SUBIECTE MODEL PENTRU EXAMEN

OBS: In functie de grupa pot fi mai simple sau mai grele, pot lipsi sau pot fi adaugate subpuncte.

Problema 1. *Fie* $\Omega = \{x \in \mathbb{R}^3, |x| < 2\}$ *si* $u : \Omega \to \mathbb{R}$ *o functie cu simetrie radiala, adica exista* $v : \mathbb{R} \to \mathbb{R}$ *astfel incat*

$$u(x) = v(|x|).$$

(1) *Calculati/Scrieti* Δu *in functie de* v

(2) *Aratati ca*

$$u(x) = \left(\frac{2\sqrt{3}}{4 - |x|^2}\right)^{1/2}, x \in \Omega$$

satisface ecuatia $\Delta u = u^5$.

Problema 2. *Fie* $\Omega = (0, \pi) \times (0, \pi) \in \mathbb{R}^2$. *Consideram problema*

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in \Omega, \\ u(x,0) = \sin(x), & x \in (0,\pi), \\ u(x,\pi) = \sin(3x) + \sin(5x), & x \in (0,\pi), \\ u(0,y) = u(\pi,y) = 0, & y \in (0,\pi). \end{cases} \tag{0.1}$$

(1) *Scrieti formula lui Green pentru doua functii* $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$

(2) *Considerati* u_1 *si* u_2 *doua solutii in clasa* $C^2(\Omega) \cap C(\overline{\Omega})$ *ale ecuatiei* (0.2). *Scrieti ecuatia satisfacuta de* $v = u_1 - u_2$.

(3) *Aratati ca ecuatia* (0.2) *are o unica solutie in clasa* $C^2(\Omega) \cap C(\overline{\Omega})$

(4) *Calculati o solutie* u *a ecuatiei* (0.2).

(5) *Aratati ca solutia gasita la punctul anterior este de clasa* $C^2(\Omega) \cap C(\overline{\Omega})$

(6) *Calculati*

$$\max_{(x,y)\in\overline{\Omega}} u(x,y), \qquad \max_{(x,y)\in\overline{\Omega}} u(x,y)$$

si gasiti macar un exemplu de puncte unde se ating cele doua extreme.

(7) *Definim*

$$D = \left\{ u \in C^2(\Omega) \cap C(\overline{\Omega}), \begin{cases} u(x,0) = \sin(x), x \in (0,\pi), \\ u(x,\pi) = \sin(3x) + \sin(5x), x \in (0,\pi), \\ u(0,y) = u(\pi,y) = 0, y \in (0,\pi), \end{cases} \right\}$$

si functionala

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2.$$

Aratati ca E *are un minim in* D *si calculati*

$$\min_{v \in D} E(v).$$

Problema 3. *Fie* $\Omega = (0,1) \times (0,1) \in \mathbb{R}^2$. *Consideram problema*

$$
\begin{cases}
\Delta u(x,y) = 0, & (x,y) \in \Omega, \\
u(x,0) = u(x,1) = \cos(\pi x), & x \in (0,1) \\
u_x(0,y) = u_x(1,y) = 0, & y \in (0,1).
\end{cases} \tag{0.2}
$$

(1) *Scrieti formula lui Green pentru doua functii* $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$
(2) *Presupunem ca ecuatia* (0.2) *admite o solutie* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. *Aplicati formula lui Green pentru functia* u *si functia* $v \equiv 1$
(3) *Aratati ca* u *verifica*
$$\int_0^1 u_y(x,0)dx + \int_0^1 u_y(x,1)dx = 0$$
(4) *Considerati* u_1 *si* u_2 *doua solutii in clasa* $C^2(\Omega) \cap C^1(\overline{\Omega})$ *ale ecuatiei* (0.2). *Scrieti ecuatia satisfacuta de* $v = u_1 - u_2$.
(5) *Aratati ca ecuatia* (0.2) *are o unica solutie in clasa* $C^2(\Omega) \cap C^1(\overline{\Omega})$
(6) *Calculati o solutie* u *a ecuatiei* (0.2).
(7) *Aratati ca solutia gasita la punctul anterior este de clasa* $C^2(\Omega) \cap C^1(\overline{\Omega})$
(8) *Calculati*
$$\max_{(x,y)\in\overline{\Omega}} u(x,y), \qquad \max_{(x,y)\in\overline{\Omega}} u(x,y)$$
si gasiti macar un exemplu de puncte unde se ating cele doua extreme.
(9) *Fie* $\Gamma_1 = \{(x,y), x \in (0,1), y \in \{0,1\}\}$ *si* $\Gamma_2 = \{(x,y), y \in (0,1), x \in \{0,1\}\}$. *Definim*
$$D = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}), u(x,y) = \cos(\pi x), (x,y) \in \Gamma_1, \frac{\partial u}{\partial \nu}(x,y) = 0\}$$
si functionala
$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2.$$
Aratati ca E *are un minim in* D *si calculati*
$$\min_{v\in D} E(v).$$

Problema 4. *Fie* $\Omega = (0,1) \times (0,1) \in \mathbb{R}^2$. *Consideram problema*

$$
\begin{cases}
\Delta u(x,y) = 0, & (x,y) \in \Omega, \\
u(x,0) = u(x,1) = \cos(\pi x), & x \in (0,1) \\
u_x(0,y) = u_x(1,y) = \sin(\pi y), & y \in (0,1).
\end{cases} \tag{0.3}
$$

(1) *Scrieti formula lui Green pentru doua functii* $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$
(2) *Presupunem ca ecuatia* (0.3) *admite o solutie* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. *Aplicati formula lui Green pentru functia* u *si functia* $v \equiv 1$
(3) *Aratati ca* u *verifica*
$$\int_0^1 u_y(x,0)dx + \int_0^1 u_y(x,1)dx = 0$$

(4) *Considerati* u_1 *si* u_2 *doua solutii in clasa* $C^2(\Omega) \cap C^1(\overline{\Omega})$ *ale ecuatiei* (0.3). *Scrieti ecuatia satisfacuta de* $v = u_1 - u_2$.

(5) *Aratati ca ecuatia* (0.3) *are o unica solutie in clasa* $C^2(\Omega) \cap C^1(\overline{\Omega})$

(6) *Scriind eventual* $u = u_1 + u_2$ *unde* u_1 *si* u_2 *sunt solutii ale ecuatiilor*

$$\begin{cases} \Delta u(x.y) = 0, & (x,y) \in \Omega, \\ u(x,0) = u(x,1) = \cos(\pi x), & x \in (0,1) \\ u_x(0,y) = u_x(1,y) = 0, & y \in (0,1). \end{cases} \tag{0.4}$$

si

$$\begin{cases} \Delta u(x.y) = 0, & (x,y) \in \Omega, \\ u(x,0) = u(x,1) = 0, & x \in (0,1) \\ u_x(0,y) = u_x(1,y) = \sin(\pi y), & y \in (0,1). \end{cases} \tag{0.5}$$

calculati o solutie u *a ecuatiei* (0.3).

(7) *Aratati ca solutia gasita la punctul anterior este de clasa* $C^2(\Omega) \cap C^1(\overline{\Omega})$

(8) *Calculati*

$$\max_{(x,y) \in \overline{\Omega}} u(x,y), \quad \max_{(x,y) \in \overline{\Omega}} u(x,y)$$

si gasiti macar un exemplu de puncte unde se ating cele doua extreme.

(9) *Fie* $\Gamma_1 = \{(x,y), x \in (0,1), y \in \{0,1\}\}$ *si* $\Gamma_2 = \{(x,y), y \in (0,1), x \in \{0,1\}\}$. *Definim*

$$D = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}), u(x,y) = \cos(\pi x), (x,y) \in \Gamma_1, \frac{\partial u}{\partial \nu}(x,y) = \sin(\pi y)\}$$

si functionala

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Gamma_2} \sin(\pi y) u(x,y) dS(x,y).$$

Aratati ca E *are un minim in* D *si calculati*

$$\min_{v \in D} E(v).$$