26 mai 2022

Seminar 14

Perpendiculara comună a două drepte necoplanare:

$\mathbb{R}^3 \quad d_k : \dfrac{x-x_k}{\alpha_k} = \dfrac{y-y_k}{\beta_k} = \dfrac{z-z_k}{\gamma_k} = (t_k),\ k=\overline{1,2}$

$P_k(x_k, y_k, z_k) < \begin{matrix} P_1 \in d_1 \\ P_2 \in d_2 \end{matrix} \quad \overrightarrow{P_1P_2} = (x_2-x_1, y_2-y_1, z_2-z_1)$

$\vec{v}_{d_k} = (\alpha_k, \beta_k, \gamma_k),\ k=\overline{1,2}$

d_1, d_2 - dr. necoplanare $\Leftrightarrow \begin{vmatrix} \alpha_1 & \alpha_2 & x_2-x_1 \\ \beta_1 & \beta_2 & y_2-y_1 \\ \gamma_1 & \gamma_2 & z_2-z_1 \end{vmatrix} \neq 0$

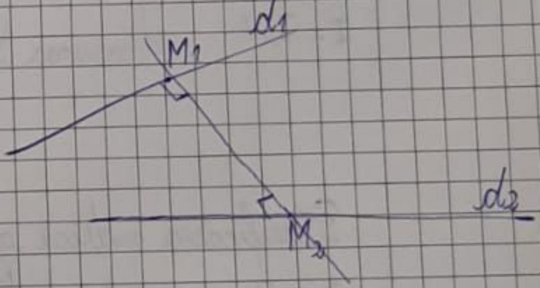
Perp. com. a 2 dr. necoplanare $(\exists!)$

Algoritm $\cancel{P_1P_2}$ $\begin{cases} \langle \overrightarrow{P_1'P_2'}, \vec{v}_{d_1} \rangle = 0 \\ \langle \overrightarrow{P_1'P_2'}, \vec{v}_{d_2} \rangle = 0 \end{cases} \Rightarrow P_1', P_2'$ Ec. ⊥ com.: $\overrightarrow{P_1'P_2'}$

$d(d_1, d_2) = |\overrightarrow{P_1'P_2'}|$

Apl. $d_1: \dfrac{x-2}{1} = \dfrac{y}{2} = \dfrac{z-3}{1} (= t_1)$

$d_2: \dfrac{x-1}{2} = \dfrac{y-3}{1} = \dfrac{z}{1} (= t_2)$



a) d_1, d_2 necopl.

b) ec. ⊥ com.

c) $dist(d_1, d_2)$

a) $P_1(2,0,3)$

$P_2(1,3,0)$

$\overrightarrow{P_1P_2} = (-1, 3, -3)$

$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & -3 \end{vmatrix} = -3-2+6+1-3+12 = 11 \neq 0 \Rightarrow d_1, d_2$ necopl.

b) Luăm $M_1 \in d_1$, cu $M_1(t_1+2, 2t_1, t_1+3)$, $\underset{M_2\in d_2}{\forall} M_2(2t_2+1, t_2+3, t_2)$

$\overrightarrow{M_1M_2} = (2t_2 - t_1 - 1,\ t_2 - 2t_1 + 3,\ t_2 - t_1 - 3)$

$\begin{cases} \langle \overrightarrow{M_1M_2}, \vec{v}_{d_1} \rangle = 0 \\ \langle \overrightarrow{M_1M_2}, \vec{v}_{d_2} \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} 2t_2 - t_1 - 1 + 2t_2 - 4t_1 + 6 + t_2 - t_1 - 3 = 0 \\ 4t_2 - 2t_1 - 2 + t_2 - 2t_1 + 3 + t_2 - t_1 - 3 = 0 \end{cases}$ $\quad \vec{v}_{d_1} = (1, 2, 1)$, $\vec{v}_{d_2} = (2, 1, 1)$

$\Leftrightarrow \begin{cases} -6t_1 + 5t_2 + 2 = 0 \\ -5t_1 + 6t_2 = 2 \end{cases} \Leftrightarrow \begin{cases} 36t_1 - 30t_2 = 12 \\ -25t_1 + 30t_2 = 10 \end{cases} \overset{(+)}{\Longrightarrow} t_1 = 2,\ t_2 = 2$

$M_1(4,4,5)$, $M_2(5,5,2)$

$M_1M_2: \frac{x-4}{1}=\frac{y-4}{1}=\frac{z-5}{-3}$

c) $d(d_1,d_2)=|M_1M_2|=\sqrt{(5-4)^2+(5-4)^2+(2-5)^2}=\sqrt{1+1+9}=\sqrt{11}$

Conice

$\mathbb{R}^2$ $\quad E_2=(\mathbb{R}^2/\mathbb{R}, \underbrace{\langle,\rangle}_{p.sc.})$

$\Gamma = f(x,y)=0$, $f(x,y)=a_{11}x^2+a_{22}y^2+2a_{12}xy+2a_{10}x+2a_{20}y+a_{00}$

$a_{ij}\in\mathbb{R}, (\forall) i,j=\overline{0,2}$

ec. gen. a unei conice (hipercuadr. în dim. 2)

F. mat.: $f(X)={}^tXaX+2bX+c$, $a=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, $b=\begin{pmatrix}a_{10}\\ a_{20}\end{pmatrix}$, $c=(a_{00})$

$A=\begin{pmatrix}a & {}^tb\\ b & c\end{pmatrix}$ $\quad$ m. sim. $(a={}^ta)$

$\left.\begin{array}{l}\delta=\det a\\ \Delta=\det A\\ I=Tr\, A\end{array}\right\}$ - invarianți metrici (rămân nemodificați la act. unei izometrii)

$f:\mathbb{R}^2\to\mathbb{R}^2$

$d(A,B)=d(f(A),f(B))\ \forall A,B$

Clasificarea metrică a conicelor

$\Delta\neq 0 \Leftrightarrow$ conica este nedeg.

$\delta\neq 0 \Leftrightarrow$ conica are centru unic

$\Delta\neq 0$ (nedeg)
- $\delta>0 \to$ elipsă: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$
- $\delta=0 \to$ parabolă: $y^2=2px$
- $\delta<0 \to$ hiperbolă: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

$\Delta=0$ (deg)
- $\delta>0 \to$ pct. sau $\emptyset$
- $\delta=0 \to$ 2 dr. $\parallel$
- $\delta<0 \to$ 2 dr. concurente

Apl. Fie Γ de ec. $x_1^2-3x_1x_2+x_2^2-4x_1+2x_2-1=0$

Să se clasifice izom conica Γ

(Să se aducă la o formă can Γ, prin izom)

Rez. $a=\begin{pmatrix}1 & -\frac{3}{2}\\ -\frac{3}{2} & 1\end{pmatrix}$ ~~$\delta=|a|$~~ $\delta=\det a=1-\frac{9}{4}=-\frac{5}{4}<0$

$$A=\begin{pmatrix}1 & -\frac{3}{2} & -2\\ -\frac{3}{2} & 1 & 1\\ -2 & 1 & -1\end{pmatrix}\to\Delta=-1+3+3-4+\frac{9}{4}-1=\frac{9}{4}\neq 0$$

$\Delta\neq 0$ și $\delta<0\Rightarrow\Gamma$ hiperbolă

$\delta\neq 0\Rightarrow$ conica Γ are centru unic

Coord. centrului unic (x_1^0,x_2^0) se det ca sol unică $\begin{cases}\frac{\partial f}{\partial x_1}=0\\ \frac{\partial f}{\partial x_2}=0\end{cases}$

Apl. 2. $\Gamma: x_1^2-2x_1x_2+x_2^2-2x_1+4x_2+1=0$

Clasif. izometrică (prin izometrii) pt conica Γ

$a=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\to\delta=\det a=0$

$$A=\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 2\\ -1 & 2 & 1\end{pmatrix}\to\Delta=\det A=-1\neq 0$$

$\left.\begin{matrix}\Delta\neq 0\\ \delta=0\end{matrix}\right\}\Rightarrow\Gamma$ parabolă

Det. val. proprii coresp. lui a:

Rezolv ec. caract. $\det(a-\lambda I_2)=0$, în $\mathbb{R}$

$$\begin{vmatrix}1-\lambda & -1\\ -1 & 1-\lambda\end{vmatrix}=0\iff(1-\lambda)^2-1=0\iff-\lambda(2-\lambda)=0 \begin{cases}\lambda_1=0\\ \lambda_2=2\end{cases}$$

Det. subsp. pr coresp.

$V_{\lambda_1}=\{v\in\mathbb{R}^2 \mid av=\underset{0}{\lambda_1}v\}$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x - y = 0$$

$$\begin{cases} x = \alpha \\ y = \alpha \end{cases}, \alpha \in \mathbb{R}$$

$V_{\lambda_1} = \{\alpha(1,1) \mid \alpha \in \mathbb{R}\}$

$V_{\lambda_2} = \{ v \in \mathbb{R}^2 \mid av = \lambda_2 v\}$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x + y = 0$$

$$\begin{cases} x = -\alpha \\ y = \alpha \end{cases}, \alpha \in \mathbb{R}$$

$V_{\lambda_1} = \{\alpha(1,1) \mid \alpha \in \mathbb{R}\}$

$V_{\lambda_2} = \{\alpha(-1,1) \mid \alpha \in \mathbb{R}\}$

Cons.: $\begin{cases} f_1 = (1,1) \\ f_2 = (-1,1) \end{cases} \xrightarrow{f_1 + f_2} \begin{cases} e_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt{2}}(1,1) \\ e_2 = \frac{f_2}{\|f_2\|} = \frac{1}{\sqrt{2}}(-1,1) \end{cases}$

$r: \begin{cases} x_1' = \frac{1}{\sqrt{2}}(x_1 + x_2) \\ x_2' = \frac{1}{\sqrt{2}}(-x_1 + x_2) \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{\sqrt{2}}(x_1' - x_2') \\ x_2 = \frac{1}{\sqrt{2}}(x_1' + x_2') \end{cases}$

$R^{-1} = R^t$