14 apr. 2022

# Seminar 9

1. Vectori și valori proprii
2. Sp. vectoriale euclidiene

Aplicații:

1. Fie $f: \mathbb{R}^3 \to \mathbb{R}^3$, dat prin $f(x,y,z) = (2x-y+2z, -x+2y-z, x+y+z), \forall x,y,z \in \mathbb{R}^3$.

temă: a) $f$ apl. lin. (endm.)

b) $M$ asoc. lui $f$ (în rap. cu b. can. din $\mathbb{R}^3$)

c) Det. val. proprii și subspațiile proprii coresp. v.p.

d) Stabiliți dacă endm. e diag.

e) $(\exists)$? m. diag. $C$ și m. diag. $D$

f) Verificare

g) $A^n$, $n \in \mathbb{N}^*$.

b) $A_f = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \to$ m. asoc. endm. $f$ în rap. cu b. can. din $\mathbb{R}^3$

$\begin{matrix} f(e_1) & f(e_2) & f(e_3) \end{matrix}$

$\{e_1, e_2, e_3\} \subset \mathbb{R}^3$
b. can.

c) Val. proprii:

Ec. caracteristică: $P(\lambda)=0 \iff \det(A_f - \lambda I_3) = 0$ ($\mathbb{R}$)

$P(\lambda) \downarrow$ pol. caract.

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \underset{L_2'=L_2+L_3}{\overset{L_1'=L_1+(\lambda-2)L_3}{\iff}} \cancel{(2-\lambda)^2(1-\lambda)=0} \begin{vmatrix} 0 & \lambda-3 & 3\lambda-\lambda^2 \\ 0 & 3-\lambda & -\lambda \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \iff$$

$$\iff (-1)^{3+1}(\lambda-3)(-\lambda)\begin{vmatrix} 1 & \lambda-3 \\ -1 & 1 \end{vmatrix} = 0 \iff (\lambda-3)(-\lambda)(1+\lambda-3) = 0 \iff \lambda(\lambda-2)(\lambda-3) = 0 \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda_1 = 0 & m_a(\lambda_1) = 1 \\ \lambda_2 = 2 & m_a(\lambda_2) = 1 \\ \lambda_3 = 3 & m_a(\lambda_3) = 1 \end{cases}$$

Spec $f = \{0, 2, 3\}$

Subspațiile proprii:

$\cancel{V_{\lambda_1} = 0}$ $V_{\lambda_1=0} = \{v \in \mathbb{R}^3 \mid f(v) = \lambda_1 v\}$ $(V_{\lambda_1=0} = \operatorname{Ker} f)$

$A_f v = \lambda_1 v$

$(A_f - \lambda_1 I_3)\, v = 0_{(3,1)}$, $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}}_{} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det A_f = 0 \Rightarrow$ rang $A_f \leq 2 \to$; $\Delta = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \neq 0 \Rightarrow$ rang $A = 2$

$$\begin{cases} 2x-y=-2\alpha \\ -x+2y=\alpha \\ z=\alpha,\ \alpha\in\mathbb{R}\end{cases} \iff \begin{cases}2x-y=-2\alpha\\ 3y=0\\ z=\alpha\end{cases} \iff \begin{cases} x=-\alpha\\ y=0\\ z=\alpha\end{cases},\ \alpha\in\mathbb{R}$$

$V_{\lambda_1=0}=\{(-\alpha,0,\alpha)\mid\alpha\in\mathbb{R}\}=\{\alpha\underbrace{(-1,0,1)}_{v_1}\mid\alpha\in\mathbb{R}\}$

$V_{\lambda_2=2}=\{v\in\mathbb{R}^3 \mid f(v)=\lambda_2 v\}$

$A_f v=\lambda_2 v$

$(A_f-\lambda_2 I_3)v=O_{(3,1)}$

$$A_f-\lambda_2 I_3 = \begin{pmatrix}0&-1&2\\-1&0&-1\\1&1&-1\end{pmatrix}$$

rang$(A_f-\lambda_2 I_3)=2$; $\Delta_2=\begin{vmatrix}0&-1\\-1&0\end{vmatrix}=-1\neq0 \Rightarrow x,y$ nec. pr.

$z=\alpha\in\mathbb{R}$ nec. sec.

$$\begin{cases} -y=-2\alpha \\ -x=\alpha \\ z=\alpha\end{cases},\ \alpha\in\mathbb{R} \Rightarrow \begin{cases} x=-\alpha\\ y=2\alpha\\ z=\alpha\end{cases},\ \alpha\in\mathbb{R}$$

$V_{\lambda_2=2}=\{(-\alpha,2\alpha,\alpha)\mid\alpha\in\mathbb{R}\}=\{\alpha\underbrace{(-1,2,1)}_{v_2}\mid\alpha\in\mathbb{R}\}$

$V_{\lambda_3=3}=\{v\in\mathbb{R}^3 \mid f(v)=\lambda_3 v\}$

$A_f v=\lambda_3 v \iff (A-\lambda_3 I_3)v=O_{(3,1)}$

$$A_f-\lambda_3 I_3 = \begin{pmatrix}-1&-1&2\\-1&-1&-1\\1&1&-2\end{pmatrix}$$

~~det~~ rang$(A_f-\lambda_3 I_3)=2$; $\Delta_2=\begin{vmatrix}-1&2\\-1&-1\end{vmatrix}=\cancel{-1+2}\ 1+2=3\neq0$

$$\begin{cases} x=\alpha \\ -y\cancel{+z}=\alpha \\ z=\alpha\end{cases} \iff \begin{cases} x=\alpha\\ y=-\alpha\\ z=\cancel{\alpha}\,0\end{cases},\ \alpha\in\mathbb{R}$$

$V_{\lambda_3=3}=\{(\alpha,-2\alpha,\alpha)\mid\alpha\in\mathbb{R}\}=\{\alpha\underbrace{(1,-1,0)}_{v_3}\mid\alpha\in\mathbb{R}\}$

d) Th.: $f: V\to V$ endm. diag.

$\iff$ 1) Toate răd. ec. caract. sunt în corpul scalarilor.

2) Multiplicitatea algebrică, resp. geometrică coincid pt. toate răd.

1) $m_a(\lambda_1)+m_a(\lambda_2)+\ldots+m_a(\lambda_r)=n$

2) $m_a(\lambda_i)=\underbrace{m_g(\lambda_i)}_{\dim V_{\lambda_i}},\ \forall i=\overline{1,r}$

$V_{\lambda_1=0} = \langle(-1,0,1)\rangle$

$B_1 = \{(-1,0,1)\} \subset V_{\lambda_1}$

bază $\Rightarrow \dim V_{\lambda_1} = 1$

$V_{\lambda_2=2} = \langle(-1,2,1)\rangle$

$B_2 = \{(-1,2,1)\} \subset V_{\lambda_2}$

bază $\Rightarrow \dim V_{\lambda_2} = 1$

$V_{\lambda_3=3} = \langle(1,-1,0)\rangle$

$B_3 = \{(1,-1,0)\} \subset V_{\lambda_3}$

bază $\Rightarrow \dim V_{\lambda_3} = 1$

În cazul nostru,
$$\left.\begin{array}{l} 1)\ 1+1+1=3\ (=\dim \mathbb{R}^3) \\ 2)\ m_a(\lambda_i) = m_g(\lambda_i) = 1, (\forall)\, i=\overline{1,3} \end{array}\right\} \Rightarrow f \text{ endm. diag.}$$

e)
$$C = \begin{array}{c} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \end{array} \rightarrow \text{m. diagonalizatoare}$$
(face trecerea de la $B_0$ la baza în care se realizează ~~matricea~~ ~~diago~~ forma diagonală)

$$\underset{\text{b. can}}{B_0} \xrightarrow[\text{m. trecere}]{C} B = \{v_1, v_2, v_3\}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$
m. diag

f) Verif.

$$\begin{array}{ccc} B_0 & \xrightarrow{C} & B \\ \downarrow & & \downarrow \\ A_f & & D = C^{-1} A_f C \end{array}$$

g) $D = C^{-1} A_f C \Rightarrow A_f = C D C^{-1}$

$$A_f^n = \underbrace{A_f \cdot A_f \cdot \ldots \cdot A_f}_{\text{de } n \text{ ori}} = \underbrace{(CDC^{-1})(CDC^{-1}) \ldots (CDC^{-1})}_{\text{de } n \text{ ori}} = C D^n C^{-1}$$

$$D^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

Apl. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (3x+y-z, 2y, x+y+z)$, $\forall x,y,z \in \mathbb{R}^3$.

Temă: Aceleași cerințe, pt. următoarele endomorfisme:

1) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (-x+3y-z, -3x+5y-z, -3x+3y+z)$

2) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (9y, 4x, 3z)$, $\forall x,y,z \in \mathbb{R}^3$.

Apl. a) f apl. lin. (endm)

$$A_f = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \to \text{m. asoc. endm. } f \text{ în rap. cu}$$

Sp. vect. euclidian:

Fie $V_{/\mathbb{R}}$ sp. vect. real

Def.: O apl. ~~lin~~ $g: V \times V \to \mathbb{R}$ s.n. f. bilin. dacă

1) $g(\alpha_1 x_1 + \alpha_2 x_2, y) = \cancel{x_1\ \alpha}\ \alpha_1 g(x_1, y) + \alpha_2 g(x_2, y)$ , $\forall x_1, x_2, y \in V$, $\alpha_1, \alpha_2 \in \mathbb{R}$

2) $g(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 g(x, y_1) + \beta_2 g(x, y_2)$ , $\forall y_1, y_2, x \in V$

Dacă, în plus, $g(x,y) = g(y,x)$, $\forall x, y \in V$, atunci $g$ s.n. f. b. sim.

Obs.: $\left.\begin{array}{l} 1)+3) \\ 2)+3) \end{array}\right\} \Rightarrow$ f. b. sim.

Def.: $g$ - f. b. s. pozitiv def., dc:

$Q(x) = g(x,x) > 0$, $\forall x \in V^*$

$g(x,x) = 0 \Leftrightarrow x = 0$

Def.: S.n. produs scalar pe V o f. b. s. pozitiv def.

Def.: Un sp. vect dotat cu un prod. scalar s.n. sp. vect eucl.

Not.: $(E_{/\mathbb{R}}, \langle\rangle) \to$ sp. vect. euclidian