M537 Final Exam

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1 (20pts) Consider the following ODE:

a. Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Briefly explain whether or not this example requires a logarithmic element for the second solution, *i.e.*, provide some justification for which case in the method of Froebenius you are using and some specifics about the case for this example. Determine expressions for the coefficients for these two solutions.
b. In Part a, $y_1(x)$ should be easily recognizable as a basic function. One of the earlier homework problems introduced the **reduction of order** method. Use this technique to find $y_2(x)$ for (1) and compare this solution to your series solution in Part a.

Part a)

$$xy'' - y' + 4x^3y = 0$$

First, we'll determine if $x = 0$ is a singular point. If we use the following notation:

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

We note that $P(x) = x, Q(x) = -1, R(x) = 4x^3y$. Clearly $P(0) = 0$, and therefore $\frac{Q(x)}{P(x)}, \frac{R(x)}{P(x)}$ and $\frac{F(x)}{P(x)}$ cannot be analytic at zero. Hence, $x = 0$ is a singular point. Now, to be a regular singular point, we must have that the following are analytic functions:

$$(x - x_0)\frac{Q(x)}{P(x)}, \qquad (x - x_0)^2\frac{R(x)}{P(x)}$$

So:

$$(x)\frac{-1}{x} = -1, \qquad (x)^2\frac{4x^3}{x} = 4x^4$$

These are both clearly analytic functions.

We now will find the indicial equation. We attempt solutions in the form of:

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \qquad y_2(x) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n+r_1}$$

Placing this generic form into our equation:

$$xy'' - y' + 4x^3y = 0$$

$$x\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)(a_n x^{n+r-2})\right) - \left(\sum_{n=0}^{\infty}(n+r)(a_n x^{n+r-2})\right) + 4x^3\left(\sum_{n=0}^{\infty}(a_n x^{n+r-2})\right) = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)(a_n x^{n+r-1}) - \sum_{n=0}^{\infty}(n+r)(a_n x^{n+r-1}) + \sum_{n=0}^{\infty}4(a_n x^{n+r+3}) = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)(a_n x^{n+r-1}) - \sum_{n=0}^{\infty}(n+r)(a_n x^{n+r-1}) + \sum_{n=4}^{\infty}4(a_n x^{n+r-1}) = 0$$

The start of finding our recurrence relation will then be (with all of our x^{n+r-1} values being the same, it's implied that we're simply dividing it out):

$$(n+r)(n+r-1)a_n - (n+r)a_n + 4a_{n-4} = 0$$

Now, we let $n = 0$ to find our recurrence relation, noting that a_{n-2} will be zero for $n = 0$:

$$(r)(r-1)a_0 - (r)a_0 = 0$$
$$((r)(r-1) - r)\, a_0 = 0$$
$$(r)(r-1) - r = 0$$
$$r^2 - r - r = 0$$
$$r^2 - 2r = 0$$
$$r(r-2) = 0$$
$$r_1 = 0, r_2 = 2$$

Therefore, we observe we have the most difficult case, namely, two distinct roots that are separated by an integer, i.e.,

$$r_1 - r_2 = N, \text{ where } N \in \mathbb{Z}$$

Clearly, we have $N = 2$. We also observe that a_0 is arbitrary from the equation above. Doing our general recurrence relation:

$$n = 0 : a_0$$
$$n = 1 : a_1 = 0$$
$$n = 2 : a_2 = 0$$
$$n = 3 : a_3 = 0$$
$$n = 4 : a_4 = \frac{-4a_0}{4(6)} = \frac{-a_0}{6}$$
$$n = 5 : a_5 = -\frac{4a_1}{5(5+2)} = 0$$

We see an interesting pattern, wherein we will only have every fourth term. So,

$$n = 6 \rightarrow a_6 = 0$$
$$n = 7 \rightarrow a_7 = 0$$

So, unless n is divisible by four, it will be zero.

$$n = 8 \rightarrow a_8 = -\frac{4a_4}{8(10)} = \frac{4a_0}{6(8)(10)}$$
$$n = 12 \rightarrow a_12 = -\frac{4a_8}{12(14)} = -\frac{4(4)a_0}{6(8)(10)(12)(14)}$$

Therefore, we can see the general relation as $a_n = 0 \iff n \mod 4 \neq 0$, but that for multiples of four:

$$a_0, -\frac{a_0}{6}, \frac{4a_0}{(6)(8)(10)}, -\frac{4(4)a_0}{6(8)(10)(12)(14)}$$

Therefore, if we instead treat each of those terms as the indices (to reduce the complexity of finding an expression for multiples of 4) we'll have:

$$a_n = \frac{(-1)^n a_0}{(2n+1)!}$$

Placing this back into our expansion (the x terms are occurring every fourth term so we know they'll be at x^0, x^4, x^6, x^8 etc.):

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^{2n})^2$$

But, this looks familiar, and we can see that it's simply the Taylor Series expansion of $\sin(x^2)$! Therefore, we have:

$$y_1(x) = a_0 \sin(x^2)$$

Therefore, we now have $y_1(x)$, and we'll find $y_1'(x)$ as we'll need it later:

$$y_1'(x) = a_0 \sin(x^2) = 2a_0 x \cos(x^2)$$

We now need to determine if we need the $\ln(x)$ term for our other solution (we do this to make sure it's linearly independent). We note again that $r_2 = 0$. Finding this recurrence relation now for a separate sequence b_n:

$$\begin{aligned}
(n+r)(n+r-1)b_n - (n+r)b_n + 4b_{n-4} &= 0 \\
((n+0)(n+0-1) - (n+0))\, b_n + 4b_{n-4} &= 0 \\
((n)(n-1) - (n))b_n &= -4b_{n-4} \\
n(n-2)b_n &= -4b_{n-4} \\
b_n &= -\frac{4b_{n-4}}{n(n-2)}
\end{aligned}$$

Finding various values for n:

$$\begin{aligned}
&n = 0 : b_0 \\
&n = 1 : 0 \\
&n = 2 : 0 \\
&n = 3 : 0 \\
&n = 4 : b_4 = -\frac{4b_0}{4(4-2)} = -\frac{4b_0}{4(2)} \\
&n = 5 : b_5 = -\frac{4a_1}{5(5-2)} = 0
\end{aligned}$$

Again, we observe the same pattern as above. Therefore, we then have:

$$n = 6 \to b_6 = 0$$
$$n = 7 \to b_7 = 0$$
$$n = 8 \to b_8 = -\frac{4b_4}{8(8-2)} = \frac{4 \cdot 4b_0}{8(6)(4)(2)}$$

Then, we observe:

$$(n+r)(n+r-1)a_n - (n+r)a_n + 4a_{n-4} = 0$$
$$a_n = -\frac{4a_{n-4}}{(n+r)(n+r-1)-(n+r)}$$

So, (where $N = r_2 - r_1$)

$$\lim_{r \to r_2} a_N(r) = \lim_{r \to 0} a_2(r) = \lim_{r \to 0} -\frac{4a_{n-4}}{(n+r)(n+r-1)-(n+r)} = \lim_{r \to 0} -\frac{0}{(n+r)(n+r-1)-(n+r)} = 0$$

Therefore, we get that the limit does exist and we can thankfully conclude that our solutions are linearly independent. Then, further evaluating our b_n sequence, we see it gives:

$$b_0, -\frac{4b_0}{4(2)}, \frac{4 \cdot 4b_0}{8(6)(4)(2)}, \cdots$$

This then becomes similar to the above, where we re-index and start counting the above sequence in order:

$$b_n = \frac{b_0(-1)^n}{(2n)!}$$

Placing this back into our solution for $y_2(x)$ gives us:

$$y_2(x) = \sum_{n=0}^{\infty} \frac{b_0(-1)^n}{(2n)!}(x^{2n})^2 = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^{2n})^2$$

Again, we notice that this is the Taylor Series expansion of $\cos(x^2)$. Therefore, we have that:

$$y_2(x) = b_0 \cos(x^2)$$

Altogether then our solution becomes:

$$y(x) = a_0 \sin(x^2) + b_0 \cos(x^2)$$

We confirm this with MAPLE:

> $g := diff(y(x), x, x) \cdot x - diff(y(x), x) + 4\,x^3 y(x) = 0$

$$g := \left(\frac{d^2}{dx^2}\, y(x)\right) x - \frac{d}{dx}\, y(x) + 4\, x^3\, y(x) = 0 \qquad \textbf{(1)}$$

> $Order := 14$

$$Order := 14 \qquad \textbf{(2)}$$

> $dsolve(g, y(x), series)$

$$y(x) = \_C1\, x^2 \left(1 - \frac{1}{6}\, x^4 + \frac{1}{120}\, x^8 - \frac{1}{5040}\, x^{12} + O\left(x^{14}\right)\right) \qquad \textbf{(3)}$$
$$+ \_C2 \left(-2 + x^4 - \frac{1}{12}\, x^8 + \frac{1}{360}\, x^{12} + O\left(x^{14}\right)\right)$$

> $dsolve(g, y(x))$

$$y(x) = \_C1 \sin\left(x^2\right) + \_C2 \cos\left(x^2\right) \qquad \textbf{(4)}$$

> $series\left(\sin\left(x^2\right), x = 0\right)$

$$x^2 - \frac{1}{6}\, x^6 + \frac{1}{120}\, x^{10} + O\left(x^{14}\right) \qquad \textbf{(5)}$$

> $series\left(\cos\left(x^2\right), x = 0\right)$

$$1 - \frac{1}{2}\, x^4 + \frac{1}{24}\, x^8 - \frac{1}{720}\, x^{12} + O\left(x^{16}\right) \qquad \textbf{(6)}$$

Part b)

We recall that:

3. **Reduction of Order** (Jean D'Almbert (1717-1783)): If $y_1(x)$ is known for the linear ODE:

$$y'' + p(x)y' + q(x)y = 0$$

Then one attempts a solution of the form $y(x) = v(x)y_1(x)$. Provided $y_1(x) \neq 0$, show that

$$\frac{dv}{dx} = \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds}$$

Therefore,we adjust the main equation to be in this form:

$$\begin{aligned} xy'' - y' + 4x^3y &= 0 \\ y'' - \frac{1}{x}y' + 4x^2y &= 0 \end{aligned}$$

Also, from above we've already noticed that $y_1(x) = a_0 \sin(x^2)$.

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{[a_0 \sin(x^2)]^2} e^{-\int^x -\frac{1}{s}ds} \\ \frac{dv}{dx} &= \frac{1}{[a_0 \sin(x^2)]^2} e^{\ln(x)} \\ \frac{dv}{dx} &= \frac{x}{[a_0 \sin(x^2)]^2} \end{aligned}$$

Using MAPLE to do this integral, we get:

$$v(x) = -\frac{\cot(x^2)}{2a_0^2}$$

But we now have:

$$\begin{aligned} y_2(x) &= v(x)y_1(x) \\ &= -\frac{\cot(x^2)}{2a_0^2} \cdot a_0 \sin(x^2) \\ &= -\frac{\cos(x^2)}{2a_0} \end{aligned}$$

But, now that we have this solution, we also note that $-1/2a_0$ is just some arbitrary constant, which we can just call b_0. Therefore, we arrive at the second linearly independent solution again, just like we did in Part a) of $y_2(x) = b_0 \cos(x^2)$.

2 (20pts) Singular Perturbation Methods: BVP

a. Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\varepsilon y'' + 2y' + e^y = 0, \qquad y(0) = 0, \qquad y(1) = 0, \qquad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived.

b. Provide computer simulations of these solutions for $\varepsilon = 0.1, 0.05$, and 0.01. Briefly discuss the observed behavior and explain what happens to the inner, outer, and uniform solutions as $\varepsilon \to 0$.

Part a)

We observe the initial ODE:

$$\varepsilon y'' + 2y' + e^y = 0$$

Wanting to find an outer solution first, we'll start by setting $\varepsilon = 0$ which gives:

$$2y' + e^y = 0$$

Using MAPLE to solve this separable ODE to avoid mistakes with $y(1) = 0$ (we select this initial condition as we're finding the boundary condition with the boundary being at 1):

$$\begin{aligned}
y_o(x) &= \ln \frac{2}{C+x} \\
0 &= \ln \frac{2}{C+x} \\
1 &= \frac{2}{C+1} \\
C &= 1
\end{aligned}$$

Therefore,

$$y_o(x) = \ln \frac{2}{1+x}$$

Now, attempting to get the inner solution. We'll need to re-scale time, therefore if we re-scale using:

$$\begin{aligned}
Y(\xi) &= y(\delta(\varepsilon)\xi) \\
Y'(\xi) &= y'(\delta(\varepsilon)\xi)(\delta(\varepsilon)\xi)' = y'(\delta(\varepsilon)\xi)\delta(\varepsilon) \to \frac{Y'(\xi)}{\delta(\varepsilon)} = y'(\delta(\varepsilon)\xi) \\
Y''(\xi) &= y''(\delta(\varepsilon)\xi)\delta(\varepsilon)(\delta(\varepsilon)\xi)' = y''(\delta(\varepsilon)\xi)\delta(\varepsilon)^2 \to \frac{Y''(\xi)}{\delta(\varepsilon)^2} = y''(\delta(\varepsilon)\xi)
\end{aligned}$$

Then, placing this into the equation shows us:

$$\begin{aligned}
\varepsilon y'' + 2y' + e^y &= 0 \\
\varepsilon \frac{Y''(\xi)}{\delta(\varepsilon)^2} + 2\frac{Y'(\xi)}{\delta(\varepsilon)} + e^{Y(\xi)} &= 0
\end{aligned}$$

Then, this gives that our first term is of the order:

$$\begin{aligned}
&1 : \frac{1}{\delta(\varepsilon)^2}\\
&2 : \frac{1}{\delta(\varepsilon)}\\
&3 : 1
\end{aligned}$$

Here, if we assume that $1, 3$ have the same order, clearly, we'll have that:

$$\delta(\varepsilon) = \mathcal{O}(\sqrt{\varepsilon})$$

But, if this is the case, then we'll see that 2 will become unbounded, which cannot be the case. Therefore, if we assume that

$$\delta(\varepsilon) = \mathcal{O}(\varepsilon)$$

then, $1, 2$ will both be equivalent and 3 which is constant will stay bounded. Hence, we choose that $\delta(\varepsilon) = \varepsilon$.

This then leads to the scaled ODE:

$$\begin{aligned}
\varepsilon\frac{Y''(\xi)}{\delta(\varepsilon)^2} + 2\frac{Y'(\xi)}{\delta(\varepsilon)} + e^{Y(\xi)} &= 0\\
\varepsilon\frac{Y''(\xi)}{\varepsilon^2} + 2\frac{Y'(\xi)}{\varepsilon} + e^{Y(\xi)} &= 0\\
Y''(\xi) + 2Y'(\xi) + \varepsilon e^{Y(\xi)} &= 0
\end{aligned}$$

Letting $\varepsilon = 0$ then gives the simple ODE:

$$Y''(\xi) + 2Y'(\xi) = 0$$

Which gives the immediate solution of:

$$Y(\xi) = C_1 + C_2 e^{-2\xi}$$

With $Y(0) = 0$ (which we'll select because it's the inner solution), then:

$$C_1 = -C_2$$

Which leads us to:

$$Y(\xi) = -C_2 + C_2 e^{-2\xi}$$

Now, substituting back in, we'll get (we note that $\xi\delta(\varepsilon) = \xi\varepsilon = x$, so $\xi = x/\varepsilon$):

$$Y(\xi) = y\left(\delta(\varepsilon)\frac{x}{\varepsilon}\right) = y\left(\varepsilon \cdot \frac{x}{\varepsilon}\right) = y_i(x) = -C_2 + C_2 e^{\frac{-2x}{\varepsilon}}$$

Then, we observe the two values of the matched $\delta(\varepsilon)$ scaling. This was: ε^0 and ε^1. Finding the geometric mean, this would be:

$$(\varepsilon^0\varepsilon^1)^{\frac{1}{2}} = \sqrt{\varepsilon}$$

Now, we need to use the fact that $\eta = x/\sqrt{\varepsilon}$.

$$\begin{aligned}
\lim_{\varepsilon\to 0} y_i(\sqrt{\varepsilon}\eta) &= \lim_{\varepsilon\to 0} y_o(\sqrt{\varepsilon}\eta) \\
\lim_{\varepsilon\to 0^+} -C_2 + C_2 e^{-\frac{2\sqrt{\varepsilon}\eta}{\varepsilon}} &= \lim_{\varepsilon\to 0^+} \ln\left(\frac{2}{1+\sqrt{\varepsilon}\eta}\right) \\
-C_2 &= \ln(2) \\
C_2 &= -\ln(2)
\end{aligned}$$

This then gives us,

$$y_i(x) = -C_2 + C_2 e^{-\frac{2x}{\varepsilon}} = \ln(2) - \ln(2)e^{-\frac{2x}{\varepsilon}}$$

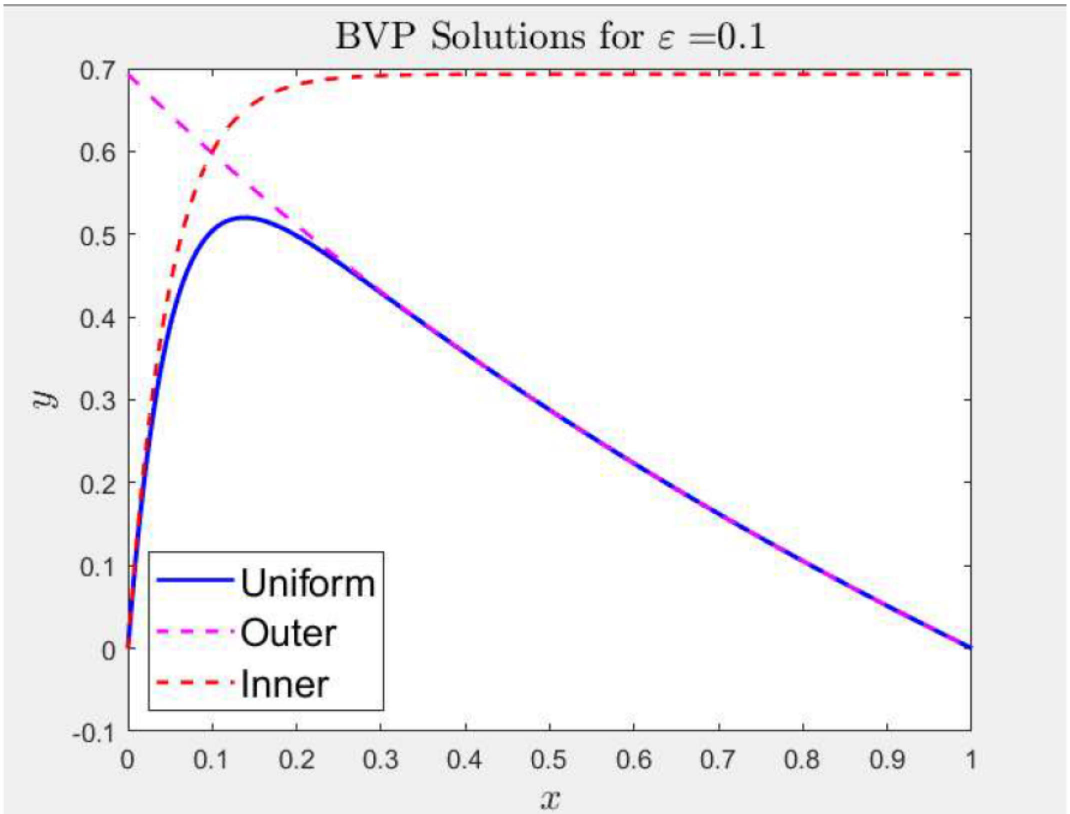
Then, altogether we have:

$$\begin{aligned}
y_o(x) &= \ln\frac{2}{1+x} \\
y_i(x) &= \ln(2) - \ln(2)e^{-\frac{2x}{\varepsilon}}
\end{aligned}$$

Our uniform solutions become (subtracting our equal limits from the combined solutions):

$$\begin{aligned}
y_u(x) &= y_o(x) + y_i(x) - 1 \\
y_u(x) &= \ln\left(\frac{2}{1+x}\right) + \ln(2) - \ln(2)e^{-\frac{2x}{\varepsilon}} - \ln(2) \\
y_u(x) &= \ln\left(\frac{2}{1+x}\right) - \ln(2)e^{-\frac{2x}{\varepsilon}}
\end{aligned}$$

Part b)


BVP Solutions for ε =0.1
Uniform
Outer
Inner
y
x
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
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0.2
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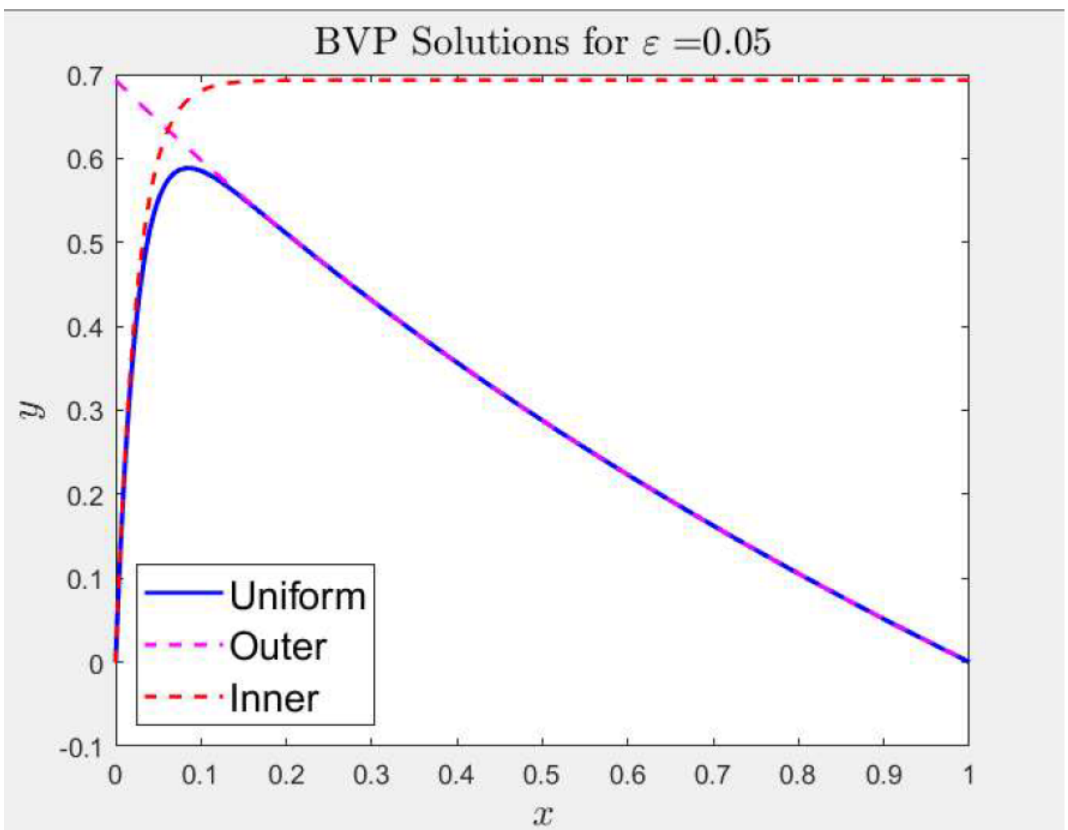
0.8
0.9
1


BVP Solutions for ε =0.05
Uniform
Outer
Inner
y
x
0.7
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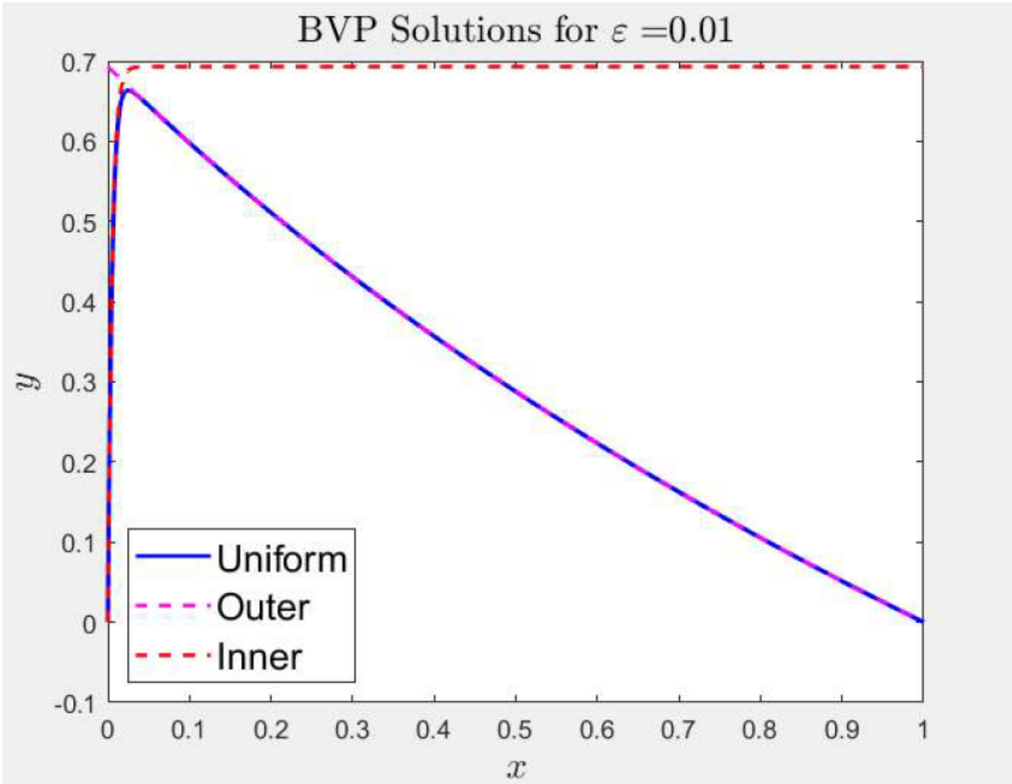


We see that as $\varepsilon \to 0$ the outer and inner solutions are more closely mapped to the uniform solution. Also, we note that the intersection of the two inner and outer curves starts to approach $(x, y) = (0, \ln(2))$. Also, as $\varepsilon \to 0$, we notice that the outer solution is more rapidly approached by the uniform curve.

3 (20pts) Singular Perturbation Methods: IVP

a. Use singular perturbation methods to obtain a uniform approximation to the solution of the IVP:

$$\frac{du}{dt} = v, \qquad \varepsilon\frac{dv}{dt} = -u^2 - v, \qquad u(0) = 1, \qquad v(0) = 0, \qquad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived for both $u(t)$ and $v(t)$.

b. Provide computer simulations of these solutions for $\varepsilon = 0.1, 0.05$, and 0.01 with $t \in [0, 5]$. Briefly discuss the observed behavior and explain what happens to the inner, outer, and uniform solutions as $\varepsilon \to 0$.

Part a)

We'll first attempt to find our outer solutions. If we start by assuming that:

$$\begin{aligned} u &= u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots \\ v &= v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \cdots \end{aligned}$$

Then, if we assume instead that we're only using $\mathcal{O}(1)$, then our initial equation becomes:

$$\begin{aligned} u' = v &\Rightarrow u_0'(t) = v_0(t) + \mathcal{O}(\varepsilon) \Rightarrow u_0'(t) = v_0(t) \\ \varepsilon\frac{dv}{dt} = -u^2 - v &\Rightarrow \mathcal{O}(\varepsilon) = -\left(u_0(t) + \mathcal{O}(\varepsilon)\right)^2 - v_0(t) \Rightarrow 0 = -u_0^2(t) - v_0(t) \end{aligned}$$

So, we have:

$$\begin{aligned} u_0'(t) &= v_0(t) \\ 0 &= -u_0^2(t) - v_0(t) \end{aligned}$$

Solving the second equation shows us that:

$$-u_0^2(t) = v_0(t)$$

And then:

$$u_0'(t) = -u_0^2(t)$$

Solving this separable first order differential equation yields:

$$u(t) = \frac{1}{t + C}$$

Now, we also nkow from our initial conditions that $u(0) = 1$, so this means that $u_0(0) = 1$. Therefore, we see that $C = 1$, giving one outer solution as:

$$u(t) = \frac{1}{t + 1}$$

Then, placing this back into the relationship between u_0 and v_0 above gives:

$$v_0(t) = -\left(\frac{1}{t+1}\right)^2$$

Now, to find our inner solutions. If we change our time scale to create a faster time scale for the quick behavior of the inner solution, we can then reevaluate our system of differential equations. Letting:

$$T = \frac{t}{\varepsilon}, \qquad U(T) = u(\varepsilon T), \qquad V(T) = v(\varepsilon T)$$

Meaning (and we observe the same for $V(t)$ without loss of generality) taking the derivative with respects to T,

$$U'(T) = u'(\varepsilon T)\varepsilon) \Rightarrow \frac{U'(T)}{\varepsilon} = u'(\varepsilon T)$$

Hence, the scaled system is:

$$\begin{aligned} \frac{U'(T)}{\varepsilon} &= V(T) \\ \frac{\varepsilon V'(T)}{\varepsilon} &= -U^2(T) - V(T) \end{aligned}$$

Then,

$$\begin{aligned} U'(T) &= \varepsilon V(T) \\ V'(T) &= -U^2(T) - V(T) \end{aligned}$$

Again, we pick $\mathcal{O}(1)$ from:

$$\begin{aligned} U(T) &= U_0(T) + \varepsilon U_1(T) + \varepsilon^2 U_2(T) + \cdots \\ V(T) &= V_0(T) + \varepsilon V_1(T) + \varepsilon^2 V_2(T) \end{aligned}$$

Hence,

$$\begin{aligned} U_0'(T) &= \mathcal{O}(\varepsilon) \Rightarrow U_0'(T) = 0 \\ V_0'(T) &= -U_0^2(T) - V_0(T) \end{aligned}$$

But we observe that this simply makes $U_0'(T) = 0 \Rightarrow U_0(t) = C$. But $u(0) = 1$, meaning $U_0(0) = 1$. So,

$$\begin{aligned} V_0'(T) &= -1^2 - V_0(T) = -1 - V_0(T) \\ V_0(T) &= -1 + Be^{-T} \end{aligned}$$

Using $v(0) = 0 \Rightarrow V(0) = 0$:

$$V_i(T) = -1 + e^{-T}$$

Now, we replace our scaling:

$$V_i(T) = v_i(\varepsilon T) = v_i\left(\varepsilon\frac{t}{\varepsilon}\right) = v_i(t) = -1 + e^{-\frac{t}{\varepsilon}}$$

As such, our inner approximation is:

$$v_i(t) = -1 + e^{-\frac{t}{\varepsilon}}$$
$$u_i(t) = 1$$

For completeness we display our outer solution again:

$$v_0(t) = -\left(\frac{1}{t+1}\right)^2$$
$$u_o(t) = \frac{1}{t+1}$$

Now, finding the limits for u:

$$\lim_{t\to 0} u_o(t) = \lim_{\varepsilon\to 0} u_i(t)$$

But we already have that $u_i = 1$ so the limit is also one. This means that we have ($f \equiv$ uniform):

$$u_f(t) = \frac{1}{t+1} + 1 - 1$$
$$u_f(t) = \frac{1}{t+1}$$

For v:

$$\lim_{t\to 0} v_0(t) = \lim_{\varepsilon\to 0} y_i(t)$$
$$\lim_{t\to 0} -\left(\frac{1}{t+1}\right)^2 = \lim_{\varepsilon\to 0} -1 + e^{-\frac{t}{\varepsilon}}$$
$$-1 = -1$$

Then,

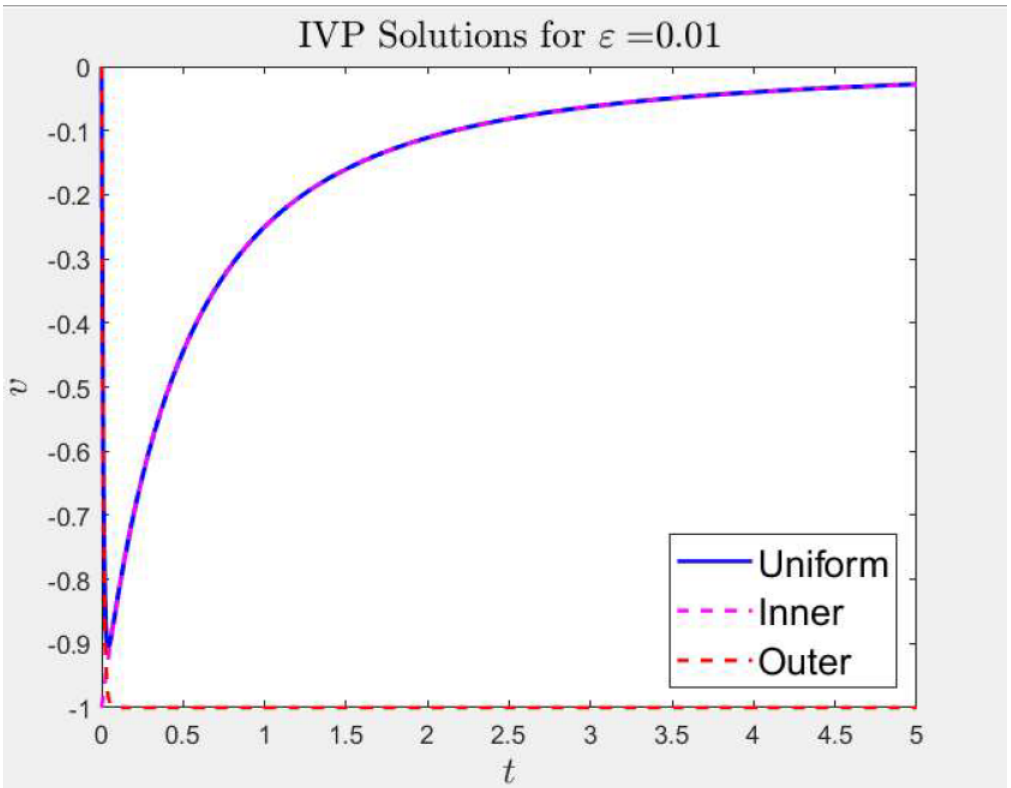
$$v_f(t) = -1 - e^{-\frac{t}{\varepsilon}} - \left(\frac{1}{t+1}\right)^2 + 1$$
$$v_f(t) = e^{-\frac{t}{\varepsilon}} - \left(\frac{1}{t+1}\right)^2$$

Altogether then, we have:

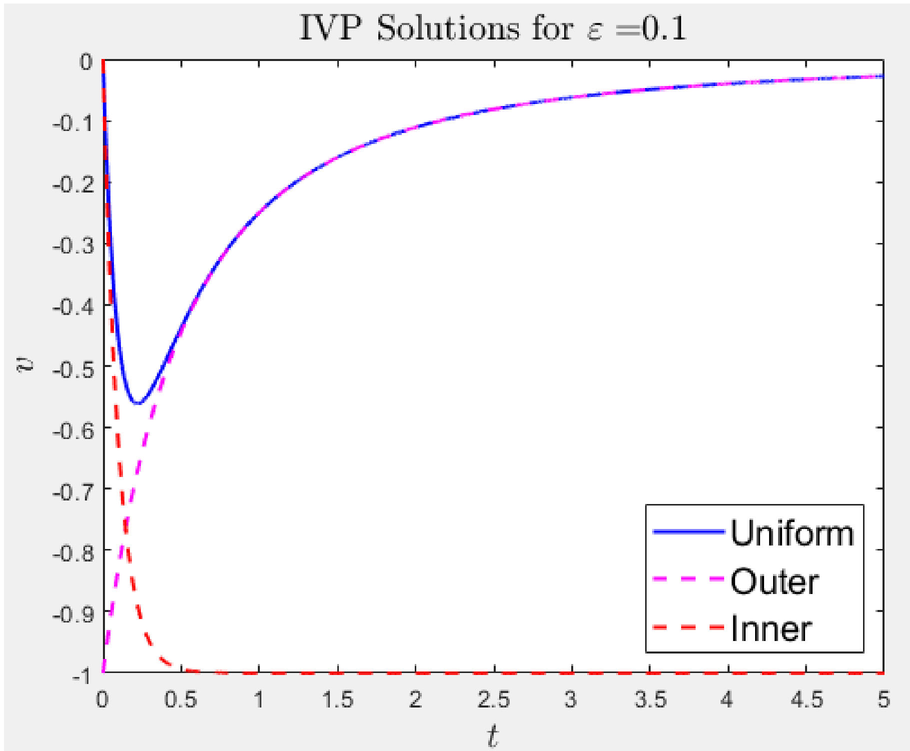
$$u_f(t) = \frac{1}{t+1}$$
$$v_f(t) = e^{-\frac{t}{\varepsilon}} - \left(\frac{1}{t+1}\right)^2$$

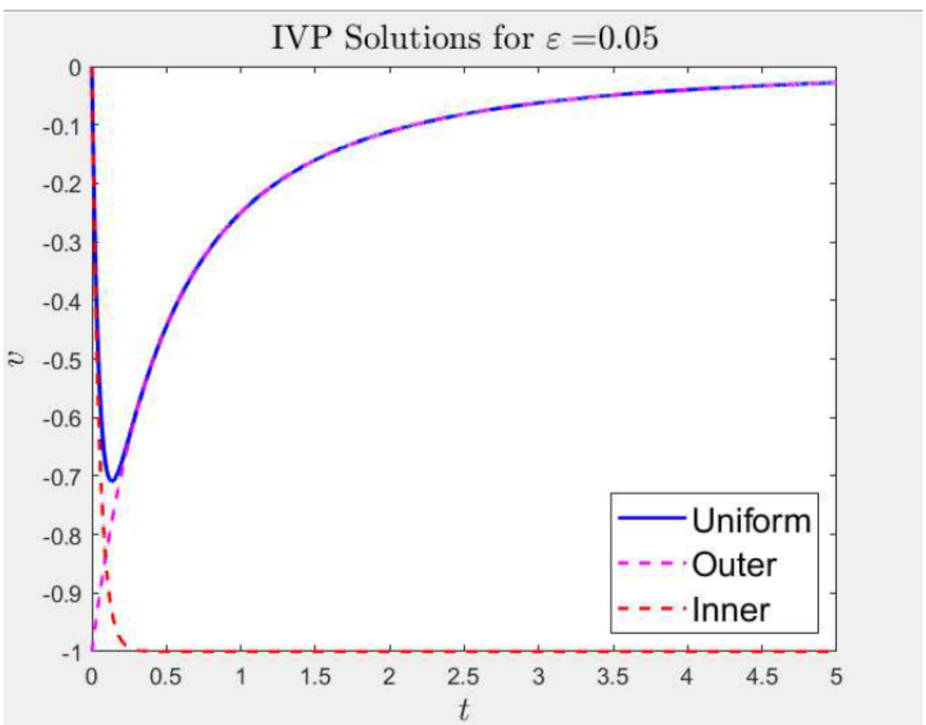
Part b)

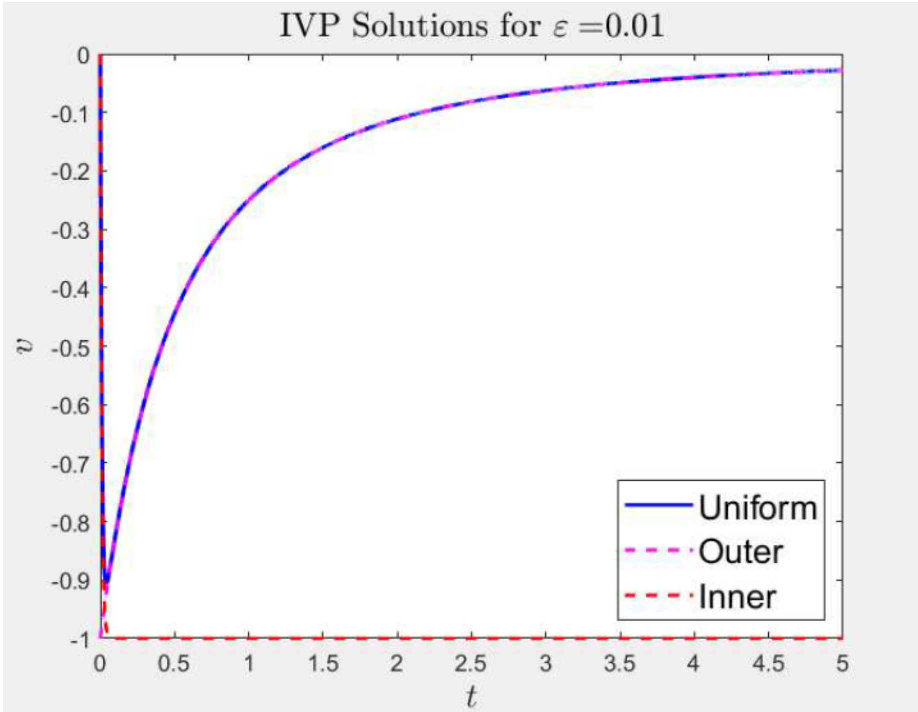
For our $u(t)$ equations, we notice that they don't depend on ε. Therefore, the only solution graph wise that we're going to have is the one graph below. Also, given how we defined our solutions and only took our approximations to $\mathcal{O}(1)$, we see there is no difference between our approximation and our uniform solution. We should expect that a numerical solver, however, would show us more of a delta between them.



For v however, we should see convergence with $\varepsilon \to 0$ for our v inner, outer and uniform approximations:





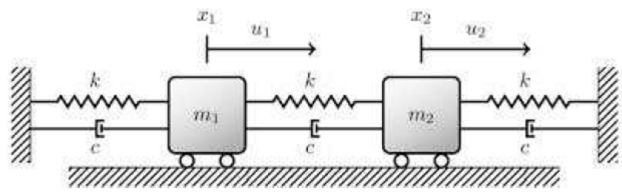


Clearly, we can see that as we decrease epsilon, we get uniform solutions that more and more closely hug the inner and outer approximations. Also, we notice that as $\varepsilon \to 0$ for v, the inner and outer approximations' intersection starts to approach $(t, v) = (0, 0)$. Lastly, we notice that the uniform approximation, regardless of the ε used, rapidly approaches the outer solution's behavior. This is caused by having an $\exp\left(-\frac{t}{\varepsilon}\right)$ term.

4 (30pts) Mass-Spring Problem - Jordan Canonical Form

Consider the mass-spring problem, where we consider two identical masses connected by three identical springs. Assuming a general Hooke's law spring, using Newton's law of forces and ignoring the viscous damping between the two springs, the following system of second order linear ODEs can be written:

$m\ddot{x}_1 = -c\dot{x} - kx_1 + k(x_2 - x_1),$
$m\ddot{x}_2 = -c\dot{x} - kx_2 + k(x_1 - x_2),$
where $u_1 = \dot{x}_1$ and $u_2 = \dot{x}_2$.



a. For this part of the problem, we assume no damping, so $c = 0$. Define the **natural frequency** by the constant $\omega^2 = \frac{k}{m}$. Define new state variables $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$, where $y_1 = x_1, y_2 = u_1, y_3 = x_2$, and $y_4 = u_2$. Rewrite the above system of second order linear ODEs into a system of first order linear ODEs:

$$\dot{\mathbf{y}} = A\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_0$$

Transform A into J_a, where J_a is a matrix in **real Jordan canonical form**. Show how you obtain your nonsingular transformation matrix, P, and give its inverse, P^{-1}. (Hint: You may want to use the theorem described on Slide 36 of the Fundamental Solutions lecture notes, which was added after the lecture.) Write the fundamental solution, $\Psi(t) = e^{J_a t}$.

b. In lecture we noted that the eigenspaces of A are invariant subspaces for the flow, $\Phi(t) = e^{At}$, dividing $\mathbb{R}$ into E^s (stable), E^u (unstable), and E^c (center) subspaces. With $c = 0$, give the dimension of each of these subspaces for this example. What is the equilibrium point and is it hyperbolic? Briefly explain these implications for the qualitative behavior of this system. Are there limitations to this theory for this particular model?

c. Again with $c = 0$, assume this system is initially at rest, and the two masses are displaced by x_{10} and x_{20}, respectively, *i.e.*, we have:

$$\mathbf{y}(0) = (x_{10}, 0, x_{20}, 0)^T.$$

Write the unique solution to this initial value problem, $\mathbf{y}(t)$. Briefly discuss the solution describing the motion when the two masses are equally displaced to the right, $x_{10} = x_{20} > 0$ (symmetric motion). Also, briefly discuss the solution describing the motion when the two masses are equally displaced in opposite directions, $x_{10} = -x_{20} > 0$ (anti-symmetric motion).

d. For this part of the problem, we assume damping, $c > 0$. Define $\omega^2 = \frac{k}{m}, 2\gamma = \frac{c}{m} \ll \omega$, and new state variables $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$, where $y_1 = x_1, y_2 = u_1, y_3 = x_2$, and $y_4 = u_2$. Rewrite the above system of second order linear ODEs into a system of first order linear ODEs:

$$\dot{\mathbf{y}} = B\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_0.$$

Transform B into J_b, where J_b is a matrix in **real Jordan canonical form**. Show how you obtain your nonsingular transformation matrix, P. Write the fundamental solution, $\mathbf{\Psi}(t) = e^{J_b t}$.

e. The eigenspaces of B are invariant subspaces for the flow, $\mathbf{\Phi}(t) = e^{Bt}$. With $c > 0$ and $2\gamma = \frac{c}{m} \ll \omega$, give the dimension of each of these eigenspaces for this example. What is the equilibrium point and is it hyperbolic? Briefly explain these implications for the qualitative behavior of this system. Are there limitations to this theory for this particular model?

Part a)

First, we'll go ahead and rewrite the equations with $c = 0$ to remove the damping terms. From this, we immediately expect that we'll get purely imaginary eigenvalues.

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 + k(x_1 - x_2)$$

We also note that $u_1 = \dot{x}_1 = y_2$ and $u_2 = \dot{x}_2 = y_4$. Now dividing through by m to be able to utilize our omega substitution of $\omega^2 = \frac{k}{m}$ gives:

$$\ddot{x}_1 = -\frac{k}{m}x_1 + \frac{k}{m}(x_2 - x_1) = \ddot{x}_1 = -\omega^2 x_1 + \omega^2(x_2 - x_1) = -2\omega^2 x_1 + \omega^2 x_2$$
$$\ddot{x}_2 = -\frac{k}{m}x_2 + \frac{k}{m}(x_1 - x_2) = \ddot{x}_2 = -\omega^2 x_2 + \omega^2(x_1 - x_2) = -2\omega^2 x_2 + \omega^2 x_1$$

Therefore, we can now note the following:

$$y_1 = x_1 \Rightarrow \dot{y}_1 = \dot{x}_1$$
$$y_2 = \dot{x}_1 \Rightarrow \dot{y}_2 = \ddot{x}_1$$
$$y_3 = x_2 \Rightarrow \dot{y}_3 = \dot{x}_2$$
$$y_4 = \dot{x}_2 \Rightarrow \dot{y}_4 = \ddot{x}_2$$

Finally, our matrix in the form $\dot{\mathbf{y}} = A\mathbf{y}$ becomes:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Now, we'll want to transform A into real Jordan Canonical form. Using MAPLE we'll perform the following steps. We'll find the eigenvectors of A, then, we'll use the real and imaginary values of those to make our vectors in P. Then we'll use $P^{-1}AP$ to get J. After getting J, we'll then take e^{Jt} which will give us $\mathbf{\Psi}(t) = e^{Jt}$.

> $A := Matrix\left(\left[[0, 1, 0, 0], \left[-2\cdot\omega^2, 0, \omega^2, 0\right], [0, 0, 0, 1], \left[\omega^2, 0, -2\cdot\omega^2, 0\right]\right]\right)$

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\,\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\,\omega^2 & 0 \end{bmatrix}$$

> $CharacteristicPolynomial(A, z)$

$$3\,\omega^4 + 4\,\omega^2\,z^2 + z^4$$

> $Eigenvectors(A)$

$$\begin{bmatrix} I\,\omega \\ -I\,\omega \\ I\sqrt{3}\,\omega \\ -I\sqrt{3}\,\omega \end{bmatrix}, \begin{bmatrix} \frac{-I}{\omega} & \frac{I}{\omega} & \frac{\frac{I}{3}\sqrt{3}}{\omega} & \frac{-\frac{I}{3}\sqrt{3}}{\omega} \\ 1 & 1 & -1 & -1 \\ \frac{-I}{\omega} & \frac{I}{\omega} & \frac{-\frac{I}{3}\sqrt{3}}{\omega} & \frac{\frac{I}{3}\sqrt{3}}{\omega} \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

> $P := Matrix\left(\left[\left[0, \frac{1}{\omega^1}, 0, \frac{-\frac{1}{3}\sqrt{3}}{\omega}\right], [1, 0, -1, 0], \left[0, \frac{1}{\omega^1}, 0, \frac{\frac{1}{3}\sqrt{3}}{\omega}\right], [1, 0, 1, 0]\right]\right)$

$$P := \begin{bmatrix} 0 & \frac{1}{\omega} & 0 & -\frac{\sqrt{3}}{3\,\omega} \\ 1 & 0 & -1 & 0 \\ 0 & \frac{1}{\omega} & 0 & \frac{\sqrt{3}}{3\,\omega} \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

> $Pinv := MatrixInverse(P)$

$$Pinv := \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\omega}{2} & 0 & \frac{\omega}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{\sqrt{3}\,\omega}{2} & 0 & \frac{\sqrt{3}\,\omega}{2} & 0 \end{bmatrix}$$

> $Ja := Pinv \cdot A \cdot P$

$$Ja := \begin{bmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}\,\omega \\ 0 & 0 & \sqrt{3}\,\omega & 0 \end{bmatrix}$$

> $PsiMat := MatrixExponential(Ja, t)$

$$PsiMat := \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 & 0 \\ 0 & 0 & \cos(\sqrt{3}\,\omega t) & -\sin(\sqrt{3}\,\omega t) \\ 0 & 0 & \sin(\sqrt{3}\,\omega t) & \cos(\sqrt{3}\,\omega t) \end{bmatrix}$$

Part b)

We notice from above, in regards to the matrix A, that the various eigenvalues of our eigenvectors are purely imaginary – we expect this as explicitly set $c = 0$ meaning we no longer have a damping term. This completely removes decay or growth from our dynamical system. Therefore, we have:

$$\lambda_1 = \sqrt{3}\omega i, \qquad \lambda_2 = -\sqrt{3}\omega i, \qquad \lambda_3 = \omega i, \qquad \lambda_4 = -\omega i$$

As such, $E^c = \text{span}\{v_1, v_2, v_3, v_4\}, E^s = \{\emptyset\}, E^u = \{\emptyset\}$. The only time we have stability will be in the trivial case when we assume that $\mathbf{y}(0) = [0, 0, 0, 0]^T$. It becomes trivial to note then then that:

$$E^c \in \mathbb{R}^4 \wedge E^s, E^u \in \mathbb{R}^0$$

If we want to find the equilibrium point, then we want to find when $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ if we assume f_1 and f_2 are our two functions respectively:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 + k(x_1 - x_2)$$

Therefore, the only set of initial conditions that gives us stability is $\mathbf{y}(0) = [0, 0, 0, 0]^T$. We can examine this by the following:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

We note that y_2 and y_4 must be zero. Therefore, we can write the above as:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix}$$

Now, this meas that we have:

$$-2\omega^2 y_1 + \omega^2 y_3 = 0$$
$$\omega^2 y_1 - 2\omega^2 y_3 = 0$$

Dividing out ω^2 gives:

$$-2y_1 + y_3 = 0 \Rightarrow y_3 = 2y_1$$
$$y_1 - 2y_3 = 0 \Rightarrow y_1 = 2y_3$$

But, clearly, the only solution to this is $y_1 = y_3 = 0$. Hence, we have that $\mathbf{y}(0) = [0, 0, 0, 0]^T$ is our only equilibrium solution. This also makes intuitive sense. If we assume that $x_1(t)$ and $x_2(t)$ are displacements of the masses m_1 and m_2 respectively, then zero displacement of the masses with no initial velocities, corresponds to no movement, and hence we have an equilibrium solution.

We now note that this is not a hyperbolic equilibrium. Any perturbation in either initial position of either mass, or any initial velocity given to it, will result in an infinitely oscillatory solution. Therefore, there is no offsetting that can occur that will give either mass zero velocity, or zero displacement. We can also see that we've effectively already computed the Jacobian above. With:

$$\det\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right)$$

But these are exactly the eigenvalues we found previously, and all of them have no real components, hence $x_e = [0, 0, 0, 0]^T$ is ***not*** a hyperbolic fixed point.

The limitations are clearly that this system is physically impossible. Slight vibrations would cause the system to move, friction always exists in both the springs, and also between the surfaces upon which the masses are moving. Therefore, our arbitrary setting of $c = 0$ initially gives us a physically impossible system.

Part c)

Now, if we have the initial condition $\mathbf{y}(t) = [x_{10}, 0, x_{20}, 0]$, this means that we have displacement of both m_1 and m_2 while assuming their initial velocities are zero. So, our solution should then be $\mathbf{\Phi}(t) = P\mathbf{\Psi}(t)P^{-}1\mathbf{y}(0)$. Utilizing MAPLE gives:

> PhiMat := P • PsiMat • Pinv

$$PhiMat := \begin{bmatrix} \frac{\cos(\sqrt{3}\,\omega t)}{2} + \frac{\cos(\omega t)}{2} & \frac{\sqrt{3}\sin(\sqrt{3}\,\omega t)}{6\,\omega} + \frac{\sin(\omega t)}{2\,\omega} & \frac{\cos(\omega t)}{2} - \frac{\cos(\sqrt{3}\,\omega t)}{2} & -\frac{\sqrt{3}\sin(\sqrt{3}\,\omega t)}{6\,\omega} + \frac{\sin(\omega t)}{2\,\omega} \\ -\frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2} & \frac{\cos(\sqrt{3}\,\omega t)}{2} + \frac{\cos(\omega t)}{2} & \frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2} & \frac{\cos(\omega t)}{2} - \frac{\cos(\sqrt{3}\,\omega t)}{2} \\ \frac{\cos(\omega t)}{2} - \frac{\cos(\sqrt{3}\,\omega t)}{2} & -\frac{\sqrt{3}\sin(\sqrt{3}\,\omega t)}{6\,\omega} + \frac{\sin(\omega t)}{2\,\omega} & \frac{\cos(\sqrt{3}\,\omega t)}{2} + \frac{\cos(\omega t)}{2} & \frac{\sqrt{3}\sin(\sqrt{3}\,\omega t)}{6\,\omega} + \frac{\sin(\omega t)}{2\,\omega} \\ \frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2} & \frac{\cos(\omega t)}{2} - \frac{\cos(\sqrt{3}\,\omega t)}{2} & -\frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2} & \frac{\cos(\sqrt{3}\,\omega t)}{2} + \frac{\cos(\omega t)}{2} \end{bmatrix}$$

> PhiMat := P • PsiMat • Pinv • Matrix([[x10], [0], [x20], [0]])

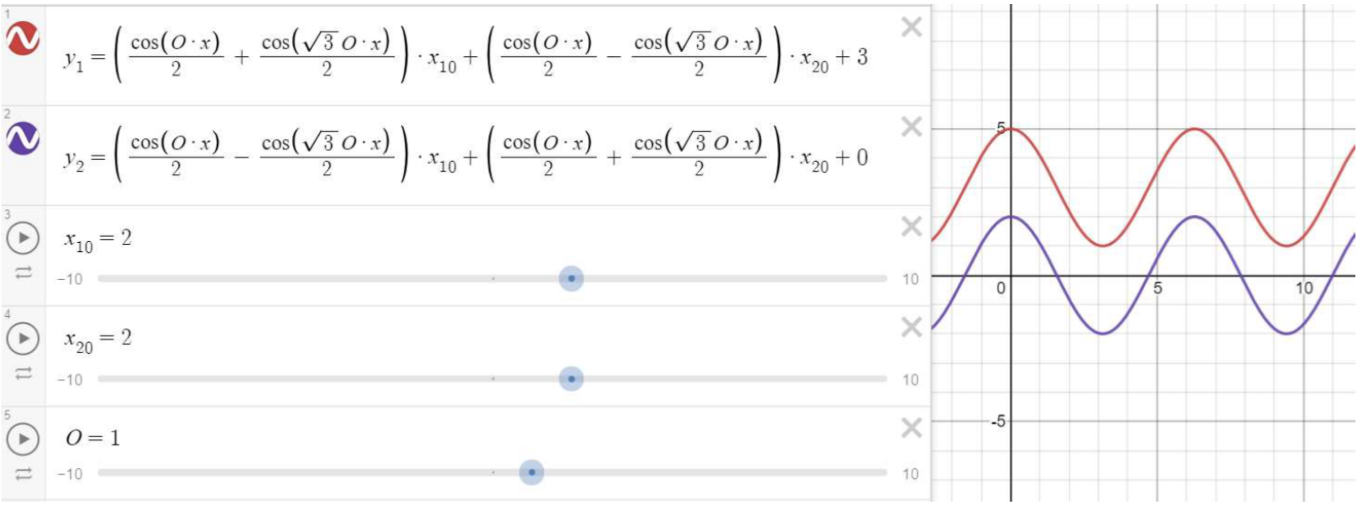
$$PhiMat := \begin{bmatrix} \left(\frac{\cos(\sqrt{3}\,\omega t)}{2} + \frac{\cos(\omega t)}{2}\right) x10 + \left(\frac{\cos(\omega t)}{2} - \frac{\cos(\sqrt{3}\,\omega t)}{2}\right) x20 \\ \left(-\frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2}\right) x10 + \left(\frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2}\right) x20 \\ \left(\frac{\cos(\omega t)}{2} - \frac{\cos(\sqrt{3}\,\omega t)}{2}\right) x10 + \left(\frac{\cos(\sqrt{3}\,\omega t)}{2} + \frac{\cos(\omega t)}{2}\right) x20 \\ \left(\frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2}\right) x10 + \left(-\frac{\sin(\sqrt{3}\,\omega t)\sqrt{3}\,\omega}{2} - \frac{\sin(\omega t)\,\omega}{2}\right) x20 \end{bmatrix}$$

We understand the first element in the vector to be the position of m_1 the second to be the velocity of m_1 and the third and fourth are the same for m_2 respectively. This is clear by simply looking at the first two rows. We see that the derivative of row one equals row two, and we know generally that

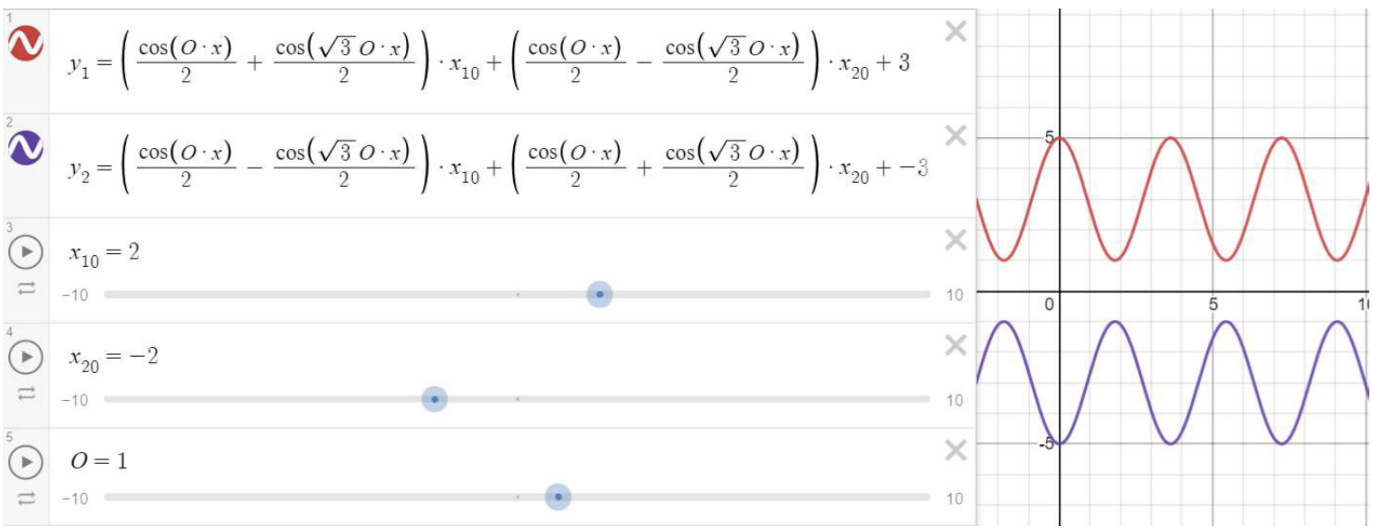
$$Velocity = (Displacement)'$$

If the masses are equally displaced to one direction, then we will see that they will move together in periodic oscillatory behavior. We note that this is obviously not possible in reality as even the slightest perturbation would likely cause erratic behavior. Also, if we pulled them equally far apart, they would come together and separate at equal periodic intervals. We can illustrate this behavior with two plots below. Here we use Demos as the sliding functions allow for ease of adjustment. Also, because we note again that $x_1(t)$ and $x_2(t)$ are displacements, we must set them physically apart manually by adding a constant to the equations. We can view the horizontal axis as time and the vertical axis as universal displacement as opposed to relative displacement of just one of the masses.

Equal Displacement in the same direction:



Equal Displacement in the opposite direction:



Part d)

Now we move to assuming that $c > 0$. Define $\omega^2 = \frac{k}{m}, 2\gamma = \frac{c}{m} \ll \omega$ with

$$\mathbf{y(t)} = [y_1, y_2, y_3, y_4]^T$$

where $y_1 = x_1, y_2 = u_1, y_3 = x_2$, and $y_4 = u_2$. We wish to find $\dot{\mathbf{y}} = A\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{y}_0$.

Reevaluating our equations from above:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 + k(x_1 - x_2)$$

Multiplying through:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -c\dot{x}_1 - kx_1 + kx_2 = -c\dot{x}_1 - 2kx_1 + kx_2$$
$$m\ddot{x}_2 = -kx_1 + k(x_2 - x_1) = -c\dot{x}_2 - kx_2 + kx_1 = -c\dot{x}_2 - 2kx_2 + kx_1$$

Then, dividing out the m:

$$\ddot{x}_1 = -\frac{c}{m}\dot{x}_1 - \frac{2k}{m}x_1 + \frac{k}{m}x_2$$
$$\ddot{x}_2 = -\frac{c}{m}\dot{x}_2 - \frac{2k}{m}x_2 + \frac{k}{m}x_1$$

Lastly, we replace the above with our variables:

$$\ddot{x}_1 = -2\gamma\dot{x}_1 - 2\omega^2 x_1 + \omega^2 x_2$$
$$\ddot{x}_2 = -2\gamma\dot{x}_2 - 2\omega^2 x_2 + \omega^2 x_1$$

Therefore, we can now note the following:

$$y_1 = x_1 \Rightarrow \dot{y}_1 = \dot{x}_1$$
$$y_2 = \dot{x}_1 \Rightarrow \dot{y}_2 = \ddot{x}_1$$
$$y_3 = x_2 \Rightarrow \dot{y}_3 = \dot{x}_2$$
$$y_4 = \dot{x}_2 \Rightarrow \dot{y}_4 = \ddot{x}_2$$

Finally, our matrix in the form $\dot{\mathbf{y}} = B\mathbf{y}$ becomes:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & -2\gamma & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & -2\gamma \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Using MATLAB, we then go ahead and find the eigenvectors (see code in last section). Writing the outputs more nicely than what is produced from MATLAB:

$$v_1 = \begin{bmatrix} -\frac{\gamma - \sqrt{(\gamma+\omega)(\gamma-\omega)}}{\omega^2} \\ 1 \\ -\frac{\gamma - \sqrt{(\gamma+\omega)(\gamma-\omega)}}{\omega^2} \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \frac{\gamma - \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ -1 \\ -\frac{\gamma - \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -\frac{\gamma + \sqrt{(\gamma+\omega)(\gamma-\omega)}}{\omega^2} \\ 1 \\ -\frac{\gamma + \sqrt{(\gamma+\omega)(\gamma-\omega)}}{\omega^2} \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} \frac{\gamma + \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ -1 \\ -\frac{\gamma + \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} \frac{-\gamma + \sqrt{(\gamma^2-\omega^2)}}{\omega^2} \\ 1 \\ \frac{-\gamma + \sqrt{(\gamma^2-\omega^2)}}{\omega^2} \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \frac{\gamma - \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ -1 \\ -\frac{\gamma - \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} \frac{-\gamma - \sqrt{(\gamma^2-\omega^2)}}{\omega^2} \\ 1 \\ \frac{-\gamma - \sqrt{(\gamma^2-\omega^2)}}{\omega^2} \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} \frac{\gamma + \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ -1 \\ -\frac{\gamma + \sqrt{\gamma^2 - 3\omega^2}}{3\omega^2} \\ 1 \end{bmatrix}$$

In order to generally get these into the form of $\alpha \pm \beta i$ we have to realize that $\sqrt{\gamma^2 - \omega^2} = \sqrt{-1(\omega^2 - \gamma^2)} = i\sqrt{\omega^2 - \gamma^2}$. This will then allow us to build our matrix P:

$$P = \{Re(v_1), Im(v_1), Re(v_2), Im(v_2)\}$$

$$P = \begin{bmatrix} -\frac{\gamma}{\omega^2} & \frac{\sqrt{\omega^2-\gamma^2}}{\omega^2} & \frac{\gamma}{3\omega^2} & -\frac{\sqrt{3\omega^2-\gamma^2}}{3\omega^2} \\ 1 & 0 & -1 & 0 \\ -\frac{\gamma}{\omega^2} & \frac{\sqrt{\omega^2-\gamma^2}}{\omega^2} & -\frac{\gamma}{3\omega^2} & \frac{\sqrt{3\omega^2-\gamma^2}}{3\omega^2} \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Now, we'll use MAPLE to find P^{-1} and then $J_b = P^{-1}BP$!

> $P4dinv := MatrixInverse(P4d)$

$$P4dinv := \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\omega^2}{2\sqrt{-\gamma^2+\omega^2}} & \frac{\gamma}{2\sqrt{-\gamma^2+\omega^2}} & \frac{\omega^2}{2\sqrt{-\gamma^2+\omega^2}} & \frac{\gamma}{2\sqrt{-\gamma^2+\omega^2}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3\omega^2}{2\sqrt{-\gamma^2+3\omega^2}} & -\frac{\gamma}{2\sqrt{-\gamma^2+3\omega^2}} & \frac{3\omega^2}{2\sqrt{-\gamma^2+3\omega^2}} & \frac{\gamma}{2\sqrt{-\gamma^2+3\omega^2}} \end{bmatrix}$$

> $Jb := P4dinv \cdot B \cdot P4d$

$$Jb := \begin{bmatrix} -\gamma & -\sqrt{-\gamma^2+\omega^2} & 0 & 0 \\ -\frac{\gamma^2}{\sqrt{-\gamma^2+\omega^2}} + \frac{\omega^2}{\sqrt{-\gamma^2+\omega^2}} & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & -\sqrt{-\gamma^2+3\omega^2} \\ 0 & 0 & -\frac{\gamma^2}{\sqrt{-\gamma^2+3\omega^2}} + \frac{3\omega^2}{\sqrt{-\gamma^2+3\omega^2}} & -\gamma \end{bmatrix}$$

In the above equations, because $2\gamma \ll \omega$, we see that this is in real Jordan Form. Lastly, we can finally use this to find $\mathbf{\Psi}(t) = e^{J_b t}$:

> $PsiMat4d := MatrixExponential(Jb \cdot t)$

$$PsiMat4d := \left[\left[\frac{e^{-\left(\gamma+\sqrt{\gamma^2-\omega^2}\right)t}}{2} + \frac{e^{-\left(\gamma-\sqrt{\gamma^2-\omega^2}\right)t}}{2}, \frac{\sqrt{-\gamma^2+\omega^2}\left(-e^{-\left(\gamma-\sqrt{\gamma^2-\omega^2}\right)t} + e^{-\left(\gamma+\sqrt{\gamma^2-\omega^2}\right)t}\right)}{2\sqrt{\gamma^2-\omega^2}}, 0, 0\right],\right.$$

$$\left[\frac{\sqrt{\gamma^2-\omega^2}\left(-e^{-\left(\gamma-\sqrt{\gamma^2-\omega^2}\right)t} + e^{-\left(\gamma+\sqrt{\gamma^2-\omega^2}\right)t}\right)}{2\sqrt{-\gamma^2+\omega^2}}, \frac{e^{-\left(\gamma+\sqrt{\gamma^2-\omega^2}\right)t}}{2} + \frac{e^{-\left(\gamma-\sqrt{\gamma^2-\omega^2}\right)t}}{2}, 0, 0\right],$$

$$\left[0, 0, \frac{e^{-\left(\gamma+\sqrt{\gamma^2-3\omega^2}\right)t}}{2} + \frac{e^{-\left(\gamma-\sqrt{\gamma^2-3\omega^2}\right)t}}{2}, -\frac{\sqrt{-\gamma^2+3\omega^2}\left(e^{-\left(\gamma-\sqrt{\gamma^2-3\omega^2}\right)t} - e^{-\left(\gamma+\sqrt{\gamma^2-3\omega^2}\right)t}\right)}{2\sqrt{\gamma^2-3\omega^2}}\right],$$

$$\left.\left[0, 0, -\frac{\sqrt{\gamma^2-3\omega^2}\left(e^{-\left(\gamma-\sqrt{\gamma^2-3\omega^2}\right)t} - e^{-\left(\gamma+\sqrt{\gamma^2-3\omega^2}\right)t}\right)}{2\sqrt{-\gamma^2+3\omega^2}}, \frac{e^{-\left(\gamma+\sqrt{\gamma^2-3\omega^2}\right)t}}{2} + \frac{e^{-\left(\gamma-\sqrt{\gamma^2-3\omega^2}\right)t}}{2}\right]\right]$$

Part e)

From our problem setup, we logically expect that because there is always a damping term acting on both masses, that no matter what our initial conditions are, that the system will collapse eventually to being at rest. If we arbitrarily select our $\mathbf{y}(0) = [x_{10}, u_{10}, x_{20}, u_{20}]^T$, then multiply it by our solution, $\mathbf{y}(t) = P\mathbf{\Psi}(t)P^{-1}\mathbf{y}(0)$, we will evaluate $t \to \infty$. Performing this operation in MAPLE gives an absolutely enormous set of equations, but not surprisingly, all of the terms are multiplied by some form of e^{-r} where $r > 0$. Therefore, as $t \to \infty$ every solution tends towards zero. Hence, the system is a non-hyperbolic sink. Again, we then have that:

$$E^s \in \mathbb{R}^4 \wedge E^c, E^u \in \mathbb{R}^0$$

The limits of the model are that we can exceed the physical limits of our setup. For example, if we have these two masses connected by springs where we assume they're physically attached to a wall/barrier while resting on a surface that's say a meter in length, and we give m_1 an initial velocity to the right of one million units with m_2 at its starting position without any initial velocity, then m_1 will physically pass through m_2 and go through the physical barrier of the wall and the springs will still be pulling it back without issue. This clearly is a nonsensical situation that would require drastically different modeling. Effectively, we have to make sure we pick initial conditions that don't lead negate the validity of the model's generic setup.

Lastly, the equilibrium points. If this needs to equal zero, then again, the Jacobian is effectively already given to us in this form. So, we know from $\dot{\mathbf{y}} = B\mathbf{y}$ becomes:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & -2\gamma & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & -2\gamma \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{0}$$

that $y_2 = y_4 = 0$. So,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega^2 & -2\gamma & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -2\omega^2 & -2\gamma \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix} = \mathbf{0}$$

Then, we again arrive at $-2\omega^2 y_1 + \omega^2 y_3 = 0$ and $\omega^2 y_1 - 2\omega^2 y_3 = 0$ as before. But we know this gave us only zeros before. Hence, the only stable equilibrium is the trivial one where both masses are sitting at rest.

5 (25pts) Variation of Parameters and Regular Perturbation Methods

This problem examines periodic forcing of the basic harmonic oscillator, using both the **variation of parameters method** and two forms of **regular perturbation methods**.

a. Consider the second order linear ODE given by:

$$\ddot{y} + y = \varepsilon \sin(\omega t), \qquad y(0) = 1, \qquad \dot{y}(0) = 0,$$

where $0 < \varepsilon \ll 1$ and ω are two positive parameters. Let $y(t) = x_1(t)$ and $\dot{y} = x_2(t)$. Transform this second order nonhomogeneous linear ODE into a system of first order linear ODEs:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \varepsilon \boldsymbol{f}(t), \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where A is in **real Jordan canonical form** and $f(t)$ is the appropriate forcing function. Write the fundamental solution, $\boldsymbol{\Psi}(t) = e^{At}$. Use $\boldsymbol{\psi}(t)$ and the **variation of parameters method** to find your unique solution $\mathbf{x}(t)$ to the IVP above. Be sure to note any special cases.

b. Now consider the second order nonlinear ODE given by:

$$\ddot{y} + y = \varepsilon y(1 - \dot{y}^2), \qquad y(0) = 1, \qquad \dot{y}(0) = 0,$$

where $0 < \varepsilon \ll 1$. Once again the right hand side is a small periodic forcing term. Use a **regular perturbation method** to find an approximate solution to this nonlinear problem. Determine an expansion to $\mathcal{O}(\varepsilon^2)$, *i.e.*, find a two term ε expansion of $y(t)$. s this approximate solution bounded? Explain.

c. In this part we again consider the second order nonlinear IVP given in Part b, (4). However, this time we find an approximation using the **Poincaré-Lindstedt perturbation method**. You rescale the time, t, in an ε expansion ($\tau = \omega t$ with $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$ along with ε expansion of $y(\tau)$. Apply the Poincaré-Lindstedt perturbation method to (4) and remove any **secular terms** in your order ε term for the $y(\tau)$ approximation. Give your two term approximate solution $y(\tau)$ along with your two term time scaling τ. Is this approximate solution necessarily bounded? Explain.

d. Let $\varepsilon = 0.1$ and 0.02 and use a numerical differential equation solver (like MatLab's ODE45) to find an accurate numerical solution of (4) for $t \in [0, 50]$. Create a graph comparing the **regular perturbation approximation** of Part b, the **Poincaré-Lindstedt perturbation approximation** of Part c, and the numerical solution. Briefly describe what you observe and how these various methods compare.

Part a)

We start by observing the differential equation:

$$\ddot{y}(t) + y(t) = \varepsilon \sin(\omega t), \qquad y(0) = 1, \qquad \dot{y}(0) = 0$$

with $0 < \varepsilon \ll 1$ and $0 < \omega$. Then, letting $y(t) = x_1(t)$ and $\dot{y}(t) = x_2(t)$ we transform the second order ODE above into a system of two ODEs:

$$\dot{\mathbf{x}} = A\mathbf{x} - \varepsilon \boldsymbol{f}(t)$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ \sin(\omega t) \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We see that this matrix is already in real Jordan form, so $A = J$, then, clearly $P = I = P^{-1}$

$$\begin{aligned} \boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t) = e^{At} &= e^{Jt} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Now, using the Variation of Constants Formula, we have:

$$\begin{aligned} \mathbf{x}(t) &= e^{Jt}\mathbf{x}_0 + \int_0^t e^{A(t-s)\boldsymbol{g}(s)ds} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t e^{A(t-s)\boldsymbol{g}(s)ds} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int_0^t \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon \sin(\omega s) \end{bmatrix} ds \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int_0^t \begin{bmatrix} -\varepsilon \sin(s)\sin(\omega s) \\ \varepsilon \cos(s)\sin(\omega s) \end{bmatrix} ds \end{aligned}$$

We then offload this work to MAPLE as computing it by hand would get rather excessive. The left terms will be:

> $Jexp := MatrixExponential(J \cdot t)$

$$Jexp := \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

> $LeftTerms := Jexp \cdot Matrix([[1], [0]])$

$$LeftTerms := \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

Now, for the more complicated terms on the right.

```
> IntNegJ := MatrixExponential(-J·s)
```

$$IntNegJ := \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix}$$

```
> g_of_s := Matrix([[0], [varepsilon·sin(omega·s)]])
```

$$g\_of\_s := \begin{bmatrix} 0 \\ \varepsilon \sin(\omega s) \end{bmatrix}$$

```
> InsideIntegrand := IntNegJ · g_of_s
```

$$InsideIntegrand := \begin{bmatrix} -\varepsilon \sin(s) \sin(\omega s) \\ \varepsilon \cos(s) \sin(\omega s) \end{bmatrix}$$

```
> IntTopRow := -ε sin(s) sin(ω s)
```

$$IntTopRow := -\varepsilon \sin(s) \sin(\omega s)$$

```
> IntBottomRow := ε cos(s) sin(ω s)
```

$$IntBottomRow := \varepsilon \cos(s) \sin(\omega s)$$

```
> ITR := simplify(int(IntTopRow, s = 0 ..t))
```

$$ITR := -\frac{((-\omega+1)\sin((\omega+1)t) + (\omega+1)\sin((\omega-1)t))\,\varepsilon}{2\omega^2-2}$$

```
> IBR := simplify(int(IntBottomRow, s = 0 ..t))
```

$$IBR := -\frac{((\omega+1)\cos((\omega-1)t) + \cos((\omega+1)t)(\omega-1) - 2\omega)\,\varepsilon}{2\omega^2-2}$$

```
> IntsDoneMatrix := simplify(Matrix([[ITR], [IBR]]))
```

$$IntsDoneMatrix := \begin{bmatrix} -\dfrac{((-\omega+1)\sin((\omega+1)t) + (\omega+1)\sin((\omega-1)t))\,\varepsilon}{2\omega^2-2} \\ -\dfrac{((\omega+1)\cos((\omega-1)t) + \cos((\omega+1)t)(\omega-1) - 2\omega)\,\varepsilon}{2\omega^2-2} \end{bmatrix}$$

```
> RightTerm := simplify(Jexp · IntsDoneMatrix)
```

$$RightTerm := \begin{bmatrix} \dfrac{\varepsilon(\sin(t)\,\omega - \sin(\omega t))}{\omega^2-1} \\ \dfrac{\varepsilon\,\omega(-\cos(\omega t) + \cos(t))}{\omega^2-1} \end{bmatrix}$$

```
> Sol := simplify(LeftTerm + RightTerm)
```

$$Sol := \begin{bmatrix} \dfrac{\sin(t)\,\omega\,\varepsilon + \cos(t)\,\omega^2 - \varepsilon\sin(\omega t) - \cos(t)}{\omega^2-1} \\ \dfrac{-\omega\,\varepsilon\cos(\omega t) - \sin(t)\,\omega^2 + \cos(t)\,\omega\,\varepsilon + \sin(t)}{\omega^2-1} \end{bmatrix}$$

As far as special cases, we note that we cannot have $\omega = \pm 1$ as this would make the denominator of our solution undefined. If this is the case, then we have much simpler ODEs to deal with (left equation: $\omega = 1$, right equation $\omega = -1$:

$$\ddot{y} + y = \varepsilon \sin(t), \qquad \ddot{y} + y = \varepsilon \sin(-t) \Rightarrow \ddot{y} + y = -\varepsilon \sin(t)$$

Both of these are easily solved with techniques from elementary differential equations (i.e., variation of parameters, which we could trivially do in MAPLE, or any other computing software).

Part b)

We now consider

$$\ddot{y} + y = \varepsilon y(1 - \dot{y}^2), \qquad y(0) = 1, \qquad \ddot{y}(0) = 0$$

where, $0 < \varepsilon \ll 1$. Then, we want to approximate a solution utilizing:

$$y(t) = y_1(t) + \varepsilon y_2(t) + \mathcal{O}(\varepsilon^2)$$

Therefore, we then take this expression and plug it into our original equation:

$$\begin{aligned}
\left(y_1''(t) + \varepsilon y_2''(t) + \mathcal{O}(\varepsilon^2)\right) + \left(y_1(t) + \varepsilon y_2(t) + \mathcal{O}(\varepsilon^2)\right) &= \varepsilon\left(y_1(t) + \varepsilon y_2(t) + \mathcal{O}(\varepsilon^2)\right)\left(1 - \left(y_1'(t) + \varepsilon y_2'(t) + \mathcal{O}(\varepsilon^2)\right)^2\right) \\
(y_1''(t) + \varepsilon y_2''(t)) + (y_1(t) + \varepsilon y_2(t)) &= \varepsilon\left(y_1(t) + \varepsilon y_2(t)\right)\left(1 - (y_1'(t) + \varepsilon y_2'(t))^2\right) \\
(y_1'' + \varepsilon y_2'') + (y_1 + \varepsilon y_2) &= \varepsilon(y_1 + \varepsilon y_2)\left(1 - \left(y_1'^{\,2} + 2\varepsilon y_1' y_2' + \varepsilon^2 y_2'^2\right)\right) \\
&= \varepsilon(y_1 + \varepsilon y_2)\left(1 - y_1'^2 - 2\varepsilon y_1' y_2'\right) \\
&= \left(\varepsilon y_1 + \varepsilon^2 y_2\right)\left(1 - y_1'^2 - 2\varepsilon y_1' y_2'\right) \\
&= (\varepsilon y_1)\left(1 - y_1'^2 - 2\varepsilon y_1' y_2'\right) \\
&= \left(\varepsilon y_1 - \varepsilon y_1'^2 - 2\varepsilon y_1' y_2'\right) \\
&= \left(\varepsilon y_1 - \varepsilon y_1'^2 y_1 - 2\varepsilon^2 y_1' y_2'\right) \\
&= \left(\varepsilon y_1 - \varepsilon y_1'^2 y_1\right) \\
y_1'' + \varepsilon y_2'' + y_1 + \varepsilon y_2 - \varepsilon y_1 + \varepsilon y_1'^2 y_1 = 0 &
\end{aligned}$$

Finally, we arrive at these as our two distinct ε^0 and ε^1 equations:

$$(y_1'' + y_1) + \varepsilon\left(y_2'' + y_2 - y_1 + y_1'^2 y_1\right) = 0$$

Therefore, we move to solve our first equation:

$$\begin{aligned}
y_1'' + y_1 &= 0 \\
y_1(t) &= C_1 \cos(t) + C_2 \sin(t)
\end{aligned}$$

With the first initial condition:

$$\begin{aligned}
1 &= C_1 \cos(0) + C_2 \sin(0) \\
C + 1 &= 1
\end{aligned}$$

So, we now have:

$$\begin{aligned} y_1(t) &= \cos(t) + C_2 \sin(t) \\ y_1'(t) &= -\sin(t) + C_2 \cos(t) \end{aligned}$$

With the second initial condition:

$$\begin{aligned} 0 &= -\sin(0) + C_2 \cos(0) \\ C_2 &= 0 \end{aligned}$$

All in all then, we have that:

$$y_1(t) = \cos(t)$$

Moving to the second equation:

$$\begin{aligned} (y_1'' + y_1) + \varepsilon\left(y_2'' + y_2 - y_1 + y_1'^2 y_1\right) &= 0 \\ y_2'' + y_2 - \cos(t) + \left((\cos'(t))^2 \cos(t)\right) &= 0 \\ y_2'' + y_2 - \cos(t) + (-\sin(t))^2 \cos(t) &= 0 \\ y_2'' + y_2 - \cos^3(t) &= 0 \end{aligned}$$

We then again have a situation where we need to utilize methods beyond things we learned in elementary differential equations. So, setting this up as a system of ODEs:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos^3(t) \end{bmatrix}$$

Using the variation of constants, we note the matrix is already in Jordan Form, and because we're now dealing with the ε terms, our initial conditions become $y(0) = 0, y'(0) = 0$:

$$\begin{aligned} \boldsymbol{x}(t) &= e^{Jt}\boldsymbol{x}_0 + \int_0^t e^{A(t-s)}\boldsymbol{g}(s)ds \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-A} \begin{bmatrix} 0 \\ \cos^3(s) \end{bmatrix} ds \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int_0^t \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ \cos^3(s) \end{bmatrix} ds \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int_0^t \begin{bmatrix} -\sin(s)\cos^3(s) \\ \cos^4(s) \end{bmatrix} ds \end{aligned}$$

Leaving the rest to MAPLE:

> $FirstRow := (-\sin(s))\cdot((\cos(s))^3)$

$$FirstRow := -\sin(s)\cos(s)^3$$

> $SecondRow := (\cos(s))^4$

$$SecondRow := \cos(s)^4$$

> $IntFirstRow := int(FirstRow, s = 0..t)$

$$IntFirstRow := -\frac{1}{4} + \frac{\cos(t)^4}{4}$$

> $IntSecondRow := int(SecondRow, s = 0..t)$

$$IntSecondRow := \frac{\sin(t)\cos(t)^3}{4} + \frac{3\cos(t)\sin(t)}{8} + \frac{3t}{8}$$

> $IntMatrix := Matrix([[IntFirstRow], [IntSecondRow]])$

$$IntMatrix := \begin{bmatrix} -\frac{1}{4} + \frac{\cos(t)^4}{4} \\ \frac{\sin(t)\cos(t)^3}{4} + \frac{3\cos(t)\sin(t)}{8} + \frac{3t}{8} \end{bmatrix}$$

> $Prob5bSol := simplify(Jexpo \cdot IntMatrix)$

$$Prob5bSol := \begin{bmatrix} \frac{\sin(t)\,(\cos(t)\sin(t) + 3t)}{8} \\ \frac{3\cos(t)^2\sin(t)}{8} + \frac{3\cos(t)\,t}{8} + \frac{\sin(t)}{4} \end{bmatrix}$$

Therefore, we utilize the first equation as $x_1(t) = y_2(t)$, so then,

$$y_2(t) = \frac{\sin^2(t)\cos(t)}{8} + \sin(t)\frac{3t}{8}$$

Altogether then our approximation becomes:

$$y(t) = \cos(t) + \varepsilon\left(\frac{\sin^2(t)\cos(t)}{8} + \sin(t)\frac{3t}{8}\right)$$

Clearly, we have a secular term. We see that $\varepsilon 3t/8$ after large enough t will tend towards infinity.

Part c)

Considering part b's equation again:

$$\ddot{y} + y = \varepsilon y(1 - \dot{y}^2), \qquad y(0) = 1, \qquad \ddot{y}(0) = 0$$

But now, we'll go ahead and let

$$\tau = \omega t \Rightarrow t = \frac{\tau}{\omega}$$
$$\omega = 1 + \varepsilon\omega_1 + \mathcal{O}(\varepsilon^2)$$
$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \mathcal{O}(\varepsilon^2)$$

So, observing $y(\tau) = y(\omega t)$ this implies that $\omega\dot{y}(\tau) = \omega\dot{y}(\omega t)$ and $\omega^2\ddot{y}(\tau) = \omega^2\ddot{y}(\omega t)$. Now,

$$\omega^2\ddot{y}(t) + y(\tau) = \varepsilon y(\tau)\left(1 - \omega^2\dot{y}(\tau)^2\right)$$

Now, adding in our ε approximations for $y(\tau)$ and ω:

$$(1 + \varepsilon\omega_1)^2(\ddot{y}_0 + \varepsilon\ddot{y}_1) + y_0 + \varepsilon y_1 = \varepsilon(y_0 + \varepsilon y_1)\left(1 - (1 + \varepsilon\omega_1)^2(\dot{y}_0(\tau) + \varepsilon\dot{y}_1)^2\right)$$

Performing tedious algebra yields:

$$(2\varepsilon\omega_1\ddot{y}_0 + \varepsilon\ddot{y}_1) + y_0 + \varepsilon y_1 = \left(\varepsilon y_0 - \varepsilon y_0\dot{y}_0^2\right)$$

Now, we collect epsilon terms and solve various ODEs to get our approximation.

$$\ddot{y}_0 + y_0 + \varepsilon\left(2\omega_1\ddot{y}_0 + \ddot{y}_1 + y_1 - y_0 + y_0\dot{y}_0^2\right) = 0$$

Now, solving the first equation:

$$\ddot{y}_0 + y_0 = 0$$
$$y_0(\tau) = C_1\sin(\tau) + C_2\cos(\tau)$$

Utilizing the initial conditions $y(0) = 1,\ \dot{y}(0) = 0$:

$$1 = C_1\sin(0) + C_2\cos 0 \tag{1}$$
$$C_2 = 1 \tag{2}$$

So, with the initial condition we first integrate and then substitute. We also note that our substitutions above were to get things into terms of τ, but now that we've finished our rescaling, we're taking derivatives with respect to τ.

$$\begin{aligned}
y_0(\tau) &= C_1 \sin(\tau) + \cos(\tau) \\
\dot{y}_0 &= C_1 \cos(\tau) - \sin(\tau) \\
0 &= C_1(1) + 0 \\
C_1 &= 0
\end{aligned}$$

Hence, $y_0(\tau) = \cos(\tau)$, which is effectively the same as our first solution from part b. Continuing with the second ε equation and plugging in our values and simplifying we arrive at:

$$\ddot{y}_1 + y_1 - \cos^3(\tau) - 2\omega_1 \cos(\tau) = 0$$

Setting this up as a system of ODEs, $x_1(\tau) = y(\tau),\ x_2 = \dot{y}(\tau)$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos^3(\tau) + 2\omega_1 \cos(\tau) \end{bmatrix}$$

Using the variation of constants, we note the matrix is already in Jordan Form, and because we're now dealing with the ε terms, our initial conditions become $y(0) = 0,\ y'(0) = 0$:

$$\begin{aligned}
\boldsymbol{x}(t) &= e^{J\tau}\boldsymbol{x}_0 + \int_0^\tau e^{A(\tau - s)}\boldsymbol{g}(s)ds \\
&= \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \int_0^\tau \begin{bmatrix} -\sin(s)\cos^3(s) - 2\omega_1 \cos(s)\sin(s) \\ \cos^4(s) + 2\omega_1 \cos^2(s) \end{bmatrix} ds
\end{aligned}$$

Leaving the rest to MAPLE:

> $FR := -\sin(s)\cos^3(s) - 2\cdot\text{omega}\cdot\cos(s)\sin(s)$

$$FR := -\sin(s)\cos(s)^3 - 2\,\omega\cos(s)\sin(s)$$

> $SR := \cos^4(s) + 2\cdot\text{omega}\cdot\cos^2(s)$

$$SR := \cos(s)^4 + 2\,\omega\cos(s)^2$$

> $FRI := int(FR, s = 0..\text{tau})$

$$FRI := -\frac{1}{4} + \frac{\cos(\tau)^4}{4} - \omega\sin(\tau)^2$$

> $SRI := int(SR, s = 0..\text{tau})$

$$SRI := \frac{\sin(\tau)\cos(\tau)^3}{4} + \frac{3\cos(\tau)\sin(\tau)}{8} + \frac{3\,\tau}{8} + \omega\cos(\tau)\sin(\tau) + \omega\,\tau$$

```
> IntegralMat := Matrix([[FRI], [SRI]])
```

$$IntegralMat := \begin{bmatrix} -\frac{1}{4} + \frac{\cos(\tau)^4}{4} - \omega \sin(\tau)^2 \\ \frac{\sin(\tau)\cos(\tau)^3}{4} + \frac{3\cos(\tau)\sin(\tau)}{8} + \frac{3\tau}{8} + \omega\cos(\tau)\sin(\tau) + \omega\tau \end{bmatrix}$$

```
> Sol5d := simplify(Jexp · IntegralMat)
```

$$Sol5d := \begin{bmatrix} \frac{\cos(\tau)\sin(\tau)^2}{8} + \frac{\tau(8\omega+3)\sin(\tau)}{8} \\ \frac{(3\cos(\tau)^2 + 8\omega + 2)\sin(\tau)}{8} + \cos(\tau)\tau\left(\omega + \frac{3}{8}\right) \end{bmatrix}$$

Then, we know that we want to eliminate our secular term, therefore we can see in both equations that if we let $\omega_1 = \frac{-3}{8}$, then both of our solutions will go to zero! Again, we're only taking the first solution as this will give us:

$$y_1(\tau) = \frac{\cos(\tau)\sin^2\tau}{8}$$

Then the solution is all put together as:

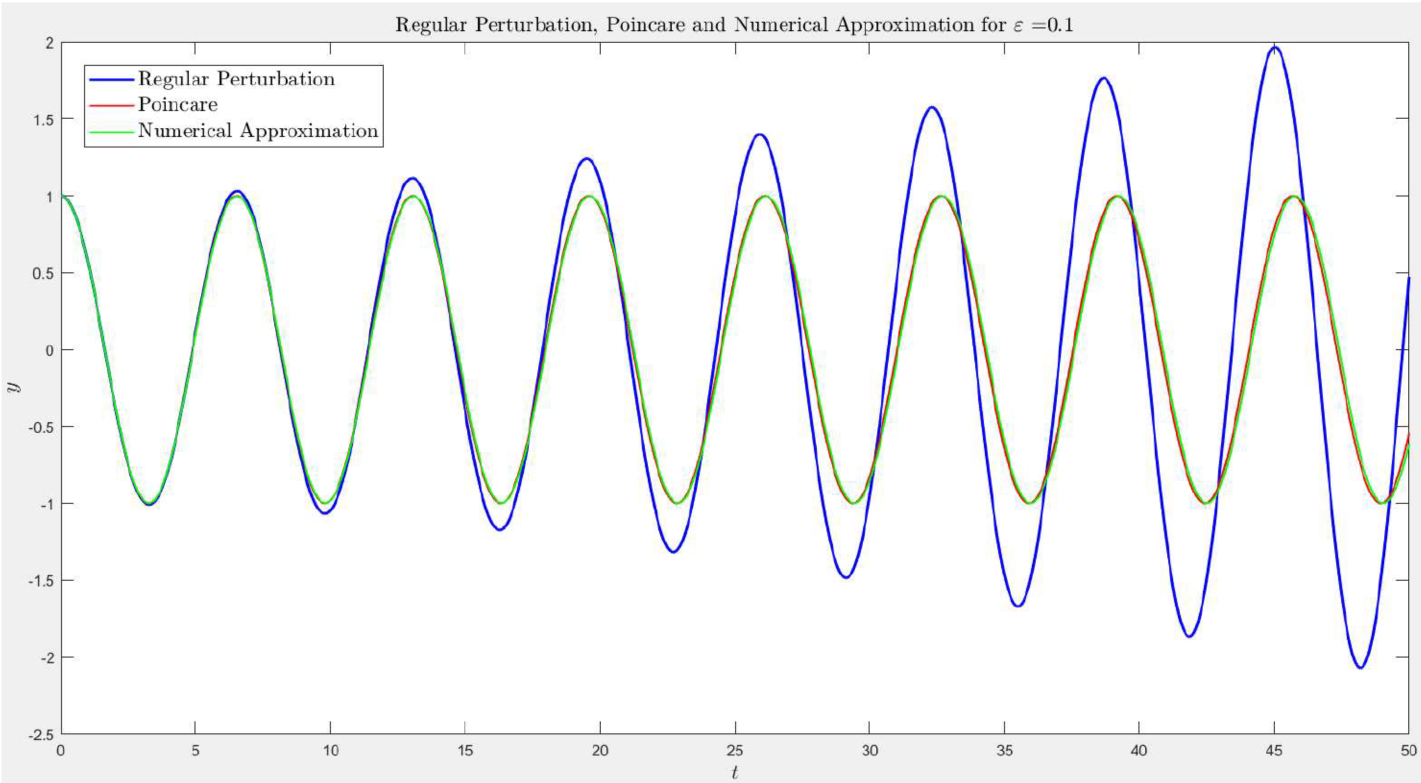
$$y(\tau) = \cos(\tau) + \varepsilon\left(\frac{\cos(\tau)\sin^2\tau}{8}\right)$$

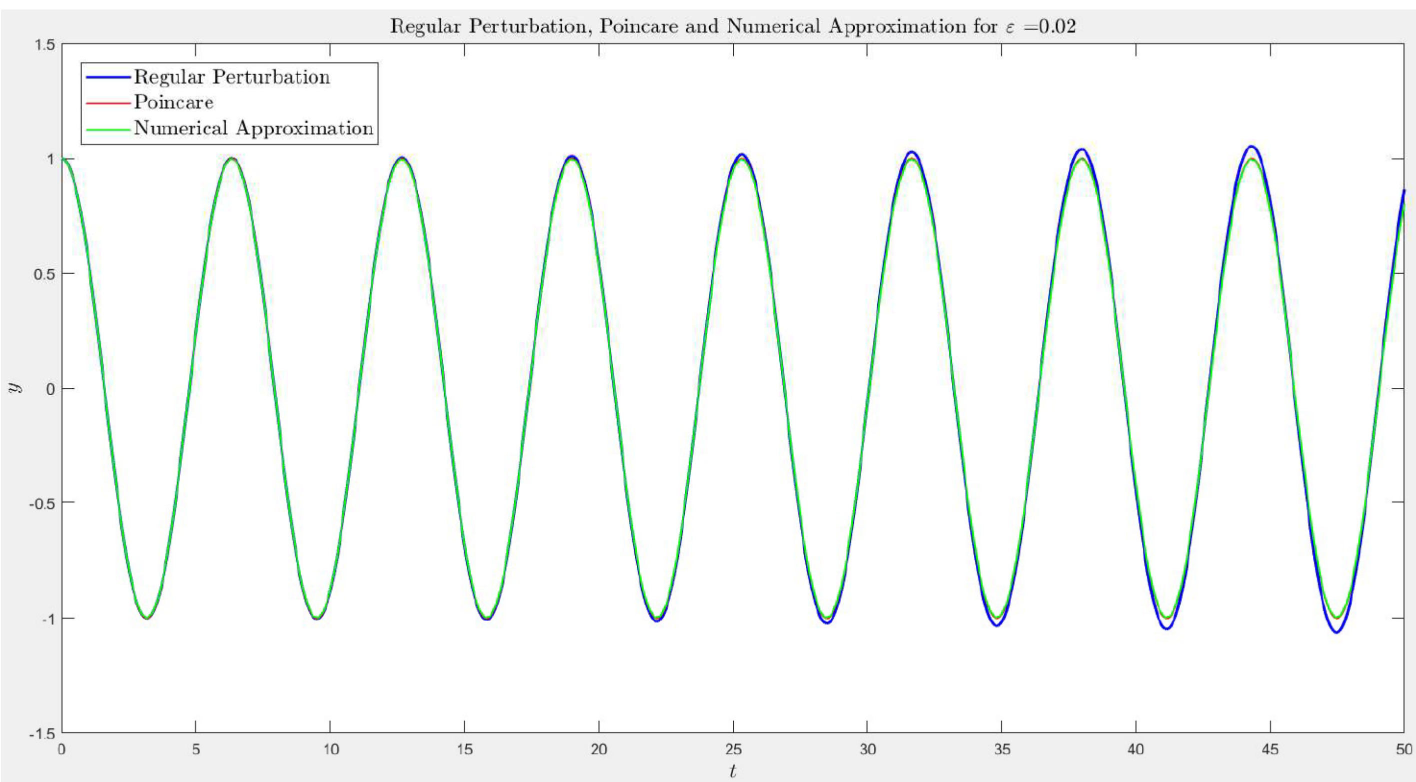
Again, we know $\tau = \omega t = (1 + \varepsilon\omega_1)t$, so $\tau = t - \frac{3}{8}t\varepsilon$.

Lastly, we note that this solution is necessarily bounded because we've gotten rid of our secular terms. The lowest values we can have are at $-1 - \frac{\varepsilon}{8}$ and the largest are $1 + \frac{\varepsilon}{8}$.

Part d)

We now provide simulations using a numerical solver, and our two approximations above.





We observe that the Poincare method, with the removed unbounded term, is a better approximation overall, regardless of our choice in epsilon. We do notice that with a better epsilon, the phase drift doesn't occur as quickly for both approximation types. We also see that with a smaller epsilon we have a better approximation for longer for both methods. Interestingly, we also see that the unbounded solution still holds decently well even out to $t = 50$ with the smaller epsilon. As such, if one were using this method and required lower overall computational overhead, the unbounded approximation would be worth considering.

6 Code Used

```
%% FINAL EXAM M537

%% prob3

close all
clear all
clc

eps = 0.1; %eps
t = linspace(0,5,10000);

% % u solutions "x"
% u_o = 1./(t+1);
% u_i = 1+0.*t;
% u_f = 1./(t+1);
% plot(t,u_f,'-b','LineWidth',1.5)
% hold on
% plot(t,u_o,'--m','LineWidth',1.5)
% plot(t,u_i,'--r','LineWidth',1.5)

% v solutions "y"
v_o = -((1./(t+1)).^2);
v_i = -1+exp(-t./eps);
v_f = exp(-t./eps)-((1./(t+1)).^2);
plot(t,v_f,'-b','LineWidth',1.5)
hold on
plot(t,v_o,'--m','LineWidth',1.5)
plot(t,v_i,'--r','LineWidth',1.5)

legend('Uniform','Outer','Inner','FontSize',15,'location','southeast')
%title("IVP Solutions for any $\varepsilon$",'FontSize',15,'interpreter','latex')
title("IVP Solutions for $\varepsilon =$" + num2str(eps),'FontSize',15,'interpreter','latex')
xlabel('$t$','FontSize',15,'interpreter','latex')
ylabel('$v$','FontSize',15,'interpreter','latex')
%ylabel('$u$','FontSize',15,'interpreter','latex')

%% prob2
```

```matlab

close all
clear all
clc

%eps = 0.10; %eps
%eps = 0.05;
eps = 0.00001;

x = linspace(0,1,1000);

% y solutions
y_o = log(2./(1+x));
y_i = log(2)-log(2).*exp(-2.*x./eps);
y_u = log(2./(1+x))-log(2).*exp(-2.*x./eps);
plot(x,y_u,'-b','LineWidth',1.5)
hold on
plot(x,y_o,'--m','LineWidth',1.5)
plot(x,y_i,'--r','LineWidth',1.5)

legend('Uniform','Outer','Inner','FontSize',15,'location','southwest')
title("BVP Solutions for $\varepsilon =$" + num2str(eps),'FontSize',15,'interpreter','latex')
xlabel('$x$','FontSize',15,'interpreter','latex')
ylabel('$y$','FontSize',15,'interpreter','latex')

%% prob 5 eigenvalue help

syms g w
B = [0 1 0 0; -2*w^2, -2*g, w^2, 0; 0 0 0 1; w^2 0 -2*w^2, -2*g];
[v,d] = eig(B)
v = simplify(v)

%% number 5 numerical solver

clear all
close all
clc

%Parameters
```

```
eps = 0.1;
tmax = 200;

t = linspace(0,tmax,1000);
y_PM = cos(t)+eps*((((sin(t)).^2.*cos(t))/8)+sin(t)*3.*t/8);
y_PL = cos(t-eps*3/8.*t)+eps.*((cos(t-eps*3/8.*t).*(sin(t-eps*3/8.*t)).^2)/8);

plot(t,y_PM,'b','Linewidth',1.5);
hold on
plot(t,y_PL ,'r','Linewidth',1);

% Now solving numerically using the ODE solver

eps1 = eps;
y0 = [1,0];
tspan = [0 tmax];
[T,y] = ode45(@(T,y) ODE5(T,y,eps1), tspan, y0);
plot(T,y(:,1),'-g','Linewidth',1)

legend('Regular Perturbation','Poincare','Numerical Approximation','interpreter','latex','FontSize',14)
title("Regular Perturbation, Poincare and Numerical Approximation for $\varepsilon$ =" + num2str(eps),'interpreter','latex','FontSize',14)
xlabel('$t$','interpreter','latex','FontSize',14)
ylabel('$y$','interpreter','latex','FontSize',14)

function dydt = ODE5(T,y,eps1)
    dydt    = zeros(2,1);
    dydt(1) = y(2);
    dydt(2) = -y(1)+eps1*y(1)-eps1*y(1)*y(2)^2;
end
```