\langle\{v_1,v_2\}\rangle}_{\text{dir } L_1}$ $\text{dir } L_1=\langle\{v_1,v_2\}\rangle$

$\text{dir } L_1 \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow \dim \text{dir } L_1=2$

$\Rightarrow \dim L_1=2 \Rightarrow L_1$ plan

$$\begin{cases} x_1=s-t+1 \\ x_2=s-t+2 \\ x_3=s-3 \end{cases}, s,t\in\mathbb{R} \Rightarrow \begin{cases} x_1=x_3+3-t+1=x_3-t+4 \\ x_2=x_3+3-t+2=x_3-t+5 \\ s=x_3+3 \end{cases} \Rightarrow \begin{cases} t=x_3-x_1+4 \\ t=x_3-x_2+5 \\ s=x_3+3 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_3-x_1+4=x_3-x_2+5 \\ t=x_3-x_1+4 \\ s=x_3+3 \end{cases} \Rightarrow -x_1+x_2=1 \text{ plan}$$

Ec impl. pt. L_1

$$\begin{array}{c} \\ v_1 \\ v_2 \end{array}\begin{vmatrix} x_1-1 & x_2-2 & x_3+3 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{vmatrix} = (x_1-1)\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - (x_2-2)\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + \cancel{(x_3+3)}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0 \iff$$

$$\iff (x_1-1) - x_2 + 2 = 0 \iff x_1 - x_2 = -1$$

b) $L_2 = \{x \in \mathbb{R}^3 \mid x = (1,-2,-3) + s(1,1,1), s \in \mathbb{R}\} = (1,-2,-3) + sv, s \in \mathbb{R} =$

$= (1,-2,-3) + \underbrace{\langle v \rangle}_{dir L_2}$

dim dir $L_2 = 1 \Rightarrow$ dim $L_2 = 1$ (dreaptă)

$$\begin{cases} x_1 = s+1 \\ x_2 = s-2 \\ x_3 = s-3 \end{cases}, \cancel{s \in \mathbb{R}} \Rightarrow \begin{cases} s = x_1 - 1 \\ s = x_2 + 2 \\ s = x_3 + 3 \end{cases} \Rightarrow \frac{x_1-1}{1} = \frac{x_2+2}{1} = \frac{x_3+3}{1}$$

$$\begin{cases} x_1 - 1 = x_2 + 2 \\ x_2 + 2 = x_3 + 3 \end{cases} \iff \begin{cases} x_1 - x_2 - x_3 = 0 \\ x_2 - x_3 - 1 = 0 \end{cases}$$

3a) Scrieți ec. param. și implicite pt. dr. care trece prin $p(2,0,4)$ și dir $d = \langle v \rangle$, $v = (1,0,-1)$

b) —"— $p_1 = (1,-1,1)$, $p_2 = (3,1,-4)$

$d = (2,0,4) + t(1,0,-1), t \in \mathbb{R}$

$= \cancel{2+t}\ \{x \in \mathbb{R}^3 \mid x = (2+t, 0, 4-t), t \in \mathbb{R}\}$

$$d: \begin{cases} x_1 = 2+t \\ x_2 = 0 \\ x_3 = 4-t \end{cases}, t \in \mathbb{R} \text{ ec. param.} \Rightarrow \begin{cases} t = x_1 - 2 \\ t = 4 - x_3 \end{cases} \Rightarrow x_1 - 2 = 4 - x_3; x_2 = 0$$

$\iff x_1 + 2 = x_3 - 4 \iff$

$\iff x_1 + x_3 - 6 = 0$

$$\frac{x_1-2}{1} = \underbrace{\frac{x_2-0}{0}}_{\text{convenție}} = \frac{x_3-4}{-1}$$

$$d: \begin{cases} x_2 = 0 \\ x_1 + x_3 - 6 = 0 \end{cases}$$

$\underbrace{\overrightarrow{P_1P_2}}_{v_d} = P_2 - P_1 = (2,2,-5) =$

$\text{dir } d = \langle v_d \rangle = \langle (2,2,-5) \rangle$

$d' = (1,-1,1) + t(3,1,-4),\ t \in \mathbb{R}$

$d' = \{x \in \mathbb{R}^3 \mid (1+3t, -1+t, 1-4t), t \in \mathbb{R}\}$

$d' : \begin{cases} x_1 = 1+3t \\ x_2 = -1+t \\ x_3 = 1-4t \end{cases},\ t \in \mathbb{R}$ ec param.

$\frac{x_1-1}{3} = \frac{x_2+1}{1} = \frac{x_3-1}{-4}$

$\begin{cases} (x_1-1) = 3(x_2+1) \\ -4(x_2+1) = x_3+1 \end{cases} \Leftrightarrow \begin{cases} x_1 - 3x_2 - 4 = 0 \\ -4x_2 - x_3 - 3 = 0 \end{cases}$

temă 4. $P = (0,1,-1),\ Q = (1,2,-2),\ R = (3,1,2) \in \mathbb{R}^3$

Arătați că sunt necol. și scrieți ec. impl. pt. planul det. de ele

5. Poziția relativă a planelor $\pi = \{(2-s-t, 3-s+t, 3s-2t) \mid s,t \in \mathbb{R}\}$

$\pi' = \{(\cancel{1+2s}, 1-2s, 2+2t, 3+s-5t) \mid s,t \in \mathbb{R}\}$

ec. impl. (tb să iasă că coincid)

4. $v_1 = \overrightarrow{PQ} =$

$v_2 = \overrightarrow{PR} =$