Seminar 7

Alg Gauss-Jordan pt. determinarea inversei unei matrice ~~păt~~ (dacă $\exists$)

$A \in \mathcal{M}_n(\mathbb{R})$. Consider $(A|I_n) \xrightarrow[\text{alg G-J}]{\text{aplic}} (B|C) \underbrace{\sim (I_n|A^{-1})}_{\text{dacă } \exists A^{-1}}$

aduc la forma E-R

Dacă A m. inversabilă $\Longrightarrow \begin{cases} B = I_n \\ C = A^{-1} \end{cases}$

Dacă A nu e inversabilă $\Longrightarrow B \neq I_n$

① Determinați dacă $\exists$ inversele urm. matrice, utilizând alg G-J

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 2 \\ 4 & 3 & -2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 4 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \underbrace{\substack{L_2' = L_2 + L_1 \\ L_3' = L_3 - 4L_1}} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & -1 & -10 & -4 & 0 & 1 \end{array}\right) \underbrace{L_2' = \tfrac{1}{4} L_2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -1 & -10 & -4 & 0 & 1 \end{array}\right) \underbrace{\substack{L_1' = L_1 - L_2 \\ L_3' = L_3 + L_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{15}{4} & \frac{1}{4} & 1 \end{array}\right) \underbrace{L_3' = -\tfrac{1}{9} L_3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{36} & -\frac{1}{9} \end{array}\right) \underbrace{\substack{L_1' = L_1 - L_3 \\ L_2' = L_2 - L_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \cancel{-\frac{5}{18}}\,\frac{2}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{36} & -\frac{1}{9} \end{array}\right)$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

$$A^{-1} = \frac{1}{-36} \begin{pmatrix} -12 & 8 & -4 \\ 6 & -10 & -4 \\ -15 & 1 & 4 \end{pmatrix}$$

Apl. lin (morf. de sp. vect)

Fie $V, W /_K$ 2 sp. vect.

O apl. liniară $f: V \to W$ s.n. apl. lin. (morf. de sp. vect), dacă

i) $f(v_1+v_2) = f(v_1)+f(v_2)$, $\forall v_1, v_2 \in V$

ii) $f(\lambda v) = \lambda f(v)$, $\forall v \in V, \lambda \in K$

$\iff \begin{cases} \forall v_1, v_2 \in V \\ \lambda_1, \lambda_2 \in K \end{cases}, f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$

$\text{Ker } f = \{ v \in V \mid f(v) = 0_W \} \subset V$

$\text{Im } f = \{ w \in W \mid \exists v \in V, \text{ a.î. } f(v) = w \} \subset W$

(T) Th. rg-def

Fie $f: V \to W$ apl. lin (morf. de sp. vect.)

Atunci $\dim_K \text{Ker } f + \dim_K \text{Im } f = \dim_K V$

Fie $A \in \mathcal{M}_{(m,n)}(K)$

$f: K^n \to K^m$

$f_A(X) = AX \to$ apl. liniară

(1) Fie vectorii $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,-1)$

a) $\{v_1, v_2, v_3\} \subset \mathbb{R}^3$ bază?

b) ~~Baza~~ $\underset{\{e_1, e_2, e_3\}}{\overset{}{B_0}} \underset{\leftarrow}{\to} \mathcal{B} = \{v_1, v_2, v_3\}$

a) $\begin{array}{c} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \end{array} = -1-1+1 = -1 \neq 0$

$\Rightarrow \underset{\text{bază}}{\mathcal{B} = \{v_1, v_2, v_3\}} \subset \mathbb{R}^3$

Matricea de trecere de la baza canonică la o bază arbitrară a unui spațiu de tipul $\mathbb{R}^n /_{\mathbb{R}}$ se obține scriind vectorii bazei arbitrare pe coloane

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ \quad v_1 \quad v_2 \quad v_3 \end{array} \Rightarrow$$ m. de trecere de la B la E

(P) $B_1 \xrightarrow{A} B_2 \Rightarrow B_2 \xrightarrow{A^{-1}} B_1$

$\{B_1, B_2$ - baze$\}$

$C \xrightarrow{A^{-1}} B_0$

Apl. Matricea morfismului,

Ker f, dim Ker f, Im f, dim Im f

$f: \mathbb{R}^2 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+2y \\ -x+y \end{pmatrix}$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \in \mathcal{M}_{(3,2)}(\mathbb{R})$$

$\downarrow$

matricea morfismului

$f(X) = AX \to$ apl. lin.

$\operatorname{Ker} f = \{ v \in \mathbb{R}^2 \mid f(v) = 0_{\mathbb{R}^3} \}$

$$f(x,y) = (0,0,0) \Leftrightarrow \begin{cases} x - y = 0 \\ x + 2y = 0 \\ -x + y = 0 \end{cases}$$

$x = y = 0$

$\operatorname{rg} A = 2 \Rightarrow$ Sol. unică $\Rightarrow \operatorname{Ker} f = \{0_{\mathbb{R}^2}\}$,

$\dim \operatorname{Ker} f = 0$

$\operatorname{Im} f = \{ \underset{(x', y', z')}{\overset{\parallel}{w}} \in \mathbb{R}^3 \mid \exists \underset{(x,y)}{\overset{\parallel}{v}} \in \mathbb{R}^2$, a.î. $f(v) = w \}$

$$f(v) = w \Leftrightarrow \begin{cases} x - y = x' \\ x + 2y = y' \\ -x + y = z' \end{cases} \Leftrightarrow \begin{cases} 3y = y' - x' \Rightarrow y = -\frac{1}{3}x' + \frac{1}{3}y' \\ \qquad x = \frac{2}{3}x' + \frac{1}{3}y' \\ -\frac{2}{3}x' - \frac{1}{3}y' + \frac{1}{3}x' + \frac{1}{3}y' = z' \Leftrightarrow -x' = z' \end{cases}$$

$\operatorname{Im} f = \{ (x', y', z') \in \mathbb{R}^3 \mid x' - z' = 0 \}$

$(x', y', z') \ni \operatorname{Im} f$

$(x', y', x') = x'(1,0,1) + y'(0,1,0) = \langle (1,0,1), (0,1,0) \rangle \subset$ bază $\Rightarrow \dim \operatorname{Im} f = 2$

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Seminar 8

Vectori și valori proprii

Apl. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x \quad -z \end{pmatrix}$

a) f morf. de sp. vect. (apl. lin.)

b) Ker f, dim Ker f, Im f, dim Im f

a) V1 $f: V \to W$ apl. lin (morf. sp. vect.) $\Leftrightarrow \forall v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in K$, $f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$

V2 $f(X) = AX$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$, rang $A = 2$, $D = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$

$\downarrow$

m. asociată apl. lin. rap. la bazele canonice de pe $\mathbb{R}^3$, resp. $\mathbb{R}^2$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $f(X) = A \cdot X$

f e apl. lin. $\Leftrightarrow \forall x_1, x_2 \in \mathbb{R}^3, \forall \alpha_1, \alpha_2 \in \mathbb{R}$, $f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$

$$f(\alpha_1 x_1 + \alpha_2 x_2) = A(\alpha_1 x_1 + \alpha_2 x_2) = (A\alpha_1)x_1 + (A\alpha_2)x_2 = (\alpha_1 A)x_1 + (\alpha_2 A)x_2 =$$
$$= \alpha_1(Ax_1) + \alpha_2(Ax_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

$\Rightarrow f$ apl. lin. (morf. sp. vect.)

b) $\operatorname{Ker} f = \{v \in \mathbb{R}^3 \mid f(v) = 0_{\mathbb{R}^2}\}$

$$f(v) = 0_{\mathbb{R}^2} \Leftrightarrow f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x + y + z = 0 \\ x - z = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{rang } A = 2, \quad \Delta = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

x, y nec. pr.

$z = \lambda \in \mathbb{R}$ nec. sec.

$$\begin{cases} x + y = -\lambda \\ x = \lambda \end{cases} \Leftrightarrow \begin{cases} y = -\lambda - x = -2\lambda \\ x = \lambda \end{cases} \Leftrightarrow \begin{cases} x = \lambda \\ y = -2\lambda \\ z = \lambda \end{cases}, \lambda \in \mathbb{R}$$

$\operatorname{Ker} f = \{(\lambda, -2\lambda, \lambda) \mid \lambda \in \mathbb{R}\} = \{\lambda(1, -2, 1) \mid \lambda \in \mathbb{R}\} = \langle (1, -2, 1) \rangle$

$\Rightarrow B = \{(1, -2, 1)\}$ bază pt. $\operatorname{Ker} f \Rightarrow \dim \operatorname{Ker} f = 1 \Rightarrow f$ nu e inj (1)

$\operatorname{Im} f = \{w \in \mathbb{R}^2 \mid \exists v \in \mathbb{R}^2$, a.î. $f(v) = w\}$

$$f(v) = w \Leftrightarrow f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + y + z \\ x - z \end{pmatrix} \Rightarrow \begin{cases} x + y + z = x' \\ x - z = y' \end{cases} \Rightarrow (x', y') = (x + y + z, x - z)$$

$\Leftrightarrow (x', y') = x(1, 1) + y(1, 0) + z(1, -1)$

$\operatorname{Im} f = \{x(1, 1) + y(1, 0) + z(1, -1) \mid (x, y, z) \in \mathbb{R}^3\} \Rightarrow S_1 = \{(1, 1), (1, 0), (1, -1)\}$ sist. gen. pt. $\operatorname{Im} f$

Dar $\operatorname{Im} f \subset \mathbb{R}^2 \Rightarrow \dim \operatorname{Im} \leq \dim \mathbb{R}^2 = 2$

Aplicăm tg. rg.-def.: $\underbrace{\dim \operatorname{Ker} f}_{1} + \dim \operatorname{Im} f = \dim \mathbb{R}^3 \Rightarrow \left.\begin{array}{l} \dim \operatorname{Im} f = 2 \\ \operatorname{Im} f \subseteq \mathbb{R}^2 \\ \text{sp. vect.} \end{array}\right\} \Rightarrow$

$\Rightarrow$ ~~dim~~ $\operatorname{Im} f = \mathbb{R}^2 \Rightarrow f$ surj. (2)

c) f inj, f surj, f bij

Th. $f: V \to W$ apl. lin.

a) f inj. $\Leftrightarrow \operatorname{Ker} f = \{0_V\}$

b) f surj. $\Leftrightarrow \operatorname{Im} f = W$

c) f bij. $\Leftrightarrow \begin{cases} \operatorname{Ker} f = \{0_V\} \\ \operatorname{Im} f = W \end{cases}$

Apl $\exists$ morf. inj. $\tau: \mathbb{R}^3 \to \mathbb{R}^2$?

Pp. RA $(\exists)\, \tau: \mathbb{R}^3 \to \mathbb{R}^2$ morf. inj.

Apl. Th. rg-def, ~~dim Ker f~~ $\underbrace{\dim \operatorname{Ker} \tau}_{0} + \dim \operatorname{Im} \tau = \underbrace{\dim \mathbb{R}^3}_{3}$

$$\left.\begin{array}{l} \Rightarrow \dim \operatorname{Im} \tau = 3 \\ \text{Dar } \operatorname{Im} \tau \subseteq \mathbb{R}^2 \end{array}\right\} \times$$

Deci $(\nexists)\, \tau: \mathbb{R}^3 \to \mathbb{R}^2$ morf. inj.

Apl Fie $A \in \mathcal{M}_n(\mathbb{R})$ fixată.

$f_A: \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$, $f_A(X) = AX - XA$

S-a demonstrat că f_A morf. vectorial (apl. lin.)

Considerăm subspațiul $sl_2(\mathbb{R}) = \{A \in \mathcal{M}_2(\mathbb{R}) \mid Tr(A) = 0\}$, $E = E_{12}$, $F = E_{21}$, $H = E_{11} - E_{22} \in sl_2(\mathbb{R})$

subsp. vect

Dem. că $B = \{E, F, H\} \subset sl_2(\mathbb{R})$.

bază

$(\dim sl_2(\mathbb{R}) = 3)$

$A \in \mathcal{M}_2(\mathbb{R})$

$Tr(A) = 0 \Rightarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}$

$\Updownarrow$

$a_{11} + a_{22} = 0 \Leftrightarrow a_{22} = -a_{11}$

$$sl_2(\mathbb{R}) \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} = a_{12} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}} + a_{21} \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{21}} + a_{11} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{E_{11} - E_{22}} = a_{12} E + a_{21} F + a_{11} H \Rightarrow$$

$\Rightarrow B = \{E, F, H\} \subset sl_2(\mathbb{R})$ sist. gen. pt. $sl_2(\mathbb{R})$

Dem. că B s.v.l.i

Fie $\alpha_1 E + \alpha_2 F + \alpha_3 H = O_2 \Rightarrow \alpha_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\alpha_i \in \mathbb{R}, \forall i = \overline{1,3}$

$\Rightarrow \begin{pmatrix} \alpha_3 & \alpha_1 \\ \alpha_2 & -\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow B \subset sl_2(\mathbb{R})$ s.v.l.i (2)

Din (1), și (2) $\Rightarrow B = \{E, F, H\} \subset sl_2(\mathbb{R})$ bază $\Rightarrow \dim sl_2(\mathbb{R}) = 3$

Generalizare $sl_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid Tr(A) = 0\}$

$\sum_{i=1}^{n} a_{ii}$

$\underbrace{sl_n(\mathbb{R})}_{n^2-1} \subseteq \underset{n^2}{M_n(\mathbb{R})}$

Am dem. că: $\underset{n^2-1}{sl_n(\mathbb{R})} \oplus \overset{1}{D} = \overset{n^2}{M_n(\mathbb{R})}$

$D = \{B \in M_n(\mathbb{R}) \mid B = \lambda I_n, \lambda \in \mathbb{R}\}$

~~$sl_n(\mathbb{R})$~~

$sl_n(\mathbb{R}) \ni A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & -\sum_{i=1}^{n-1} a_{ii} \end{pmatrix}$

$B = \{E_{ij}, \forall 1 \le i \ne j \le n\} \cup \{F_{ii}, \forall i = \overline{1, n-1}\}$

$F_{ii} = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix}$

Vectori și valori proprii

Apl. temă. Fie $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x + 2y, 2y, -2y + z)$

a) f morfism vectorial

b) matricea asoc. lui f în rap. cu b. can. din $\mathbb{R}^3$

c) det. val. proprii și vect. proprii corep. lui $f(A_f)$

d) stab. dacă f este diagonalizabil

e) Stabiliți dacă $\exists$ mat. diagonalizatoare C și m. diagonală D

f) A_f^n, $n \in \mathbb{N}^*$

b)

$A_f = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

$\quad f(e_1) \; f(e_2) \; f(e_3)$

$B_0 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$

$e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$

$\begin{cases} f(e_1) = f(1,0,0) = (1,0,0) \\ f(e_2) = f(0,1,0) = (2,2,-2) \\ f(e_3) = f(0,0,1) = (0,0,1) \end{cases}$