An Attempt to Find an Optimal Wavelet for an Arbitrary Input Signal

Stefan Cline

San Diego State University, Department of Mathematics and Statistics

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Section Outline

Project Goal

I wanted to see if, given an arbitrary input signal, I could use an ML algorithm to 'learn' the best parameters for a predefined wavelet shape.

Project Goal

Generic **mother wavelet** form

$$\tilde{\psi}(t) = \exp\left(\frac{-t^2}{2}\right)(\varepsilon t + A\sin(Bt))$$

By using only odd functions being damped by a Gaussian, we satisfy (odd function times even is odd) the first condition. Dividing by the L^2 Norm we satisfy the second.

$$\underbrace{\int_{\mathbb{R}} \psi \, dt = 0, \qquad ||\psi||_{L^2} = 1}_{\text{Conditions}} \qquad \Rightarrow \qquad \psi(t) = \frac{\tilde{\psi}(t)}{||\tilde{\psi}(t)||_{L^2}}$$

General Equation Form

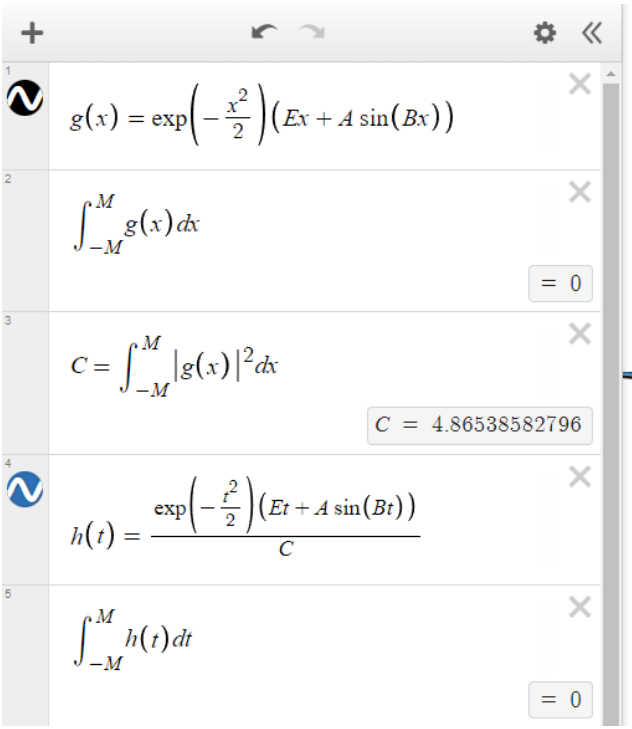
Analytical expression for a **daughter wavelet** indexed by $j \in (J \subset \mathbb{Z}^+)$:

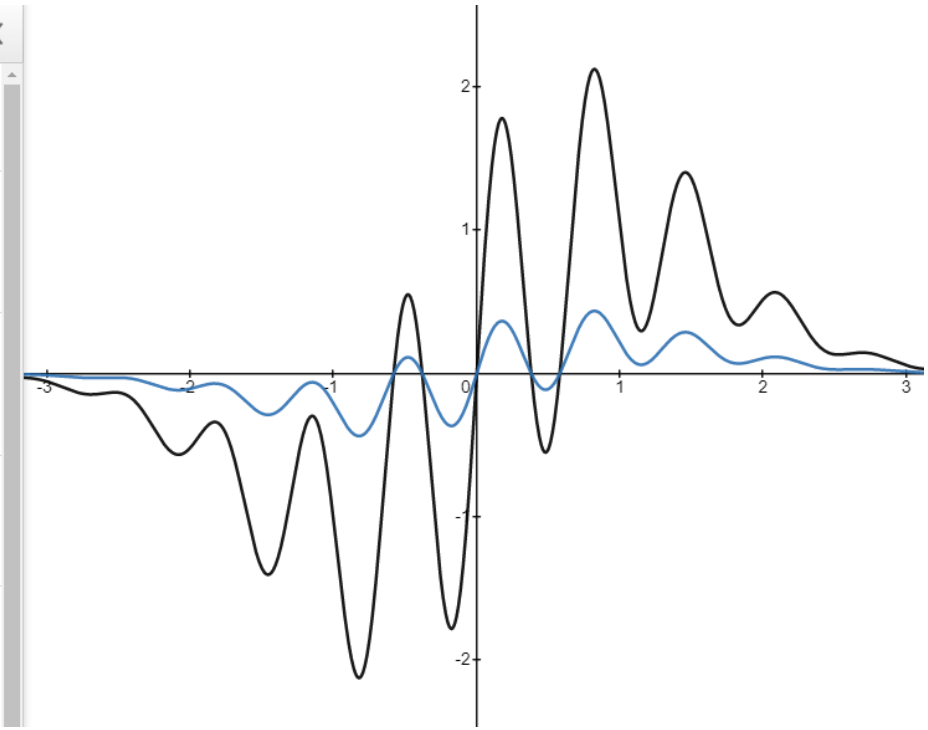
$$\varphi_j(t) = \frac{1}{\sqrt{a^j}}\psi\left(\frac{t}{a^j}\right)$$

$$= \frac{1}{\sqrt{a^j}} \frac{\exp\left(-\frac{1}{2}\frac{t^2}{a^{2j}}\right)\left(\frac{\varepsilon t}{a^j} + A\sin\left(\frac{Bt}{a^j}\right)\right)}{\left(\int_{\mathbb{R}} \exp\left(-\frac{t^2}{a^{2j}}\right)\left(\frac{\varepsilon t}{a^j} + A\sin\left(\frac{Bt}{a^j}\right)\right)^2 \, dt\right)^{1/2}}$$

Example Mother Wavelet

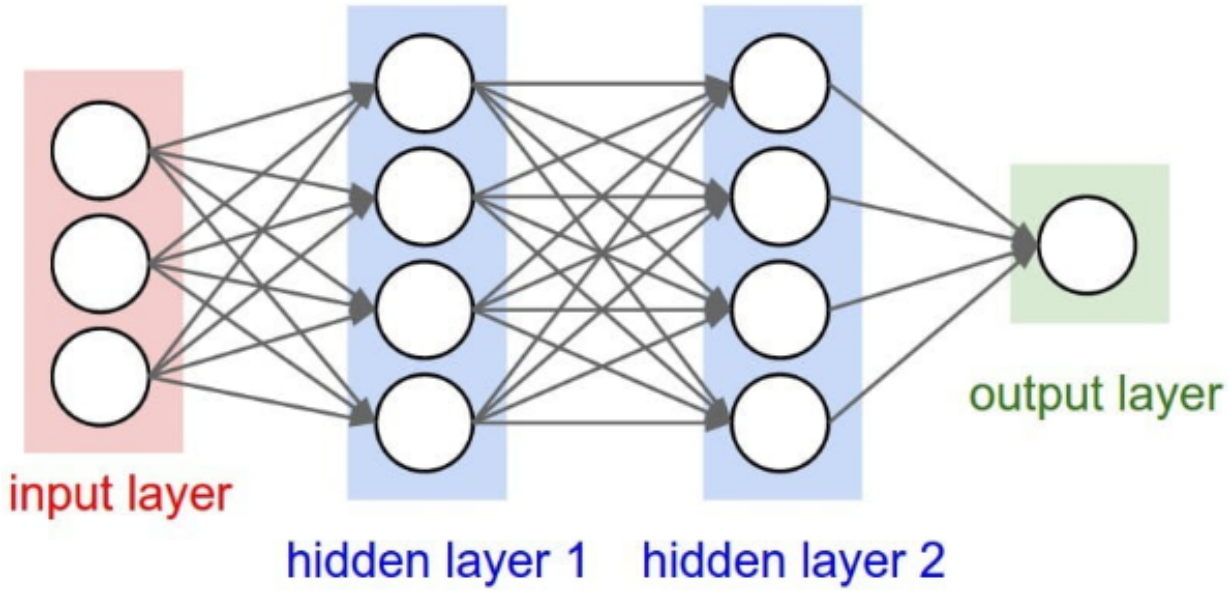
Choosing: $\varepsilon = 1.8$, $A = 1.5$ and $B = 9.5$. The blue curve is the normalization of the black curve.





How ML Works (1)

Think of each node (circle) as containing some value and the (lines) mathematical operations performed on them produce new nodes.



How ML Works (2)

The input $\equiv \mathbf{x}$, is fed through the NN $\equiv \mathcal{N}$, produces some output $\mathbf{y}$. So,

$$\mathcal{N}(\mathbf{x}) = \mathbf{y}$$

How well did $\mathbf{y}$ match up to our expectations?

This is determined by a **cost function**. In *supervised learning*, the 'answer' is known. Using least squares ($D \equiv$ Desired)

$$C = (\mathcal{N}(\mathbf{x}) - \mathcal{N}_D(\mathbf{x}))^2 = (\mathbf{y} - \mathbf{y}_D)^2$$

How ML Works (3)

The NN must learn. Using **gradient descent**, each partial derivative per parameter is calculated to get $-\nabla C$ where

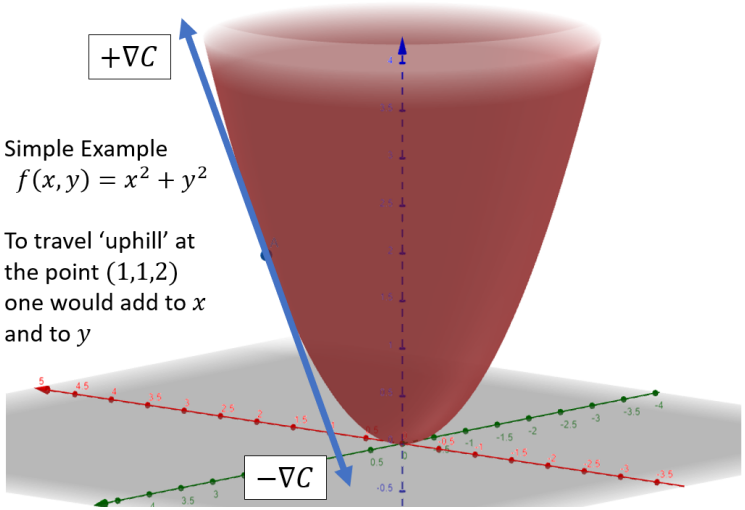
$$\nabla C = \begin{bmatrix} \frac{\partial C}{\partial A} \\ \\ \frac{\partial C}{\partial B} \end{bmatrix} \Rightarrow \frac{\partial C}{\partial \Gamma} = \begin{bmatrix} \partial_{j_0,t_0} & \partial_{j_0,t_1} & \cdots & \partial_{j_0,t_M} \\ \partial_{j_1,t_0} & \ddots & & \\ \vdots & & \ddots & \\ \partial_{j_N,t_0} & \cdots & & \partial_{j_N,t_M} \end{bmatrix}$$ Discretized

The big take-away here is that we are trying to **minimize our cost function**[1].

[1] Note, $\{j_0, ..., j_N\} \in J$, $\{t_0, t_M\} \in \Omega$, and Γ can be a stand-in for A or B

How ML Works (4)

Gradients give steepest ascent for a given parameter[2]. Then, simply add or subtract a small value known as the **learning rate** from the parameter to move it 'downhill'.



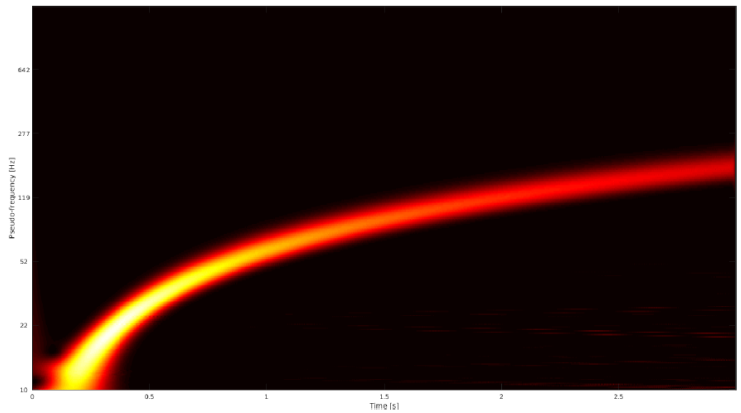
This whole process is known as **back propagation**.

[2]This is what's given to us from finding ∇C.

Challenge with Supervised Learning

What is a 'good' wavelet?

I noticed this classic example seemed to be zero almost everywhere except in a tight band.



I told the algorithm to learn to minimize the field, i.e.,
$\forall j \in (J \subset \mathbb{Z}^+), \forall t \in \Omega,\ \mathcal{N}_D(\mathbf{x}) = \mathbf{y}_D = 0.$

Choice of $\tilde{\psi}$

Recall:

$$\tilde{\psi}(t) = \exp\left(\frac{-t^2}{2}\right)(\varepsilon t + A\sin(Bt))$$

This brings us back to the εt term. This will ensure that our wavelet simply doesn't learn to become zero.

We needed a robust way to ensure the algorithm doesn't learn to minimize our parameters to be zero.

Enhanced Cost Function

Keeping the zero field idea

$$C = (\mathcal{N}(\mathbf{x}) - \mathcal{N}_D(\mathbf{x}))^2$$

but now add in two biasing terms that heavily penalize the algorithm if it tries to make our parameters too small.

$$C = (\mathcal{N}(\mathbf{x}) - \mathcal{N}_D(\mathbf{x}))^2 + \mathcal{Q}_A + \mathcal{Q}_B$$

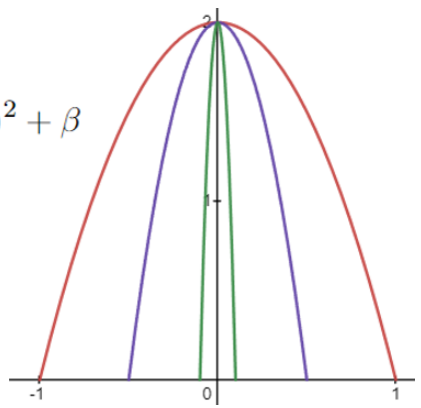
Again note that from $\hat{\psi}(t)$ the only two parameters for this example are A and B ($\Gamma \in \{A, B\}$).

$\mathcal{Q}_\Gamma = -(l^{-1}\sqrt{\beta}\Gamma)^2 + \beta$

$l \equiv$ limit.

$\beta \equiv$ added error.

$\Gamma \equiv$ parameter.



Forward Propagation (1)

We know that in the forward direction we'll generate a **scalogram** by computing

$$\forall j \in J, \qquad W_j = \varphi_j(t) * f(t) = \int_{\mathbb{R}} \varphi_j(t-\tau) * f(\tau)\, d\tau$$

$$W := \frac{e^{-\frac{(t-\tau)^2}{2a^{2j}}}\left(\frac{\varepsilon(t-\tau)}{a^j} + A\sin\left(\frac{B(t-\tau)}{a^j}\right)\right) f(\tau)}{\sqrt{a^j}\sqrt{\int_{-\infty}^{\infty} e^{-\frac{(t-\tau)^2}{a^{2j}}}\left(\frac{\varepsilon(t-\tau)}{a^j} + A\sin\left(\frac{B(t-\tau)}{a^j}\right)\right)^2 d\tau}}$$

Forward Propagation (2)

Each discrete point in the field/scalogram then becomes

$$\begin{array}{c} j_0 \\ \downarrow \\ J \end{array} \begin{bmatrix} W_{t_0,j_0} = \int_\Omega \varphi(t_0-\tau)f(\tau)d\tau & \cdots & W_{t_0,j_0} = \int_\Omega \varphi(t_T-\tau)f(\tau)d\tau \\ W_{t_0,j_1} = \int_\Omega \varphi(t_0-\tau)f(\tau)d\tau & & \\ \vdots & \ddots & \\ W_{t_0,j_J} = \int_\Omega \varphi(t_0-\tau)f(\tau)d\tau & & W_{t_T,j_J} = \int_\Omega \varphi(t_T-\tau)f(\tau)d\tau \end{bmatrix}$$

$$t_0 \longrightarrow T$$

Each point was easily calculated because nicely we know $\hat{h}[k] = \hat{f}[k]\hat{g}[k] = \widehat{f \otimes g}[k]$, which is required for DWTs

Analytical Back Propagation - Finding ∇C (1)

Note here we still need to do $\int_{\mathbb{R}} EQA \; d\tau$

$$> \; EQA := diff(W, A)$$

$$EQA := \frac{e^{-\frac{(t-\tau)^2}{2a^{2j}}} \sin\left(\frac{B(t-\tau)}{a^j}\right) f(\tau)}{\sqrt{a^j}\sqrt{\int_{-\infty}^{\infty} e^{-\frac{(t-\tau)^2}{a^{2j}}} \left(\frac{\varepsilon(t-\tau)}{a^j} + A\sin\left(\frac{B(t-\tau)}{a^j}\right)\right)^2 d\tau}} - \left(e^{-\frac{(t-\tau)^2}{2a^{2j}}} \left(\frac{\varepsilon(t-\tau)}{a^j} + A\sin\left(\frac{B(t-\tau)}{a^j}\right)\right) f(\tau) \left(\int_{-\infty}^{\infty} 2\, e^{-\frac{(t-\tau)^2}{a^{2j}}} \left(\frac{\varepsilon(t-\tau)}{a^j} + A\sin\left(\frac{B(t-\tau)}{a^j}\right)\right) \sin\left(\frac{B(t-\tau)}{a^j}\right) d\tau\right)\right) \Bigg/ \left(2\sqrt{a^j}\left(\int_{-\infty}^{\infty} e^{-\frac{(t-\tau)^2}{a^{2j}}} \left(\frac{\varepsilon(t-\tau)}{a^j} + A\sin\left(\frac{B(t-\tau)}{a^j}\right)\right)^2 d\tau\right)^{3/2}\right)$$

Analytical Back Propagation - Finding ∇C (2)

Again we still need to do $\int_{\mathbb{R}} EQB \; d\tau$

$$> \; EQB := diff(W, B)$$

$$EQB := \frac{e^{-\frac{(t-\tau)^2}{2\alpha^{2j}}} A(t-\tau)\cos\left(\frac{B(t-\tau)}{\alpha^j}\right) f(\tau)}{(\alpha^j)^{3/2}\sqrt{\int_{-\infty}^{\infty} e^{-\frac{(t-\tau)^2}{\alpha^{2j}}}\left(\frac{\varepsilon(t-\tau)}{\alpha^j} + A\sin\left(\frac{B(t-\tau)}{\alpha^j}\right)\right)^2 d\tau}} - \left(e^{-\frac{(t-\tau)^2}{2\alpha^{2j}}}\left(\frac{\varepsilon(t-\tau)}{\alpha^j} + A\sin\left(\frac{B(t-\tau)}{\alpha^j}\right)\right) f(\tau)\left(\int_{-\infty}^{\infty} \frac{2\, e^{-\frac{(t-\tau)^2}{\alpha^{2j}}}\left(\frac{\varepsilon(t-\tau)}{\alpha^j} + A\sin\left(\frac{B(t-\tau)}{\alpha^j}\right)\right) A(t-\tau)\cos\left(\frac{B(t-\tau)}{\alpha^j}\right)}{\alpha^j} d\tau\right)\right) \Bigg/ \left(2\sqrt{\alpha^j}\left(\int_{-\infty}^{\infty} e^{-\frac{(t-\tau)^2}{\alpha^{2j}}}\left(\frac{\varepsilon(t-\tau)}{\alpha^j} + A\sin\left(\frac{B(t-\tau)}{\alpha^j}\right)\right)^2 d\tau\right)^{3/2}\right)$$

Analytical Back Propagation - Finding ∇C (3)

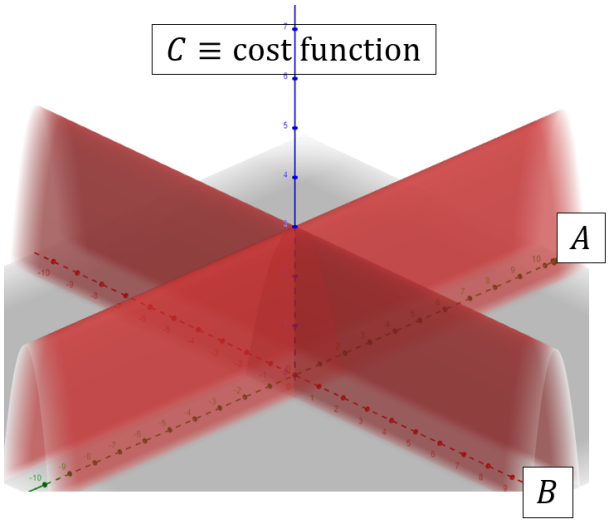
```
# num = numerator, dem = denominator
def EQA(neg_oh,t,tau,two,thd2,a_to_j,A,B,f_t,delta_t,eps):
  Q1 = (t-tau)/a_to_j
  a_to_j2 = a_to_j**two
  S1 = A*np.sin(B*Q1)
  # usin em
  num1 = np.exp(neg_oh*(t-tau)**two/(a_to_j2))*np.sin(B*Q1)*f_t
  dem1_intg = np.exp(-(t-tau)**two/a_to_j2)*(eps*Q1+S1)**two
  dem1_integral = integrate.simpson(dem1_intg,dx=delta_t)
  dem1 = np.sqrt(a_to_j)*np.sqrt(dem1_integral)
  T1 = num1/dem1
  LT = np.exp(neg_oh*(t-tau)**two/(two*a_to_j2))*(eps*Q1+S1)*f_t
  RT_intg = two*np.exp(-(t-tau)**two/a_to_j2)*(eps*Q1+S1)*np.sin(B*Q1)
  RT = integrate.simpson(RT_intg,dx=delta_t)
  dem2_intg = np.exp(-(t-tau)**two/a_to_j2)*(eps*Q1+S1)**two
  dem2 = two*np.sqrt(a_to_j)*(integrate.simpson(dem2_intg,dx=delta_t))**(thd2)
  A_curve = T1-LT*RT/dem2
  return integrate.simpson(A_curve,dx=delta_t)


def EQB(neg_oh,t,tau,two,thd2,a_to_j,A,B,f_t,delta_t,eps):
  Q1 = B*(t-tau)/a_to_j
  Q2 = eps*(t-tau)/a_to_j
  Trigs = A*np.sin(Q1)
  Trigc = A*(t-tau)*np.cos(Q1)
  # now to use em
  num1 = np.exp(neg_oh*(t-tau)**two)*Trigc*f_t
  dem1_intg = np.exp(-(t-tau)**two/a_to_j**two)*(Q2+Trigs)**two
  dem1 = a_to_j**thd2*np.sqrt(integrate.simpson(dem1_intg,dx=delta_t))
  LT = np.exp(neg_oh*(t-tau)**two/a_to_j**two)*(Q2+Trigs)*f_t
  RT_intg = two*np.exp(-(t-tau)/a_to_j)*(Q2+Trigs*Trigc)
  RT = integrate.simpson(RT_intg,dx=delta_t)
  dem2_intg = np.exp(-(t-tau)**two/a_to_j**two)*(Q2+Trigs)**two
  dem2 = two*np.sqrt(a_to_j)*(integrate.simpson(dem2_intg,dx=delta_t))**thd2
  B_curve = num1/dem1-LT*RT/dem2
  return integrate.simpson(B_curve,dx=delta_t)
```

Gross $\rightarrow$

Quick Note on Parameter Space

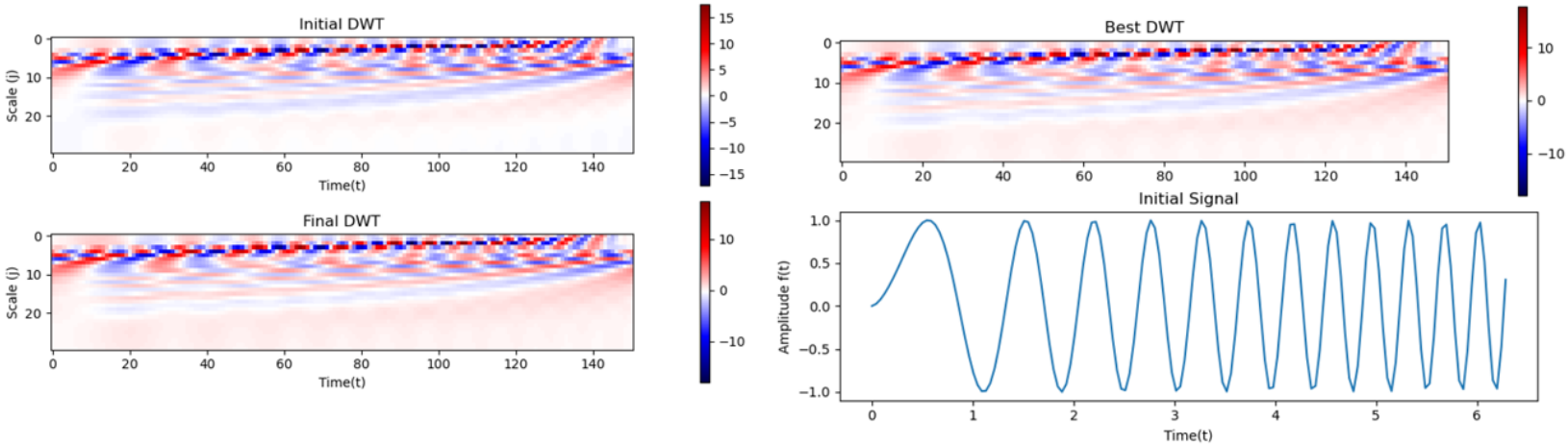
By forcing $A \neq 0,\ B \neq 0$ via $\mathcal{Q}_\Gamma$, the parameter space is partitioned.

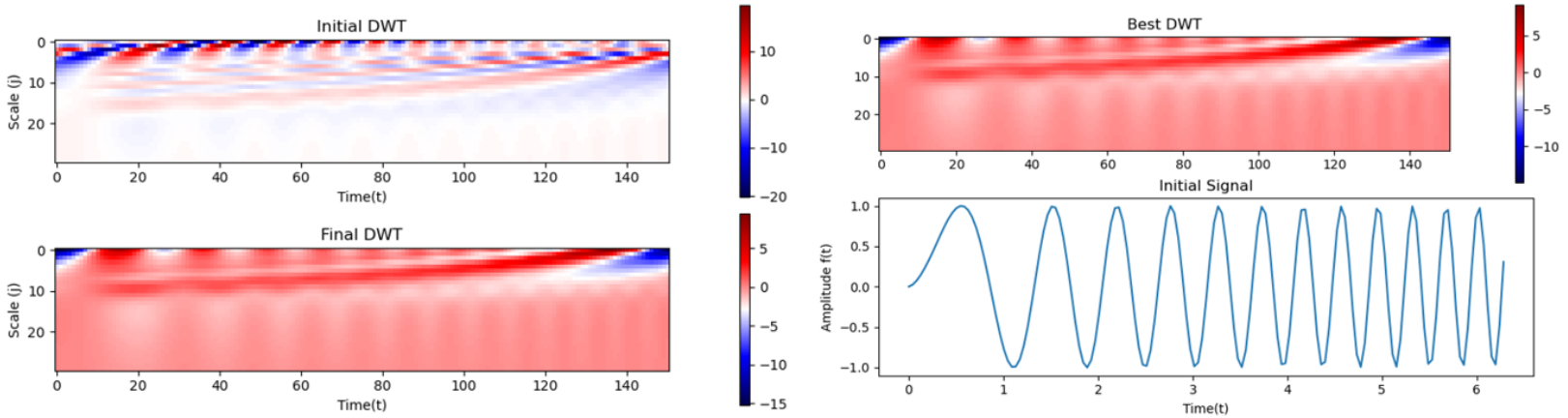


Therefore, while testing, I had to be cognizant of which of the 4 sections I was starting in $(A^+B^+,\ A^-B^+,\ A^+B^-,\ A^-B^-)$

Mixed Results for $f(t) = \sin\left(4t^{1.6}\right)$ (1)

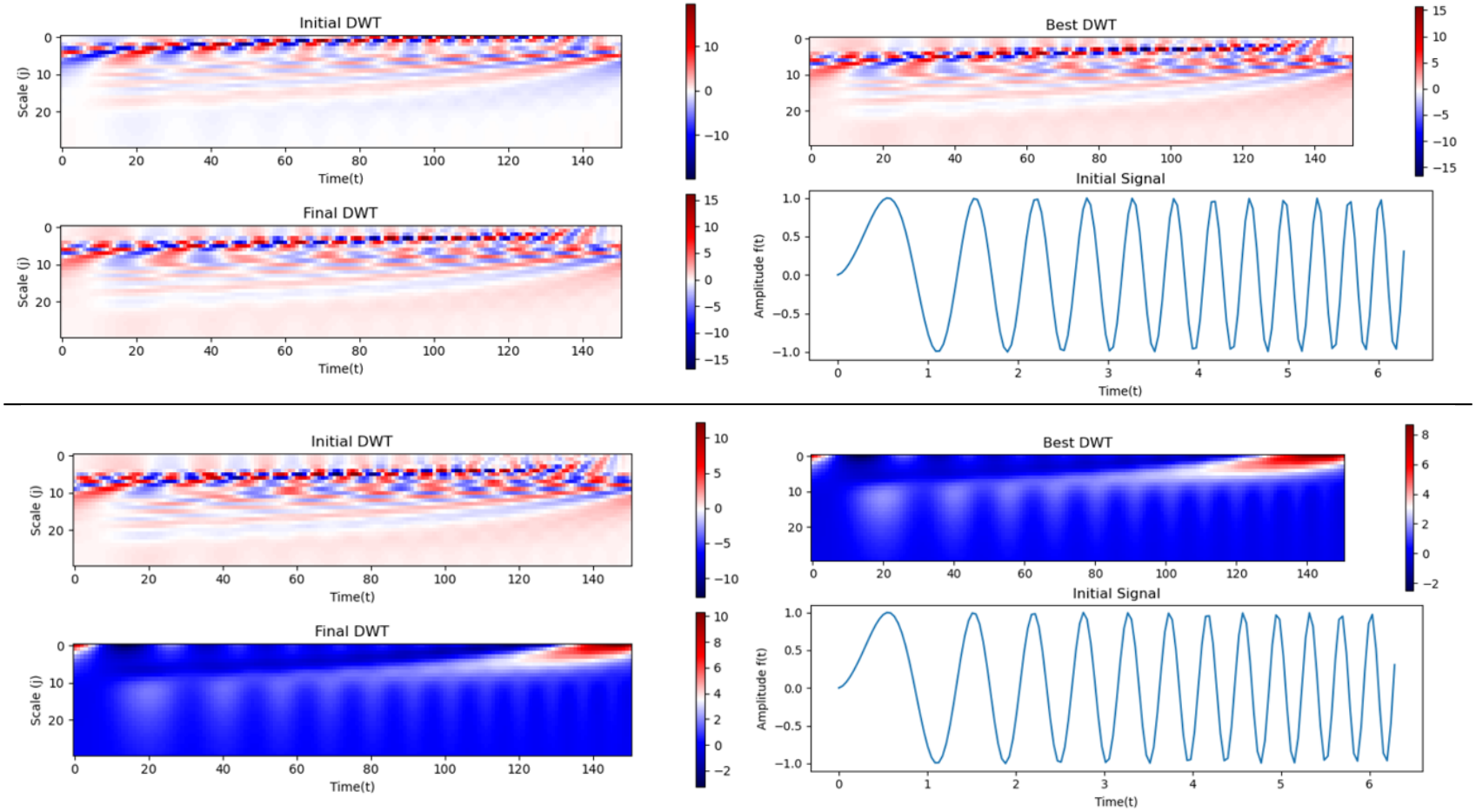
Top: A^+, B^+, Bottom: A^+, B^-





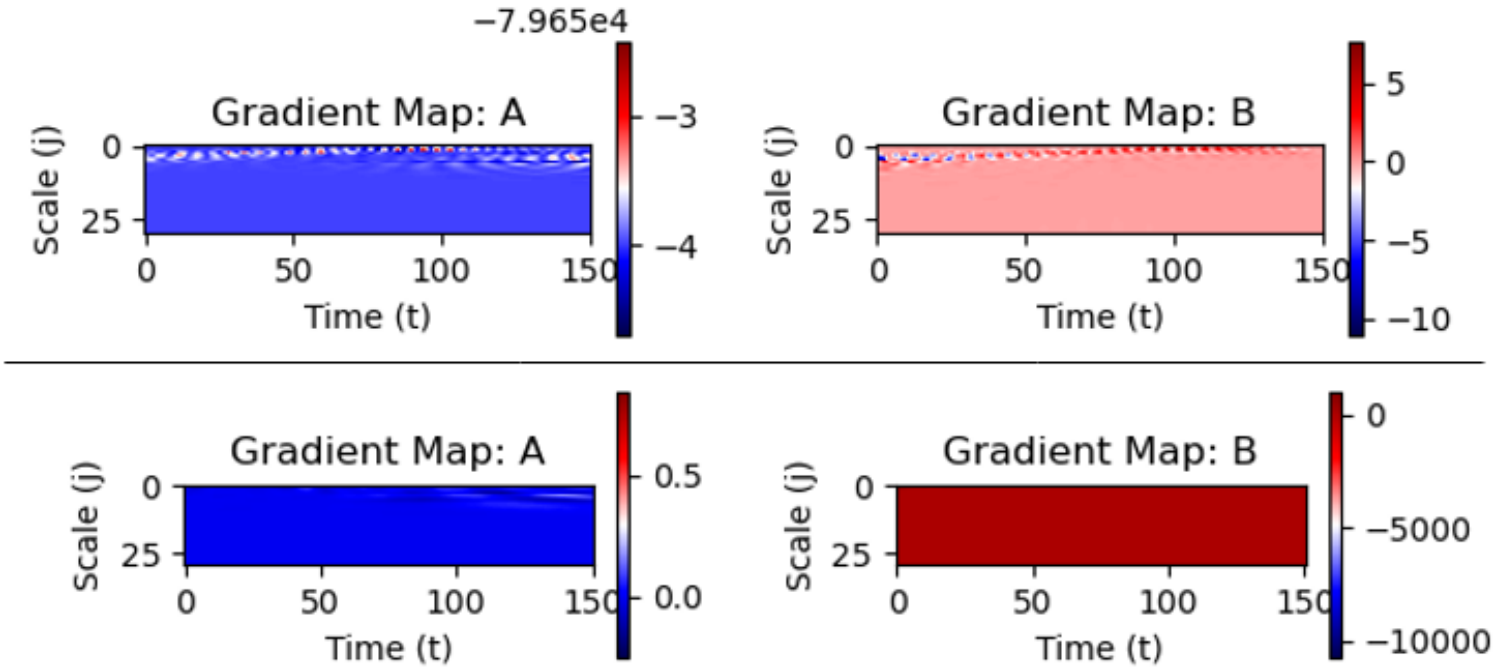
Mixed Results for $f(t) = \sin\left(4t^{1.6}\right)$ (2)

Top: A^-, B^+, Bottom: A^-, B^-



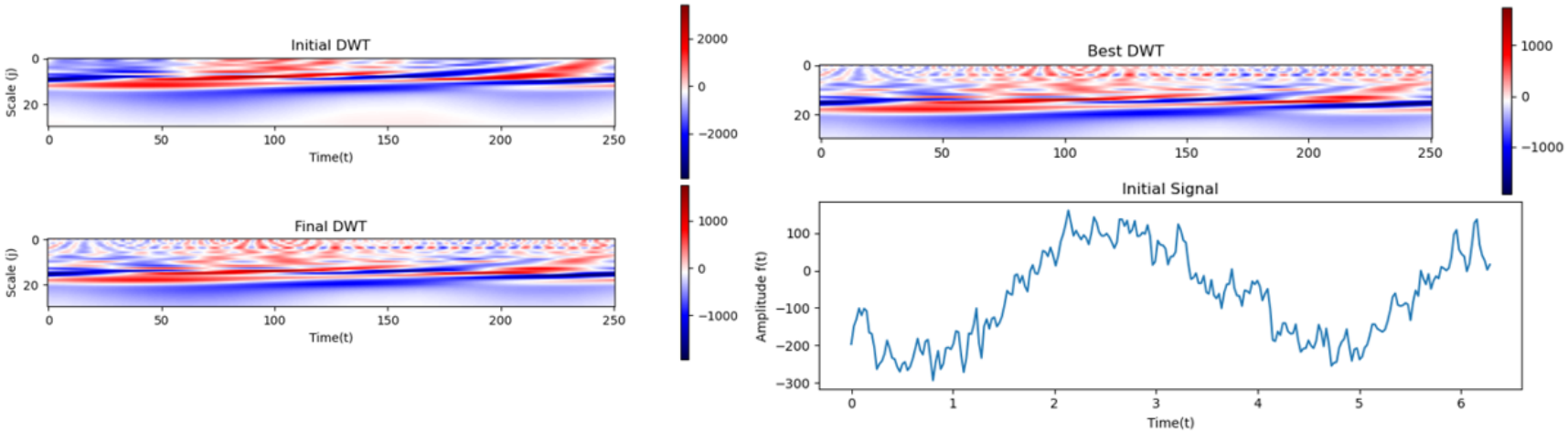
Verification of Minimization

Did the algorithm actually do what I wanted it to?
Top: A^+, B^+, Bottom: A^+, B^-



Arbitrary Signal - Seismic Data

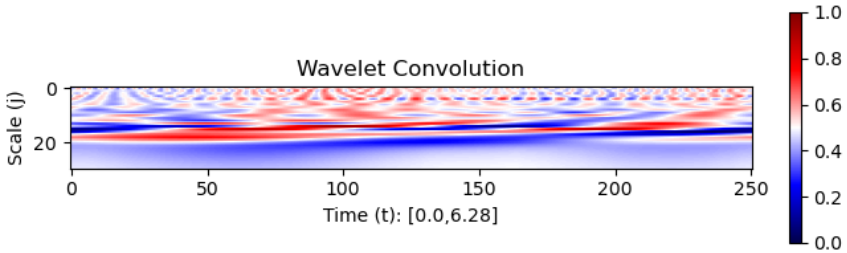
I took this signal from Dr. Gilles' seismic data (available in MatLab). From an eyeball perspective, this did seem to work decently well, and gave the DWT a more structured appearance while also minimizing the field.

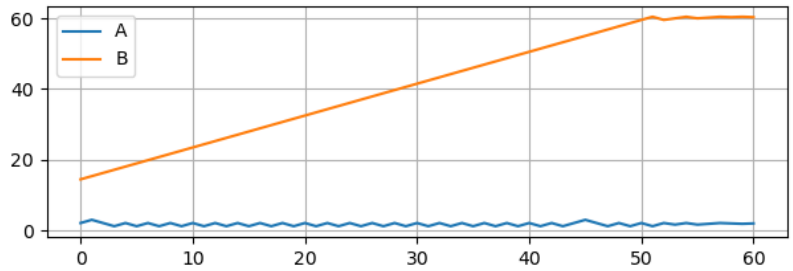


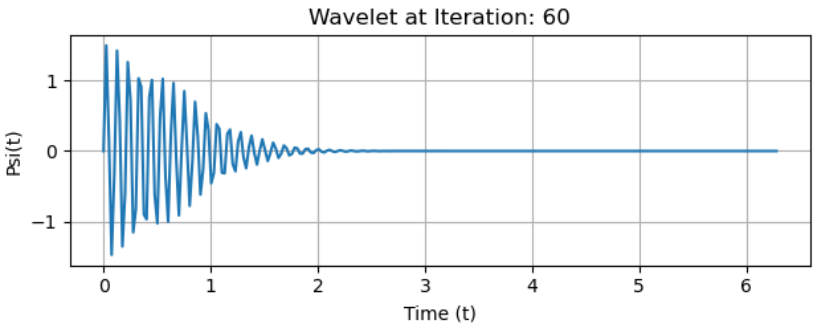
Optimized to a Non-Trivial Curve

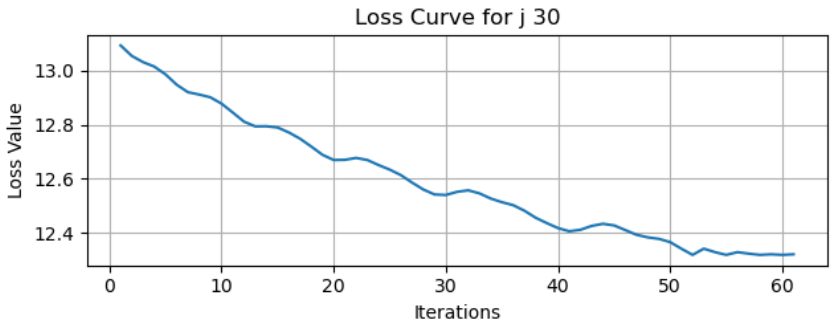
Here we see all of this nicely come together.

1. The loss curve goes down.
2. Clear difference in wavelet shape from $A_0, B_0 \to A_{63}, B_{63}$
3. The scalogram looks more organized









Concluding Thoughts

o This method was finicky and required several iterations. Manual verification was often required. However, some results did seem promising/interesting.

o I would be curious to know what an expert in this space would have to say about my final results.

o More complex potential wavelet forms both real and complex possible (where $\mathcal{O} \equiv$ odd functions)

$$\hat{\psi}(t) = \exp\left(\frac{-t^2}{2}\right)\left(\sum_{n=1}^{N} A_n \sin\left(B_n t^n\right) + \sum_{n=1}^{N} E_n t^{2n-1} + \sum \mathcal{O}\right)$$

References

1. Jerome Gilles (2023). Empirical Wavelet Transforms (https://www.mathworks.com/matlabcentral/fileexchange/42141-empirical-wavelet-transforms), MATLAB Central File Exchange. Retrieved April 25, 2023.
2. Jerome Gilles (2023). Course Reader for Math 668 - Fourier Analysis. San Diego State University.

Links

This presentation, the code, and support paper can be found here:

Click Here.

Questions?