C_6 - GAL

25.03.2022.

Vectori proprii. Valori proprii. Diagonalizare

Problemă

$(V,+,\cdot)/\mathbb{K}$ sp vect, $f \in End(V)$

$\exists R = \{e_1, .., e_n\}$ reper în V ai $[f]_{R,R} = A = diag = \begin{pmatrix} \lambda_1 & \dots & 0 \\ 0 & & \lambda_n \end{pmatrix}$?

$f(e_1) = \lambda_1 e_1$

$f(e_n) = \lambda_n e_n$.

Def $f \in End(V)$

$x \in V$ ($x \neq 0_V$) s.n. vector propriu $\Leftrightarrow \exists \lambda \in \mathbb{K}$ aî $f(x) = \lambda x$

λ = valoare proprie coresp. vect. propriu x.

OBS $f(0_V) = 0_V = \lambda \cdot 0_V$

Not $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$ spațiu propriu coresp valorii proprii λ.

Prop $f \in End(V)$, $x \neq 0$ vector propriu, λ = valoare proprie

a) $V_\lambda \subseteq V$ subsp. vect

b) V_λ subsp. invariant al lui f i.e $f(V_\lambda) \subseteq V_\lambda$

Dem

a) $\forall x, y \in V_\lambda, \forall a, b \in \mathbb{K} \overset{?}{\Rightarrow} ax + by \in V_\lambda$

$f(ax+by) = af(x) + bf(y) = a\lambda x + b\lambda y = \lambda(ax+by)$

b) Fie $x \in V_\lambda \Rightarrow f(x) = \lambda x \in V_\lambda$.

Polinom caracteristic

$f \in End(V)$, $R = \{e_1, .., e_n\}$ reper in V, $[f]_{R,R} = A$

$f(x) = \lambda x$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \underbrace{f(e_i)}_{\sum_{j=1}^{n} a_{ji} e_j} = \underline{\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji} x_i\right) e_j}$$

$$\underline{\lambda x = \lambda \sum_{j=1}^{n} x_j e_j}$$

$$\Rightarrow \sum_{i=1}^{n} a_{ji} x_i = \lambda x_j, \ \forall j = \overline{1,n}$$

$$\sum_{i=1}^{n} (\underline{a_{ji} - \delta_{ji}\lambda}) x_i = 0, \ \forall j = \overline{1,n} \quad (*)$$

$(*)$ SLO și are și sol nenule $\Rightarrow$

$$\varphi(\lambda) = \det(A - \lambda I_n) = 0$$

Prop Polinomul caracteristic este invariant la schimbarea de reper

Dem

$$\begin{array}{ccc} R = \{e_1, .., e_n\} & \longrightarrow & R = \{e_1, .., e_n\} \quad [f]_{R,R} = A_f \\ C \downarrow & & C \downarrow \\ R' = \{e_1', .., e_n'\} & \longrightarrow & R' = \{e_1', .., e_n'\} \quad [f]_{R',R'} = A_f' \end{array}$$

$A' = C^{-1} A C$, $C \in GL(n, \mathbb{K})$

$$\det(A' - \lambda I_n) = \det(C^{-1} A C - \lambda C^{-1} I_n C)$$

$$= \det\left[C^{-1}(A - \lambda I_n) C\right] = \det(C^{-1}) \det(A - \lambda I_n) \det C$$

$$= \det(A - \lambda I_n)$$

OBS a) Valorile proprii = <u>răd din $\mathbb{K}$</u> ale pol. caract.

b) $P(\lambda) = \det(A - \lambda I_n) = (-1)^n [\lambda^n - \sigma_1 \lambda^{n-1} + \ldots + (-1)^n \sigma_n] = 0$

σ_k = suma minorilor diagonali de ord k

$\sigma_1 = Tr(A), \ldots, \sigma_n = \det(A)$

$(\lambda - \lambda_1)^{m_1} \cdot \ldots \cdot (\lambda - \lambda_r)^{m_r} = 0$

$\lambda_1, \ldots, \lambda_r$ răd. dist. ale pol. caract.

$m_1, \ldots, m_r$ = multiplicitățile, $m_1 + \ldots + m_r = n$.

- $\sigma(f) = \{\lambda_1, \ldots, \lambda_r\}$
- $Spec(f) = \{\underbrace{\lambda_1 = \ldots = \lambda_1}_{m_1 \text{ ori}} < \underbrace{\lambda_2 = \ldots \lambda_2}_{m_2 \text{ ori}} < \ldots < \underbrace{\lambda_r = \ldots = \lambda_r}_{m_r \text{ ori}}\}$

Exemplu $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $J : \mathbb{R}^2 \to \mathbb{R}^2$, $J(x) = (-x^2, x^1)$.

$R_0 = \{e_1, e_2\}$ reperul canonic

$[J]_{R_0, R_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad J(e_1) = J(1,0) = (0,1) = 0e_1 + 1 \cdot e_2$

$\quad J(e_2) = J(0,1) = (-1,0) = -e_1 + 0e_2$

$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i \notin \mathbb{R}$.

$J \circ J(x) = J(-x^2, x^1) = (-x^1, -x^2) = -id_{\mathbb{R}^2}$

Prop Vectorii proprii coresp. la valori proprii dist. formează un SLI.

Dem Dem prin ind după nr de vectori proprii

$n = 1$. $\exists$ vector propriu $\neq 0$ $\Rightarrow \{x\}$ SLI

Pp. prop. adevărată pt $n-1$ vectori proprii

Dem că P_n este adev,

$\{v_1, \ldots, v_n\}$ mult de vect pr. coresp la val pr dist $\Rightarrow$ SLI

Fie $a_1, .., a_n \in \mathbb{K}$ a.î. $a_1 v_1 + ... + a_n v_n = 0_V$ (1) $| f$

$f(a_1 v_1 + ... + a_n v_n) = f(0_V)$

$a_1 \underset{\lambda_1 v_1}{\underset{\parallel}{f(v_1)}} + ... + a_n \underset{\lambda_n v_n}{\underset{\parallel}{f(v_n)}} = 0_V \Rightarrow \boxed{a_1 \lambda_1 v_1 + ... + \underline{a_n \lambda_n v_n} = 0_V}$ (*)

Pp. (fără a restrânge generalitatea) $\lambda_n \neq 0_{\mathbb{K}}$.

(1) $\cdot | \lambda_n \Rightarrow$ (**) $\boxed{\lambda_n a_1 v_1 + .. + \lambda_n a_{n-1} v_{n-1} + \underline{\lambda_n a_n v_n} = 0_V}$

(*) $-$ (**) $(\overset{\neq 0}{\overbrace{\lambda_1 - \lambda_n}}) a_1 v_1 + ... + (\overset{\neq 0}{\overbrace{\lambda_{n-1} - \lambda_n}}) a_{n-1} v_{n-1} = 0_V$.

$\{v_1, .., v_{n-1}\}$ vect. pr. coresp. la val pr dist $\lambda_1, .., \lambda_{n-1}$

$\xRightarrow{P_{n-1}}$ SLI și $a_1 = .. = a_{n-1} = 0_{\mathbb{K}}$. $\xRightarrow{(1)} a_n v_n = 0_V$

$\Rightarrow a_n = 0_{\mathbb{K}}$

Deci $a_1 = .. = a_n = 0_{\mathbb{K}} \Rightarrow \{v_1, .., v_n\}$ SLI

P_n este adev, $\forall n \geq 1$.

<u>Prop</u> $f \in End(V)$, λ = valoare proprie.

$\Rightarrow \dim V_\lambda \leq m_\lambda$

m_λ = multiplicitatea lui λ.

<u>Dem</u> Notăm $n_\lambda = \dim V_\lambda$, $V_\lambda \subseteq V$ subsp. vect.

$R_0 = \{e_1, .., e_{n_\lambda}\}$ reper în V_λ. Extindem la reperul $R = \{e_1, .., e_{n_\lambda}, e_{n_\lambda + 1}, .., e_n\}$ reper în V, $n = \dim V$.

$A = [f]_{R,R}$.
$$\begin{cases} f(e_1) = \lambda e_1 \\ \vdots \\ f(e_{n_\lambda}) = \lambda e_{n_\lambda} \\ f(e_i) = \sum_{j=1}^{n} a_{ji} e_j, \ \forall i = \overline{n_\lambda + 1, n} \end{cases}$$

$$A = \left(\begin{array}{c|c} \begin{matrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{matrix} & \\ \hline O & \end{array}\right) \in \mathcal{M}_n(\mathbb{K})$$

$$P(X) = \det(A - XI_n) = \left|\begin{array}{c|c} \begin{matrix} \lambda - X & & 0 \\ & \ddots & \\ 0 & & \lambda - X \end{matrix} & \\ \hline O & \end{array}\right|$$

$$P(X) = (\lambda - X)^{n_\lambda} Q(X) \Rightarrow m_\lambda \geq \underset{\dim V_\lambda}{\overset{\shortparallel}{n_\lambda}}.$$

<u>Teorema de diagonalizare</u>

$f \in End(V)$

$\exists \mathcal{R} = \{e_1, \dots, e_n\}$ reper în V a.î. $[f]_{\mathcal{R},\mathcal{R}}$ = diagonală

$\Leftrightarrow$ 1) răd. pol. caract $\in \mathbb{K}$.

$[\lambda_1, \dots, \lambda_r \in \mathbb{K}$, $\lambda_1, \dots, \lambda_r$ răd. distincte$]$

2) dim. subsp. proprii = multiplicitățile valorilor proprii coresp.

$[\dim V_{\lambda_i} = m_i$, $i = \overline{1,r}$

$m_1, \dots, m_r$ = multiplicitățile pt $\lambda_1, \dots, \lambda_r$ și

$m_1 + \dots + m_r = n]$.

<u>Dem</u>

"$\Rightarrow$" Ip: $\exists \mathcal{R} = \{e_1, \dots, e_n\}$ reper în V a.î.

$$A = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$$

Eventual renumerotând, considerăm.

$$A = \begin{pmatrix} \overbrace{\lambda_1 \dots \lambda_1}^{m_1 \text{ ori}} & & 0 \\ & \ddots & \\ O & & \overbrace{\lambda_r \dots \lambda_r}^{m_r \text{ ori}} \end{pmatrix} \in \mathcal{M}_n(\mathbb{K}) \Rightarrow \lambda_1, \dots, \lambda_r \in \mathbb{K}.$$
(distincte)

$$P(\lambda) = \det(A - \lambda I_n) = \begin{vmatrix} \lambda_1 - \lambda & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_1 - \lambda & & & \\ & & & \lambda_r - \lambda & & \\ & & & & \ddots & \\ O & & & & & \lambda_r - \lambda \end{vmatrix} =$$

$P(\lambda) = (\lambda-\lambda_1)^{m_1} \cdot \ldots \cdot (\lambda-\lambda_r)^{m_r}$, $m_1+\ldots+m_r = n$.

$\lambda_1, \ldots, \lambda_r$ răd. din $\mathbb{K}$ ale pol caract $\Rightarrow$ valori proprii. (1)

$f(e_1) = \lambda_1 e_1$ $\quad \mathcal{R}_1 = \{e_1, \ldots, e_{m_1}\} \subset V_{\lambda_1}$.

$\vdots$

$f(e_{m_1}) = \lambda_1 e_{m_1}$ $\quad \mathcal{R}_1 \subset \underset{SLI}{\mathcal{R}} \Rightarrow \mathcal{R}_1$ este un SLI

$\Rightarrow \dim V_{\lambda_1} \geq m_1$

dar $\dim V_{\lambda_1} \leq m_1$ (cf prop) $\Rightarrow \dim V_{\lambda_1} = m_1$.

Analog $\dim V_{\lambda_i} = m_i$, $\forall i = \overline{1,r}$ (2)

"$\Leftarrow$" Ip: $f \in End(V)$ a.î.

1) $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ (răd. dist. ale pol caract)

2) $\dim V_{\lambda_i} = m_i$, $\forall i = \overline{1,r}$, $m_1 + \ldots + m_r = n$.

Fie R_i reper în V_{λ_i}, $i = \overline{1,n}$

Considerăm $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_r$. Dem că $\mathcal{R}$ reper.

$|\mathcal{R}| = m_1 + \ldots + m_r = n = \dim V$

Este suficient să dem că $\mathcal{R}$ este un SLI

$\mathcal{R} = \{e_1, \ldots, e_n\}$

$$\underbrace{\sum_{i=1}^{m_1} a_i e_i}_{\substack{f_1 \\ \cap \\ V_{\lambda_1}}} + \ldots + \underbrace{\sum_{j=m_1+\ldots+m_{r-1}+1}^{n} a_j e_j}_{\substack{f_r \\ \cap \\ V_{\lambda_r}}} = 0$$

Dem că $f_1 = \ldots = f_r = 0_V$.

Pp. prin absurd că $f_{i_1}, \ldots, f_{i_p}$ nenule dintre $\{f_1, \ldots, f_r\}$. $\Rightarrow f_{i_1}, \ldots, f_{i_p}$ sunt vect proprii corisp la valori proprii dist. $\Rightarrow \{f_{i_1}, \ldots, f_{i_p}\}$ SLI

$f_{i_1} + \ldots + f_{i_p} = 0_V$ Contrad.

Ip. este falsă $\Rightarrow$ $f_1 = 0 \Rightarrow \sum_{i=1}^{m_1} a_i e_i = 0 \xRightarrow{\mathcal{R}_1 \text{ SLI}} a_1 = \dots = a_{m_1} = 0$

$\vdots$

$f_r = 0 \Rightarrow \sum_{j=m_1+\dots+m_{r-1}+1}^{n} a_j e_j = 0 \xRightarrow[\text{SLI}]{\mathcal{R}_r}$

$a_{m_1+\dots+m_{r-1}+1} = \dots = a_n = 0$

$\Rightarrow a_i = 0, \forall i = \overline{1,n} \Rightarrow \mathcal{R} = \{e_1, \dots, e_n\}$ SLI

$\Rightarrow \mathcal{R}$ este reper în V

$\mathcal{R} = \mathcal{R}_1 \cup \dots \cup \mathcal{R}_r$.

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} \overbrace{\lambda_1 \dots \lambda_1}^{m_1 \text{ ori}} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\lambda_r \dots \lambda_r}^{m_r \text{ ori}} \end{pmatrix}$$

<u>Aplicatie</u>

$f: \mathbb{R}^3 \xrightarrow{\mathcal{L}} \mathbb{R}^3$, $f(x) = (x_1, x_2 + x_3, 2x_3)$

Să se det. un reper $\mathcal{R}$ în $\mathbb{R}^3$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ = diagonală.

<u>SOL</u> $\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reperul canonic din $\mathbb{R}^3$

$$A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 (2-\lambda) = 0$

$\lambda_1 = 1$, $m_1 = 2$ $\quad \lambda_1, \lambda_2 \in \mathbb{R}$

$\lambda_2 = 2$, $m_2 = 1$

$(3,3)(3,1) \to (3,1)$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = x\}$.

$AX = X \Leftrightarrow (A - I_3)X = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = 0$

Scanned with CamScanner

$V_{\lambda_1} = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\} = \langle \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}\} \rangle$

$R_1 = \{e_1, e_2\}$ reper în V_{λ_1} $\left(SG, \ R_1 \subset \underset{SLI}{R_0} \Rightarrow R_1 \in SLI \right)$

$\dim V_{\lambda_1} = m_1 = 2$.

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = 2x\} = \{(0, x_2, x_2) \mid x_2 \in \mathbb{R}\} = \langle \{(0,1,1)$

$$AX = 2X \Rightarrow (A - 2I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 0$

$x_2 = x_3$

$R_2 = \{e_2 + e_3\}$ reper în V_{λ_2}

$\dim V_{\lambda_2} = m_2 = 1$

$R = R_1 \cup R_2 = \{e_1, e_2, e_2 + e_3\}$ reper în care

$$[f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Forme biliniare. Forme pătratice

Def a) $g : V \times V \to \mathbb{K}$ s.n. *formă biliniară* $\Leftrightarrow$ g este liniară în fiecare argument

1) $g(ax + by, z) = a\,g(x, z) + b\,g(y, z)$

2) $g(x, ay + bz) = a\,g(x, y) + b\,g(x, z)$, $\forall x, y, z \in V$, $\forall a, b \in \mathbb{K}$.

b) g s.n. formă biliniară simetrică dacă în plus $g(x, y) = g(y, x)$, $\forall x, y \in V$

c) g s.n. formă biliniară antisimetrică dacă în plus, $g(x, y) = -g(y, x)$, $\forall x, y \in V$.

OBS $\left(L(V,V;\mathbb{K}) = \{g: V\times V \longrightarrow \mathbb{K} \,/\, g \text{ formă biliniară}\}, +, \cdot\right)_{/\mathbb{K}}$ spațiu vectorial

$L^s(V,V;\mathbb{K}) = \{g \in L(V,V;\mathbb{K}) \,/\, g \text{ simetrică}\}$,

$L^a(V,V;\mathbb{K}) = \{g \in L(V,V;\mathbb{K}) \,/\, g \text{ antisimetrică}\}$

subspații vectoriale în $L(V,V;\mathbb{K})$

OBS $g: V\times V \longrightarrow \mathbb{K}$.

Dacă $g(x,y) = g(y,x)$ și g liniară într-un argument, at g este biliniară

Matricea asociată unei forme biliniare

Fie $\mathcal{R} = \{e_1, .., e_n\}$ reper în V

$g(e_i, e_j) = g_{ij}$; $G = (g_{ij})_{i,j=\overline{1,n}}$

$$g(x,y) = g\left(\sum_{i=1}^{n} x_i e_i , \sum_{j=1}^{n} y_j e_j\right) =$$

$$= \sum_{i,j=1}^{n} g_{ij}\, x_i y_j = X^T G\, Y$$

$$= (x_1 \dots x_n) \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \\ g_{n1} & \dots & g_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Modificarea matricei la schimbarea reperului

$\mathcal{R} = \{e_1, .., e_n\} \xrightarrow[C \in GL(n,\mathbb{K})]{C} \mathcal{R}' = \{e_1', .., e_n'\}$ repere în V

$G = (g_{ij})_{i,j=\overline{1,n}}$ $\qquad G' = (g'_{ij})_{i,j=\overline{1,n}}$

$g_{ij} = g(e_i, e_j)$ $\boxed{G' = C^T G C}$ $g'_{ij} = g(e'_i, e'_j)$

Prop $\operatorname{rg} G = \operatorname{rg} G'$ (invariant la sch. reperului)

Dem $\operatorname{rg}(G') = \operatorname{rg}(C^T G C) = \operatorname{rg} G$, $C \in GL(n, \mathbb{K})$

OBS $g: V \times V \longrightarrow \mathbb{K}$.

a) g simetrică $\Leftrightarrow G = G^T$

$G' = C^T G C$

$(G')^T = (C^T G C)^T = C^T G^T (C^T)^T = C^T G C = G'$

b) g antisimetrică $\Leftrightarrow G = -G^T$

Def Fie $g \in L^s(V, V; \mathbb{K})$.

$\operatorname{Ker} g = \{x \in V \,/\, g(x, y) = 0_{\mathbb{K}},\ \forall y \in V\}$.

g nedegenerată $\Leftrightarrow \operatorname{Ker} g = \{0_V\}$.

OBS $R = \{e_1, \dots, e_n\}$ reper în V, $x \in \operatorname{Ker} g$

$$\begin{cases} g(x, e_1) = 0 \\ \vdots \\ g(x, e_n) = 0 \end{cases} \Rightarrow \begin{cases} g\left(\sum_{i=1}^{n} x_i e_i, e_1\right) = 0 \\ \vdots \\ g\left(\sum_{i=1}^{n} x_i e_i, e_n\right) = 0 \end{cases} \Rightarrow (*) \begin{cases} \sum_{i=1}^{n} g_{i1} x_i = 0 \\ \vdots \\ \sum_{i=1}^{n} g_{in} x_i = 0 \end{cases}$$

$(*)$ SLO. Are sol unică nulă $\Leftrightarrow \det(G) \neq 0 \Leftrightarrow$ G nedegenerată.

Exemplu $g: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$, $g(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$.

$R_0 = \{e_1, e_2, e_3\}$ reper canonic.

$g(e_1, e_1) = 1$, $g(e_2, e_2) = 1$, $g(e_3, e_3) = 1$

$g(e_i, e_j) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ $\quad G = I_3$, $\quad \det G = 1$.

g nedeg $\Rightarrow \operatorname{Ker} g = \{0_{\mathbb{R}^3}\}$.

$$g(x,y) = X^T I_3 Y$$

$$\underbrace{(x_1 \quad x_2 \quad x_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$(1,3)\ (3,1)$

$\downarrow$

$(1,1)$