

Forcing the Π_3^1 -Reduction Property and a Failure of Π_3^1 -Uniformization

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Abstract

We generically construct a model in which the Π_3^1 -reduction property is true and the Π_3^1 -uniformization property is wrong, thus producing a model which separates these two principles for the first time.

1 Introduction

The reduction property was introduced by K. Kuratowski in 1936 and is one of the three regularity properties of subsets of the reals which were extensively studied by descriptive set theorists, along with the separation and the uniformization property.

Definition 1.1. *We say that a universe has the Π_n^1 -reduction property if every pair A_0, A_1 of Π_n^1 -subsets of the reals can be reduced by a pair of Π_n^1 -sets D_0, D_1 , which means that $D_0 \subset A_0$, $D_1 \subset A_1$, $D_0 \cap D_1 = \emptyset$ and $D_0 \cup D_1 = A_0 \cup A_1$.*

The reduction property for Π_n^1 is implied by the stronger uniformization property for Π_n^1 -sets. Recall that for an $A \subset 2^\omega \times 2^\omega$, we say that f is a uniformization (or a uniformizing function) of A if there is a function $f : 2^\omega \rightarrow 2^\omega$, $\text{dom}(f) = \text{pr}_1(A)$ (where $\text{pr}_1(A)$ is A 's projection on the first coordinate) and the graph of f is a subset of A . In other words, f chooses exactly one point of every non-empty section of A in a definable way.

Definition 1.2. *We say that the Π_n^1 -uniformization property is true, if every set $A \subset 2^\omega \times 2^\omega$, $A \in \Pi_n^1$ has a uniformizing function f_A whose graph is Π_n^1 .*

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Classical work of M. Kondo, building on ideas of Novikov, shows that the Π_1^1 -uniformization (and equivalently the Σ_2^1 -uniformization) property is true. This is as much as ZFC can prove about the uniformization and the reduction property. In Gödel's constructible universe L , the Σ_n^1 -uniformization-property for $n \geq 3$ is true, as L admits a good Σ_2^1 -wellorder of its reals. On the other hand, due to Y. Moschovakis' celebrated result, the axiom of projective determinacy PD outright implies the Π_{2n+1}^1 -uniformization property for every $n \in \omega$, indeed Δ_{2n}^1 -determinacy implies the stronger Π_{2n+1}^1 -scale property for every $n \in \omega$. Due to D. Martin and J. Steel, n -many Woodin cardinals and a measurable above outright imply Π_{n+1}^1 -determinacy, in particular under the assumption of ω -many Woodin cardinals PD becomes true which fully settles the behaviour of the uniformization and reduction property for projective pointclasses.

Despite the extensive list of deep results that has been produced in the last 60 years on this topic, there are still some basic and natural questions concerning the reduction or the uniformization property which remained open. Note e.g. that in the scenarios above the reduction property holds because the uniformization property does so. As these are the only known examples in which the reduction property holds, it is possible that reduction and uniformization for projective pointclasses are in fact equivalent principles over ZFC. So the very natural question, which surely has been asked already much earlier, arises whether one can produce universes of set theory where the reduction property holds for some pointclass, yet the corresponding uniformization property fails. The goal of our article is to show that this can be done.

Theorem. *There is a generic extension of L in which the Π_3^1 -reduction property is true and the Π_3^1 -uniformization property is wrong.*

We expect the arguments to be applicable to the universes M_n as well, which would yield models in which the Π_{n+3}^1 -reduction property holds, and the Π_{n+3}^1 -uniformization property fails (see [9] for a paradigmatic example of how to carefully lift the argument designed for L to work for M_n as well).

This article builds on ideas first introduced in [8] and [9]. The proof of the theorem is, however, far from a mere application of the two mentioned articles.

The main theme which organises the proof is that the problem of forcing the Π_3^1 -reduction property can be rephrased as a fixed point problem for certain sets of \aleph_1 -sized proper forcings. This fixed point problem can be solved, which unlocks a seemingly self-referential definition of an iteration which will produce a universe of the Π_3^1 -reduction property. A closer inspection shows that in this universe the Π_3^1 -uniformization property is wrong.

2 Preliminaries

The forcings which we will use in the construction are all well-known. We nevertheless briefly introduce them and their main properties.

Definition 2.1. (see [3]) For a stationary $R \subset \omega_1$ the club-shooting forcing for R , denoted by \mathbb{P}_R consists of conditions p which are countable functions from $\alpha + 1 < \omega_1$ to R whose image is a closed set. \mathbb{P}_R is ordered by end-extension.

The club shooting forcing \mathbb{P}_R is the paradigmatic example for an R -proper forcing, where we say that \mathbb{P} is R -proper if and only if for every condition $p \in \mathbb{P}$, every $\theta > 2^{|\mathbb{P}|}$ and every countable $M \prec H(\theta)$ such that $M \cap \omega_1 \in R$ and $p, \mathbb{P} \in M$, there is a $q < p$ which is (M, \mathbb{P}) -generic; and a condition $q \in \mathbb{P}$ is said to be (M, \mathbb{P}) -generic if $q \Vdash \dot{G} \cap M$ is an M -generic filter", for \dot{G} the canonical name for the generic filter. See also [6].

Lemma 2.2. Let $R \subset \omega_1$ be stationary, co-stationary. Then the club-shooting forcing \mathbb{P}_R generically adds a club through the stationary set $R \subset \omega_1$. Additionally \mathbb{P}_R is R -proper, ω -distributive and hence ω_1 -preserving. Moreover R and all its stationary subsets remain stationary in the generic extension.

Proof. We shall just show the ω -distributivity of \mathbb{P}_R , the rest can be found in [6], Fact 3.5, 3.6 and Theorem 3.7. Let $p \in \mathbb{P}_R$ and \dot{x} be such that $p \Vdash \dot{x} \in 2^\omega$. Without loss of generality we assume that \dot{x} is a nice name for a real, i.e. given by an ω -sequence of \mathbb{P}_R -maximal antichains. We shall find a real x in the ground model and a condition $q < p$ such that $q \Vdash \dot{x} = x$. For this, fix a $\theta > 2^{|\mathbb{P}|}$ and a countable elementary submodel $M \prec H(\theta)$ which contains \mathbb{P} , \dot{x} and p as elements and which additionally satisfies that $M \cap \omega_1 \in R$. Note that we can always assume that such an M exists by the stationarity of R . We recursively construct a descending sequence $(p_n)_{n \in \omega} \subset M$ of conditions below $p = p_0$ such that every p_n decides the value of $\dot{x}(n)$ and such that the sequence of $\max_{n \in \omega} \text{range}(p_n)$ converges to $M \cap \omega_1$. We let $x(n) \in 2$ be the value of \dot{x} as forced by p_n , and let $x = (x(n))_{n \in \omega} \in 2^\omega \cap V$.

Let $q' = \bigcup_{n \in \omega} p_n \subset (M \cap \omega_1)$. We set $q := q' \cup \{(\omega, M \cap \omega_1)\}$, which is function from ω to R with closed image, and hence a condition in \mathbb{P}_R which forces that $\dot{x} = x$ as desired. □

We will choose a family of R_β 's so that we can shoot an arbitrary pattern of clubs through its elements such that this pattern can be read off from the stationarity of the R_β 's in the generic extension. For that it is crucial to recall that for stationary, co-stationary $R \subset \omega_1$, R -proper posets can be iterated with countable support and always yield an R -proper forcing again. This is proved exactly as in the well-known case for plain proper forcings (see [6], Theorem 3.9 and the subsequent discussion).

Fact 2.3. *Let $R \subset \omega_1$ be stationary, co-stationary. Assume that $(\mathbb{P}_\alpha : \alpha < \gamma)$ is a countable support iteration of length γ , let \mathbb{P}_γ denote the resulting partial order and assume also that at every stage α , $\mathbb{P}_\alpha \Vdash \dot{\mathbb{P}}(\alpha)$ is R -proper. Then \mathbb{P}_γ is R -proper.*

Once we decide to shoot a club through a stationary, co-stationary subset of ω_1 , this club will belong to all ω_1 -preserving outer models. This hands us a robust method of coding arbitrary information into a suitably chosen sequence of sets which has been used several times already (see e.g. [5]).

Lemma 2.4. *Let $(R_\alpha : \alpha < \omega_1)$ be a partition of ω_1 into \aleph_1 -many stationary sets, let $r \in 2^{\omega_1}$ be arbitrary, and let \mathbb{P} be a countable support iteration $(\mathbb{P}_\alpha : \alpha < \omega_1)$, inductively defined via*

$$\mathbb{P}(\alpha) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{2 \cdot \alpha}} \text{ if } r(\alpha) = 1$$

and

$$\mathbb{P}(\alpha) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{(2 \cdot \alpha) + 1}} \text{ if } r(\alpha) = 0.$$

Then in the resulting generic extension $V[\mathbb{P}]$, we have that $\forall \alpha < \omega_1 :$

$$r(\alpha) = 1 \text{ if and only if } R_{2 \cdot \alpha} \text{ is nonstationary,}$$

and

$$r_\alpha = 0 \text{ iff } R_{(2 \cdot \alpha) + 1} \text{ is nonstationary.}$$

Proof. Assume first without loss of generality that $r(0) = 1$, then the iteration will be R_1 -proper, hence ω_1 -preserving. Now let $\alpha < \omega_1$ be arbitrary and assume that $r(\alpha) = 1$ in $V[\mathbb{P}]$. Then by definition of the iteration we must have shot a club through the complement of $R_{2\alpha}$, thus it is nonstationary in $V[\mathbb{P}]$.

On the other hand, if $R_{2\alpha}$ is nonstationary in $V[\mathbb{P}]$, then we assume for a contradiction that we did not use $\mathbb{P}_{\omega_1 \setminus R_{2 \cdot \alpha}}$ in the iteration \mathbb{P} . Note that for $\beta \neq 2 \cdot \alpha$, every forcing of the form $\mathbb{P}_{\omega_1 \setminus R_\beta}$ is $R_{2 \cdot \alpha}$ -proper as $\mathbb{P}_{\omega_1 \setminus R_\beta}$ is $\omega_1 \setminus R_\beta$ -proper and $R_{2 \cdot \alpha} \subset \omega_1 \setminus R_\beta$. Hence the iteration \mathbb{P} will be $R_{2 \cdot \alpha}$ -proper, thus the stationarity of $R_{2 \cdot \alpha}$ is preserved. But this is a contradiction. \square

The second forcing we use is the almost disjoint coding forcing due to R. Jensen and R. Solovay. We will identify subsets of ω with their characteristic function and will use the word reals for elements of 2^ω and subsets of ω respectively. Let $D = \{d_\alpha : \alpha < \aleph_1\}$ be a family of almost disjoint subsets of ω , i.e. a family such that if $r, s \in D$ then $r \cap s$ is finite. Let $X \subset \kappa$ for $\kappa \leq 2^{\aleph_0}$ be a set of ordinals. Then there is a ccc forcing, the almost disjoint coding $\mathbb{A}_D(X)$ which adds a new real x which codes X relative to the family D in the following way

$$\alpha \in X \text{ if and only if } x \cap d_\alpha \text{ is finite.}$$

Definition 2.5. *The almost disjoint coding $\mathbb{A}_D(X)$ relative to an almost disjoint family D consists of conditions $(r, R) \in \omega^{<\omega} \times D^{<\omega}$ and $(s, S) < (r, R)$ holds if and only if*

1. $r \subset s$ and $R \subset S$.
2. If $\alpha \in X$ and $d_\alpha \in R$ then $r \cap d_\alpha = s \cap d_\alpha$.

For the rest of this paper we let $D \in L$ be the definable almost disjoint family of reals one obtains when recursively adding the $<_L$ -least real to the family which is almost disjoint from all the previously picked reals. Whenever we use almost disjoint coding forcing, we assume that we code relative to this fixed almost disjoint family D .

The last two forcings we briefly discuss are Jech's forcing for adding a Suslin tree with countable conditions and, given a Suslin tree T , the associated forcing which adds a cofinal branch through T . Recall that a set theoretic tree $(T, <)$ is a Suslin tree if it is a normal tree of height ω_1 and has no uncountable antichain. As a result, forcing with a Suslin tree S , where conditions are just nodes in S , and which we always denote with S again, is a ccc forcing of size \aleph_1 . Jech's forcing to generically add a Suslin tree is defined as follows.

Definition 2.6. *Let \mathbb{P}_J be the forcing whose conditions are countable, normal trees ordered by end-extension, i.e. $T_1 < T_2$ if and only if $\exists \alpha < \text{height}(T_1) T_2 = \{t \restriction \alpha : t \in T_1\}$*

It is wellknown that \mathbb{P}_J is σ -closed and adds a Suslin tree. In fact more is true, the generically added tree T has the additional property that for any Suslin tree S in the ground model $S \times T$ will be a Suslin tree in $V[G]$. This can be used to obtain a robust coding method (see also [7] for more applications)

Lemma 2.7. *Let V be a universe and let $S \in V$ be a Suslin tree. If \mathbb{P}_J is Jech's forcing for adding a Suslin tree, if $g \subset \mathbb{P}_J$ is generic and if $T = \bigcup g$ is the generic tree, and if we let $T \in V[g]$ be the forcing which adds an ω_1 -branch through T , then*

$$V[g][T] \models S \text{ is Suslin.}$$

Proof. Let \dot{T} be the \mathbb{P}_J -name for the generic Suslin tree. We claim that $\mathbb{P}_J * \dot{T}$ has a dense subset which is σ -closed. As σ -closed forcings will always preserve ground model Suslin trees, this is sufficient. To see why the claim is true consider the following set:

$$\{(p, \check{q}) : p \in \mathbb{P}_J \wedge \text{height}(p) = \alpha + 1 \wedge \check{q} \text{ is a node of } p \text{ of level } \alpha\}.$$

It is easy to check that this set is dense and σ -closed in $\mathbb{P}_J * \dot{T}$.

□

A similar observation shows that we can add an ω_1 -sequence of such Suslin trees with a countably supported iteration.

Lemma 2.8. *Let S be a Suslin tree in V and let \mathbb{P} be a countably supported product of length ω_1 of forcings \mathbb{P}_J with G its generic filter. Then in $V[G]$ there is an ω_1 -sequence of Suslin trees $\vec{T} = (T_\alpha : \alpha \in \omega_1)$ such that for any finite $e \subset \omega$ the tree $S \times \prod_{i \in e} T_i$ will be a Suslin tree in $V[G]$.*

These sequences of Suslin trees will be used for coding in our proof and get a name.

Definition 2.9. *Let $\vec{T} = (T_\alpha : \alpha < \kappa)$ be a sequence of Suslin trees. We say that the sequence is an independent family of Suslin trees if for every finite set $e = \{e_0, e_1, \dots, e_n\} \subset \kappa$ the product $T_{e_0} \times T_{e_1} \times \dots \times T_{e_n}$ is a Suslin tree again.*

2.1 The ground model W of the iteration

We have to first create a suitable ground model W over which the actual iteration will take place. W will be a generic extension of L , which has no new reals and has the crucial property that in W there is an ω_1 -sequence \vec{S} of independent Suslin trees which is $\Sigma_1(\omega_1)$ -definable over $H(\omega_2)^W$. The sequence \vec{S} will enable a coding method we will use throughout this article all the time.

To form W , we start with Gödel's constructible universe L as our ground model. We first fix an appropriate sequence of stationary, co-stationary subsets of ω_1 as follows. Recall that \diamond holds in L , i.e. over L_{ω_1} there is a Σ_1 -definable sequence $(a_\alpha : \alpha < \omega_1)$ of countable subsets of ω_1 such that any set $A \subset \omega_1$ is guessed stationarily often by the a_α 's, i.e. $\{\alpha < \omega_1 : a_\alpha = A \cap \alpha\}$ is a stationary and co-stationary subset of ω_1 . The \diamond -sequence can be used to produce an easily definable sequence of stationary, co-stationary subsets: we list the reals in L in an ω_1 sequence $(r_\alpha : \alpha < \omega_1)$, and let $\tilde{r}_\alpha \subset \omega_1$ be the unique element of 2^{ω_1} which copies r_α on its first ω -entries followed by ω_1 -many 0's. Then, identifying $\tilde{r}_\alpha \in 2^{\omega_1}$ with the according subset of ω_1 , we define for every $\beta < \omega_1$ a stationary, co-stationary set in the following way:

$$R'_\beta := \{\alpha < \omega_1 : a_\alpha = \tilde{r}_\beta \cap \alpha\}.$$

It is clear that $\forall \alpha \neq \beta (R'_\alpha \cap R'_\beta \in \text{NS}_{\omega_1})$ and we obtain a sequence of pairwise disjoint stationary sets as usual via setting for every $\beta < \omega_1$

$$R_\beta := R'_\beta \setminus \bigcup_{\alpha < \beta} R'_\alpha.$$

and let $\vec{R} = (R_\alpha : \alpha < \omega_1)$. Via picking out one element of \vec{R} and re-indexing we assume without loss of generality that there is a stationary, co-stationary

$R \subset \omega_1$, which has pairwise empty intersection with every $R_\beta \in \vec{R}$. Note that for any $\beta < \omega_1$, membership in R_β is uniformly Σ_1 -definable over the model L_{ω_1} , i.e. there is a Σ_1 -formula $\psi(x, y)$ such that for every $\beta < \omega_1$ $\alpha \in R_\beta \Leftrightarrow L_{\omega_1} \models \psi(\alpha, \beta)$.

We proceed with adding \aleph_1 -many Suslin trees using of Jech's Forcing \mathbb{P}_J . We let

$$\mathbb{Q}^0 := \prod_{\beta \in \omega_1} \mathbb{P}_J$$

using countable support. This is a σ -closed, hence proper notion of forcing. We denote the generic filter of \mathbb{R} with $\vec{S} = (S_\alpha : \alpha < \omega_1)$ and note that by Lemma 2.8 \vec{S} is independent. We fix a definable bijection between $[\omega_1]^\omega$ and ω_1 and identify the trees in $(S_\alpha : \alpha < \omega_1)$ with their images under this bijection, so the trees will always be subsets of ω_1 from now on.

We work in $L[\mathbb{Q}^0]$ and will define the second block of forcings as follows: we let

$$\mathbb{Q}^1 := \prod_{\beta < \omega_1} S_\beta$$

in other words, we add to each generically created tree from \vec{S} an ω_1 -branch, via forcing with the tree. Note that by the argument from the proof of lemma 2.10, this forcing has a dense subset which is σ -closed. Hence $L[\mathbb{Q}^0][\mathbb{Q}^1]$ is a proper and ω -distributive generic extension of L .

In a third step we code the trees from \vec{S} into the sequence of L -stationary subsets \vec{R} we produced earlier, using Lemma 2.4. It is important to note, that the forcing we are about to define does preserve Suslin trees, a fact we will show later. The forcing used in the second step will be denoted by \mathbb{Q}^2 and will itself be a countable support iteration of length $\omega_1 \cdot \omega_1$. Fix first a definable bijection $h \in L_{\omega_2}$ between $\omega_1 \times \omega_1$ and ω_1 and write \vec{R} from now on in ordertype $\omega_1 \cdot \omega_1$ making implicit use of h , so we assume that $\vec{R} = (R_\alpha : \alpha < \omega_1 \cdot \omega_1)$. We let $\alpha < \omega_1$ and consider the tree $S_\alpha \subset \omega_1$. We let \mathbb{Q}_α^2 be the countable support iteration which codes the characteristic function of S_α into the α -th ω_1 -block of the R_β 's just as in Lemma 2.4. So \mathbb{Q}_α^2 is a countable support iteration, defined via

$$\forall \gamma < \omega_1 \ (\mathbb{R}_\alpha(\gamma) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{\omega_1 \cdot \alpha + 2\gamma + 1}}) \text{ if } S_\alpha(\gamma) = 0$$

and

$$\forall \gamma < \omega_1 \ (\mathbb{R}_\alpha(\gamma) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{\omega_1 \cdot \alpha + 2\gamma}}) \text{ if } S_\alpha(\gamma) = 1.$$

Recall that we let R be a stationary, co-stationary subset of ω_1 which is disjoint from all the R_α 's which are used. It is obvious that for every $\alpha < \omega_1$, \mathbb{Q}_α^2 is an R -proper forcing which additionally is ω -distributive. Then we let \mathbb{Q}^2 be the countably supported iteration,

$$\mathbb{Q}^2 := \star_{\alpha < \omega_1} \mathbb{Q}_\alpha^2$$

which is again R -proper (and ω -distributive as we shall see later). This way we can turn the generically added sequence of trees \vec{S} into a definable sequence of trees. Indeed, if we work in $L[\vec{S} * \vec{b} * G]$, where $\vec{S} * \vec{b} * G$ is $\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2$ -generic over L , then, as seen in Lemma 2.4

$$\begin{aligned} \forall \alpha, \gamma < \omega_1 (\gamma \in S_\alpha &\Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma} \text{ is not stationary and} \\ \gamma \notin S_\alpha &\Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma + 1} \text{ is not stationary}) \end{aligned}$$

Note here that the above formula can be written in a $\Sigma_1(\omega_1)$ -way, as it reflects down to \aleph_1 -sized, transitive models of \mathbf{ZF}^- which contain a club through exactly one element of every pair $\{(R_\alpha, R_{\alpha+1}) : \alpha < \omega_1\}$.

Our goal is to use \vec{S} for coding again. For this it is essential, that the sequence remains independent in the inner universe $L[\mathbb{Q}^0 * \mathbb{Q}^2]$. Note that this is reasonable as $\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2$ can be written as $\mathbb{Q}^0 * (\mathbb{Q}^1 \times \mathbb{Q}^2)$, hence one can form the inner model $L[\mathbb{Q}^0 * \mathbb{Q}^2]$ without problems.

The following line of reasoning is similar to [7]. Recall that for a forcing \mathbb{P} and $M \prec H(\theta)$, a condition $q \in \mathbb{P}$ is (M, \mathbb{P}) -generic iff for every maximal antichain $A \subset \mathbb{P}$, $A \in M$, it is true that $A \cap M$ is predense below q . The key fact is the following (see [12] for the case where \mathbb{P} is proper)

Lemma 2.10. *Let T be a Suslin tree, $R \subset \omega_1$ stationary and \mathbb{P} an R -proper poset. Let θ be a sufficiently large cardinal. Then the following are equivalent:*

1. $\Vdash_{\mathbb{P}} T$ is Suslin
2. *if $M \prec H_\theta$ is countable, $\eta = M \cap \omega_1 \in R$, and \mathbb{P} and T are in M , further if $p \in \mathbb{P} \cap M$, then there is a condition $q < p$ such that for every condition $t \in T_\eta$, (q, t) is $(M, \mathbb{P} \times T)$ -generic.*

Proof. For the direction from left to right note first that $\Vdash_{\mathbb{P}} T$ is Suslin implies $\Vdash_{\mathbb{P}} T$ is ccc, and in particular it is true that for any countable elementary submodel $N[\dot{G}_{\mathbb{P}}] \prec H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$, $\Vdash_{\mathbb{P}} \forall t \in T (t \text{ is } (N[\dot{G}_{\mathbb{P}}], T)\text{-generic})$. Now if $M \prec H(\theta)$ and $M \cap \omega_1 = \eta \in R$ and $\mathbb{P}, T \in M$ and $p \in \mathbb{P} \cap M$ then there is a $q < p$ such q is (M, \mathbb{P}) -generic. So $q \Vdash \forall t \in T (t \text{ is } (M[\dot{G}_{\mathbb{P}}], T)\text{-generic})$, and this in particular implies that (q, t) is $(M, \mathbb{P} \times T)$ -generic for all $t \in T_\eta$.

For the direction from right to left assume that $\Vdash \dot{A} \subset T$ is a maximal antichain. Let $B = \{(x, s) \in \mathbb{P} \times T : x \Vdash_{\mathbb{P}} \dot{s} \in \dot{A}\}$, then B is a predense subset in $\mathbb{P} \times T$. Let θ be a sufficiently large regular cardinal and let $M \prec H(\theta)$ be countable such that $M \cap \omega_1 = \eta \in R$ and $\mathbb{P}, B, p, T \in M$. By our assumption there is a $q <_{\mathbb{P}} p$ such that $\forall t \in T_\eta ((q, t) \text{ is } (M, \mathbb{P} \times T)\text{-generic})$. So $B \cap M$ is predense below (q, t) for every $t \in T_\eta$, which yields that $q \Vdash_{\mathbb{P}} \forall t \in T_\eta \exists s <_T t (s \in \dot{A})$ and hence $q \Vdash \dot{A} \subset T \restriction \eta$, so $\Vdash_{\mathbb{P}} T$ is Suslin. \square

In a similar way, one can show that Theorem 1.3 of [12] holds true if we replace proper by R -proper for $R \subset \omega_1$ a stationary subset.

Theorem 2.11. *Let $(\mathbb{P}_\alpha)_{\alpha < \eta}$ be a countable support iteration of length η , let $R \subset \omega_1$ be stationary and suppose that for every $\alpha < \eta$, for the α -th factor of the iteration $\dot{\mathbb{P}}(\alpha)$ it holds that \Vdash_α “ $\dot{\mathbb{P}}(\alpha)$ is R -proper and preserves every Suslin tree.” Then \mathbb{P}_η is R -proper and preserves every Suslin tree.*

So in order to argue that our forcing \mathbb{Q}^2 preserves Suslin trees when used over the ground model $W[\mathbb{Q}^0]$, it is sufficient to show that every factor preserves Suslin trees. This is indeed the case.

Lemma 2.12. *Let $R \subset \omega_1$ be stationary, co-stationary, then the club shooting forcing \mathbb{P}_R preserves Suslin trees.*

Proof. Because of Lemma 2.10, it is enough to show that for any regular and sufficiently large θ , every $M \prec H_\theta$ with $M \cap \omega_1 = \eta \in R$, and every $p \in \mathbb{P}_R \cap M$ there is a $q < p$ such that for every $t \in T_\eta$, (q, t) is $(M, (\mathbb{P}_R \times T))$ -generic. Note first that as T is Suslin, every node $t \in T_\eta$ is an (M, T) -generic condition. Further, as forcing with a Suslin tree is ω -distributive, $M[t]$ has the same $M[t]$ -countable sets as M . It is not hard to see that if $M \prec H(\theta)$ is such that $M \cap \omega_1 \in R$ then an ω -length descending sequence of \mathbb{P}_R -conditions in M whose domains converge to $M \cap \omega_1$ has a lower bound as $M \cap \omega_1 \in R$.

We construct an ω -sequence of elements of \mathbb{P}_R which has a lower bound which will be the desired condition. We list the nodes on T_η , $(t_i : i \in \omega)$ and consider the according generic extensions $M[t_i]$. In every $M[t_i]$ we list the \mathbb{P}_R -dense subsets of $M[t_i]$, $(D_n^{t_i} : n \in \omega)$ and write the so listed dense subsets of $M[t_i]$ as an $\omega \times \omega$ -matrix and enumerate this matrix in an ω -length sequence of dense sets $(D_i : i \in \omega)$. If $p = p_0 \in \mathbb{P}_R \cap M$ is arbitrary we can find, using the fact that $\forall i (\mathbb{P}_R \cap M[t_i] = M \cap \mathbb{P}_R)$, an ω -length, descending sequence of conditions below p_0 in $\mathbb{P}_R \cap M$, $(p_i : i \in \omega)$ such that $p_{i+1} \in M \cap \mathbb{P}_R$ is in D_i . We can also demand that the domain of the conditions p_i converge to $M \cap \omega_1$. Then the (p_i) 's have a lower bound $p_\omega \in \mathbb{P}_R$ and (t, p_ω) is an $(M, T \times \mathbb{P}_R)$ -generic conditions for every $t \in T_\eta$ as any $t \in T_\eta$ is (M, T) -generic and every such t forces that p_ω is $(M[T], \mathbb{P}_R)$ -generic; moreover $p_\omega < p$ as desired. \square

Let us set $W := L[\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2]$ which will serve as our ground model for a second iteration of length ω_1 . To summarize the above:

Theorem 2.13. *The universe $W = L[\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2]$ is an ω_1 -preserving, ω -distributive generic extension of L . In W there is a $\Sigma_1(\omega_1)$ -definable, independent sequence of trees \vec{S} which are Suslin in the inner model $L[\mathbb{Q}^0][\mathbb{Q}^2]$, yet no tree is Suslin in W .*

Proof. The first assertion should be clear from the above discussion. The second assertion holds by the following standard argument. As $\mathbb{Q}^0 * \mathbb{Q}^1$ does not add any reals it is sufficient to show that \mathbb{Q}^2 is ω -distributive. Let $p \in \mathbb{Q}^2$ be a condition and assume that $p \Vdash \dot{r} \in 2^\omega$. We shall find a stronger $q < p$

and a real r in the ground model such that $q \Vdash \check{r} = \dot{r}$. Let $M \prec H(\omega_3)$ be a countable elementary submodel which contains p, \mathbb{Q}^2 and \dot{r} and such that $M \cap \omega_1 \in R$, where R is our fixed stationary set from above. Inside M we recursively construct a decreasing sequence p_n of conditions in \mathbb{Q}^2 , such that for every n in ω , $p_n \in M$, p_n decides $\dot{r}(n)$ and for every α in the support of p_n , the sequence $\sup_{n \in \omega} \max(p_n(\alpha))$ converges towards $M \cap \omega_1$ which is in R . Now, $q' := \bigcup_{n \in \omega} p_n$ and for every $\alpha < \omega_1$ such that $q'(\alpha) \neq 1$ (where 1 is the weakest condition of the forcing), in other words for every α in the support of q' we define $q(\alpha) := q'(\alpha) \cup \{(\omega, \sup(M \cap \omega_1))\}$ and $q(\alpha) = 1$ otherwise. Then $q = (q(\alpha))_{\alpha < \omega_1}$ is a condition in \mathbb{Q}^2 , as can be readily verified and $q \Vdash \dot{r} = \check{r}$, as desired. \square

The independent sequence \vec{S} will be split into two $\Sigma_1(\omega_1)$ -definable sequences via letting

$$\vec{S}^1 := (S_\alpha \in \vec{S} : \alpha \text{ is even})$$

and

$$\vec{S}^2 := ((S_\alpha \in \vec{S} : \alpha \text{ is odd}).$$

These two sequences will be used for defining the Π_3^1 -sets witnessing the reduction property, as we will see soon.

We end with a straightforward lemma which is used later in coding arguments.

Lemma 2.14. *Let T be a Suslin tree and let $\mathbb{A}_D(X)$ be the almost disjoint coding which codes a subset X of ω_1 into a real with the help of an almost disjoint family of reals D of size \aleph_1 . Then*

$$\mathbb{A}_D(X) \Vdash T \text{ is Suslin}$$

holds.

Proof. This is clear as $\mathbb{A}_D(X)$ has the Knaster property, thus the product $\mathbb{A}_D(X) \times T$ is ccc and T must be Suslin in $V[\mathbb{A}_D(X)]$. \square

3 Main Proof

3.1 Informal discussion of the idea

We proceed with an informal discussion of the main ideas of the proof. We focus on reducing one fixed, arbitrary pair A_m and A_k of Π_3^1 -sets. The arguments will be uniform, so that reducing every pair of Π_3^1 -sets will follow immediately.

The ansatz is to use the two definable sequences of Suslin trees \vec{S}^1 and \vec{S}^2 for coding and a bookkeeping function F which lists all possible reals in our iteration. We use a mixed support iteration over W of length ω_1 . At

stages β , where $F(\beta)$ is (the name of) a real number x , we decide whether to code x into the \vec{S}^1 -sequence or the \vec{S}^2 -sequence. Coding here means that we write the characteristic function of x into \aleph_1 -many ω -blocks of elements of \vec{S}^i , $i \in \{1, 2\}$ in a way such that the statement “ x is coded into \vec{S}^i ” is a $\Sigma_3^1(x, i)$ -statement and hence a $\Sigma_3^1(x)$ -statement. Our goal is that eventually, after ω_1 stages of our iteration, the resulting universe satisfies

$$\forall x \in A_m \cup A_k, \text{ either } x \text{ is coded into } \vec{S}^1 \text{ or } x \text{ is coded into } \vec{S}^2.$$

This dichotomy allows the following construction of reducing sets. The set of elements of A_m which are *not* coded somewhere into \vec{S}^1 , will be equivalent to the set of elements which are coded somewhere into \vec{S}^2 , and shall form our reducing set $D_{m,k}^1$ which should be a subset of A_m .

$$\begin{aligned} D_{m,k}^1 := & \{x \in A_m \cup A_k : x \text{ is coded somewhere into } \vec{S}^2\} = \\ & \{x \in A_m \cup A_k : x \text{ is not coded somewhere into } \vec{S}^1\} \end{aligned}$$

Note that the second definition of $D_{m,k}^1$ witnesses that $D_{m,k}^1$ is Π_3^1 , as being coded is Σ_3^1 , hence not being coded is Π_3^1 .

On the other hand, reals in $A_m \cup A_k$ which are *not* coded into \vec{S}^2 form the set $D_{m,k}^2$ which shall eventually reduce A_k :

$$\begin{aligned} D_{m,k}^2 := & \{x \in A_m \cup A_k : x \text{ is coded somewhere into } \vec{S}^1\} = \\ & \{x \in A_m \cup A_k : x \text{ is not coded somewhere into } \vec{S}^2\} \end{aligned}$$

This set-up has the following difficulties one has to overcome: the evaluation of Π_3^1 -sets changes as we use coding forcings, yet deciding to code a real into the \vec{S} -sequence, once performed, can not be undone in future extensions, by the upwards absoluteness of Σ_3^1 -formulas. In particular it could well be that at some stage β of the iteration we decide to code the real x into \vec{S}^2 , which is equivalent to put x into $D_{m,k}^1$. Now it could happen that this coding forcing, or some later coding forcing we will use, actually puts x out of A_m , while x remains in A_k . The consequence of this is that x witnesses that $D_{m,k}^1$ and $D_{m,k}^2$ do not reduce A_m and A_k as $x \notin D_{m,k}^1 \cup D_{m,k}^2$, yet $x \in A_m \cup A_k$. One can try to repair this misery, in trying to use an additional iteration of coding forcings to kick x out of A_k . But it is possible that no such further iteration exists. Once we hit such a pathological situation in our attempt to force the Π_3^1 -reduction property, we have lost and we need to start a new attempt with a different placement strategy for which, of course, a similar pathological situation could arise.

If we look closer at the possibly arising pathologies, we see that they actually give nontrivial information which can be exploited to make progress on the problem of the right placement. Assume, for an illustration of this,

that x is a real in the ground model W which we coded into \vec{S}^2 , and then, as the iteration proceeds, we eventually produce a forcing \mathbb{P} for which we encounter the pathological situation described above, i.e. $W[\mathbb{P}] \models x \notin A_m$ and there is no further placement forcing $\mathbb{Q} \in W[\mathbb{P}]$, such that $W[\mathbb{P}] \models \Vdash_{\mathbb{Q}} x \notin A_k$. We can interpret the pathological situation in a different way, in noting that there is a forcing $\mathbb{P} \in W$, such that $x \in W$ can not be kicked out of the Π_3^1 -set A_k any more, by forcings which are seen to be placement forcings in the virtual universe $W[\mathbb{P}]$. In particular, we can code x into \vec{S}^1 and, as long as the forcings \mathbb{Q} we use are such that $W[\mathbb{P}]$ thinks that \mathbb{Q} is some placement forcing, x will forever remain an element of A_k , which is in accordance with our strategy of reducing A_m and A_k with $D_{m,k}^1$ and $D_{m,k}^2$. Thus we made progress, the real x will be placed properly, at the cost of storing the information of the forcing \mathbb{P} , and use \mathbb{P} in all future considerations. These forcings \mathbb{P} will never be used in the iteration, nevertheless they play a key role in the definitions.

Applying the above reasoning for every real we encounter in our iteration will lead to a new set of rules of where to place an arbitrary real x , which in turn leads to a new set of forcings we want to use for finally getting a universe in which the Π_3^1 -reduction property holds. For this new set of forcings, the above mentioned pathological situations can arise again, thus we can use these pathologies to define again a new set of rules, yielding a new set of forcings and so on. This leads to an inductive definition of an operator which acts on sets of forcings. The operator, which can be seen as some sort of derivation, will be uniformly definable over W .

The main idea is now, to set things up in such a way that this whole process converges to a fixed and non-empty set of forcings, which we will denote the ∞ -allowable forcings. Here we make crucial use of the fact that both sequences \vec{S}^1 and \vec{S}^2 of Suslin trees are from the ground model W , justifying therefore the move from L to W . All of these considerations are preliminary.

After we obtained this set, which is a fixed point of the derivation operator, we can use the fact that it is a fixed point to argue in a seemingly self-referential way which will take care of all the pathological situations. We then use an iteration of ∞ -allowable forcings to produce a model of the Π_3^1 -reduction property.

3.2 ∞ -allowable Forcings

We continue with the construction of the appropriate notions of forcing which we want to use in our proof. The goal is to iteratively shrink the set of notions of forcing we want to use until we reach a fixed point. All forcings will belong to a certain class, which we call allowable. These are just forcings which iteratively code reals into ω -blocks of \vec{S}^1 or \vec{S}^2 . We first want to present the coding method, which we use to code a real x up, using the definable

sequence of Suslin trees, and subsequently introduce the notion allowable.

Our ground model shall be W . Let x be a real, let $m, k \in \omega$ and let $\gamma < \omega_1$ be an arbitrary ordinal. The forcing $\mathbb{P}_{((x,m,k),1)}$, which codes the real w , which in turn codes the triple (x, m, k) into \vec{S}^1 is defined as a two step iteration $\mathbb{P}_{(x,m,k),1} := (\mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}(\dot{Y})$, where $(\mathbb{C}(\omega_1))^L$ is the usual ω_1 -Cohen forcing, as defined in L , and $\dot{\mathbb{A}}(\dot{Y})$ is the (name of) an almost disjoint coding forcing, coding a particular set into a real. We shall describe the second factor $\dot{\mathbb{A}}(\dot{Y})$ now in detail.

We let $g \subset \omega_1$ be a $(\mathbb{C}(\omega_1))^L$ -generic filter over W , and let $\rho : [\omega_1]^\omega \rightarrow \omega_1$ be some canonically definable, constructible bijection between these two sets. We use ρ and g to define the set $h \subset \omega_1$, which eventually shall be the set of indices of ω -blocks of \vec{S} , where we code up the characteristic function of the real $((x, y, m))$. Let $h := \{\rho(g \cap \alpha) : \alpha < \omega_1\}$ and let $X \subset \omega_1$ be the $<$ -least set (in some previously fixed well-order of $H(\omega_2)^W[g]$ which codes the following objects:

- The $<$ -least set of ω_1 -branches in W through elements of \vec{S} which code (x, y, m) at ω -blocks which start at values in h , that is we collect $\{b_\beta \subset S_\beta : \beta = \omega\gamma + 2n, \gamma \in h \wedge n \in \omega \wedge n \notin (x, y, m)\}$ and $\{b_\beta \subset S_\beta : \beta = \omega\gamma + 2n + 1, \gamma \in h \wedge n \in \omega \wedge n \in (x, y, m)\}$.
- The $<$ -least set of $\omega_1 \cdot \omega \cdot \omega_1$ -many club subsets through \vec{R} , our $\Sigma_1(\omega_1)$ -definable sequence of L -stationary subsets of ω_1 from the last section, which are necessary to compute every tree $S_\beta \in \vec{S}$ which shows up in the above item, using the $\Sigma_1(\omega_1)$ -formula from the previous section before Lemma 2.10.

Note that, when working in $L[X]$ and if $\gamma \in h$ then we can read off w and hence (x, m, k) via looking at the ω -block of \vec{S}^1 -trees starting at γ and determine which tree has an ω_1 -branch in $L[X]$:

- (*) $n \in w$ if and only if $S_{\omega \cdot \gamma + 2n+1}$ has an ω_1 -branch, and $n \notin w$ if and only if $S_{\omega \cdot \gamma + 2n}$ has an ω_1 -branch.

Indeed if $n \notin w$ then we added a branch through $S_{\omega \cdot \gamma + 2n}$. If on the other hand $S_{\omega \cdot \gamma + 2n}$ is Suslin in $L[X]$ then we must have added an ω_1 -branch through $S_{\omega \cdot \gamma + 2n+1}$ as we always add an ω_1 -branch through either $S_{\omega \cdot \gamma + 2n+1}$ or $S_{\omega \cdot \gamma + 2n}$ and adding branches through some S_α 's will not affect that some S_β is Suslin in $L[X]$, as \vec{S} is independent.

We note that we can apply an argument resembling David's trick in this situation. We rewrite the information of $X \subset \omega_1$ as a subset $Y \subset \omega_1$ using the following line of reasoning. It is clear that any transitive, \aleph_1 -sized model M of ZF^- which contains X will be able to correctly decode out of X all the information. Consequentially, if we code the model (M, \in) which contains

X as a set $X_M \subset \omega_1$, then for any uncountable β such that $L_\beta[X_M] \models \mathbf{ZF}^-$ and $X_M \in L_\beta[X_M]$:

$L_\beta[X_M] \models$ “The model decoded out of X_M satisfies $(*)$ for every $\gamma \in h$ ”.

In particular there will be an \aleph_1 -sized ordinal β as above and we can fix a club $C \subset \omega_1$ and a sequence $(M_\alpha : \alpha \in C)$ of countable elementary submodels such that

$$\forall \alpha \in C (M_\alpha \prec L_\beta[X_M] \wedge M_\alpha \cap \omega_1 = \alpha)$$

Now let the set $Y \subset \omega_1$ code the pair (C, X_M) such that the odd entries of Y should code X_M and if $Y_0 := E(Y)$ where the latter is the set of even entries of Y and $\{c_\alpha : \alpha < \omega_1\}$ is the enumeration of C then

1. $E(Y) \cap \omega$ codes a well-ordering of type c_0 .
2. $E(Y) \cap [\omega, c_0) = \emptyset$.
3. For all β , $E(Y) \cap [c_\beta, c_{\beta+1})$ codes a well-ordering of type $c_{\beta+1}$.
4. For all β , $E(Y) \cap [c_\beta + \omega, c_{\beta+1}) = \emptyset$.

We obtain

- (**) For any countable transitive model M of \mathbf{ZF}^- such that $\omega_1^M = (\omega_1^L)^M$ and $Y \cap \omega_1^M \in M$, M can construct its version of the universe $L[Y \cap \omega_1^M]$, and the latter will see that there is an \aleph_1^M -sized transitive model $N \in L[Y \cap \omega_1^M]$ which models $(*)$ for w and every $\gamma \in h$.

Thus we have a local version of the property $(*)$.

In the next step $\dot{\mathbb{A}}(\dot{Y})$, working in $W[g]$, for $g \subset \mathbb{C}(\omega_1)$ generic over W , we use almost disjoint forcing $\mathbb{A}_D(Y)$ relative to the $<_L$ -least almost disjoint family of reals $D \in L$ to code the set Y into one real r . This forcing is well-known, has the ccc and its definition only depends on the subset of ω_1 we code, thus the almost disjoint coding forcing $\mathbb{A}_D(Y)$ will be independent of the surrounding universe in which we define it, as long as it has the right ω_1 and contains the set Y .

We finally obtained a real r such that

- (***) For any countable, transitive model M of \mathbf{ZF}^- such that $\omega_1^M = (\omega_1^L)^M$ and $r \in M$, M can construct its version of $L[r]$ which in turn thinks that there is a transitive \mathbf{ZF}^- -model N of size \aleph_1^M such that N believes $(*)$ for w and every $\gamma \in h$.

Note that (***) is a Π_2^1 -formula in the parameters r and w , as the set $h \subset \omega_1^M$ is coded into r . We will often suppress the reals r, w when referring to (***) as they will be clear from the context. We say in the above situation that

the real w , which codes (x, m, k) is written into \vec{S}^1 , or that w is coded into \vec{S}^1 and r witnesses that w is coded. Likewise a forcing $\mathbb{P}_{(x,m,k),2}$ is defined for coding the real w which codes (x, m, k) into \vec{S}^2 . Next we define the set of forcings which we will use in our proof. We aim to iterate the coding forcings. As the first factor is always $(\mathbb{C}(\omega_1))^L$, the iteration is actually a hybrid of an iteration and a product. We shall use a mixed support, that is we use countable support on the coordinates which use $(\mathbb{C}(\omega_1))^L$, and finite support on the coordinates which use almost disjoint coding.

Definition 3.1. *A mixed support iteration $\mathbb{P} = (\mathbb{P}_\beta : \beta < \alpha)$ is called allowable if $\alpha < \omega_1$ and there exists a bookkeeping function $F : \alpha \rightarrow H(\omega_2)^2$ such that \mathbb{P} is defined inductively using F as follows:*

- *If $F(0) = (x, i)$, where x is a real, $i \in \{1, 2\}$, then $\mathbb{P}_0 = \mathbb{P}(x, i)$. Otherwise \mathbb{P}_0 is the trivial forcing.*
- *If $\beta > 0$ and \mathbb{P}_β is defined, $G_\beta \subset \mathbb{P}_\beta$ is a generic filter over W , $F(\beta) = (\dot{x}, i)$, where \dot{x} is a \mathbb{P}_β -name of a real, $i \in \{1, 2\}$ and $\dot{x}^{G_\beta} = x$ then, working in $W[G_\beta]$ we let $\mathbb{P}(\beta) := \mathbb{P}_{x,i}$, that is we code x into the \vec{S}^i , using our coding forcing. Note that we use finite support on the parts where almost disjoint coding is used and countable support on the parts where ω_1 -Cohen forcing, as computed in L and seen as a product, is used.*

Informally speaking, an allowable forcing just decides to code the reals which the bookkeeping F provides into either \vec{S}^1 or \vec{S}^2 . Note further that the notion of allowable can be defined in exactly the same way over any $W[G]$, where G is a \mathbb{P} -generic filter over W for an allowable forcing.

We obtain the following first properties of allowable forcings:

- Lemma 3.2.** *1. If $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta) \in W$ is allowable then for every $\beta < \delta$, $\mathbb{P}_\beta \Vdash |\mathbb{P}(\beta)| = \aleph_1$, thus every factor of \mathbb{P} is forced to have size \aleph_1 .*
- 2. Every allowable forcing over W preserves CH and ω_1 .*
- 3. The product of two allowable forcings is allowable again.*

Proof. The first assertion follows immediately from the definition.

To see the second item we exploit some symmetry. Indeed, every allowable $\mathbb{P} = \star_{\beta < \delta} \mathbb{P}(\beta) = \star_{\beta < \delta} (((\mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}(\dot{Y}_\beta)))$ can be rewritten as $(\prod_{\beta < \delta} (\mathbb{C}(\omega_1))^L) * \star_{\beta < \delta} \dot{\mathbb{A}}_D(\dot{Y}_\beta)$ (again with countable support on the $(\mathbb{C}(\omega_1))^L$ part and finite support on the almost disjoint coding forcings). The latter representation is easily seen to be of the form $\mathbb{P} * \star_{\beta < \delta} \dot{\mathbb{A}}_D(\dot{Y}_\beta)$, where \mathbb{P} is σ -closed and the second part is a finite support iteration of ccc forcings, hence ω_1 is preserved. That CH holds is standard.

To see that the third item is true, we note that the definition of an allowable forcing just depends on F and is independent of the surrounding universe $V \subset W$ over which it is applied, so we immediately see that a two step iteration $\mathbb{P}_1 * \mathbb{P}_2$ of two allowable $\mathbb{P}_1, \mathbb{P}_2 \in W$ is in fact a product. As the iteration of two allowable forcings (in fact the iteration of countably many allowable forcings) is allowable as well, the proof is done. \square

The second assertion of the last lemma immediately gives us the following:

Corollary 3.3. *Let $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta) \in W$ be an allowable forcing over W . Then $W[\mathbb{P}] \models \text{CH}$. Further, if $\mathbb{P} = (\mathbb{P}(\alpha) : \alpha < \omega_1) \in W$ is an ω_1 -length iteration such that each initial segment of the iteration is allowable over W , then $W[\mathbb{P}] \models \text{CH}$.*

We define next a derivative of the class of allowable forcings. Inductively we assume that for an ordinal α and an arbitrary bookkeeping function F mapping to $H(\omega_2)^2$, we have already defined the notion of δ -allowable with respect to F for every $\delta \leq \alpha$, and the definition works uniformly for every model $W[G]$, where G is a generic filter for an allowable forcing. Now we aim to define the derivation of the α -allowable forcings which we call $\alpha + 1$ -allowable.

A $\delta \leq \omega_1$ -length iteration \mathbb{P} is called $\alpha + 1$ -allowable if it is recursively constructed following a bookkeeping function $F : \delta \rightarrow H(\omega_2)^2$, such that for every $\beta < \delta$, $F(\beta)$ is a pair $((F(\beta))_0, (F(\beta))_1)$, and two rules at every stage $\beta < \delta$ of the iteration. We assume inductively that we already created the forcing iteration up to β , \mathbb{P}_β . We shall now define the next forcing of our iteration $\mathbb{P}(\beta)$. Using the bookkeeping F we split into two cases.

- (a) We assume first that the first coordinate of $F(\beta)$, $(F(\beta))_0 = (\dot{x}, m, k)$, where \dot{x} is the \mathbb{P}_β -name of a real and m, k are natural numbers. Further we assume that $\dot{x}^{G_\beta} = x$, for G_β a \mathbb{P}_β -generic filter over W and $W[G_\beta] \models x \in A_m \cup A_k$. We assume that in $W[G_\beta]$, the following is true:

There is an ordinal $\zeta \leq \alpha$, which is chosen to be minimal for which

- (i) for every ζ -allowable forcing $\mathbb{Q} \in W[G_\beta]$ we have that, over $W[G_\beta]$:

$$\mathbb{Q} \Vdash x \in A_m$$

- (ii) if (i) for ζ is wrong but the dual situation is true, i.e. for every ζ -allowable forcing $\mathbb{Q} \in W[G_\beta]$, we have that $W[G_\beta]$ thinks that

$$\mathbb{Q} \Vdash x \in A_k$$

We emphasize that if the minimal such ζ is such that both (i) and (ii) are true, then we give case (i) preference, and suppress case (ii).

Now if (a) and (i) is the true, then we define the forcing we use at stage β , $\mathbb{P}(\beta)$ to be $\mathbb{P}_{(x,y,m),2}$.

If, on the other hand (a) (i) is wrong and (a) (ii) is true, then we define the next forcing in the dual way, i.e. we let $\mathbb{P}(\beta) := \mathbb{P}_{(x,m,k),1}$.

Note that if (a) (i) was true at stage β , $G_{\beta+1} = G_\beta * G(\beta)$ and $G(\beta)$ is a filter for $\mathbb{P}(\beta)$, then for every forcing $\mathbb{Q} \in W[G_{\beta+1}]$ such that $W[G_{\beta+1}] \models \text{“}\mathbb{Q} \text{ is } \zeta\text{-allowable”}$, we have that $W[G_{\beta+1}] \models \mathbb{Q} \Vdash x \in A_m$, by construction. Likewise, if ζ is minimal for which (a) (i) or (a) (ii) is true, but (a) (i) is wrong, then $W[G_{\beta+1}] \models \forall \mathbb{Q}(\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow \mathbb{Q} \Vdash (x, y) \in A_k)$.

- (b) Else F guesses where we code x , i.e. we code x into $\vec{S}^{F(\beta)_1}$, provided $F(\beta)_1 \in \{1, 2\}$ (otherwise we redefine $F(\beta)_1 := 1$).

This ends the definition of \mathbb{P} being $\alpha + 1$ -allowable with respect to F at successor stages $\beta + 1$. To define the limit stages β of an $\alpha + 1$ -allowable forcing, we assume that we have defined already $(\mathbb{P}_\gamma : \gamma < \beta)$ and let

$$\mathbb{P}_\beta := \text{inv lim}_{\gamma < \beta} \mathbb{P}_\gamma$$

We finally have finished the definition of an $\alpha + 1$ -allowable forcing relative to the bookkeeping function F . In the following we often drop the reference to F and simply say that some forcing \mathbb{P} is $\alpha + 1$ -allowable, in which case we always mean that there is some F such that \mathbb{P} is $\alpha + 1$ -allowable relative to F .

For limit stages α , we will define the notion of α -allowable, relative to $F : \delta \rightarrow H(\omega_2)^2$ as follows. First we assume that we know already, what β -allowable with respect to F means, for any $W[G]$ and any arbitrary bookkeeping $F \in W[G]$, where G is generic for some allowable forcing. Then α -allowable will be defined as in (a) and (b) above, using the bookkeeping F in the background, but in (a) we do not ask for a $\zeta \leq \alpha$, but a $\zeta < \alpha$. Thus, at every stage β of an α -allowable forcing, α a limit ordinal, we ask whether there exists for $\zeta = 0$ a \mathbb{P} and a γ such that (a)(i) becomes true. If not then we ask the same question for (a)(ii). If both are wrong, we pass to $\zeta = 1$, and so on. If (a) (i) or (a) (ii) never applies for any $\zeta < \alpha$, we pass to (b). We will see in a moment that the notion of α -allowable becomes harder and harder to satisfy as α increases, thus case (a) in the definition becomes easier and easier to satisfy, which leads in turn to more restrictions of how an $\alpha + 1$ -allowable forcing can look like.

Lemma 3.4. *For any ordinal α , the notion α -allowable is definable over the universe W .*

Lemma 3.5. *If \mathbb{P} is β -allowable and $\alpha < \beta$, then \mathbb{P} is α -allowable. Thus the sequence of α -allowable forcings is decreasing with respect to the \subset -relation.*

Proof. Let $\alpha < \beta$, let \mathbb{P} be a β -allowable forcing and let F be the bookkeeping function which, together with the rules (a)+(b) from above determine \mathbb{P} . We will show that there is a bookkeeping function $F' \in W$ such that \mathbb{P} can be seen as an α -allowable forcing determined by F' . The first coordinate of F' should always coincide with the first coordinate of F , i.e. $\forall \gamma ((F(\gamma))_0 = F'(\gamma)_0)$. The second coordinate, which determines which of the \vec{S}^i -sequence is used for coding when in case (b) is defined via simulating the reasoning for a β -allowable forcing. This means that at every stage γ of the iteration, we pretend that we are working with β -allowable forcings, we do the reasoning described in (a) and (b) for β -allowable using F . If case (a) does apply, and $\mathbb{P}(\gamma)$ is some $\mathbb{P}_{(x,m,k),1}$, then we simply let $(F'(\gamma))_1 = 1$. That is, we let F' simulate the reasoning we would apply if \mathbb{P} would be a β -allowable forcing using F , and then forget about β -allowable and just keep the result of the reasoning. The new bookkeeping F' is definable from F , and clearly \mathbb{P} is α -allowable using F' . □

Lemma 3.6. *Let α be an arbitrary ordinal, let F_1, F_2 be two bookkeeping functions, $F_1 : \delta_1 \rightarrow W^2, F_2 : \delta_2 \rightarrow W^2$, and let $\mathbb{P}_1 = (\mathbb{P}_\beta^1 : \beta < \delta_1)$ and $\mathbb{P}_2 = (\mathbb{P}_\beta^2 : \beta < \delta_2)$ be the α -allowable forcings one obtains when using F_1 and F_2 respectively.*

Then $\mathbb{P}_1 \times \mathbb{P}_2$ is α -allowable over W , as witnessed by some $F : (\delta_1 + \delta_2) \rightarrow W^2$, which is definable from $\{F_1, F_2\}$.

Proof. By induction on α . For $\alpha = 0$, this follows immediately from the definition of 0-allowable.

Now suppose the Lemma is true for α and we want to show it is true for $\alpha + 1$. Given F_1 and F_2 , we define $F(\gamma) := F_1(\gamma)$ for $\gamma < \delta_1$ and $F(\delta_1 + \gamma) := F_2(\gamma)$ for $\gamma < \delta_2$. We claim that \mathbb{P} is $\alpha + 1$ -allowable with respect to F over W . This is shown via induction on the stages $\beta < \delta_1 + \delta_2$, i.e. we shall show that for each $\beta < \delta_1 + \delta_2$, \mathbb{P}_β is $\alpha + 1$ -allowable over W . For $\beta < \delta_1$, this follows immediately from the fact that \mathbb{P}_1 is $\alpha + 1$ -allowable.

For $\beta \in [\delta_1, \delta_1 + \delta_2)$, we assume by induction hypothesis that $\mathbb{P}_{\delta_1 + \beta}$ is $\alpha + 1$ -allowable over W , and want to see that also $\mathbb{P}_{\delta_1 + \beta} * \mathbb{P}(\delta_1 + \beta + 1)$ is $\alpha + 1$ -allowable.

Let us work over the model $W[\mathbb{P}_{\delta_1}^1][\mathbb{P}_\beta^2]$. Assume that $F(\delta_1 + \beta + 1)_0 = F_2(\beta + 1)_0 = (\dot{x}, m, k)$, let $x = \dot{x}^{G_\beta}$, and assume that, when defining $\mathbb{P}(\delta_1 + \beta)$ over $W[\mathbb{P}_{\delta_1}^1][\mathbb{P}_\beta^2]$ we are in case (a) (i). We shall show that in this situation, we are in case (a) (i) at stage β , when defining $\mathbb{P}^2(\beta)$ over $W[\mathbb{P}_\beta^2]$.

Indeed, as we are in case (a) (i) when defining $\mathbb{P}(\delta_1 + \beta)$ over $W[\mathbb{P}_{\delta_1}^1][\mathbb{P}_\beta^2]$, there is a minimal $\zeta < \alpha + 1$ such that $\forall \mathbb{Q} \in W[\mathbb{P}_{\delta_1}^1][\mathbb{P}_\beta^2]$ it holds that

$W[G\mathbb{P}_{\delta_1}^1][\mathbb{P}_\beta^2] \models \mathbb{Q} \Vdash x \in A_m$. If we assume for a contradiction, that we are not in case (a) (i), at stage β when defining $\mathbb{P}^2(\beta)$ over $W[\mathbb{P}_\beta^2]$, then there is a $\mathbb{R} \in W[\mathbb{P}_\beta^2]$ such that \mathbb{R} is ζ -allowable and $\mathbb{R} \Vdash x \notin A_m$.

But now, by induction hypothesis, $\mathbb{R} \in W[\mathbb{P}_\delta^1][\mathbb{P}_\beta^2]$ is ζ -allowable there as well. Indeed, \mathbb{P}^1 is $\alpha + 1$ -allowable, hence ζ -allowable over W , and so is the iteration $\mathbb{P}_\beta^2 * \mathbb{R}$. By induction hypothesis, $(\mathbb{P}_\beta^2 * \mathbb{R}) \times \mathbb{P}^1$ is ζ -allowable, and so \mathbb{R} is, in $W[\mathbb{P}^1][\mathbb{P}_\beta^2]$, a ζ -allowable forcing which kicks x out of A_m . Hence we can not be in case (a) (i) at stage $\delta_1 + \beta$, when defining \mathbb{P} which is a contradiction.

The dual reasoning yields that if we are in case (a) (ii) at stage β in the definition of \mathbb{P} using F over W , then we must be in case (a) (ii) as well at stage β in the definition of $\mathbb{P}^2(\beta)$ over $W[\mathbb{P}_\beta^2]$.

Last, if we are in case (b) at stage β in the definition of \mathbb{P} using F over $W[\mathbb{P}_{\delta_1+\beta}]$, then we shall show that we are in case (b) as well at stage β in the definition of $\mathbb{P}^2(\beta)$ over $W[\mathbb{P}_\beta^2]$. Under our assumption, for every $\zeta < \alpha + 1$ there are ζ -allowable forcings \mathbb{R}_1^ζ and $\mathbb{R}_2^\zeta \in W[\mathbb{P}_{\delta_1+\beta}]$ such that $\mathbb{R}_1^\zeta \Vdash x \notin A_m$ and $\mathbb{R}_2^\zeta \Vdash x \notin A_k$.

But by induction hypothesis, $\mathbb{P}^1 * \mathbb{R}_i^\zeta$ is ζ -allowable over $W[\mathbb{P}_\beta^2]$, hence these forcings show that we are in case (b) at stage β in the definition of $\mathbb{P}^2(\beta)$ over $W[\mathbb{P}_\beta^2]$.

To summarize, the above shows that if we define the $\alpha + 1$ -allowable forcing \mathbb{P} with F as our bookkeeping function, the outcome will be $\mathbb{P}^1 \times \mathbb{P}^2$, so the latter is indeed $\alpha + 1$ -allowable.

Finally if α is a limit ordinal, then $\mathbb{P}_1 \times \mathbb{P}_2$ will be ξ -allowable for every $\xi < \alpha$, but this implies that $\mathbb{P}_1 \times \mathbb{P}_2$ is α -allowable. \square

Lemma 3.7. *For any α , the set of α -allowable forcings is nonempty.*

Proof. By induction on α . If there are α -allowable forcings, then the rules (a) and (b) above, together with some bookkeeping F will create an $\alpha + 1$ -allowable forcing. For limit ordinals α , an α -allowable forcing always exists, as for any given bookkeeping function F there will be an α -allowable, non-trivial forcing with F as its bookkeeping function. \square

As a direct consequence of the last two observations we obtain that there must be an ordinal α such that for every $\beta > \alpha$, the set of α -allowable forcings must equal the set of β -allowable forcings. Indeed every allowable forcing is an \aleph_1 -sized partial order, thus there are only set-many of them, and the decreasing sequence of α -allowable forcings must eventually stabilize at a set which also must be non-empty.

Definition 3.8. *Let α be the least ordinal such that for every $\beta > \alpha$, the set of α -allowable forcings is equal to the set of β -allowable forcings. We say that some forcing \mathbb{P} is ∞ -allowable if and only if it is α -allowable.*

The set of ∞ -allowable forcings can also be described in the following way. An $\delta < \omega_1$ -length iteration $\mathbb{P} = (\mathbb{P}_\alpha : \alpha < \delta)$ is ∞ -allowable if it is recursively constructed following a bookkeeping function F and two rules at every stage $\beta < \delta$ of the iteration:

- (a) If the first coordinate of $F(\beta)$, $(F(\beta))_0 = (\dot{x}, m, k)$, where \dot{x} is the \mathbb{P}_β -name of a real. Further we assume that $\dot{x}^{G_\beta} = x$, for G_β a \mathbb{P}_β -generic filter over W and $W[G_\beta] \models x \in A_m \cup A_k$. We assume that in $W[G_\beta]$, the following is true:

There is an ordinal ζ , which is chosen to be minimal for which

- (i) in the universe $W[G_\beta]$, the following holds:

$$\forall \mathbb{Q} (\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow \mathbb{Q} \Vdash x \in A_m)$$

- (ii) Or if (a) (i) is not true, but it holds in $W[G_\beta]$ that

$$\forall \mathbb{Q} (\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow \mathbb{Q} \Vdash x \in A_k)$$

We give case (a) (i) preference over (a) (ii) if both are true for the minimal ζ . If this is the case, then we define the β -th factor of our iteration as $\mathbb{P}(\beta) := \mathbb{P}_{(x,m,k),2}$ if (a) (i) is true. We let $\mathbb{P}(\beta) := \mathbb{P}_{(x,m,k),1}$ if case (a) (ii) is true.

- (b) Otherwise, we let $F(\beta)_1 \in \{1,2\}$ decide which \vec{S}^i -sequence to use and define $\mathbb{P}(\beta) := \mathbb{P}_{(x,m,k),F(\beta)_1}$.

For every ω_1 -length iteration following some F and the rules above we can compute the supremum of the α 's which appear in the cases (a) of the definition of the iteration. As there are only set many such iterations, there will be an ordinal α_0 such that we can replace item (a) in the definition with the stronger (a') below and still end up with exactly the same set of forcings.

- (a') We assume that $F(\beta) = (\dot{x}, m, k)$ and $\dot{x}^{G_\beta} = x$ $W[G_\beta] \models x \in A_m \cup A_k$. Assume that the allowable forcing \mathbb{P}_β has already been defined. We demand that there is an ordinal $\zeta < \alpha_0$ which is chosen to be minimal for which the following is true:

- (i) we have that in $W[G_\beta]$:

$$\forall \mathbb{Q} (\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow \mathbb{Q} \Vdash x \in A_m),$$

- (ii) or if (a') (i) is not true but it holds that in $W[G_\beta]$:

$$\forall \mathbb{Q} (\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow \mathbb{Q} \Vdash x \in A_k)$$

This α_0 is exactly the ordinal where the notion of α -allowable starts to stabilize.

The next Lemma follows immediately from the definitions of ∞ -allowable and tells us, when an iteration results in an ∞ -allowable notion of forcing.

Lemma 3.9. *Let $(\mathbb{P}_\beta : \beta < \delta < \omega_1)$ be an ∞ -allowable forcing over W . Let $(\mathbb{Q}_\beta \mid \beta < \delta') \in W[G_\delta]$ be such that $W[G_\delta] \models (\mathbb{Q}_\beta \mid \beta < \delta')$ is ∞ -allowable. Then $(\mathbb{P}_\beta : \beta < \delta < \omega_1) * (\mathbb{Q}_\beta \mid \beta < \delta')$ is ∞ -allowable, over W .*

As a consequence, blocks of ∞ -allowable iterations $(\mathbb{P}_\beta^i \mid \beta < \delta^i)$, $i < \eta < \omega_1$ can be concatenated to one ∞ -allowable forcing over W . This will be used to see that the upcoming iteration is indeed an ∞ -allowable iteration over W .

3.3 Definition of the universe in which the Π_3^1 reduction property holds

The notion of ∞ -allowable will be used now to define the universe in which the Π_3^1 -reduction property is true. We let W be our ground model and start an ω_1 -length iteration of ∞ -allowable forcings, whose initial segments are all ∞ -allowable themselves, using four rules and some bookkeeping F .

1. We assume that we are at stage $\beta < \omega_1$, the ∞ -allowable forcing \mathbb{P}_β has been defined, $F(\beta) = (\dot{x}, m, k)$ and $\dot{x}^{G_\beta} = x$ and in $W[G_\beta]$, $x \in A_m \cup A_k$. If in $W[G_\beta]$, there is a minimal $\zeta < \alpha_0$ such that

- (i) $W[G_\beta] \models \forall \mathbb{Q} \in W[G_\beta] (\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow \mathbb{Q} \Vdash x \in A_m)$, then force with $\mathbb{P}(\beta) := \mathbb{P}_{(x, m, k), 2}$.

Note that this has as a direct consequence, that if we restrict ourselves from now on to forcings $\mathbb{Q} \in W[G_{\beta+1}]$ such that \mathbb{Q} is ζ -allowable, then x will remain an element of A_m . In particular, the pathological situation that $x \notin A_m$, $x \in A_k$ while x is coded into \vec{S}^2 is ruled out for (x, m, k) .

- (ii) If we can kick x out of A_m with a ζ -allowable forcing over $W[G_\beta]$, yet it is true that

$$W[G_\beta] \models \forall \mathbb{Q} \in W[G_\beta] (\mathbb{Q} \text{ is } \zeta\text{-allowable} \rightarrow x \in A_k)$$

then force with $\mathbb{P}(\beta) := \mathbb{P}_{(x, m, k), 1}$.

2. If $F(\beta) = (x, m, k)$ and $W[G_\beta] \models x \in A_m \cap A_k$ and neither case 1 (i) nor 1 (ii) applies, then we obtain that

$$W[G_\beta] \models \exists \mathbb{Q} (\mathbb{Q} \text{ is } \infty\text{-allowable}) \text{ and } \mathbb{Q} \Vdash x \notin A_m).$$

With the same argument we also obtain that

$$W[G_\beta] \models \exists \mathbb{R} (\mathbb{R} \text{ is } \infty\text{-allowable}) \text{ and } \mathbb{R} \Vdash x \notin A_k).$$

In this situation, we let \mathbb{Q} and \mathbb{R} the $<$ -least ∞ -allowable forcings as above and use

$$\mathbb{P}(\beta) := \mathbb{Q} \times \mathbb{R}$$

which is an ∞ -allowable forcing over $W[G_\beta]$ and which forces that $x \notin A_m \cup A_k$.

This ends the definition of the iteration and we shall show that the resulting universe $W[G_{\omega_1}]$ satisfies the Π_3^1 -reduction property. For every pair $(m, k) \in \omega^2$, we define

$$D_{m,k}^1 := \{x \in 2^\omega : (x, m, k) \text{ is coded into the } \vec{S}^2\text{-sequence}\}$$

and

$$D_{m,k}^2 := \{x \in 2^\omega : (x, m, k) \text{ is coded into the } \vec{S}^1\text{-sequence}\}$$

Our goal is to show that for every pair (m, k) the sets $D_{m,k}^1 \cap A_m$ and $D_{m,k}^2 \cap A_k$ reduce the pair of Π_3^1 -sets A_m and A_k .

Lemma 3.10. *In $W[G_{\omega_1}]$, for every pair (m, k) , $m, k \in \omega$ and corresponding Π_3^1 -sets A_m and A_k :*

- (a) $D_{m,k}^1 \cap A_m$ and $D_{m,k}^2 \cap A_k$ are disjoint.
- (b) $(D_{m,k}^1 \cap A_m) \cup (D_{m,k}^2 \cap A_k) = A_m \cup A_k$.
- (c) $D_{m,k}^1 \cap A_m$ and $D_{m,k}^2 \cap A_k$ are Π_3^1 -definable.

Proof. We prove (a) first. If x is an arbitrary real in $A_m \cap A_k$ there will be a stage β , such that $F(\beta) = (x, m, k)$. As $x \in A_m \cap A_k$, we know that case 1 (i) or 1 (ii) must have applied. We argue for case 1 (i) as case (ii) is similar. In case 1 (i), $\mathbb{P}_{(x,m,k),2}$ does code (x, m, k) into \vec{S}^2 , while ensuring that for all future ∞ -allowable extensions, x will remain an element of A_m . Thus $x \in D_{m,k}^1 \cap A_m$. The rules of the iteration ensure however that (x, m, k) will never be coded into \vec{S}^1 . Thus $x \notin D_{m,k}^2$ and $D_{m,k}^1 \cap A_m$ and $D_{m,k}^2 \cap A_k$ are disjoint.

To prove (b), let x be an arbitrary element of $A_m \cup A_k$. Let β be the stage of the iteration where the triple (x, m, k) is considered first. As $x \in A_m \cup A_k$, either case 1 (i) or (ii) were applied at stage γ . Assume first that it was case 1 (i). Then, as argued above, $x \in D_{m,k}^1 \cap A_m$. If at stage γ case (ii) applied, then $x \in D_{m,k}^2 \cap A_k$ and we are finished.

To prove (c), we claim that $D_{m,k}^1$ has uniformly the following Π_3^1 -definition over $W[G_{\omega_1}]$:

$$x \in D_{m,k}^1 \cap A_m \Leftrightarrow x \in A_m \wedge \neg(\exists r((**)(x, r) \text{ does hold in } M))$$

Note that the right hand side is the conjunction of two Π_3^1 -formulas, so Π_3^1 as desired. To show the claim, it is sufficient to show that if $x \in A_m$ then

$x \in D_{m,k}^1$, i.e. (x, m, k) being coded into the \vec{S}^2 -sequence is equivalent to (x, m, k) is not coded into the \vec{S}^1 -sequence. But this follows again from the way we defined our iteration. If $x \in A_m$, then if β is some stage such that (x, m, k) is considered by the bookkeeping function, then we must have always applied either case 1 (i) or (ii) and the choice of either (i) or (ii) is constant throughout all of the iteration. Thus if $x \in A_m$ either x is coded into the \vec{S}^1 -sequence or the \vec{S}^2 -sequence. Consequentially, if $x \in A_m$, then (x, m, k) not being coded into the \vec{S}^1 -sequence is equivalent to (x, m, k) being coded into the \vec{S}^2 -sequence. \square

3.4 A Π_3^1 -set which can not be uniformized by a Π_3^1 -function

The next observations will finish the proof of our main result, namely that in $W[G_{\omega_1}]$, there is a Π_3^1 -set which can not be uniformized by a Π_3^1 function. We first fix some assumptions which will help us organizing the proof. First note that, using a homeomorphism of $2^\omega \times 2^\omega$ and 2^ω we know that the Π_3^1 -reduction property for sets in the plane holds. In particular, for a pair of Π_3^1 -sets A_m and A_k in the plane, we obtain the reducing set

$$D_{m,k}^1 = \{(x, y) : (x, y, m, k) \text{ is not coded somewhere in } \vec{S}^1\}$$

and

$$D_{m,k}^2 = \{(x, y) : (x, y, m, k) \text{ is not coded somewhere in } \vec{S}^2\}.$$

Next we note that we can change the definition in $\alpha + 1$ -allowable such that we switch the order of visiting case a (i) and case a (ii), i.e whenever we hit a stage β such that $F(\beta)_0 = (x, m, k)$ and case (a) (i) and case (a) (ii) applies, we give case (b) preference (instead of case (a)), as we did in the definition of $\alpha + 1$ -allowable). This has no effect on the structure of the proof, but allows for a simpler argument in the upcoming. We assume next, without loss of generality, that in our list of Π_3^1 -formulas, the first formula φ_0 has the following form:

$$\varphi_0(x, y) \equiv \forall x_1 \exists x_2 \forall x_3 (x_1 = x_1 \wedge x_2 = x_2 \wedge x_3 = x_3 \wedge x = x \wedge y = y).$$

It is clear that there is no allowable, indeed no forcing at all which kicks pair (x, y) out of A_0 . As a consequence, whenever we start to run our iteration to produce $W[G_{\omega_1}]$, and we hit a stage β such that $F(\beta)_0 = (x, 0, k)$, then we will find ourselves in case 1. This means that at stage β we will force with $\mathbb{P}_{(x,0,k,1)}$. In particular this means that graphs of Π_3^1 -functions are in fact Σ_3^1 as well, by the next lemma.

Lemma 3.11. *In $W[G_{\omega_1}]$, if A_m is the Π_3^1 -set of the graph of a (possibly partial) function f_m , then the complement $(2^\omega \times 2^\omega) \setminus A_m$ is Π_3^1 as well.*

Proof. Note that as $A_0(x, y)$ is, by our assumption, the full plane, whenever A_m is the graph of a function, and we are at a stage β in our iteration such that $F(\beta)_0 = (x, y, 0, m)$ and $(x, y) \in A_m$, then we must use $\mathbb{P}_{(x, 0, k, 1)}$. In particular, if we look at the sets $D_{0, m}^1$ and $D_{0, m}^2$ which reduce A_0 and A_m we find that

$$D_{m, k}^1 = \{(x, y) : (x, y, 0, m) \text{ is not coded in } \vec{S}^1\} = (2^\omega \times 2^\omega) \setminus A_m$$

and

$$D_{m, k}^2 = \{(x, y) : (x, y, 0, m) \text{ is not coded in } \vec{S}^2\} = A_m$$

Thus the complement of the graph of f_m is Π_3^1 , hence A_m is also Σ_3^1 and so Δ_3^1 . \square

Theorem 3.12. *In $W[G_{\omega_1}]$ the Π_3^1 -uniformization property is wrong.*

Proof. We use a recursive bijection

$$h : \omega \times 2^\omega \rightarrow 2^\omega$$

to partition 2^ω into ω many pairwise disjoint sets. We let $U(n, x, y)$ denote a universal Π_3^1 -set in the plane, i.e. a set which satisfies that for any Π_3^1 set B , there is an $n \in \omega$, such that $(x, y) \in B \Leftrightarrow U(n, x, y)$ holds true. Then we define a set A in the plane as follows:

$$(x, y) \in A \Leftrightarrow \forall n \forall \bar{x} (h(n, \bar{x}) = x \rightarrow U(n, x, y))$$

Note that A is Π_3^1 .

We claim that A can not be uniformized by a Π_3^1 -function. To see this, let f_m be an arbitrary Π_3^1 -function, whose graph is $A_m(x, y)$. Using our last lemma, we know that $(2^\omega \times 2^\omega) \setminus A_m$ is Π_3^1 as well and we let $k \in \omega$ be such that

$$A_k = (2^\omega \times 2^\omega) \setminus A_m$$

Now we fix a pair (x, y) such that there is a \bar{x} which satisfies $x = h(k, \bar{x})$. Then we claim that the graph of f_m will not intersect A on the k -th part of the partition induced by h . Indeed, as $x = h(k, \bar{x})$, if we assume that $(x, f_m(x)) \in A$, then

$$\begin{aligned} (x, f_m(x)) &\in A \Leftrightarrow \\ (k, x, f_m(x)) &\in U \Leftrightarrow \\ (x, f_m(x)) &\in A_k \end{aligned}$$

but $(x, f_m(x)) \in A_k$ is wrong as A_k is the complement of the graph of f_m . Thus, given an arbitrary f_m , we can find an x such that $(x, f_m(x)) \notin A$, yet the x -section of A is nonempty, as for every $y \neq f_m(x)$, we have that $(x, y) \in A_k$, and, as $x = h(k, \bar{x})$, $(x, y) \in A$. \square

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