Numerical Mathematics I, 2017/2018, Lab session 7

Deadline for discussion: Monday 11/06/18 at 5 pm

Keywords: boundary value problem, finite difference approximation

Remarks

- Make a new folder called `NM1_LAB_7` for this lab session, save all your functions in this folder.
- Whenever a new `MATLAB` function is introduced, try figuring out yourself what this function does by typing `help <function>` in the command window.
- Make sure that you have done the preparation before starting the lab session. The answers should be worked out either by pen and paper (readable) or with any text processing software (LaTeX, Word, etc.).

1 Preparation

1.1 The Poisson problem

1. Study (Textbook, Section 9.2.1 & 9.2.5).
2. Consider the following *boundary value problem* (BVP)
$$\begin{array}{rcll} -\Delta u & = & f & \text{for } x \in \Omega, \\ u & = & g & \text{for } x \in \partial\Omega, \end{array} \tag{1}$$
where $\Omega = (0, 1)$ and $\partial\Omega = \{0, 1\}$. Show that
$$\tilde{u}(x) = \sin(\pi x)$$
is a solution to the Poisson problem (1) with $f(x) = \pi^2 \tilde{u}(x)$ and $g = 0$.
3. Let $h = 1/(N + 1)$, and define a uniform grid as $x_i = ih$, for $i = 0, \dots, N + 1$. Show that when using the second-order central finite difference approximation, we get
$$-f_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \mathcal{O}(h^2), \tag{2}$$
for $i = 1, \dots, N$ (the interior nodes). The subscript i refers to evaluation at the point x_i.
4. Define the tridiagonal matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ as
$$\mathbf{A} := \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$
Show that the three-point scheme given by (2) leads to the following system of equations
$$\mathbf{Au} = \mathbf{b},$$
where $\mathbf{u} = (u_1, \dots, u_N)^T$. Give $\mathbf{b}$ in terms of f_i and the boundary values g_0 and g_{N+1}.

1.2 The heat equation

1. Study (Textbook, Section 9.2.6).

2. Consider the *initial boundary value problem* (IBVP) given by

$$\begin{array}{rcll} \partial_t v &=& \mu\Delta v + p & \text{for } (x,t) \in \Omega \times [0,T], \\ v(x,t) &=& g(x) & \text{for } (x,t) \in \partial\Omega \times [0,T], \\ v(x,0) &=& v_0(x) & \end{array} \tag{3}$$

   where μ is the (positive) diffusion coefficient and the spatial domain is given by $\Omega = (0,1)$. Show that

$$\tilde{v}(x,t) = (1 + e^{-\gamma t})\tilde{u}(x)$$

   is a solution to (3) for

$$p(x,t) = \mu f(x) = \gamma \sin(\pi x), \quad g = 0, \tag{4}$$

   and $\gamma = \pi^2\mu$.

3. Show that discretising (3) in space gives the semi-discretisation given by

$$\frac{d\mathbf{v}}{dt}(t) = -\frac{\mu}{h^2}\mathbf{A}\mathbf{v}(t) + \mathbf{b}(t), \tag{5}$$

   where $\mathbf{v}(0) = \mathbf{v}^0$ is given. $\mathbf{b}(t)$ should be expressed in terms of $p_i(t)$, g_0 and g_N.

4. Show that applying the θ-method to (5) gives

$$\left(\mathbf{I} + \frac{\mu\Delta t}{h^2}\theta\mathbf{A}\right)\mathbf{v}^{n+1} = \left(\mathbf{I} - \frac{\mu\Delta t}{h^2}(1-\theta)\mathbf{A}\right)\mathbf{v}^n + \Delta t(\theta\mathbf{b}^{n+1} + (1-\theta)\mathbf{b}^n),$$

   where the superscript n means evaluation at time $t^n = n\Delta t$.

5. The eigenvalues of $\mathbf{A}$ are given by (see (Textbook, Exercise 9.2))

$$\lambda_k = 2\left(1 - \cos\frac{k\pi}{N+1}\right), \tag{6}$$

   for $k = 1, \ldots, N$.

   (a) Show that the smallest and largest eigenvalue of $\mathbf{A}$ can be approximated as

$$\lambda_{\min}(\mathbf{A}) = (h\pi)^2 + \mathcal{O}(h^4), \quad \lambda_{\max}(\mathbf{A}) = 4 - (h\pi)^2 + \mathcal{O}(h^4). \tag{7}$$

   (b) Show that the explicit Euler method ($\theta = 0$) is absolutely stable for

$$\frac{\mu\Delta t}{h^2} < \frac{1}{2}. \tag{8}$$

2 Lab experiments

2.1 The Poisson problem

Write a function makeLaplace.m that ouputs the $N \times N$ matrix $\mathbf{A}$, the header of your function should be of the following form

```
% INPUT
% N         number of unknowns (h = 1 / (N + 1))
% OUTPUT
% L         discrete Laplace operator (3-point stencil)
function L = makeLaplace(N)
```

Write a MATLAB function called poissonSolveFD.m that solves the 1D Poisson problem on $\Omega = (0, 1)$ with Dirichlet boundary conditions. This function should use iterMethod.m from lab session 4 to solve the linear system of equations. The header of your function should be of the following form

```
% Solves the 1D Poisson problem
%       -laplace u = f on (0,1)
% with Dirichlet bdy conditions given by g
% INPUT
% f         right-hand side function
% g         Dirichlet boundary condition function
% N         number of unknowns (h = 1 / (N + 1))
% precon    'none', 'jacobi' or 'gs'
% tol       tolerance for iterative solver
% OUTPUT
% sol       (N+2) x 1 solution array
% nodes     (N+2) x 1 array with location of spatial nodes
function [sol, nodes] = poissonSolveFD(f, g, N, precon, tol)
```

For iterMethod.m* use the dynamic parameter which minimises the residual in the 2-norm (so dynamic = 2). Choose a preconditioner.

Test the consistency of the finite difference approximation by confirming (Textbook, Proposition 9.1). For $h = 2^{-i}$, where $i = 2, \dots 13$ compute the finite difference approximation to the Poisson problem with homogeneous Dirichlet boundary conditions (hence $g = 0$) and let the right-hand side function be given by

$$f(x) = \pi^2 \sin(\pi x).$$

Summarise your results in one figure where you plot the maximum error

$$\epsilon(h) = \max_{i=1,\dots,N} |u_i - \tilde{u}(x_i)|,$$

as a function of h together with the line $y = h^2$ in double logarithmic scale.

2.2 The heat equation

Test problem

Write a MATLAB function heatSolveTheta.m that solves the IBVP (3) using the semi-discretisation given by (5). The time integration should be done using odeSolveTheta.m from the previous

*The functions iterMethod, newton and odeSolveTheta are given as a *_pcode.p file. You are free to use those implementations in case you did not succeed to implement them yourself in the previous Labs.

θ	Δt	$\frac{\mu \Delta t}{(h_2)^2}$	$\epsilon(h_1, T)$	$\epsilon(h_2, T)$
0	0.15625			
0	0.3125			
0	0.3225			
1/2	100			
1/2	200			
1	100			
1	200			

Table 1: Summary of results for the heat equation.

lab session.

The header of your function should look like

```
% Solves the 1D heat eqn
%        du/dt = mu laplace u + p(x,t) on (0,1) x (0, tEnd)
% and initial condition u0 (the Dirichlet boundary
% conditions are imposed by u0)
% INPUT
% p          right-hand side forcing term (function of x and t)
% u0Func     initial condition function (function of x)
% mu         diffusion coefficient
% theta      parameter for time integration
% tEnd       end time
% N          number of unknowns (h = 1 / (N + 1))
% dt         step-size
% OUTPUT
% tArray     array containing the time points
% solArray   array containing the solution at each time level
%            (the ith row equals the solution at time tArray(i))
%            (nrTimeSteps + 1) x (N+2) array
% nodes      (N+2) x 1 array with location of spatial nodes
function [tArray, solArray, nodes] = heatSolveTheta(p,...
    u0Func , mu, theta, tEnd, N, dt)
```

Test your implementation on the IBVP where p is given by (4), $g = 0$, $\mu = 10^{-3}$, $T = 1000$ and let $v_i^0 = \tilde{v}(x_i, 0)$. For the spatial discretisation choose $h_1 = 1/20, h_2 = 1/40$. Let $\epsilon(h, T)$ be the maximum error at $t = T$

$$\epsilon(h, T) = \max_{i=1,\dots,N} |u_i^{N_t} - \tilde{u}(x_i, T)|, \tag{9}$$

where N_t is the number of time steps done to reach $t = T$. Test the combinations of $\theta, \Delta t$ and h as shown Table 1, and summarise your results in such a table.

Application

Consider heating a one-dimensional iron bar with a length of one metre. Let the heat source be located at the centre and be of length 20 centimetres which we model as the following source term

$$p(x) = \begin{cases} P^* & x \in (0.4, 0.6) \\ P^*/2 & x \in \{0.4, 0.6\} \\ 0 & x \in [0, 0.4) \cup (0.6, 1] \end{cases},$$

where $P^* = \frac{2\cdot 10^7}{450\cdot 7874}$. Moreover let the diffusion coefficient be given by $\mu = \frac{80.4}{450\cdot 7874}$. Let the initial temperature be given by $293K$, and assume that the bar is kept at constant temperature of $293K$ at both ends.

The goal is to find out long it takes for the centre of the bar to reach the melting temperature of iron, which is given by $T^* = 1811K$. We denote the time instance at which the temperature T reaches T^* by t^*. Solve the heat equation for $t \in [0, 600]$ (the bar should melt before then), using a mesh width h of 1mm. Pick one of the θ-methods, and choose your step-size Δt such that the first three significant digits of t^* no longer change upon refinement of Δt.

3 Discussion

3.1 The Poisson problem

1. Do your experiments show that the three-point stencil yields a second-order accurate approximations?

2. Explain how from (7) it follows that

$$K(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} = \frac{4}{(\pi h)^2} + \mathcal{O}(1).$$

   How is this relevant? Relate the condition number to the convergence factor of the Richardson method.

3.2 The heat equation

Test problem

1. What happens to the error $\epsilon(h_1, T)$ (as defined in (9)) in the test problem, when halving Δt (for $\theta = 0, 1/2, 1$)? What if h is halved for fixed Δt?

2. Consider using the explicit Euler method for finding the stationary solution of (5), where $\mathbf{b}$ is constant in time.

   (a) Explain why in this case integrating in time is equivalent to directly applying Richardson iteration by setting $\frac{d\mathbf{v}}{dt}(t) = 0$ in (5). Note that such a steady state leads to

$$\frac{\mu}{h^2}\mathbf{A}\mathbf{v}(t) = \mathbf{b}(t).$$

   (b) Give the iteration matrix $\mathbf{B}_\alpha$ and show that in this context, the parameter α of the Richardson iteration is given by

$$\alpha = \Delta t.$$

   (c) How does the *optimal* parameter α_{opt} relate to chosing the largest step-size Δt possible according to (8)? Hint: Recall that $\alpha_{\text{opt}}(\mathbf{M}) = \frac{2}{\lambda_{\min}(\mathbf{M}) + \lambda_{\max}(\mathbf{M})}$, use (6) to analytically find an approximation to α_{opt}.

Application

1. You were asked to choose a value of θ to solve the application problem. Motivate this value of θ (think of the stability, accuracy and efficiency properties of each of the methods).

2. How long does it take until the centre of the iron bar reaches its melting temperature?