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Talk 4

Second Half.

Conventions: - A variety is a reduced separated scheme of finite type over $\mathbb{C}$. They need not be irreducible or equidimensional in general.

- Smooth varieties are also assumed equidimensional.
- We identify $\mathbb{A}^n \times \operatorname{Spec} \mathbb{C} \cong \mathbb{C}^n$.
- All schemes are locally of finite type over $\mathbb{C}$.

Lemma: Let $f: X \to Y$ be a finite morphism of varieties.

Then (1) ${}^0 f_*: Sh(X) \to Sh(Y)$ is exact.

(2). $\mathrm{Supp}({}^0 f_* \mathcal{F}) = f(\mathrm{Supp}(\mathcal{F}))$ for all $\mathcal{F} \in Sh(X)$.

proof: (1) It suffices to show that $f_* \mathcal{F}$ is concentrated at degree zero for $\mathcal{F} \in Sh(X)$. We can check this on stalks:

$$(f_* \mathcal{F})_y \underset{\substack{\uparrow \\ \text{proper base change theorem}}}{\cong} R\Gamma(\mathcal{F}|_{f^{-1}(y)}) \overset{\substack{\Gamma \text{ is exact on finite discrete} \\ \text{space } f^{-1}(y) \\ \downarrow}}{\cong} \Gamma(\mathcal{F}|_{f^{-1}(y)}) = \prod_{x \in f^{-1}(y)} \mathcal{F}_x \qquad \text{—— } (*)$$

is indeed concentrated at degree zero.

(2) Note that f is closed since it is finite.

Let $T = \{ y \in Y : ({}^0 f_* \mathcal{F})_y \neq 0 \}$ and $S = \{ x \in X : \mathcal{F}_x \neq 0 \}$.

If $y \in T$ then $\prod_{x \in f^{-1}(y)} \mathcal{F}_x \neq 0$ by $(*)$, and so $\exists\, x \in f^{-1}(y)$ with $\mathcal{F}_x \neq 0$. If $x \in S$ then similarly $(f_* \mathcal{F})_{f(x)} \neq 0$, so $f(S) = T$.

Since f is closed, $\mathrm{supp}({}^0 f_* \mathcal{F}) = \overline{T} = \overline{f(S)} = f(\overline{S}) = f(\mathrm{supp}\, \mathcal{F})$.

$\blacksquare$

<u>Example</u>. If f is not even proper, both (1) and (2) in the above lemma is false in general: consider $\mathbb{C} \setminus \{0\} \xrightarrow{f} \mathbb{C}$.

Then (1) $\quad f_*(\underline{\mathbb{C}}_{\mathbb{C}\setminus\{0\}})_0$ has cohomology type of a circle and so ${}^\circ f_*$ is not exact.

(2) $\quad \mathrm{supp}({}^\circ f_* \underline{\mathbb{C}}_{\mathbb{C}\setminus\{0\}}) = \mathbb{C}$

$$f(\mathrm{supp}(\underline{\mathbb{C}}_{\mathbb{C}\setminus\{0\}})) = \mathbb{C}\setminus\{0\}$$

Part 3

§ Smooth Morphisms

- Recall that varieties inherit the analytic topology, which is locally compact and locally contractible.
- Smooth morphisms of varieties correspond to submersions in differential geometry. (We will see this precisely in Lemma 2.2.1)
- Indeed, a morphism of non-singular varieties is smooth iff the differential is surjective at all points.
- The following definition is the correct generalization to all varieties.

Def^n [18.1.7, Arapura] A morphism of schemes $f: X \to Y$ is smooth of relative dimension m at $p \in X$, if there are affine opens

$$\begin{tikzcd} p \in \mathcal{U} \arrow[r, "f|_{\mathcal{U}}"] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\circlearrowleft"] & V \ni f(p) \arrow[d, phantom, "\cong" rotate=90] \\ \mathrm{Spec}(\mathbb{C}[y_1,\dots,y_\ell,x_1,\dots,x_{n+m}]/(g_1,\dots,g_r,f_1,\dots,f_n)) \arrow[r] & \mathrm{Spec}(\mathbb{C}[y_1,\dots,y_\ell]/(g_1,\dots,g_r)) \end{tikzcd}$$

where $\begin{cases} f_i \in \mathbb{C}[y_1,\dots,x_{n+m}] \\ g_i \in \mathbb{C}[y_1,\dots,y_\ell] \end{cases}$ and $\underbrace{\mathrm{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{n\times(n+m)} = n}_{\text{Jacobian Criterion at } p \in \mathbb{C}^{\ell+n+m}}$.

We say that f is smooth if it is smooth at every closed point $p \in X$, and the relative dimension m is fixed for all closed $p \in X$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[rr, bend left, "f|_{\mathcal{U}}"] \arrow[d] & V \times \mathbb{C}^{n+m} \arrow[r, "pr_1"] \arrow[d] & V \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ f^{-1}(0) \arrow[r, hook] \arrow[d] & \mathbb{C}^\ell \times \mathbb{C}^{n+m} \arrow[r, "pr_1"] \arrow[d, "f = f_1\times\cdots\times f_n"] & \mathbb{C}^\ell \arrow[r, "g = g_1\times\cdots\times g_r"'] & \mathbb{C}^r \\ \{0\} \arrow[r, hook] & \mathbb{C}^n & & \end{tikzcd}$$

i.e. locally

Examples 1. The map to a point $X \longrightarrow *$ is smooth precisely when X is smooth/$\mathbb{C}$ at all points, i.e. X is covered by affine opens of the form $\operatorname{Spec}(\mathbb{C}[x_1,\dots,x_{n+m}]/(f_1,\dots,f_n))$, with $\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right)=n$, with fixed relative dimension $m=\dim X$.

2. More generally, $X\times Y \longrightarrow Y$ is smooth at $(p,q)\in X\times Y$ iff locally it is of the form

$$\operatorname{Spec}\mathbb{C}[x_1,\dots,x_{n+m}]/(f_1,\dots,f_n)\otimes\mathbb{C}[y_1,\dots,y_s]/(g_1,\dots,g_r)$$
$$\longrightarrow \operatorname{Spec}\mathbb{C}[y_1,\dots,y_s]/(g_1,\dots,g_r)$$

and $\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p,q)\right)=n$. But this is again equivalent to the variety X being smooth at p.

3. An open embedding (of schemes) $U\longrightarrow X$ is always smooth of relative dimension 0.

Non-examples: 1. The multiplication map defined by

$$\mathbb{C}^2 = \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$

defined by $\mathbb{C}[x,y] \longleftarrow \mathbb{C}[z]$ (on closed points: $(x,y) \overset{\phi}{\longmapsto} xy$)
$xy \longmapsfrom z$

is not smooth at $(0,0) \in \mathbb{C}^2$, since the Jacobian

$\left(\frac{\partial \phi}{\partial x} \; \frac{\partial \phi}{\partial y}\right) = (y \; x)$ is zero when $(x,y) = (0,0)$

2. Maps of varieties with wrong dimension:

If X and Y are varieties with $\dim X < \dim Y$,

then $\nexists$ smooth morphism of varieties $X \longrightarrow Y$.

(This is obvious in our definition, locally.)

Exercise: Smoothness is closed under base change.

- In particular, smooth morphisms have smooth fibres with dimension equal to the relative dimension of the smooth morphism.

Theorem [Generic smoothness]. Let $f: X \to Y$ be a morphism of varieties, and assume X is non-singular. Then there is a non-empty Zariski open subset $U \subseteq Y$ such that $f|_{f^{-1}(U)}: f^{-1}(U) \longrightarrow U$ is smooth.

proof: See [Hartshorne Corollary 10.7]. $\blacksquare$

Example For the multiplication map $\underset{\mathbb{C}^2}{\underset{\shortparallel}{\operatorname{Spec} \mathbb{C}[x,y]}} \longrightarrow \underset{\mathbb{C}}{\underset{\shortparallel}{\operatorname{Spec} \mathbb{C}[t]}}$,

the restriction $D(xy) \longrightarrow \mathbb{C} \setminus \{0\}$ is smooth, since the differential is surjective.

(**Remark**: If $\overline{f(X)} \neq Y$, then one has $U := Y \setminus \overline{f(X)} \neq \emptyset$, and the theorem always restrict f to $f|_{\emptyset}: \emptyset \longrightarrow U$.)

Def^n A morphism is étale if it is smooth of relative dimension zero.

Remark - Étale $\Rightarrow$ quasi-finite. (In our definition this is obvious: $\dfrac{\mathbb{C}[x_1, \dots, x_n]}{(f_1, \dots, f_n)}$ has finitely many $\mathbb{C}$-points)

- An étale morphism of varieties/$\mathbb{C}$ is a local homeomorphism. It is a covering map if it is also proper and surjective. (Lemma 2.1.14 Achar) $\leftarrow$ Recall: Prop 2.5 (2) Talk 3: pushforward on local systems is exact in this case

Part 4 § Smooth Base Change.

Theorem [Smooth Base Change, Achar Theorem 2.2.2]

Suppose the following is a cartesian square of varieties (over $\mathbb{C}$).

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that g is a smooth morphism of schemes. Then for $\mathcal{F} \in D^+(X)$, the base change map

$$g^* f_* \mathcal{F} \longrightarrow f'_* (g')^* \mathcal{F}$$

is an isomorphism.

Remark: Note that the smoothness assumption is necessary, again due to

$$\begin{tikzcd} \emptyset \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C} - \{0\} \arrow[d, "f"] \\ \{0\} \arrow[r, hook, "g"'] & \mathbb{C} \end{tikzcd}$$

(It is clear $\{0\} \to \mathbb{C}$ is not smooth).

As we have seen in the last two talks,

$$f'_* (g')^* (\underline{\mathbb{C}}) = 0$$

$$\text{but} \quad g^* f_* (\underline{\mathbb{C}}) = R\Gamma(\underline{\mathbb{C}}) \neq 0$$

Smooth morphisms of relative dimension m are topological submersions of relative real dimension $2m$.

Lemma 2.2.1

Let $\phi: X \to Y$ be a smooth morphism of relative dimension m. Then for any $p \in X$, there is an analytic neighbourhood $U \ni p$, an analytic neighbourhood $V \ni \phi(p)$, an analytic open set $M \subseteq \mathbb{C}^m$ that is homeomorphic to $\mathbb{C}^m$, and a biholomorphism $b: U \xrightarrow{\sim} V \times M$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\phi"'] & U \arrow[l, hook'] \arrow[r, "b", "\sim"'] \arrow[d, "\phi|_U"'] & V \times M \arrow[dl, "pr_1"] \\ Y & V \arrow[l, hook'] & \end{tikzcd}$$

proof: Assume $Y \subset \mathbb{C}^l$, $X \subset \mathbb{C}^{l+m+n}$, and $f_i: \mathbb{C}^{l+m+n} \to \mathbb{C}^n$ as before.

We renumber the coordinates of $\mathbb{C}^{l+m+n}$ so that the last n columns of the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{n \times (n+m)}$ is linearly independent. Then the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i \le n,\, m+1 \le j \le m+n} \Big|_{(x_j)=(p_j)}$ is invertible. Write $p = (q, p', p'')$, where $q \in \mathbb{C}^l$, $p' \in \mathbb{C}^m$, $p'' \in \mathbb{C}^n$.

from here, a proof with commutative diagrams may be clearer. See page after the remaining proof.

See P.18, Theorem 15, Several Complex Variables and the Geometry of Real Hypersurfaces, D'Angelo

By the holomorphic implicit function theorem, $\exists$ analytic open $W_1 \subset \mathbb{C}^{l+m}$ containing (q, p'), $\exists$ analytic open $W_2 \subset \mathbb{C}^n$ containing p'', and $\exists$ holomorphic map $h_0 : W_1 \longrightarrow W_2$ such that the map

$$h : W_1 \longrightarrow f^{-1}(0) \cap (W_1 \times W_2) \left(\subseteq \mathbb{C}^{l+m+n} \right)$$

defined by

$$u \longmapsto (u, h_0(u))$$

is bijective (and has inverse map the projection).

Now we replace W_1 by a smaller set of the form $V_0 \times M$, where $V_0 \subset \mathbb{C}^l$ is an analytic open neighborhood of q and $M \subset \mathbb{C}^m$ is an analytic open neighborhood of p' such that M is homeomorphic to $\mathbb{C}^m$. Then we have

$$\begin{tikzcd} (\mathbb{C}^l \times \mathbb{C}^m \times \mathbb{C}^n \supseteq) \ f^{-1}(0) \cap (V_0 \times M \times W_2) \arrow[d, "pr_1"'] & V_0 \times M \ (\subseteq \mathbb{C}^l \times \mathbb{C}^m) \arrow[l, "h"', "\cong"] \arrow[dl, "pr_1"] \\ V_0 \subseteq \mathbb{C}^l & \end{tikzcd}$$

Putting $U = (V_0 \times M \times W_2) \cap X$ and $V = V_0 \cap Y$, the above restricts to

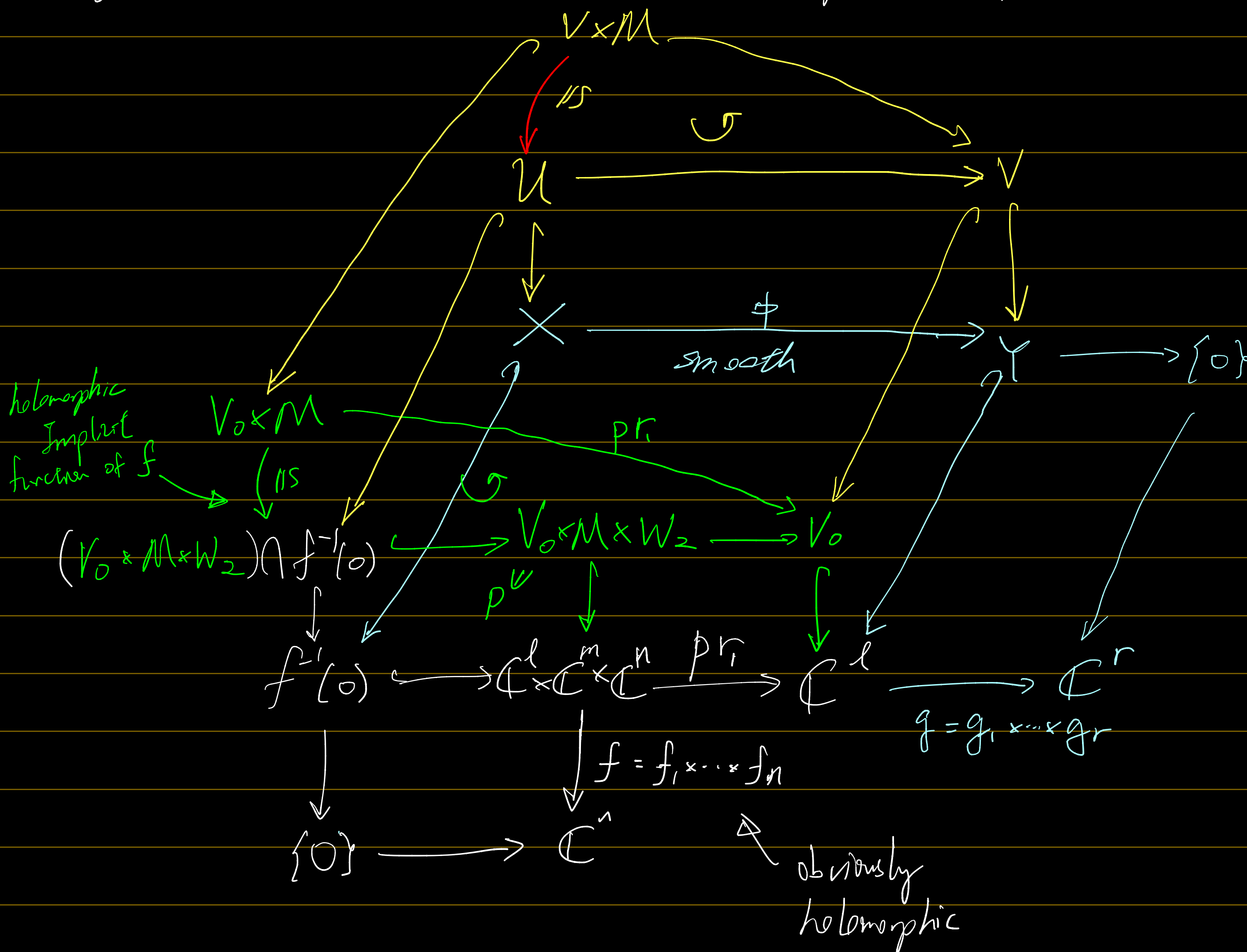
$$\begin{tikzcd} U \arrow[d] & V \times M \arrow[l, "\cong"'] \arrow[dl, "pr_1"] \\ V & \end{tikzcd}$$

, as desired. ∎

Recall that a smooth morphism is locally of the form

$$\mathrm{Spec}(\mathbb{C}[y_1,\dots,y_\ell,x_1,\dots,x_{n+m}]/(g_1,\dots,g_r,f_1,\dots,f_n)) \longrightarrow \mathrm{Spec}(\mathbb{C}[y_1,\dots,y_\ell]/(g_1,\dots,g_r)).$$

This can be expressed as the white/blue part of the everywhere cartesian diagram of topological spaces below.



The green part is supplied by the holomorphic implicit function theorem. The yellow part restricts the constructions to $X \longrightarrow Y$.

The red map is a biholomorphism since it is a restriction of one.

proof of Smooth Base Change Theorem:

Recall we want to show that for a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

of varieties with g smooth that for $F \in D^+(X)$

$$g^* f_* F \longrightarrow f'_* (g')^* F \text{ is an isomorphism.}$$

It suffices to show this locally on Y'. So let $p \in Y'$ and $p \in U \subset Y'$, $f(p) \in V \subset Y$ be analytic open neighborhoods supplied by the lemma. Then $U \cong V \times M$ as in the lemma and thus by open base change + $\underbrace{\text{trivial fibration}}_{\text{ie. "product"}}$ base change

$$\begin{tikzcd}
 & X' \arrow[rr] \arrow[dd, "f'"'] & & X \arrow[dd, "f"] \\
f^{-1}(V) \times M \arrow[r, "\cong"] & (f')^{-1}(U) \arrow[u, hook] \arrow[rr] \arrow[dd] & & f^{-1}(V) \arrow[u, hook] \arrow[dd] \\
 & Y' \arrow[rr] & & Y \\
V \times M \arrow[r, "\cong"] & U \arrow[u, hook, "\iota"] \arrow[rr] & & V \arrow[u, hook]
\end{tikzcd}$$

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and observing that the faces of the commuting cube except top and bottom faces are cartesian squares, one can check

$$\iota^* g^* f_* F \xrightarrow{\cong} \iota^* f'_* g^* F, \quad \text{hence} \quad g^* f_*(F) \xrightarrow{\cong} f'_* g^* F \;.\; \blacksquare$$

To conclude:

- Pushforward by a proper morphism/locally trivial fibration is nice when restricted to D^+_{loc}, i.e. the objects with cohomology sheaves that are local systems.

- Pullback by a smooth morphism is always nice.