Laborator 5 - solutii

Exercitiul 1

Fie $(X, Y) : \Omega \to \mathbb{R}^2$ cu densitate. Stim densitatea cuplului (X, Y):

$$p(x, y) = \begin{cases} c \cdot x^2 \cdot y, 0 \leq x, y \leq 1, x + y \leq 1 \\ 0, \text{ in rest} \end{cases}$$

- $\mathbb{E}[X], \mathbb{E}[Y]$.
- $\mathbb{E}[XY]$.
- Sunt X si Y independente?

Solutie:
Stim densitatea cuplului (X, Y).
Stim ca $\mathbb{E}[X] = \int_{\mathbb{R}} x \cdot p_X(x)\, dx$, unde $p_X : \mathbb{R} \to \mathbb{R}$ este densitatea lui X (stim din ipoteza ca X are densitate). Observam ca nu putem calcula in acest mod $\mathbb{E}[X]$ intrucat nu stim densitatea lui X, p_X.
Mai stim ca pentru orice functie masurabila $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, avem ca $\mathbb{E}[f(X, Y)] = \int_{\mathbb{R}\times\mathbb{R}} f(x, y) \cdot p(x, y)\, dx\, dy$. In cazul de fata, daca alegem $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f(x, y) = x$, atunci avem $\mathbb{E}[X] = \int_{\mathbb{R}\times\mathbb{R}} x \cdot p(x, y)\, dx\, dy = c \cdot \int_{\mathcal{D}} x \cdot x^2 \cdot y\, dx\, dy$, unde $\mathcal{D} = \{(x, y) \in [0, 1] \times [0, 1] | x + y \leq 1\}$.
Vrem sa aflam mai intai valoarea lui c. Stim ca $\int_{\mathbb{R}\times\mathbb{R}} p(x, y)\, dx\, dy = 1$, deci $\int_{\mathcal{D}} c \cdot x^2 \cdot y\, dx\, dy = 1$.
Stim (curs) ca $\int_{\mathcal{D}} f(x, y)\, dx\, dy = \int_{\mathbb{R}}(\int_{\mathcal{D}_x} f(x, y)\, dy)\, dx$, unde $\mathcal{D}_x = \{y \in [0, 1] | (x, y) \in \mathcal{D}\} = \{y \in [0, 1] | y \leq 1 - x\}$. Deci, $\int_{\mathcal{D}} p(x, y)\, dx\, dy = \int_{[0,1]}(\int_0^{1-x} c \cdot x^2 \cdot y\, dy)\, dx = c \cdot \int_0^1 x^2 \cdot (\int_0^{1-x} y\, dy)\, dx = c/2 \cdot \int_0^1 x^2 \cdot (1-x)^2\, dx = c/2 \cdot \int_0^1 (x^2 - 2x^3 + x^4)\, dx = c/2 \cdot (1/3 - 1/2 + 1/5) = c/2 \cdot 1/30$. Deci, $c = 60$.
Atunci, $\mathbb{E}[X] = 60 \cdot \int_0^1 x^3 (\int_0^{1-x} y\, dy)\, dx = 30 \cdot \int_0^1 x^3 \cdot (1-x)^2\, dx = 30 \cdot \int_0^1 x^3 \cdot (1 - 2x + x^2)\, dx = 30 \int_0^1 x^3 - 2x^4 + x^5\, dx = 30 \cdot (1/4 - 2/5 + 1/6) = 1/2$.
$\mathbb{E}[Y] = 60 \cdot \int_0^1 x^2 (\int_0^{1-x} y^2\, dy)\, dx = 20 \cdot \int_0^1 x^2 \cdot (1-x)^3\, dx = 30 \cdot \int_0^1 x^2 \cdot (1 - 3x + 3x^2 - x^3)\, dx = 20 \cdot (1/3 - 3/4 + 3/5 - 1/6) = 1/3$.
$\mathbb{E}[XY] = \int_{\mathbb{R}\times\mathbb{R}} xy \cdot p(x, y)\, dx\, dy = 60 \cdot \int_0^1 (\int_{\mathcal{D}_x} x^3 \cdot y^2\, dy)\, dx = \int_0^1 x^3 \cdot (\int_0^{1-x} y^2\, dy)\, dx = 20 \cdot \int_0^1 x^3 \cdot (1-x)^3\, dx = 20 \cdot \int_0^1 x^3 \cdot (1 - 3x + 3x^2 + x^3)\, dx = 20(1/4 - 3/5 + 3/6 - 1/7) = 1/7$.
Observam ca $\mathbb{E}[XY] = 1/7 \neq \mathbb{E}[X] \cdot \mathbb{E}[Y] = 1/2 \cdot 1/3$. Deci, X si Y dependente.

Exercitiul 2

Fie $(X,Y):\Omega\to\mathbb{R}^2$ cu

$$p(x,y)=\begin{cases}c\cdot(x^2+y), 0\le x,y\le 1\\ 0,\ \text{in rest}\end{cases}$$

Calculati:

- $\mathbb{E}[X],\mathbb{E}[Y]$.
- $\mathrm{Var}(X),\mathrm{Var}(Y)$.
- $\mathrm{Cov}(X,Y)$.

Solutie:

- Luam $f(x,y)=x$ si calculam $\mathbb{E}[f(X,Y)]$: $\mathbb{E}[X]=\int_{\mathbb{R}\times\mathbb{R}}x\cdot p(x,y)\,dx\,dy=c\cdot\int_{[0,1]\times[0,1]}x\cdot(x^2+y)\,dy\,dx=c\cdot\int_0^1(\int_0^1 x^3+xy\,dy)dx=c\cdot\int_0^1 x^3+x\cdot(1^2/2-0^2/2)\,dx=c\cdot\int_0^1 x^3+x/2\,dx=c\cdot(1/4+1/4)=c/2$.
  Calculam c. Stim ca $\int_{\mathbb{R}\times\mathbb{R}}p(x,y)\,dx\,dy=1$, deci $c\cdot\int_0^1(\int_0^1 x^2+y\,dy)\,dx=c\cdot\int_0^1 x^2+1/2\,dx=c\cdot(1/3+1/2)=5c/6=1$. Deci, $c=6/5$.
  Astfel, $\mathbb{E}[X]=3/5$.
  $\mathbb{E}[Y]=c\cdot\int_{[0,1]\times[0,1]}y\cdot(x^2+y)\,dy\,dx=c\cdot\int_0^1(\int_0^1 x^2y+y^2\,dy)dx=c\cdot\int_0^1 x^2/2+1/3\,dx=c\cdot(1/6+1/3)=6/5\cdot 1/2=3/5$.

- Calculam $\mathbb{E}[X^2]=c\int_0^1(\int_0^1(x^2(x^2+y)\,dy)\,dx=c\cdot\int_0^1 x^4+x^2/2\,dx=c\cdot(1/5+1/6)=6/5\cdot 11/30=11/25$. Deci, $\mathrm{Var}(X)=11/25-9/25=2/25\Rightarrow\sigma(X)=\frac{\sqrt{2}}{5}\approx 0.34$.
  Calculam $\mathbb{E}[Y^2]=c\int_0^1(\int_0^1(y^2(x^2+y)\,dy)\,dx=c\cdot\int_0^1 x^2/3+1/4\,dx=c\cdot(1/9+1/4)=6/5\cdot 13/36=13/30$. Deci, $\mathrm{Var}(Y)=13/30-9/25=\frac{11}{6\cdot 25}\Rightarrow\sigma(Y)=\frac{\sqrt{11}}{\sqrt{6}}\cdot\frac{1}{5}\approx 0.27$.
  Asadar, $X\approx\frac{3}{5}\pm 0.34, Y\approx\frac{3}{5}\pm 0.27$.

- $\mathrm{Cov}(X,Y)=\mathbb{E}[XY]-\mathbb{E}[X]\cdot\mathbb{E}[Y]$.
  Calculam $\mathbb{E}[XY]=c\cdot\int_0^1(\int_0^1 xy(x^2+y)\,dy)\,dx=c\cdot\int_0^1 x^3/2+x/3\,dx=6/5\cdot(1/8+1/6)=7/20$.
  Avem $\mathrm{Cov}(X,Y)=7/20-9/25=0.35-0.36=-0.1$.
  $\mathrm{Corr}(X,Y)=\frac{\mathrm{Cov}(X,Y)}{\sigma(X)\cdot\sigma(Y)}=-\frac{0.1}{0.34\cdot 0.27}\approx -0.35$.

Interpretare:

- X si Y nu sunt independente ptr ca $\mathrm{Cov}(X,Y)\neq 0$;

- X si Y nu sunt dependente liniar pentru ca $\text{Corr}(X, Y) \sim -0.35 \neq \pm 1$.
- Faptul ca $\text{Corr}(X, Y) < 0$ ne indica faptul ca X si Y au valori invers proportionale ($\mathbb{E}[XY] < \mathbb{E}[X] \cdot \mathbb{E}[Y]$ implica faptul ca daca X ia valori mari, Y ia valori mici si invers).