28 apr. 202

Seminar 10

Spații vect. euclidiene

Def: Fie $V/\mathbb{R}$ un sp. vect. real.
g o formă biliniară simetrică, poz. def pe $V \to$ produs scalar
$\langle , \rangle : V \times V \to \mathbb{R}$
$(V/\mathbb{R}, \underbrace{\langle , \rangle}_{p.s.})$ s.n. sp. vect. euclidian

Exemplu: $\mathbb{R}^n/\mathbb{R} \to$ sp. vect. real

$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$, $(\forall)\ x = (x_1, \dots, x_n) \in \mathbb{R}^n$
$y = (y_1, \dots, y_n)$

$\downarrow$
p. sc. canonic

$$G = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = I_n$$

$(\mathbb{R}^n/\mathbb{R}, \underbrace{\langle , \rangle}_{p.s.can.})$ sp. vect. euclidian, n-dim.

Orice spațiu vectorial euclidian este un sp. normat, $\|x\| = \sqrt{\langle x, x \rangle}$, $(\forall)\ x \in V$
—————— || —————— metric

$d : V \times V \to \mathbb{R}_+$,

$d(x, y) = \|y - x\| = \sqrt{\langle y-x, y-x \rangle}$, $\forall x, y \in V$

Th: Cauchy - Buniakowsky - Schwarz:
În orice sp. vect. euclidian $(E/\mathbb{R}, \underbrace{\langle , \rangle}_{p.s.})$ are loc ineg.:
$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$, $(\forall)\ x, y \in E$
„=" $\Leftrightarrow$ $\{x, y\}$ s.v. lin. dep. (x, y vect. coliniari)

Caz particular:

$(\mathbb{R}^n/_{\mathbb{R}}, \underset{p.s.c.}{\langle,\rangle})$ sp. vect. eucl.

Ineg. C-B-S: $|\langle x,y\rangle| \le \|x\|\cdot\|y\| \Rightarrow \langle x,y\rangle \le \|x\|^2 \cdot \|y\|^2 \Leftrightarrow \sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^2\right)\cdot\left(\sum_{i=1}^{n} y_i^2\right)$

$\|x\|^2 = \langle x,x\rangle = \sum_{i=1}^{n} x_i^2$

„=" $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n} = t$

$(E/_{\mathbb{R}}, \underset{p.s.}{\langle,\rangle})$ sp. vect. eucl.

$\cos\sphericalangle(x,y) = \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|}, \forall x,y \in E$

$\underbrace{x \perp y}_{\text{vect. ortog}} \Leftrightarrow \cancel{\cos}\ \sphericalangle(x,y) = \frac{\pi}{2} \Leftrightarrow \cos\sphericalangle(x,y) = 0 \Leftrightarrow \langle x,y\rangle = 0$

Def: Fie $(E/_{\mathbb{R}}, \langle,\rangle)$ sp. vect. eucl.

a) $\underbrace{\{e_1, \dots, e_n\}}_{\text{bază}} \subset E/_{\mathbb{R}}$ s.n. bază ortogonală dacă $\langle e_i, e_j\rangle = 0$, $\forall\, 0 \le i \ne j \le n$

b) $\{e_1, \dots, e_n\} \subset E/_{\mathbb{R}}$ s.n. bază ortonormată dacă $\langle e_i, e_j\rangle = \delta_{ij}, \forall i,j = \overline{1,n}$

$$\delta_{ij} = \begin{cases} 1, & \text{dacă } i = j \\ 0, & \text{dacă } i \ne j \end{cases}$$

Procedeul de orto~~gonalizare~~normalizare Gram-Schmidt:

Fie $(E/_{\mathbb{R}}, \langle,\rangle)$ sp. vect. eucl.

$(\forall)\underbrace{\{f_1, f_2, \dots, f_n\}}_{\text{arbitrară}} \subset E \Rightarrow (\exists)\underbrace{\{e_1, \dots, e_n\}}_{\text{bază ortonormată}} \subset E$,

a.î. $\langle\{f_1, \dots, f_i\}\rangle = \langle\{e_1, \dots, e_i\}\rangle, \forall i = \overline{1,n}$

(V_1) $e_1' = f_1$

$e_i' = f_i - \sum_{j=1}^{i-1} \frac{\langle f_i, e_j' \rangle}{\|e_j'\|^2} \cdot e_j'$, $(\forall) i = \overline{2,n}$ $\rightarrow \{e_1', \dots, e_n'\}$ b. ortog

$\rightarrow \left\{ e_1 = \frac{e_1'}{\|e_1'\|}, \dots, e_n = \frac{e_n'}{\|e_n'\|} \right\}$ b. orton.

(V_2) $e_1 = \frac{f_1}{\|f_1\|}$

$e_i = \frac{e_i'}{\|e_i'\|}$, unde $e_i' = f_i - \sum_{j=1}^{i-1} \langle f_i, e_j \rangle \cdot e_j$, $(\forall) i = \overline{2,n} \rightarrow \{e_1, \dots, e_n\}$ b. orton.

Apl: În sp. vect. eucl. $({\mathbb{R}^3}/_{\mathbb{R}}, \underset{p.s.c.}{\langle , \rangle})$, să se construiască o bază orton., pornind de la baza:

$B = \{f_1 = (-1,1,1), f_2 = (1,-1,1), f_3 = (1,1,-1)\} \subset \mathbb{R}^3$,

folosind P.O.G.S.

(V_1) $e_1' = f_1 = (-1,1,1)$

$e_2' = f_2 - \frac{\langle f_2, e_1' \rangle}{\|e_1'\|^2} \cdot e_1' = (1,-1,1) + \frac{1}{3} \cdot (-1,1,1) = \left(\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\right) = \frac{2}{3}(1,-1,2)$

$e_3' = f_3 - \frac{\langle f_3, e_1' \rangle}{\|e_1'\|^2} \cdot e_1' - \frac{\langle f_3, e_2' \rangle}{\|e_2'\|^2} \cdot e_2' = (1,1,-1) + \frac{1}{3}(-1,1,1) - \frac{-\frac{4}{3}}{\frac{8}{3}} \cdot \frac{2}{3}(1,-1,2) =$

$= (1,1,0) \rightarrow \{e_1', e_2', e_3'\}$ b. ortog.

$\langle f_3, e_2' \rangle = \frac{2}{3}(-2)$

$\|e_2'\|^2 = \frac{4}{9} \cdot 6$

$\rightarrow \|e_1'\| = \sqrt{3}$

$\|e_2'\| = \frac{2}{3}\sqrt{6}$

$\|e_3'\| = \sqrt{2}$

$e_1 = \frac{e_1'}{\|e_1'\|} = \frac{1}{\sqrt{3}}(-1,1,1)$

$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{3}{2\sqrt{6}} \cdot \frac{2}{3}(1,-1,2) = \frac{1}{\cancel{2}\sqrt{6}}(1,-1,2)$

$e_3 = \frac{e_3'}{\|e_3'\|} = \frac{1}{\sqrt{2}}(1,1,0)$

Verif. $\langle e_i, e_j\rangle = \delta_{ij}$, $(\forall)\ i,j=\overline{1,3}$

V_2) $e_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{3}}(-1,1,1)$

$e_2 = \frac{e_2'}{\|e_2'\|}$, $e_2' = f_2 - \langle f_2, e_1\rangle e_1 = (1,-1,1) + \frac{1}{\sqrt3}\cdot\frac{1}{\sqrt3}(-1,1,1) = \left(\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\right) = \frac{2}{3}(1,-1,2)$

$\|e_2'\| = \frac{2}{3}\sqrt{6}$

$e_2 = \frac{3}{2\sqrt6}\cdot\frac{2}{3}(1,-1,2) = \frac{1}{\sqrt6}(1,-1,2)$

$e_3 = \frac{e_3'}{\|e_3'\|}$, $e_3' = f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2 = (1,1,-1) + \frac{1}{\sqrt3}\cdot\frac{1}{\sqrt3}(-1,1,1) + \frac{2}{\sqrt6}\cdot\frac{1}{\sqrt6}(1,-1,2)$

$= (1,1,-1) + \frac{1}{3}(-1,1,1) + \frac{1}{3}(1,-1,2) = (1,1,0)$

$\|e_3'\| = \sqrt2 \Rightarrow e_3 = \frac{1}{\sqrt2}(1,1,0)$

Temă: $B = \{f_1 = (1,1,1), f_2 = (1,1,-1), f_3 = (1,-1,-1)\}$

Apl. $B = \{f_1 = (0,1,1), f_2 = (1,0,1), f_3 = (1,1,0)\} \subset E_3 = (\mathbb{R}^3/\mathbb{R}, \langle,\rangle)$

Apl. $E_3 = (\mathbb{R}^3/\mathbb{R}, \underbrace{\langle,\rangle}_{p.s.c})$

$f_1 = (2,2,1)$

$f_2 = (-2,-1,2)$

a) Calculați $\|f_1\|$, $\|f_2\|$, $\sphericalangle(f_1,f_2)$

b) Det. un vector nenul $f_3 \in E_3$, a.î. $\begin{cases} f_3 \perp f_1 \\ f_3 \perp f_2 \end{cases}$

c) Pt. f_3 obținut la b) orton. sist. $\{f_1, f_2, f_3\}$ prin P.O.G.S.

a) $\|f_1\| = \sqrt{\langle f_1, f_1\rangle} = \sqrt{2^2+2^2+1} = \sqrt9 = 3$

$\|f_2\| = \sqrt{\langle f_2, f_2\rangle} = \sqrt{2^2+1^2+2^2} = \sqrt9 = 3$

$\cos\underbrace{(\widehat{f_1, f_2})}_{\alpha} = \frac{\langle f_1, f_2\rangle}{\|f_1\|\,\|f_2\|} = \frac{-4}{3\cdot3} = -\frac{4}{9}$

$\Rightarrow \alpha = \arccos\left(-\frac{4}{9}\right) = \pi - \arccos\frac{4}{9}$

b) Fie $f_3 = (a, b, c)$. Impunem $\begin{cases} f_3 \perp f_1 \\ f_3 \perp f_2 \end{cases} \Leftrightarrow \begin{cases} \langle f_3, f_1 \rangle = 0 \\ \langle f_3, f_2 \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} 2a+2b+c= \\ -2a-b+2c= \end{cases}$

$\begin{pmatrix} 2 & 2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$ rang $A = 2$, $\Delta = \begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} = 2 \neq 0$

a, b nec. pr., $c = \lambda$ nec. sec.

$\begin{cases} 2a+2b = -\lambda \\ -2a-b = -2\lambda \end{cases} (+) \Rightarrow \begin{cases} a = \frac{5}{2}\lambda \\ b = -3\lambda \\ c = \lambda \end{cases}, \lambda \in \mathbb{R} \Rightarrow f_3 = \underset{\mathbb{R}^*}{\lambda}\left(\frac{5}{2}, -3, 1\right) \in \mathbb{R}^3$

Obs: f_3 nu este unic (e unic până la înmulțirea cu o const. nenulă

Aleg. $\lambda = 2 \Rightarrow f_3 = (5, -6, 2)$

5 mai 202

Seminar 11

Sp. vect. euclidiene

* reper = bază ordonată

Apl. $E_3 = (\mathbb{R}^3/_{\mathbb{R}}, \langle , \rangle_{p.s.c})$

(reper) baza ortonormată $B' = \{e_1 = \frac{1}{\sqrt{2}}(0,1,1),\ e_2 = \frac{1}{\sqrt{6}}(2,-1,1),\ e_3 = \frac{1}{\sqrt{3}}(1,1,-1)\}$

$\downarrow$

$\langle e_i, e_j \rangle = \delta_{ij}$

să se det. coord. următorilor vectori:

a) $v = (1,2,3)$

temă b) $w = (-1,1,2)$

Rezolvare:

$\underbrace{v = v_1 e_1 + v_2 e_2 + v_3 e_3}_{\text{scriere unică}}$ (*) $[v]_{B'} = (v_1, v_2, v_3)$

* $/ \langle \ , e_1 \rangle$

$\langle v, e_1 \rangle = v_1 \underbrace{\langle e_1, e_1 \rangle}_{=1} + v_2 \underbrace{\langle e_2, e_1 \rangle}_{=0} + v_3 \underbrace{\langle e_3, e_1 \rangle}_{=0} \Rightarrow v_1 = \langle v, e_1 \rangle$

* $/ \langle \ , e_2 \rangle$

$\langle v, e_2 \rangle = v_1 \underbrace{\langle e_1, e_2 \rangle}_{=0} + v_2 \underbrace{\langle e_2, e_2 \rangle}_{=1} + v_3 \underbrace{\langle e_3, e_2 \rangle}_{=0} = v_2 \Rightarrow v_2 = \langle v, e_2 \rangle$

2) $\exists O \in$ ...

$(A, V/_K, \varphi)$ s.n. sp. afin ...

structură afină

$\dim A = V$

$\dim A = \dim \operatorname{dir} A$

Exemplu:

1. $V/_K$ sp. vect. Să se ... de sp. afin aso...

$(V, V/_K, \varphi)$ sp. af...

apl. afină

Def. $\varphi : V \times V \to V$, $\varphi(u,$ ...

Ar. că φ - str...

1) $\varphi(\underbrace{u, v}_{v-u}) + \varphi($ ...

2) $\exists 0_V \in V$, $\varphi 0_V$ ...

$\langle v, e_3\rangle = v_1\underbrace{\langle e_1, e_3\rangle}_{0} + v_2\underbrace{\langle e_2, e_3\rangle}_{0} + v_3\underbrace{\langle e_3, e_3\rangle}_{1} \Rightarrow v_3 = \langle v, e_3\rangle$

$\langle v, e_1\rangle = \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

$\langle v, e_2\rangle = \frac{3}{\sqrt{6}}$

$\langle v, e_3\rangle = \frac{1}{\sqrt{3}}\cdot 0 = 0$

$[v]_{B'} = \left(\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{6}}, 0\right)$

Spații vectoriale. Spații afine.

Def.: $\overset{\text{pct.}}{A} \neq \emptyset$, $V/_K$ sp. vect.

$\varphi: A \times A \to V$, care satisface:

1) $\varphi(A,B) + \varphi(B,C) = \varphi(A,C)$, $(\forall)\, A,B,C \in A$ $\qquad \varphi(A,B) = \overrightarrow{AB}$

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

2) $\exists\, O \in A$, a. î. $\varphi_O: A \to V$, $\varphi_O(A) = \varphi(O, A)$ bij.

$(A, V/_K, \varphi)$ s. n. sp. afin asociat sp. vect. $V/_K$

φ — structură afina

$\operatorname{dir} A = V$

$\dim A = \dim \operatorname{dir} A = \dim_K V$

Exemplu:

1. $V/_K$ sp. vect. Să se arate că $V/_K$ poate fi dotat cu o struct. de sp. afin asociat lui însuși.

$(V, V/_K, \varphi)$ sp. afin

φ — apl. afină

Def. $\varphi: V\times V \to V$, $\varphi(u, v) = v - u$, $\forall\, u, v \in V$

Ar. că φ - str. afină

1) $\underbrace{\varphi(u, v)}_{v-u} + \underbrace{\varphi(v, w)}_{w-v} = \underbrace{\varphi(u, w)}_{w-u}$, $(\forall)\, u, v, w \in V$

2) $\exists\, O_V \in V$, $\varphi_{O_V}: V \to V$, $\varphi_{O_V}(u) = \varphi(O_V, u) = u - O_V = u \Rightarrow \varphi_{O_V} = 1_V$ bij.

Obs.: φ s.n. str. afină canonică

Definiți str. afină canonică pe sp. vect. $\mathbb{R}^n/\mathbb{R}$.

$\varphi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$\varphi((x_1, ..., x_n), (y_1, ..., y_n)) \stackrel{def}{=} (y_1 - x_1, y_2 - x_2, ..., y_n - x_n), \forall (x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n$

temă: Se verifică că φ str. afină canonică pe $\mathbb{R}^n$.

1) $\varphi(x, y) + \varphi(y, z) = \varphi(x, z), (\forall) x, y, z \in \mathbb{R}^n$

2) $\exists O \in \mathbb{R}^n$, a. î. $\varphi_{O_{\mathbb{R}^n}} : \mathbb{R}^n \to \mathbb{R}^n$ bij.

C.P.: $n = 3$

Subsp. afine (varietăți liniare):

Fie $(A, V/_K, \varphi)$ un sp. afin.

Fie $A' \subset A$, A' s.n. subsp. afin (var. lin), dacă $A' = \emptyset$ sau

$\exists O' \in A'$ a. î. $\underbrace{V' = \{\overrightarrow{O'A'} \mid A' \in A'\}}_{\text{ssp. vect.}} \subseteq V$

C.P. $V/_K$ sp. vect.

Def.: S.n. ssp. afin (var. lin.)

$\underbrace{v + U}_{\text{translatatul lui } U}, (\forall)\, v \in V$

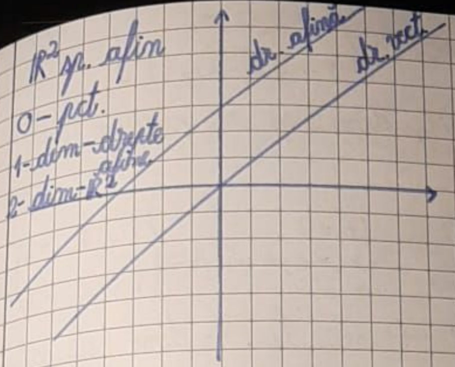
$\underbrace{U \subset V}_{\text{ssp. vect.}}$

$v + U = \{v + u \mid u \in U\}$

(!) Orice ssp. vect. este și ssp. afin (var. lin.).

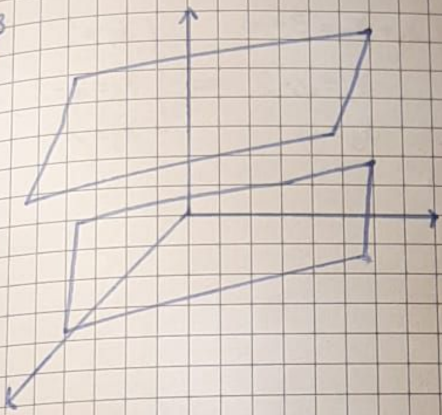
Un ssp. afin (var. lin.) cu propr. că $O_V \in U$ este și ssp. vect.

$\mathbb{R}^2$ sp. afin
0 – pct.
1 – dim – drepte afine
2 – dim – $\mathbb{R}^2$



$\mathbb{R}^2$ sp. vect.
0 – dim : $0_{\mathbb{R}^2}$
1 – dim : dr. vect. (trec prin origine)
2 – dim : $\mathbb{R}^2$

$\mathbb{R}^3$



0
1
2 – pl. vect. (conțin orig.)

Apl. (P) $\mathbb{R}^n / \mathbb{R}$

$$A = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} x_j = b_i, (\forall) i = \overline{1,m}\}$$

$AX = B$

$rg\, A = m \leq n$

$\Rightarrow A \subseteq \mathbb{R}^n$ sp. afin

În plus, dacă $A \neq \emptyset$, at. $\text{dir}\, A = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} x_j = 0, (\forall) i = \overline{1,m}\}$

$AX = 0$

$\dim A = \dim \text{dir}\, A = n - m$

Apl. $\mathbb{R}^3$ Considerăm mulțimea $L=\left\{x\in\mathbb{R}^3 \,\middle|\, \begin{cases} x_1+x_2-x_3=3 \\ x_1+2x_2-2x_3=1 \end{cases}\right\}$

Să se det. subsp. dir. al lui L, dim L și ec. param.

Rezolvare:

$\operatorname{dir} L=\left\{x\in\mathbb{R}^3 \,\middle|\, \begin{cases} x_1+x_2-x_3=0 \\ x_1+2x_2-2x_3=0 \end{cases}\right\} \to \begin{cases} x_1=0 \\ x_2=\alpha, \alpha\in\mathbb{R} \\ x_3=\alpha \end{cases}$ $A=\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \end{pmatrix}$, $rg A=2$

$\dim L=\dim \operatorname{dir} L=3-2=1 \to L$ var. lin. 1-dim. (dr. afină)

x_1, x_2 nec. pr., $x_3=\alpha\in\mathbb{R}$ nec. sec.

$\begin{cases} x_1+x_2=3+\alpha \\ x_1+2x_2=1+2\alpha \\ x_3=\alpha \end{cases}, \alpha\in\mathbb{R} \Rightarrow \begin{cases} x_1+x_2=3+\alpha \\ x_2=\alpha-2 \\ x_3=\alpha \end{cases} \Rightarrow \begin{cases} x_1=5 \\ x_2=\alpha-2 \\ x_3=\alpha \end{cases}, \alpha\in\mathbb{R}$

$\Leftrightarrow \begin{cases} x_1=5+0\alpha \\ x_2=-2+\alpha \\ x_3=\alpha \end{cases}, \alpha\in\mathbb{R}$

$(x_1,x_2,x_3)=(5,-2,0)+\alpha(0,1,1)$

~~$L=\{(5,-2,0)+\alpha(0,1,1)$~~

$L=\{\underbrace{(5,-2,0)}_{v}+\alpha\underbrace{(0,1,1)}_{u} \mid \alpha\in\mathbb{R}\}$ var. lin. $v+\langle u\rangle$

$U=\langle u\rangle$

Obs.: $\operatorname{dir} L=U=\langle u\rangle=\langle 0,1,1\rangle$

Apl. $p=(2,-1,4)\in\mathbb{R}^3$

$L_1=\{x\in\mathbb{R}^3 \mid x_1-2x_2-2x_3=2\}$

$p\in L_1$?

$L_2=\{x\in\mathbb{R}^3 \mid x_1-2x_2-2x_3=-4\}$ $p\in L_2$?

$L_1 \parallel L_2$?

$L_3=\left\{x\in\mathbb{R}^3 \,\middle|\, \begin{cases} x_1-2x_2-2x_3=-4 \\ x_1+x_2+x_3=5 \end{cases}\right\}$ $p\in L_3$?

$L_3 \parallel L_1$?

$\dim L_1 \overset{?}{\underset{2}{=}}$, $\dim L_2 \overset{?}{\underset{2}{=}}$, $\dim L_3 \overset{?}{\underset{1}{=}}$