Numerical Mathematics Lab session 5

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1 Preparation

2.

(a)

$$\phi(x^{(k)}) = \phi(\alpha) + \phi'(\xi^{(k)})(x^{(k)} - \alpha) \rightarrow \text{First Order Taylor expansion}$$
$$x^{(k+1)} - \alpha = \phi(\alpha) - \alpha + \phi'(\xi^{(k)})(x^{(k)} - \alpha) \rightarrow \text{We know that } x^{(k)} - \alpha = e^{(k)}$$
$$e^{(k+1)} = \phi(\alpha) - \alpha + \phi'(\xi^{(k)})e^{(k)} \rightarrow \text{But, } \phi(\alpha) - \alpha = 0 \text{ (Fixed point)}$$
$$e^{(k+1)} = \phi'(\xi^{(k)})e^{(k)}, \text{ where } \xi^{(k)} \text{ is between } \alpha \text{ and } x^{(k)}$$

$$\phi(x^{(k)}) = \phi(\alpha) + \phi'(\alpha)(x^{(k)} - \alpha) + \frac{\phi''(\eta^{(k)})}{2}(x^{(k)} - \alpha)^2 \rightarrow \text{Second Order Taylor expansion}$$
$$x^{(k+1)} - \alpha = \phi(\alpha) - \alpha + \phi'(\alpha)(x^{(k)} - \alpha) + \frac{\phi''(\eta^{(k)})}{2}(x^{(k)} - \alpha)^2$$
$$e^{(k+1)} = \phi'(\alpha)e^{(k)} + \frac{\phi''(\eta^{(k)})}{2}(e^{(k)})^2$$

(b)

(c) We know that $e^{(k+1)} = x^(k+1) - \alpha$. Since $\alpha = \phi(\alpha)$ and $x^(k+1) = \phi(x^{(k)})$, it follows that

$$x^{(k+1)} - \alpha = \phi(x^{(k)}) - \phi(\alpha) = \phi'(\xi^{(k)})(x^{(k)} - \alpha) \rightarrow \text{Mean-Value theorem, equation 1.10}$$
$$x^{(k+1)} - x^{(k)} + x^{(k)} - \alpha = \phi'(\xi^{(k)})(x^{(k)} - \alpha)$$
$$x^{(k+1)} - x^{(k)} = -(x^{(k)} - \alpha) + \phi'(\xi^{(k)})(x^{(k)} - \alpha)$$
$$x^{(k+1)} - x^{(k)} = (-1 + \phi'(\xi^{(k)}))(x^{(k)} - \alpha)$$
$$x^{(k+1)} - x^{(k)} = (1 - \phi'(\xi^{(k)}))(\alpha - x^{(k)})$$
$$\alpha - x^{(k)} = \frac{x^{(k+1)} - x^{(k)}}{(1 - \phi'(\xi^{(k)}))}$$
$$e^{(k)} = \frac{x^{(k+1)} - x^{(k)}}{(1 - \phi'(\xi^{(k)}))}$$

3.

(a) A "better" convergence is considered a faster convergence. This can be achieved with a small asymptotic convergence factor (page 60). The asymptotic convergence factor can be expressed by $\phi'(\xi^{(k)})$, where $\xi^{(k)} \in (\alpha, x^{(k)})$. We want thus

$$|\phi'(\xi^{(k)})| \approx 0$$

The best case scenario is achieved with perfect equality and we know that $\xi^{(k)} \rightarrow \alpha$, as $k \rightarrow \infty$, so the above relation becomes

$$|\phi'(\alpha)| = 0 \Leftrightarrow |1 + cf(\alpha)| = 0 \Leftrightarrow 1 + cf(\alpha) = 0 \Leftrightarrow c = -\frac{1}{f'(\alpha)}$$

(b) In this case, $\phi(x) = x - \frac{1}{f'(x^{(0)})} f(x) \Rightarrow \phi'(\xi^{(k)}) = 1 - \frac{f'(\xi^{(k)})}{f'(x^{(0)})}$. But $\xi^{(k)} \to \alpha$ when $k \to \infty$, so

4.

(a) Since $f'(\alpha)$ is unknown and with each iteration $x^{(k)}$ gets closer to α, we can then infer that for every step k, the best value of function $C(x)$ is $C(x) = -1/f'(x)$

(b) For a simple root, the order of convergence of 2. Also the asymptotic convergence factor is $\frac{f''(\alpha)}{2f'(\alpha)}$

2 Lab Experiments

Running the program we obtain the following figures:

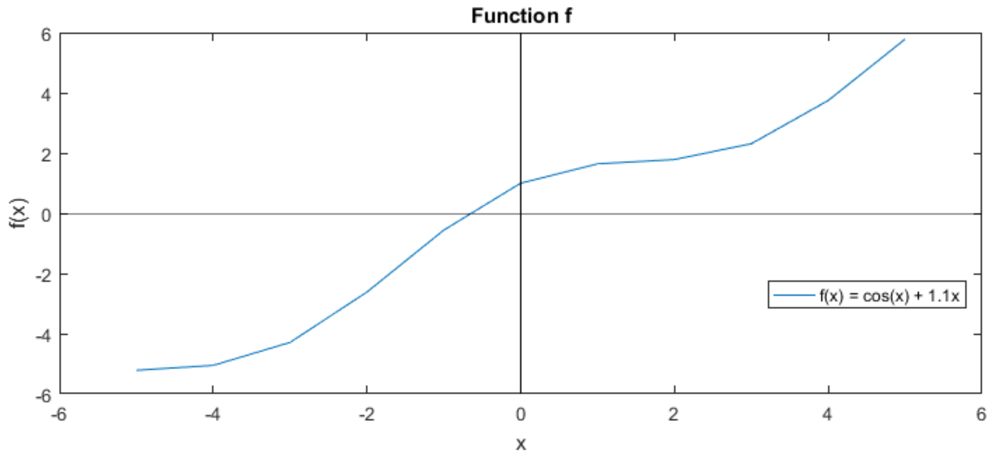

Figure 1: Function f

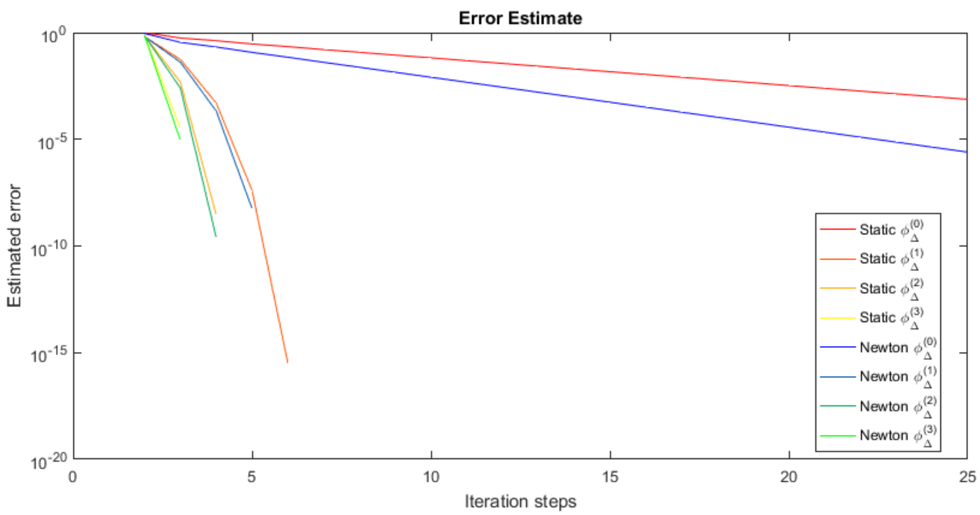

Figure 2: Error estimates

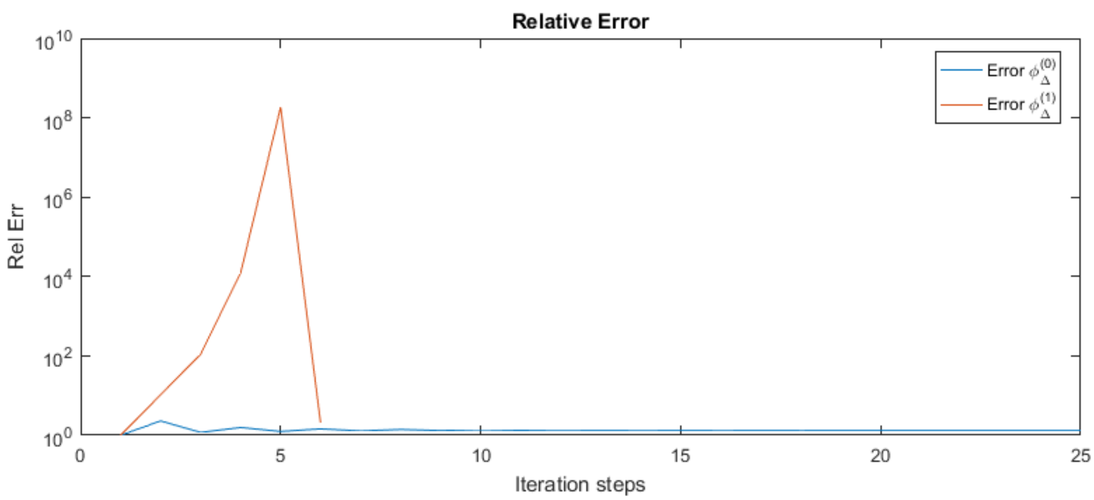

Figure 3: Relative error

3 Discussion

1.

(a) δ^k is a good estimator whenever $\phi'(\xi^k) \approx 0$. As we move fut

When $n = 0$, we do

When $n > 0$, this converges even faster and therefore it is also a good estimator.

(b)

2. We do not actually have enough iterations to clearly see the difference in iteration speed. All methods find the root too quickly to meaningfully compare the methods.
3. This can be done using timing, but we can also see it from the plots.
4. One of the major cons of Newton's method is computational expense, we need to find the derivative at every step. Also it does not converge in all scenarios. Fails when derivative is close to zero, so initial point have to be chosen carefully.

A major pro in the case of Newton's method is that it converges quadratically in the case of a single root. If convergence is worse than quadratic, other methods can be used first to approximate the target.
5. Newton's method behaves badly whenever the derivative is equal to zero, as it becomes very large. Static iteration does not depend on the derivative.