Numerical Mathematics Lab session 4

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1 Preparation

1.1 Iterative Methods

2. Having $x^{(k+1)} = x^{(k)} + \alpha_k P^{-1}(b - Ax^{(k)})$, we can subtract x in both sides to obtain a relation between $e^{(k+1)}$ and $e^{(k)}$:

$$\begin{gathered}
e^{(k+1)} = e^{(k)} + \alpha^{(k)} P^{-1}(b - Ax^{(k)}) \Leftrightarrow \\
e^{(k+1)} = e^{(k)} + \alpha^{(k)} P^{-1}(b - A(x^{(k)} - x + x)) \Leftrightarrow \\
e^{(k+1)} = e^{(k)} + \alpha^{(k)} P^{-1}(b - Ax - A(x^{(k)} - x) \Leftrightarrow \\
e^{(k+1)} = e^{(k)} + \alpha^{(k)} P^{-1}(-A(x^{(k)} - x)) \Leftrightarrow \\
e^{(k+1)} = e^{(k)} - \alpha^{(k)} P^{-1} A e^{(k)} \Leftrightarrow \\
e^{(k+1)} = (I - \alpha^{(k)} P^{-1} A) e^{(k)} \Leftrightarrow \\
e^{(k+1)} = B_{\alpha_k} e^{(k)}
\end{gathered}$$

3. A is real symmetric and positive definite matrix. The A-norm is defined as follows

$$||e^{(k)}||_A = \sqrt{(e^{(k)})^T A e^{(k)}}$$

6.

(a) $\beta_{\alpha_{opt}} = 1 - \alpha_{opt}\lambda_{min} = 1 - \frac{2}{\lambda_{min}+\lambda_{max}}\lambda_{min} = \frac{\lambda_{min}+\lambda_{max}}{\lambda_{min}+\lambda_{max}} - \frac{2\lambda_{min}}{\lambda_{min}+\lambda_{max}} = \frac{\lambda_{max}-\lambda_{min}}{\lambda_{max}+\lambda_{min}}$

(b) We start by the definition of the residual $r^{(k+1)} = b - Ax^{(k+1)} = b - A(x^{(k)} + \alpha_k P^{-1} r^{(k)})$
We know that $P = I$, so we obtain $r^{(k+1)} = b - Ax^{(k)} - \alpha_k A r^{(k)}$
Then we can add and subtract $r^{(k)}$: $r^{(k+1)} = b - Ax^{(k)} - r^{(k)} + r^{(k)} - \alpha_k A r^{(k)}$
The first 3 terms on the right side of the equation are 0, because $r^{(k)} = b - Ax^{(k)}$, obtaining:

$$r^{(k+1)} = (I - \alpha_k A) r^{(k)} \Rightarrow r^{(k+1)} = B_\alpha r^{(k)}$$

Having proven the above, we can apply the 2-norm, obtaining:

$$||r^{(k+1)}||_2 = ||B_\alpha r^{(k)}||_2 \leq ||B_\alpha||_2 \cdot ||r^{(k)}||_2$$

We have to prove $||B_\alpha||_2 \leq \rho(B_\alpha)$. We know from equation 5.29 that $||Aw|| \leq \lambda_{max}||w||, \forall w \in \mathbb{R}^n$
Having $A = B_\alpha$ and $w = r^{(k)}$, we can infer that $||B_\alpha r^{(k)}||_2 \leq \lambda_{max} \cdot ||r^{(k)}||_2$
We also know that by definition $\rho(B_\alpha) = \lambda_{max}$ and therefore $||r^{(k+1)}||_2 \leq \rho(B_\alpha) \cdot ||r^{(k)}||_2$

(c) Knowing that $||r^{(k+1)}||_2 \leq \lambda_{max}||r^{(k)}||_2$ we can infer that $||r^{(k)}||_2 \leq \lambda_{max}^k ||r^{(0)}||_2$
So, $\lambda_{max}^k \geq \frac{||r^{(k)}||_2}{||r^{(0)}||_2}$. The equality is achieved when $k = k_{min}$, therefore we obtain:

$$\lambda^{k_{min}} = \frac{||r^{(k)}||_2}{||r^{(0)}||_2} \Rightarrow k_{min} = \frac{log(\frac{||r^{(k)}||_2}{||r^{(0)}||_2})}{log(\lambda)}$$

We now have to prove that $||r^{(k)}||_2 = \epsilon||b||_2$

1.2 Power Iteration

2. $\lambda^{(k)} \to \lambda_1$ when the matrix A has one eigenvalue (λ_1) strictly greater than the other eigenvalues (so A must be a generic matrix), and the vectors $x^{(0)}$ and x_1 should NOT be orthogonal.

3. In the case of a generic matrix, the convergence rate is $||y^k - ({y^k}^H x_1)x_1|| \leq C|\frac{\lambda_2}{\lambda_1}|^k$. In the case of a Hermitian matrix, it is proportional to $||y^k - ({y^k}^H x_1)x_1|| \leq C|\frac{\lambda_2}{\lambda_1}|^{2k}$ Since A is Hermitian, it is diagonalizable. In this case, the convergence rate is $\leq C\frac{|\lambda_2|}{|\lambda_1|}^k$.

2 Lab Experiments

2.1 Iterative Methods

Running the program yields the following results:

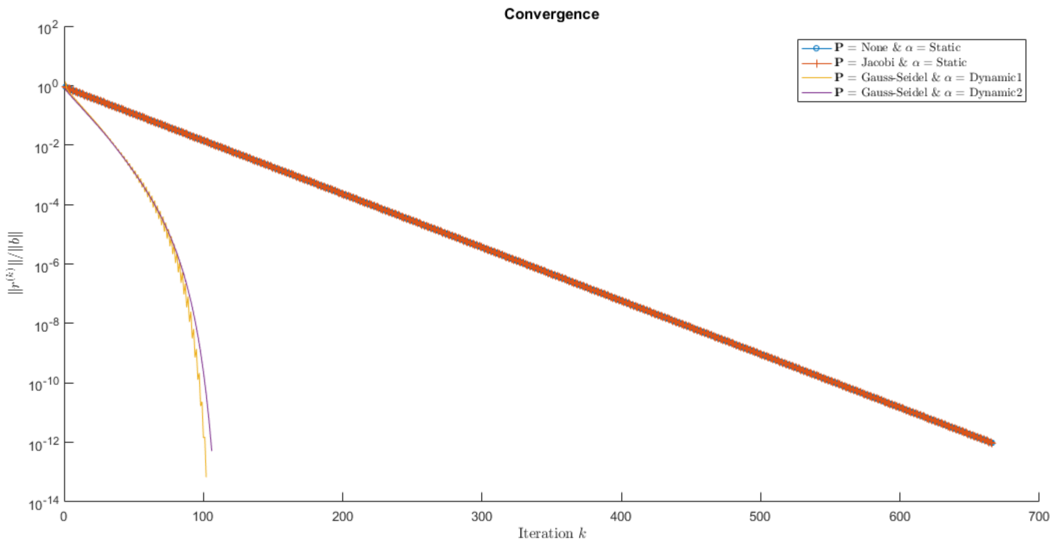

Figure 1: Convergence using different values for P and α_k

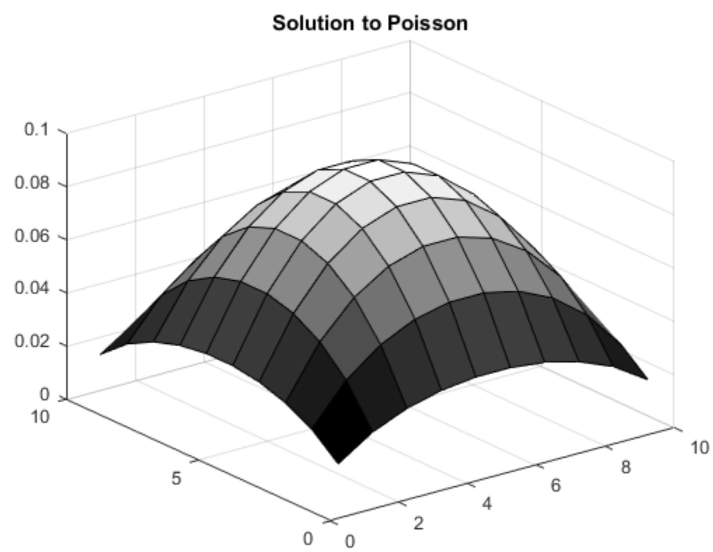

Figure 2: Solution surface for the Poisson problem

2.2 Power iteration

Running the program yields the following results

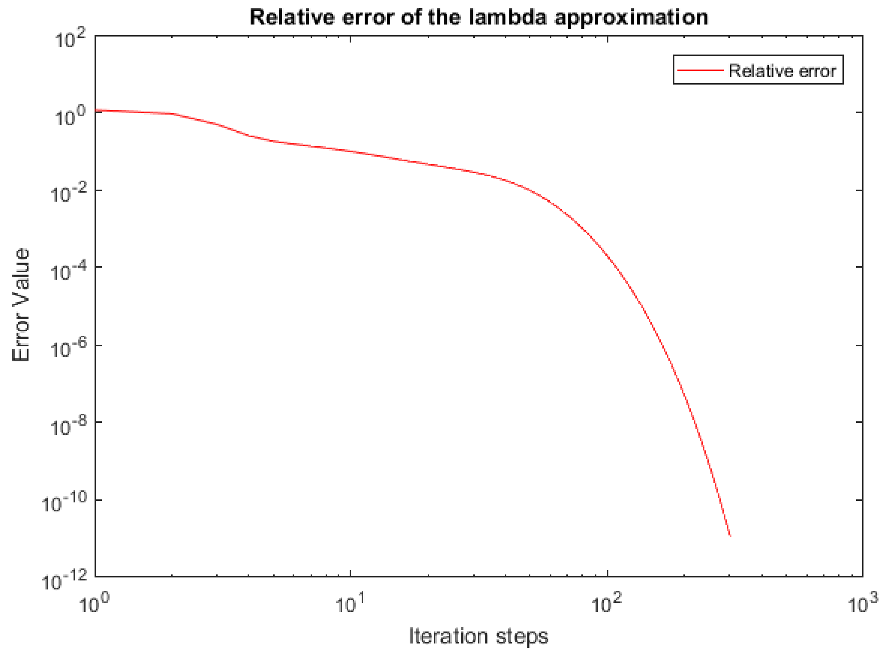

Figure 3: Relative error of the lambda approximation

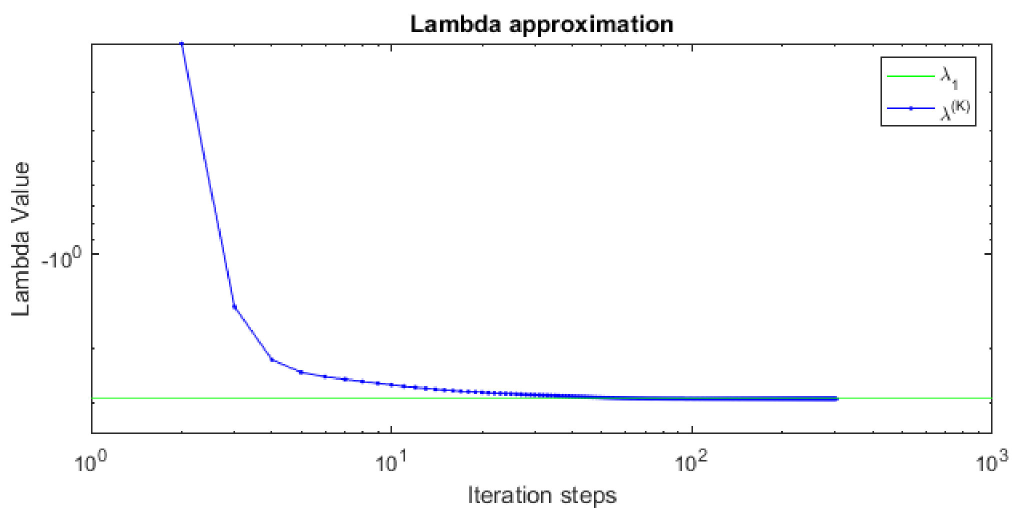

Figure 4: Lambda approximation for each iteration

3 Discussion

Running the following commands

```
Q = gallery('poisson', 100);
whos Q
fullQ = full(Q);
whos fullQ
```

Yield the following output

```
Name          Size           Bytes        Class     Attributes
 Q          10000x10000      873608       double     sparse

Name          Size           Bytes        Class     Attributes
fullQ       10000x10000    800000000      double
```

This clearly shows that having a sparse matrix utilizes nearly 100 time less space than the full matrix.

3.1 Iterative methods

1. The optimal value for α_k in (2) minimizes the spectral radius for $\rho(B_\alpha)$. When using the Gauss-Seidel preconditioner we cannot find this optimal α_k because there is not a single iteration matrix B_α, since it is constantly being updated during the calculation.

2. One of the conditions for $||\cdot||_A$ to be a norm is for its value to always be greater than or equal to zero. This only happens when $\mathbf{e}^T A\mathbf{e} \geq 0, \forall \mathbf{e} \in \mathcal{R}^n$, that is, when A is positive semidefinite. Therefore, minimizing this quantity only makes sense when using a positive semidefinite A.

3. Apart from minimizing the A norm, we can also minimize the 2-norm in the residual.
If A has full rank this means that it is invertible. The product of two invertible matrices, $A^T A$, is also invertible. Furthermore, $A^T A$ is symmetric, $(AA^T)^T = (A^T)^T A^T = AA^T$, and this tells us that its eigenvalues are real. Therefore, $||\mathbf{e}||_{A^T A} \geq 0, \forall \mathbf{e} \in \mathcal{R}^n$ is a norm.

4. Since we know that $\lambda_{min} + \lambda_{max} = 8$, we know that $\alpha_{opt} = \frac{2}{8} = \frac{1}{4}$. A general iterative method is of the form $x^{k+1} = x^k + \alpha_k P^{-1} r^k$
The preconditioner we are handed by `iterCompare` for the Jacobi method is equal to $4I$. Its inverse is therefore $\frac{1}{4}I$, such that the update step of the Jacobi method with $\alpha = 1$ is the same as the method using no preconditioner $P = I$ and an $\alpha_{opt} = \frac{1}{4}$ when substituting in the general formula.
5. We have

$$[\rho(B)]^{k_{min}} \leq \epsilon$$

with `tol` $= \epsilon = 10^{-12}$. Now we perform the MATLAB command $P_{jacobi} = A - diag(diag(A))$;, which gives us the desired preconditioner. The script `find_k_min` calculates $k_{min} = 13$, a theoretical bound which is much lower than the value of 639, so they do not coincide.
6.

3.2 Power iteration

1. Since α_k is constant, then the matrix $B = I - \alpha P^{-1} A$ will stay constant, meaning that the approximation of the solution for k iteration will be $x^{(k)} = B^k x^{(0)}$.

2. The convergence factor is 1. It agrees with the theory because the relative error is proportional to the $|\frac{\lambda_2}{\lambda_1}|^k$, so the `loglog` plot will appear (nearly) linear.

3. If no component is in the direction of x_1, then $y^{(k)}$ does not converge, so $\lambda^{(k)}$ will not converge. In practice, this is highly unlikely, because of the round-off errors.