18.12.2013

= SEMINARUL 6 =

$W^{1,p}(I) \subset L^{\infty}(I)$

$1 \le p \le \infty$

$\|u\|_{L^{\infty}(I)} < c \cdot \|u\|_{W^{1,p}(I)}$

$\|u\|_{L^p} + \|u'\|_{L^p} = \|u\|_{W^{1,p}(I)}$

$\|u\|_{L^p(I)} \le c \|u\|_{W^{1,p}(I)}$

$W^{1,p}(I) \underset{cont.}{\subset} L^{q}(I)$

$L^p(I) = \{u : I \to \mathbb{R} , \int_I |u|^p < \infty\}$

$$\|u\|_{L^p(I)} = \begin{cases} \left(\int_I |u|^p\right)^{1/p}, & 1 \le p \le \infty \\ M, & p = \infty. \end{cases}$$

$\|u\| \le M$

Cauchy: $\int_I |f||g| \le \left(\int_I |f|^2\right)^{1/2} \left(\int_I |g|^2\right)^{1/2}$

Holder: $\int_I |f||g| \le \left(\int_I |f|^p\right)^{1/p} \left(\int_I (g)^{p}\right)^{1/p'}$ $\quad \frac{1}{p} + \frac{1}{p'} = 1 , p \in [1, \infty]$

Obs: $g \equiv 1$:

$$\int_I |f| \le \left(\int_I |f|^p\right)^{1/p} \left(\int_I 1\right)^{1/p'}$$

$$\int_I |f| \le \left(\int_I |f|^p\right)^{1/p} \cdot |I|^{1/p'}.$$

(*) $\|f\|_{L^1} \le \|f\|_{L^p} \cdot |I|^{1/p'}$

Cazul 1

Ex: Am un interval I, $\underline{|I|<\infty}$, vreau: $L^p(I) \underset{cont.}{\subset} L^q(I) \Leftrightarrow$

$\Leftrightarrow \|u\|_{L^q(I)} \le c\,\|u\|_{L^p(I)}$

depinde de int., p, q.

$$\int_I |f| \le \left(\int_I |f|^2\right)^{1/2} |I|^{1/2}$$

$$\|f\|_{L^1(I)} \le \|f\|_{L^2(I)} \, |I|^{1/2}$$

$$L^2(I) \underset{cont.}{\subset} L^1(I)$$

(*) $L^p(I) \underset{cont.}{\subset} L^1(I)$, $1 \le p \le \infty$

$\boxed{p=\infty}$

$L^\infty(I) \underset{cont.}{\subset} L^1(I)$

$(A, \|\cdot\|_A)$, $(B, \|\cdot\|_B)$

$A \underset{cont}{\subset} B$

$\|u\|_B \le c \cdot \|u\|_A$.

În cazul meu, vreau: $\|u\|_{L^1(I)} < c\, \underbrace{\|u\|_{L^\infty(I)}}_{M}$

măsura intervalului

$$\int_I |u| \le \int_I M = M \cdot |I| \qquad |u| \le M$$

$$\int_I |u| \le \|u\|_{L^\infty} \cdot \underbrace{|I|}_{c}. \Rightarrow L^\infty(I) \underset{cont}{\subset} L^1(I)$$

Am arătat: $\|u\|_{L^1(I)} \le c\|u\|_{L^p(I)}$, $\forall p\in[1,\infty]$.

Vreau: $\|u\|_{L^q(I)} \le c\|u\|_{L^p}$.

$q = 1 \le p$ (am arătat)

$$\int_I |u|^q = \int_I (|u^p|)^{\frac{q}{p}}\cdot 1 \overset{?}{\le} \int_I |u|^p$$

Holder: $\int fg \le \left(\int f^r\right)^{1/r}\left(\int g^{r'}\right)^{1/r'}$, $\frac{1}{r} \le \frac{1}{r}+\frac{1}{r'} = 1$

$$\int_I (|u^p|)^{\frac{q}{p}}\cdot 1 \le \Big[\int_I (|u|^p)^{\overbrace{\frac{q}{p}\cdot r}^{1}}\Big]^{1/r}\cdot\Big[\int_I 1^{r'}\Big]^{1/r'} =$$

$1/r + 1/r' = 1$. $\quad r = \frac{p}{q} \ge 1$ $\quad p > q$ (altfel nu pot aplica Holder)

$$= \left(\int_I u^p\right)^{q/p}\cdot |I|^{1-\frac{q}{p}} = \left(\int_I u^p\right)^{q/p} |I|^{1-\frac{q}{p}}$$

Deci, $\int_I |u|^q \le \left(\int_I u^p\right)^{q/p}\cdot |I|^{1-\frac{q}{p}} \Big/^{1/q} \Rightarrow$

$$\Rightarrow \|u\|_{L^q(I)} \le \|u\|_{L^p(I)}\, |I|^{\frac{1}{q}-\frac{1}{p}}$$

Deci, $L^p(I) \underset{\text{cont.}}{\subset} L^q(I)$, $\forall I$-mărg, $1\le q\le p<\infty$

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cazul 2) $|I| = \infty$, $I = \mathbb{R}$

$$\int_{\mathbb{R}} |u|^2 \le M\int_{\mathbb{R}} |u|$$

$|u| \le M$

Dacă iau $M = \|u\|_{L^\infty(\mathbb{R})}$

$$\int_{\mathbb{R}} u^2 \leq \|u\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u|$$

$$\|u\|^2_{L^2(\mathbb{R})} \leq \|u\|_{L^\infty(\mathbb{R})} \cdot \|u\|_{L^1(\mathbb{R})}$$

$$\|u\|_{L^2(\mathbb{R})} \leq \|u\|^{1/2}_{L^\infty(\mathbb{R})} \|u\|^{1/2}_{L^1(\mathbb{R})}$$

$$L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$$

$$L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^p(\mathbb{R}), \ 1 \leq p \leq \infty.$$