Numerical Mathematics Lab session 7

Stefan Evanghelides (s2895323) and Frank te Nijenhuis (S2462575)

June 12, 2018

1 Preparation

1.1 The Poisson problem

Done on paper.

1.2 The heat equation

Done on paper.

2 Lab

2.1 The Heat Equation

Running the program will result in the following table:

θ	Δt	$\frac{\mu\Delta t}{(h_2)^2}$	$\epsilon(h_1, T)$	$\epsilon(h_2, T)$
0	0.15625	0.25	1.0031	1.0031
0	0.3125	0.5	1.0029	1.0029
0	0.3225	0.516	1.0029	2.0726e+76
0.5	100	160	0.99575	0.99546
0.5	200	320	0.97666	0.97861
1	100	160	0.99938	0.99939
1	200	320	0.9973	0.99731

An example of the heat map can be depicted from figure 1

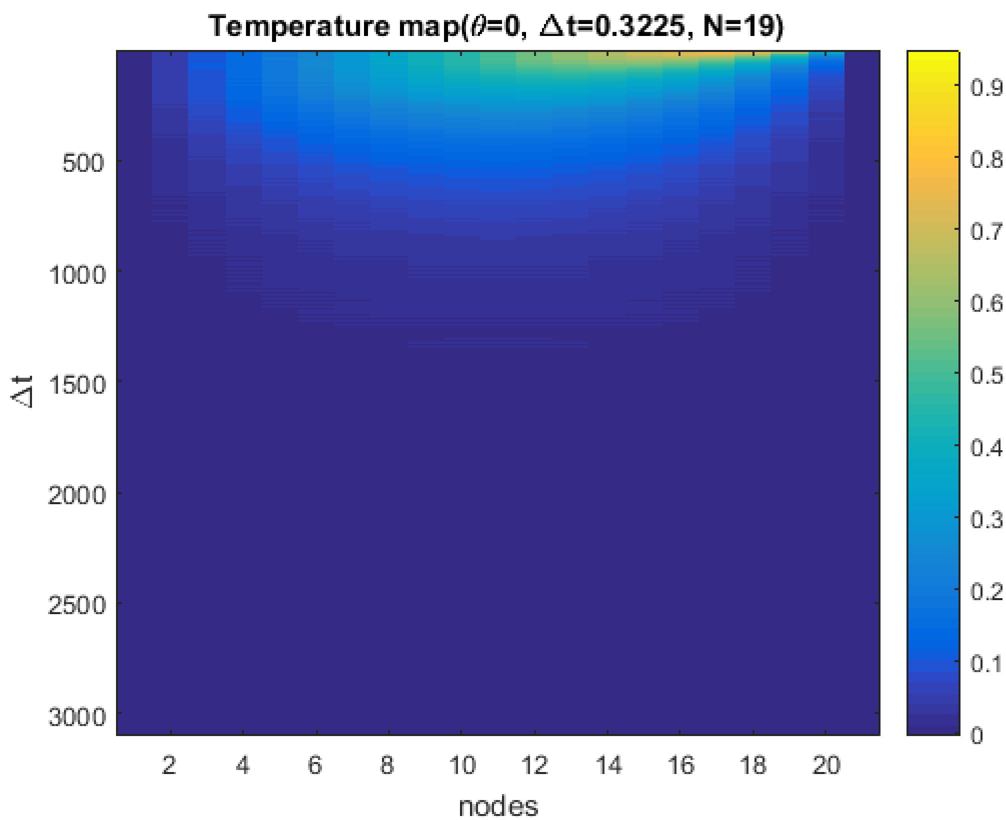

Figure 1: The maximum error as a function of h.

2.2 The Poisson problem

Running the program will output the figure 2, on the next page.

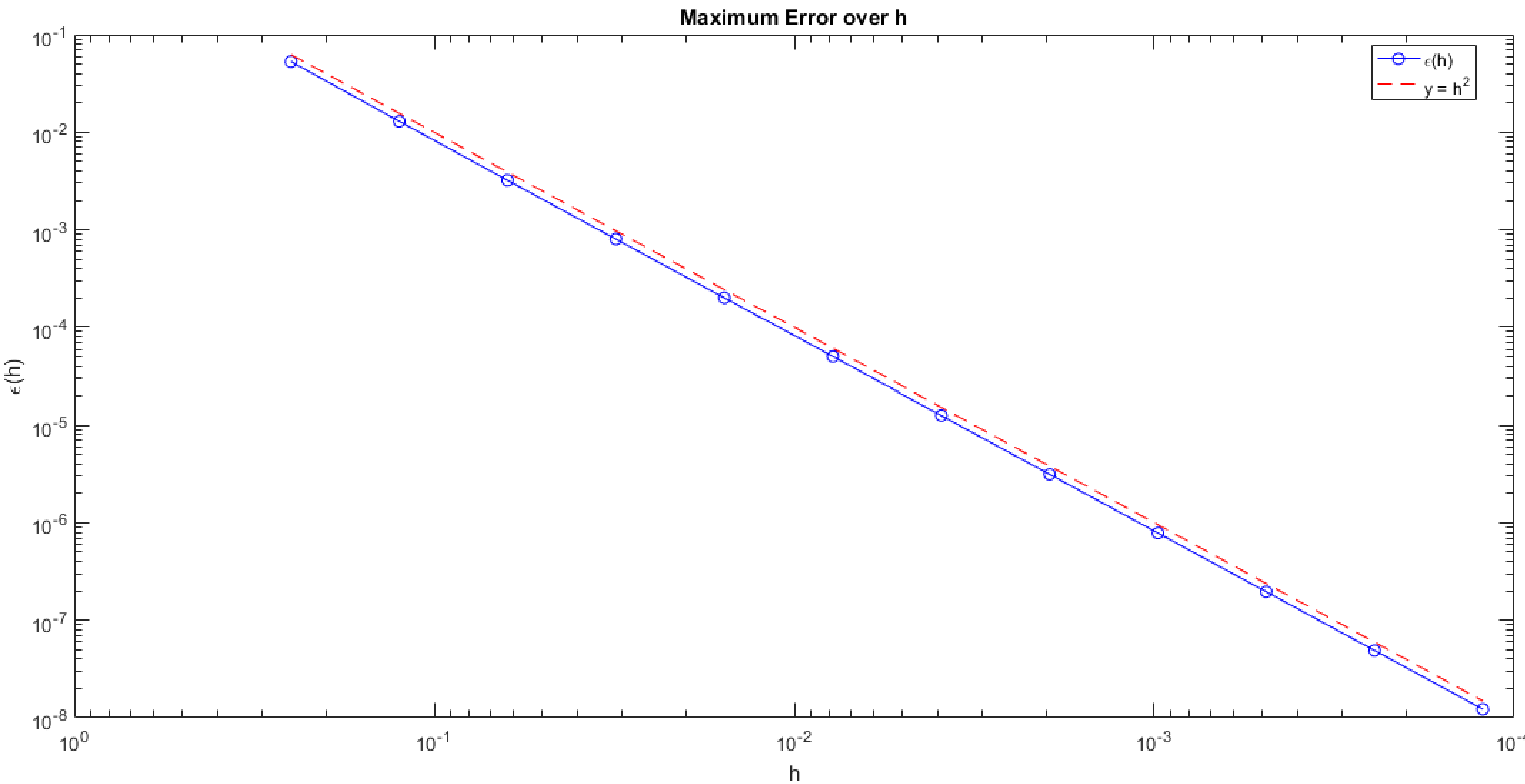

Figure 2: The maximum error as a function of h.

3 Discussion

3.1 The Poisson problem

1. Yes, the 3-point stencil yields a second-order accurate approximation, as can be clearly seen on figure **??** in the previous section.

2.
$$K(\mathbf{A}) = \frac{\lambda_{max}(\mathbf{A})}{\lambda_{min}(\mathbf{A})} = \frac{4-(h\pi)^2+\mathcal{O}(h^4)}{(h\pi)^2+\mathcal{O}(h^4)} = \frac{4}{(h\pi)^2+\mathcal{O}(h^4)} + \mathcal{O}(1)$$
Which is an approximation, becoming exact when $h \to 0$ as
$$\frac{4}{(h\pi)^2} + \mathcal{O}(1)$$
The condition number depends quadratically on the step size, becoming larger as h becomes smaller.

3.2 The heat equation

Test problem

1. Δt should decay as the square of the grid spacing h, meaning that if we halve h, Δt should decrease by a factor of 4. Conversely, halving Δt, h should decrease by a factor of $\sqrt{2}$.
Accuracy behaves like $\Delta t + h^2$ when $\theta \neq \frac{1}{2}$
Accuracy behaves like $\Delta t^2 + h^2$ when $\theta = \frac{1}{2}$

2. a. Setting the time derivative to zero and integrating gives us the formula
$$\frac{\mu}{h^2}\mathbf{A}\mathbf{u}^{k+1} = \mathbf{f}^k$$
which is analogous to the update step in the Richardson method
$$P\mathbf{z}^k = \mathbf{r}^k$$

b.
$$\mathbf{B}_\alpha = I - \alpha \cdot P^{-1} \cdot \mathbf{A}$$
is the matrix from the explicit Euler method with $\alpha = \Delta t$ and $P^{-1} = -\frac{\mu}{h^2}$

c. $\alpha_{opt}(\mathbf{M}) = \frac{2}{(h\pi)^2+4-(h\pi)^2+\mathcal{O}(h^4)} = \frac{1}{2}$ as $h \to 0$, so that this is the optimal alpha in the continuous case. Therefore the largest Δt is a tradeoff between stability and optimal alpha, so the largest value less than .5 is the optimal Δt.

Application

1. Crank-Nicolson method is unconditionally absolutely stable and is has a quadratic order of accuracy. However, it takes longer time, because the states have only an implicit representation.
Forward Euler is conditionally stable ($\Delta t < \frac{h^2}{2\mu}$, but it will be performed slightly faster because the steps are represented explicitly.

2. -