15.01.2014

= SEMINARUL 7 =

Soluții slabe

Ex: (1)

$$(*)\begin{cases} -u'' + u = f & , x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Soluție tare: $u \in \mathcal{C}^2([0,1])$ ce satisface $(*)$ punct cu punct, $\forall x \in (0,1)$

Pas 1: sol tare $\Rightarrow$ sol. slabă (def. ce e sol. slabă)

Def: Soluția slabă se obține prin înmulțirea cu φ, integrăm prin părți și obținem o sumă

$$\begin{cases} u \in H_0^1(I) \\ \underbrace{\int_0^1 u'\varphi' + \int_0^1 u\varphi}_{a} = \overbrace{\int_0^1 f\cdot\varphi}^{\langle F, v\rangle}, \ \forall \varphi \in H_0^1(I) \end{cases}$$

$H^1(I) = W^{1,2}(I) = \{u \in L^2, u' \in L^2\}$ = fcțiile din H^1 cu 0 pe frontieră $(u(0) = u(1) = 0)$

Pas 2: $\exists$ o unică sol. slabă

nu insistăm

Pas 3: Regularitatea sol. slabe

$f \in L^2(I) \Rightarrow u$-sol. slabă $(H_0^1) \Rightarrow u \in H^2(I) = \{u, u', u'' \in L^2\}$

$f \in C(\bar{I}) \Rightarrow u \in C^2(\bar{I})$

Pas 4: $u \in C^2(\bar{I})$ - sol. slabă $\Rightarrow u$ - sol. tare

Pasul 2: fol. teorema Lax-Milgram

- H - sp. Hilbert
- $a: H \times H \to \mathbb{R}$
  - biliniară
  - continuă: $|a(u,v)| \le \overset{ct.}{c}\|u\|_H \|v\|_H$
  - coercivă: $a(u,u) \ge \alpha \|u\|_H$
- $F: H \to \mathbb{R}$
  - liniară
  - continuă: $|\langle F, v\rangle| \le c\|v\|_H$

Dacă se verif. ce e mai sus, at. $\exists!\, u \in H$ aî $a(u,v) = \langle F, V\rangle\ \forall v \in H$

Identific $H, a, u, \underset{\langle F, v\rangle}{f}$: $H = H_0^1$, $a = \int_0^1 u'\varphi' + \int_0^1 u\varphi$, $f = \int_0^1 f\varphi$

Ex: (2) $\begin{cases} -u'' + u = f \in L^2(I) \\ u'(0) = 0, u'(1) = 0 \end{cases}$ → o funcție generală

Pasul 1 Dacă $u \in C^2(I)$

Iau o fcție $\varphi \in C^\infty(\mathbb{R})$. Înmulțesc cu φ și fac int. prin părți: (→ ca să ne permută integrarea)

$$-\int_0^1 u''\varphi + \int_0^1 u\varphi = \int_0^1 f\cdot\varphi \Leftrightarrow$$

$$\Leftrightarrow -u'\varphi\Big|_0^1 + \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi$$

$$\Leftrightarrow \underbrace{\underset{=0}{u'(1)}\varphi(1) + \underset{=0}{u'(0)}\varphi(0)}_{=0} + \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi \Rightarrow$$

$$\Rightarrow \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi, \forall \varphi \in C^\infty(\mathbb{R}) \text{ are sens dc: } \begin{cases} u' \in L^2 \\ \varphi' \in L^2 \\ u \in L^2 \\ \varphi \in L^2 \end{cases}$$

$f, g \in L^2(I) \rightarrow \int f\cdot g$ există și e finită (aplic Cauchy sau Hölder)

$$\left(\int f^2\right)^{1/2}\cdot\left(\int g^2\right)^{1/2}$$

def. soluției slabe

Pet defini integralele de mai sus dc* $\boxed{u \in H^1(I), \quad (**)\ \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi,\ \varphi \in H^1}$

ca să fie simetrie

Am plecat de la u-sol. tare ⇒ (*) e adv. ptr. $u \in C^\infty(\mathbb{R})$, dar vreau $u \in H^1(I)$.

Vreau să trec de la $\int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi, \varphi \in C^\infty(\mathbb{R})$ la (**).

Facem prin densitate:

$C^\infty(\mathbb{R})$, în particular $C_c^\infty(\mathbb{R})$ dens în $H^1(I) \supset H_0^1(I)$

$[C_c^\infty(I)$ - dens în $H_0^1(I)]$

$\forall \varphi \in H^1(I), \exists \varphi_n \in C_c^\infty(\mathbb{R}), \|\varphi_n - \varphi\|_{H^1(I)} \rightarrow 0$ (asta repreз. densitatea)

(**) e adv. ptr. $\varphi_n \Rightarrow \boxed{\int_0^1 u'\varphi_n' + \int_0^1 u\varphi_n = \int_0^1 f\varphi_n}$

$$\|\varphi_n - \varphi\|_{H^1} \rightarrow 0 \Leftrightarrow \underbrace{\|\varphi_n - \varphi\|_{L^2}}_{\rightarrow 0} + \underbrace{\|\varphi_n' - \varphi'\|_{L^2}}_{\rightarrow 0} \quad (1)$$

(2)

Revenim la problemă:

arătăm pe componente

• $\int_0^1 u' \varphi_n' \longrightarrow \int_0^1 u'\varphi' \Leftrightarrow \int_0^1 u'(\varphi_n' - \varphi') \longrightarrow 0.$

$$\left|\int_0^1 u'(\varphi_n' - \varphi')\right| \le \int_0^1 |u'|\cdot|\varphi_n' - \varphi'| \underset{\text{aplic cauchy}}{\le} \left(\int_0^1 u'^2\right)^{1/2} \cdot \underbrace{\left(\int_0^1 |\varphi_n' - \varphi'|^2\right)^{1/2}}_{\|\varphi_n' - \varphi'\|_{L^2} \xrightarrow{\text{din } 1} 0}$$

• $\int_0^1 u\varphi_n \longrightarrow \int_0^1 u\varphi$: $\left|\int_0^1 u(\varphi_n - \varphi)\right| \le \int_0^1 |u|\cdot|\varphi_n - \varphi| \le$

$$\le \left(\int_0^1 u^2\right)^{1/2} \cdot \underbrace{\left(\int_0^1 |\varphi_n - \varphi|^2\right)^{1/2}}_{\|\varphi_n - \varphi\|_{L^2(I)} \xrightarrow{\text{din } 1} 0}$$

Deci, $\begin{cases} u \in H^1(I) \\ \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi, \forall \varphi \in H^1(I) \end{cases}$

<u>Pasul 2</u>: Există o unică sol. slabă.

Iau $H = H^1(I)$

$$a(u,v) = \int_0^1 u'\varphi' + \int_0^1 uv$$

$$\langle F, v\rangle = \int_0^1 f\varphi$$

Dacă se verifică th. Lax-... $\Rightarrow \exists!\, u \in H_1(I)$ a.î. $a(u,v) = \langle F, v\rangle$, $\forall v \in H^1(I)$

• <u>a - biliniară</u>

- a - bine definită
- $a(\alpha u_1 + \beta u_2) = \alpha a(u_1, v) + \beta a(u_2, v)$
- a - continuă: $|a(u,v)| \le c\cdot\|u\|_{H^1}\cdot\|v\|_{H^1} \Leftrightarrow$

$$\Leftrightarrow \left|\int_0^1 u'v' + \int_0^1 uv\right| \le \int_0^1 |u'|\cdot|v'| + \int_0^1 |u|\cdot|v| \underset{\text{cauchy}}{\le} \left(\int_0^1 u'^2\right)^{1/2} \cdot$$

$$\cdot \left(\int_0^1 v'^2\right)^{1/2} + \left(\int_0^1 u^2\right)^{1/2} + \left(\int_0^1 v^2\right)^{1/2} \le$$

$$\left(\|u\|_{H_1} = \|u\|_{L^2} + \|u'\|_{L^2}\right)$$

$$\leq \|u\|_{H^1} \cdot \|v\|_{H^1} + \|u\|_{H^1} \cdot \|v\|_{H^1} = 2\|u\|_{H^1} + \|v\|_{H^1} \Rightarrow$$

$\Rightarrow$ a - continuă

- a - coercivă

$$a(u,u) \geq c\|u\|^2_{H_1}$$

$$a(u,u) = \int_0^1 (u')^2 + \int_0^1 u^2 \geq c\left(\|u\|_{L^2} + \|u'\|_{L^2}\right)^2$$

$$\int_0^1 (u')^2 + \int_0^1 u^2 = \|u'\|^2_{L^2} + \|u\|^2_{L^2} \geq \frac{1}{2}\left(\|u\|_{L^2} + \|u'\|_{L^2}\right)^2 \Rightarrow a(u,u) \geq \frac{1}{2}\|u\|^2_{H_1}$$

Vreau $a^2 + b^2 \geq c(a+b)^2 \Rightarrow c = \frac{1}{2}$.

$a = b \Rightarrow 2a^2 \geq c(2a^2) \to c \leq \frac{1}{2}$.

Arătăm: $\langle F, v \rangle = \int_0^1 f \cdot v$

- F - liniară

$$\langle F, \alpha v_1 + \beta v_2 \rangle = \alpha \langle F, v_1 \rangle + \beta \langle F, v_2 \rangle.$$

- F - cont.

$$|\langle F, v \rangle| \leq c\|v\|_{H^1}$$

$$\left|\int_0^1 f v\right| \leq c\|v\|_{H^1}$$

$$\leq \int_0^1 |f||v| \leq \left(\int_0^1 |f|^2\right)^{1/2}\left(\int_0^1 v^2\right)^{1/2} \leq c \cdot \|v\|_{H^1} \Rightarrow F\text{-cont.}$$

Deci, din th. L-M $\Rightarrow \exists!\, u \in H^1(I)$ a.î. $a(u,v) = \langle F, v \rangle, \forall v \in H^1(I)$

$\Rightarrow \exists!$ sol. slabă u.

Pasul 4 ($\approx$) dc. plec de la sol. slabă $\to$ sol. tare

$u \in C^2(\bar{I})$ sol. slabă

$$\begin{cases} u \in H^1(I) \\ \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi, \ \forall \varphi \in H^1(I) \end{cases}$$

sol. tare $\Leftrightarrow \begin{cases} -u'' + u = f, & x \in (0,1) \quad (2) \\ u'(0) = u'(1) = 0 & (3) \end{cases}$

Demonstrez ptr (2):

Fac derivarea prin părți în sens invers: de la o integrală ajung la $2\int$

Iau $\varphi \in C_c^\infty(I) \subset H^1(I)$

$$\underbrace{u'\varphi\Big|_0^1}_{0} - \int_0^1 u''\varphi + \int_0^1 u\varphi = \int_0^1 f\cdot\varphi$$

are suport compact $\Rightarrow \varphi(1) = \varphi(0) = 0$

$$\int_0^1 \underbrace{(-\overset{\text{cont}\in L^1}{u''} + \overset{\text{cont}}{u} - \overset{f\in L^1}{f})}_{g\in L^1((0,1))}\varphi = 0, \forall \varphi \in C_c^\infty(I) \Rightarrow g = 0 \text{ aproape peste tot}$$

Deci, $\underbrace{-u''+u-f}_{\text{continuă}} = 0$ a.p.t. în $(0,1)$

$\Rightarrow -u''+u-f=0, \forall x\in(0,1)$.

Înmulțesc $(-u''+u-f)$ cu $\varphi \in C^\infty(\mathbb{R}) \Rightarrow -\int_0^1 u''\varphi + \int_0^1 u\varphi = \int_0^1 f\varphi, \forall \varphi \in C^\infty(\mathbb{R})$

$\Leftrightarrow -u'\varphi\Big|_0^1 + \int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 f\varphi \Leftrightarrow$

$$\Leftrightarrow -u'(1)\varphi(1) + u'(0)\varphi(0) = \underbrace{-\int_0^1 u'\varphi' - \int_0^1 u\varphi + \int_0^1 f\varphi}_{u\text{-sol. slabă} \Rightarrow\ =0, \forall \varphi\in C^\infty(\mathbb{R})}$$

$-u'(1)\varphi(1) + u'(0)\varphi(0) = 0, \forall \varphi \in C^\infty(\mathbb{R})$

Aleg φ a.î. $\varphi(0)=0, \varphi(1)\neq 0 \Rightarrow -u'(1)\varphi(1) = 0 \Rightarrow \underline{u'(1)=0}$

φ a.î. $\varphi(0)\neq 0, \varphi(1) = 0 \Rightarrow u'(0)\varphi(0) \Rightarrow \underline{u'(0)=0}$ $\Rightarrow$ am dem. (3).

Deci, u - sol. tare

Ex: (3)
$$(S)\begin{cases} -u'' + 2u = 1+\sin(x), & x\in(0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Pasul 1:

- Definesc soluția slabă
- u - sol. tare, $u \in C^2(I)$ ce satisface (S) ptr. $\forall x \in (0,1) \Rightarrow u$ - sol. slabă

Pasul 2: $\exists!$ sol. slabă

Pasul 1:

Fie $\varphi \in C^\infty(\mathbb{R})$:

$$-\int_0^1 u''\varphi + 2\int_0^1 u\varphi = \int_0^1 (1+\sin x)\varphi \quad \text{(scap de derivatele lui } u\text{)}$$

$$\Leftrightarrow -u'\varphi\Big|_0^1 + \int_0^1 u'\varphi' + 2\int_0^1 u\varphi = \int_0^1 (1+\sin x)\varphi \Leftrightarrow$$

$$\Leftrightarrow \underbrace{-u'(1)\varphi(1) + u'(0)\varphi(0)}_{\text{vreau să scap de asta}} + \int_0^1 u'\varphi' + 2\int_0^1 u\varphi = \int_0^1 (1+\sin x)\varphi$$

tb. să ajung la $a(u,v)$

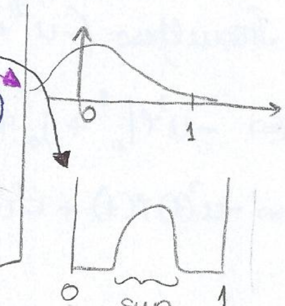
Dacă $\varphi(0)=\varphi(1)=0$:

$$\boxed{\int_0^1 u'\varphi' + 2\int_0^1 u\varphi = \int_0^1 (1+\sin x)\varphi.} \to \text{adv.}, \forall\, \varphi \in C_c^\infty(I)$$

Iau niște $\underbrace{\text{fcții}}_{\varphi}$ din $C_c^\infty(\mathbb{R})$ sau $\boxed{C_c^\infty(I)}$ a.î. $\varphi(0)=\varphi(1)=0 \Rightarrow -u'(1)=u'(0)=0$

$\Rightarrow -u'(1)\varphi(1) + u'(0)\varphi(0) = 0$

Deci, iau φ din $C_c^\infty(I)$



aleg această var. ptr. că așa suntem siguri ?

$u(0)=u(1)=0$

$$\int_0^1 u'\varphi' + 2\int_0^1 u\varphi = \int_0^1 (1+\sin x)\varphi, \forall\, \varphi \in C_c^\infty(I)$$

Aleg un sp. Hilbert ca să pot aplica L-M

$u \in H^1(I)$

$u(0)=u(1)=0$

$H_0^1 = \{u \in H^1,\ u(0)=u(1)=0\}$

În primă fază, $H_0^1 = \overline{C_c^\infty(I)}^{\|\cdot\|_{H^1}}$

$u \in H_0^1 \Leftrightarrow \exists\, \varphi_n \in C_c^\infty(I)$ a.î. $\|u-\varphi_n\|_{H^1} \to 0$

$$\boxed{\begin{array}{l} u \in H_0^1(I) \\ \displaystyle\int_0^1 u'\varphi' + 2\int_0^1 u\varphi = \int_0^1 (1+\sin x)\varphi, \ \forall\, \varphi \in H_0^1 \end{array}} \to \text{def. soluției slabe}$$

Inițial, $\varphi \in C_c^\infty(I) \xrightarrow[\text{densitate}]{} \varphi \in H_0^1(I)$.

Deci, $\forall$ sol tare este și sol. slabă

Pasul 2 : Există o unică sol. slabă

Iau $H = H_0^1$, $\|u\|_{H_0^1} = \|u\|_{L^2} + \|u'\|_{L^2}$

$$a(u,v) = \int_0^1 u'v' + 2\int_0^1 uv$$

$$<F, v> = \int_0^1 (1+\sin x)$$

• $a \to$ bine definită

integralele sunt finite: $u, v \in H_0^1 \Rightarrow u, u', v, v' \in L^2 \Rightarrow \begin{cases} \left|\int_0^1 uv\right| < \infty \\ \left|\int_0^1 u'v'\right| < \infty \end{cases}$

• $a \to$ biliniară

..

• $a \to$ continuă — poate fi și o funcție mărginită, de ex $1+\sin x$

$$|a(u,v)| \le c\|u\|_{H_0^1} \cdot \|v\|_{H_0^1}$$

$$|a(u,v)| = \left|\int_0^1 u'v' + 2\int_0^1 uv\right| \le \left|\int_0^1 u'v'\right| + 2\left|\int_0^1 uv\right| \le$$

$$\underset{\text{Cauchy}}{\le} \int_0^1 |u'|\cdot|v'| + 2\int_0^1 |u||v| \le \left(\int_0^1 u'^2\right)^{1/2}\left(\int_0^1 v'^2\right)^{1/2} + 2\left(\int_0^1 u^2\right)^{1/2}\left(\int_0^1 v^2\right)^{1/2}$$

$$\le \|u\|_{H_0^1}\|v\|_{H_0^1} + 2\|u\|_{H_0^1}\|v\|_{H_0^1} \le \underset{\substack{\| \\ c}}{3}\|u\|_{H_0^1}\|v\|_{H_0^1} \qquad \text{QED}$$

• $a \to$ coercivă

$$a(u,u) = \int_0^1 u'^2 + 2\int_0^1 u^2 \ge c\cdot\|u\|_{H_0^1} = c\Big(\underbrace{\|u\|_{L^2}}_{x} + \underbrace{\|u'\|_{L^2}}_{y}\Big)^2$$

Vreau: $\underbrace{x^2 + 2y^2}_{\ge x^2+y^2 \ge} \ge \frac{1}{2}(x+y)^2$

Deci, $c = \frac{1}{2}$.

• $F : H_0^1 \to \mathbb{R}$ → bine def.
→ liniar
→ F - continuă $\Rightarrow \forall v \in H_0^1 \to F$ - $\overset{\text{bine}}{\text{definită}}$

• F - continuă

$$\left| \int_0^1 (1+\sin x) v \right| \le C \|v\|_{H_0^1}$$

$$\int_0^1 |1+\sin x| \cdot |v| \le \left[\int_0^1 (1+\sin x)^2 \right]^{1/2} \cdot \underbrace{\left(\int_0^1 v^2 \right)^{1/2}}_{\le \|v\|_{H_0^1}} \Rightarrow$$

$\sin x < 1 \Rightarrow$

$\Rightarrow 1+\sin x < 2 \Rightarrow \int_0^1 (1+\sin x)^2 \le 4$

$$\Rightarrow \left| \int_0^1 (1+\sin x) v \right| \le 2 \|v\|_{H_0^1}$$

Deci, din L-M $\Rightarrow \exists!\, u \in H_0^1$ aî $a(u,v) = \langle F, v \rangle$, $\forall v \in H_0^1$ (1) $\Rightarrow$
$\Rightarrow \exists!\, u$ - sol. slabă.

$\boxed{\text{EXAMEN}}$

- separări de variabile
- și partea de soluții slabe