Forcing the Π^1_n-Uniformization Property

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Abstract

We generically construct a model in which the Π^1_3-uniformization property is true, thus lowering the best known consistency strength from the existence of $M_1^{\#}$ to just ZFC. The forcing construction can be adapted to work over canonical inner models with Woodin cardinals, which yields, for the first time, universes where the Π^1_{2n}-uniformization property holds, thus producing models which contradict the natural PD-induced pattern. It can also be used to obtain models for the Π^1_1-uniformization property in the generalized Baire space.

1 Introduction

The question of finding nicely definable choice functions for a definable family of sets is an old and well-studied subject in descriptive set theory. The uniformization problem, first mentioned by N. Lusin in 1930 (see [15]), asks to find choice functions which lie at the same projective level as the set they aim to uniformize. Recall that for an $A \subset 2^\omega \times 2^\omega$, we say that f is a uniformization (or a uniformizing function) of A if there is a function $f : 2^\omega \to 2^\omega$, $dom(f) = pr_1(A)$ and the graph of f is a subset of A.

Definition 1.1. *We say that a pointclass Γ has the uniformization property iff every element of Γ admits a uniformization in Γ.*

It is a classical result due to M. Kondo that lightface Π^1_1-sets do have the uniformization property, this also yields the uniformization property for Σ^1_2-sets. This is as much as ZFC can prove about uniformization. In the constructible universe L, as shown by J. Addison in [2], Σ^1_n does have the uniformization property for $n \geq 3$, which follows from the existence of a Σ^1_2-good wellorder of the reals, thus the Π^1_n-uniformization fails for $n \geq 3$. On the other hand, by the celebrated results of Y. Moschovakis (see [20],

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Theorem 1), $\mathbf{\Delta^1_{2n}}$-projective determinacy implies Π^1_{2n+1}-uniformization, yet the determinacy assumption exceeds in logical stength ZFC. It is known due to H. W. Woodin (see [18],), that $\mathbf{\Delta^1_2}$-projective determinacy implies the existence of $M_1^{\#}$, hence yields an inner model with a Woodin cardinal. As with other regularity properties of the reals like Lebesgue measurability or Baire property, which both follow from PD as well, it is natural to ask whether the Π^1_3-uniformization property bears large cardinal strength as well. We shall answer it negatively.

Theorem. *There is a generic extension of L in which the Π^1_3-uniformization property is true.*

The proof can be adapted such that it applies to canonical inner models with Woodin cardinals. This can be used to obtain better lower bounds in terms of consistency strength for the Π^1_n uniformization property for odd n. For even n we can produce for the first time models where the Π^1_n uniformization property holds true.

Theorem. *Let M_n be the canonical inner model with n Woodin cardinals. Then there is a generic extension of M_n in which the Π^1_{n+3} uniformization property holds true.*

Questions concerning the forcability of (local) consequences of PD do have a long tradition in set theory. There is a vast body of literature concerning the forcability of local levels of the projective hierarchy satisfying (Boolean combinations of) the Baire property, the perfect set property or Lebesgue measurability. There has been very little progress in the past, however, concerning similar questions for the separation, the reduction and the uniformization property. Indeed, even the question of whether one can force the Σ^1_3-separation property, which is the weakest of said properties, remained an open problem for 50 years and was considered very difficult (see [16], Problem 3029, or [13]), until it was solved recently in [10].

This article continues this line of research and provides a natural endpoint to the work which started with [10]. It is organized as follows: in the preliminaries section, we briefly introduce the forcings which we will use in the proof and produce a generic extension W of L which will be a well-suited ground model for our needs. We then start to prove the theorems from above. The main idea is to turn the problem of finding a partial order $\mathbb{P}$ which forces the Π^1_3-property into a fixed point problem. We shall define a derivation operator which acts on a specific set of ccc iterations of length ω_1. This operator will be applied transfinitely often, and will produce better and better approximations to the set of forcings we actually want to use in the end. The process is shown to converge in that eventually a fixed point, i.e. a suitable set of forcings is reached. Forcings which belong to this fixed point allow for a certain, seemingly self-referential line of argumentation which can be used to show that there is indeed a forcing, consisting of factors entirely from this fixed point, which forces the Π^1_3-uniformization property over W. We then follow up, to alter the said process such that it becomes applicable to the canonical inner models M_n with n Woodin cardinals.

There are some similarities to [10], in particular the two proofs rely on a similar ground model W, which is a generic extension of L, and use a similar coding method which relies on a suitably chosen ω_1-sequence of ω_1-Suslin trees.. However a more straightforward application of the ideas of [10] will fail badly to produce a model of the Π^1_3-uniformization property. As a consequence, a solution has to necessarily introduce several new ideas in order to succeed. The presentation of those is the goal of this paper.

2 Preliminaries

2.1 Notation

The notation we use will be mostly standard, we hope. We write $\mathbb{P} = (\mathbb{P}_\alpha : \alpha < \gamma)$ for a forcing iteration of length γ with initial segments $\mathbb{P}_\alpha$. The α-th factor of the iteration will be denoted with $\mathbb{P}(\alpha)$. Note here that we drop the dot on $\mathbb{P}(\alpha)$, even though $\mathbb{P}(\alpha)$ is in fact a $\mathbb{P}_\alpha$-name of a partial order. If $\alpha' < \alpha < \gamma$, then we write $\mathbb{P}_{\alpha'\alpha}$ to denote the intermediate forcing of $\mathbb{P}$ which happens in the interval $[\alpha', \alpha)$, i.e. $\mathbb{P}_{\alpha'\alpha}$ is such that $\mathbb{P} \cong \mathbb{P}_{\alpha'} * \mathbb{P}_{\alpha'\alpha}$.

We write $\Sigma_n(X)$, for X an arbitrary set, to denote the set of formulas which are Σ_n and use X as a parameter.

We write $\mathbb{P} \Vdash \varphi$ whenever every condition in $\mathbb{P}$ forces φ, and make deliberate use of restricting partial orders below conditions, that is, if $p \in \mathbb{P}$ is such that $p \Vdash \varphi$, we let $\mathbb{P}' := \mathbb{P}_{\leq p} := \{q \in \mathbb{P} : q \leq p\}$ and use $\mathbb{P}'$ instead of $\mathbb{P}$. This is supposed to reduce the notational load of some definitions and arguments. We also sometimes write $V[\mathbb{P}] \models \varphi$ to indicate that for every $\mathbb{P}$-generic filter G over V, $V[G] \models \varphi$, and use $V[\mathbb{P}]$ to denote the generic extension of V by $\mathbb{P}$ in case the particular choice of the generic filter does not matter in the current context.

2.2 The forcings which are used

The forcings which we will use in the construction are all well-known. We nevertheless briefly introduce them and their main properties.

Definition 2.1. *(see [3]) For a stationary $R \subset \omega_1$ the club-shooting forcing for R, denoted by $\mathbb{P}_R$ consists of conditions p which are countable functions from $\alpha + 1 < \omega_1$ to R which are increasing and continuous. $\mathbb{P}_R$ is ordered by end-extension.*

The club shooting forcing $\mathbb{P}_R$ is the paradigmatic example for an *R-proper forcing*, where we say that $\mathbb{P}$ is R-proper if and only if for every condition

$p \in \mathbb{P}$, every $\theta > 2^{|\mathbb{P}|}$ (we will utilize the common jargon and say in that situation that θ is sufficiently large) and every countable $M \prec H(\theta)$ such that $M \cap \omega_1 \in R$ and $p, \mathbb{P} \in M$, there is a $q < p$ which is $(M, \mathbb{P})$-generic; and a condition $q \in \mathbb{P}$ is said to be $(M, \mathbb{P})$-generic if $q \Vdash$ "$\dot{G} \cap M$ is an M-generic filter", for $\dot{G}$ the canonical name for the generic filter. See also [8].

Lemma 2.2. *Let $R \subset \omega_1$ be stationary, co-stationary. Then the club-shooting forcing $\mathbb{P}_R$ generically adds a club through the stationary set $R \subset \omega_1$. Additionally $\mathbb{P}_R$ is R-proper, ω-distributive and hence ω_1-preserving. Moreover R and all its stationary subsets remain stationary in the generic extension.*

Proof. Though the arguments are well-known, we shall show the ω-distributivity of $\mathbb{P}_R$ and the preservation of stationary subsets of R, the rest can be found in [8], Fact 3.5, 3.6 and Theorem 3.7.

We start with a proof of ω-distributivity first. Let $p \in \mathbb{P}_R$ and $\dot{x}$ be such that $p \Vdash \dot{x} \in 2^\omega$. Without loss of generality we assume that $\dot{x}$ is a nice name for a real, i.e. given by an ω-sequence of $\mathbb{P}_R$-maximal antichains. We shall find a real x in the ground model and a condition $q < p$ such that $q \Vdash \dot{x} = x$. For this, fix $\theta > 2^{|\mathbb{P}|}$ and a countable elementary submodel $M \prec H(\theta)$ which contains $\mathbb{P}$, $\dot{x}$ and p as elements and which additionally satisfies that $M \cap \omega_1 \in R$. Note that we can always assume that such an M exists by the stationarity of R. We recursively construct a descending sequence $(p_n)_{n\in\omega} \subset M$ of conditions below $p = p_0$ such that every p_n decides the value of $\dot{x}(n)$ and such that both sequences $(dom(p_n))_{n\in\omega}$ and $(max\, range(p_n))_{n\in\omega}$ converge to $M \cap \omega_1$. We let $x(n) \in 2$ be the value of $\dot{x}$ as forced by p_n, and let $x = (x(n))_{n\in\omega} \in 2^\omega \cap V$.

Let $q' = \bigcup_{n\in\omega} p_n \subset (M \cap \omega_1)$. We set $q := q' \cup \{((M \cap \omega_1), (M \cap \omega_1))\}$, which is a function from $(M \cap \omega_1) + 1$ to R with closed image, and hence a condition in $\mathbb{P}_R$ which forces that $\dot{x} = x$ as desired. Thus $\mathbb{P}_R$ is ω-distributive.

We shall now show the second assertion, namely that whenever $A \subset R$ is stationary, then A remains stationary in the generic extension by $\mathbb{P}_R$ which will be very similar to the previous argument. Let $p \in \mathbb{P}_R$ be an arbitrary condition, $\dot{C}$ be such that $p \Vdash$ "$\dot{C}$ is a club in ω_1". We aim to find a $q < p$ and an $\alpha \in A$ such that $q \Vdash \alpha \in \dot{C} \cap A$.

Let $\theta > 2^{\omega_1}$ be a regular cardinal and let $M \prec H(\theta)$ be such that $\{p, \mathbb{P}_R, \dot{C}\} \subset M$ and $M \cap \omega_1 \in A$. Starting with $p_0 := p$, we can define a decreasing sequence of conditions $p_n \in \mathbb{P}_R \cap M$ and a sequence of ordinals $\alpha_n \in M \cap \omega_1$ such that for every $n \in \omega$, $p_n \Vdash \alpha_n \in \dot{C}$. Using the elementarity of M we can also demand that the sequence of the α_n's is unbounded in $M \cap \omega_1$. Arguing exactly as above, we can infer that $(p_n : n \in \omega)$ has a lower bound q, hence $q < p$ and as $p \Vdash$ "$\dot{C}$ is club", we also have $q \Vdash sup(\alpha_n) \in \dot{C}$. So $q \Vdash M \cap \omega_1 \in (\dot{C} \cap A)$, and A remains a stationary set after forcing with $\mathbb{P}_R$.

$\square$

Once we decide to shoot a club through a stationary, co-stationary subset of ω_1, this club will belong to all ω_1-preserving outer models. Using an antichain $R = (R_\alpha : \alpha < \omega_1)$ in the Boolean algebra $P(\omega_1)/\mathrm{NS}_{\omega_1}$, the club shooting forcing thus becomes a tool of coding up arbitrary $\aleph_1$-sized information relative to R. The following method is well-known and has been used already several times (see e.g. [6]).

Lemma 2.3. *Let $(R_\alpha : \alpha < \omega_1)$ be a partition of ω_1 into $\aleph_1$-many stationary sets. Let $r \in 2^{\omega_1}$ be arbitrary, and set*

$$X := \bigcup\{R_{2\alpha} : \alpha \textit{ such that } r(\alpha) = 1\} \cup \bigcup\{R_{2\alpha+1} : \alpha \textit{ such that } r(\alpha) = 0\}$$

and Y the complement of X which is

$$Y := \bigcup\{R_{2\alpha+1} : \alpha \textit{ such that } r(\alpha) = 1\} \cup \bigcup\{R_{2\alpha} : \alpha \textit{ such that } r(\alpha) = 0\}.$$

Then forcing with $\mathbb{P}_Y$ will create a universe where the information r is coded into $(R_\alpha : \alpha < \omega_1)$ in the following way: in $V[\mathbb{P}_Y]$ it holds that $\forall \alpha < \omega_1$:

$$r(\alpha) = 1 \textit{ if and only if } R_{2\cdot\alpha} \textit{ is nonstationary,}$$

and

$$r(\alpha) = 0 \textit{ iff } R_{(2\cdot\alpha)+1} \textit{ is nonstationary.}$$

Proof. Forcing with $\mathbb{P}_Y$ will join a club to Y, so every stationary subset of $X = \omega_1 \backslash Y$ becomes nonstationary and as a consequence we get that if $r(\alpha) = 1$, then $R_{2\alpha}$ is nonstationary and that if $r(\alpha) = 0$, then $R_{2\alpha+1}$ is nonstationary.

On the other hand, if $R_{2\alpha}$ is nonstationary then $r(\alpha)$ can not be 0, as otherwise the stationarity of $R_{2\alpha} \subset Y$ would be preserved by the last Lemma. The same line of reasoning also shows that if $R_{2\alpha+1}$ is nonstationary, then $r(\alpha)$ can not be 1, which ends the proof.

□

The second forcing we use is the almost disjoint coding forcing due to R. Jensen and R. Solovay. We will identify subsets of ω with their characteristic function and will use the word reals for elements of 2^ω and subsets of ω respectively. Let $D = \{d_\alpha \;\; \alpha < \aleph_1\}$ be a family of almost disjoint subsets of ω, i.e. a family such that if $r, s \in D$ then $r \cap s$ is finite. Let $X \subset \kappa$ for $\kappa \leq 2^{\aleph_0}$ be a set of ordinals. Then there is a ccc forcing, the almost disjoint coding $\mathbb{A}_D(X)$ which adds a new real x which codes X relative to the family D in the following way

$$\alpha \in X \text{ if and only if } x \cap d_\alpha \text{ is finite.}$$

Definition 2.4. *The almost disjoint coding $\mathbb{A}_D(X)$ relative to an almost disjoint family D consists of conditions $(r, R) \in [\omega]^{<\omega} \times D^{<\omega}$ and $(s, S) < (r, R)$ holds if and only if*

1. $r \subset s$ *and* $R \subset S$.

2. If $\alpha \in X$ *and* $d_\alpha \in R$ *then* $r \cap d_\alpha = s \cap d_\alpha$.

We shall briefly discuss the L-definable, $\aleph_1^L$-sized almost disjoint family of reals D we will use throughout this article. The family D is the canonical almost disjoint family one obtains when recursively adding the $<_L$-least $d_\beta \subset \omega$ such that d_β is almost disjoint from all the previous d_α, $\alpha < \beta$.

The last two forcings we briefly discuss are Jech's forcing for adding a Suslin tree with countable conditions and, given a Suslin tree T, the associated forcing which adds a cofinal branch through T. Recall that a set theoretic tree $(T, <)$ is a Suslin tree if it is a normal tree of height ω_1 and has no uncountable antichain. As a result, forcing with a Suslin tree S, where conditions are just nodes in S, and which we always denote with S again, is a ccc forcing of size $\aleph_1$. Jech's forcing to generically add a Suslin tree is defined as follows.

Definition 2.5. *Let* $\mathbb{J}$ *be the forcing whose conditions are countable, normal trees ordered by end-extension, i.e.* $T_1 < T_2$ *if and only if* $\exists \alpha < height(T_1)\, T_2 = \{t \upharpoonright \alpha \, : \, t \in T_1\}$

It is wellknown that $\mathbb{J}$ is σ-closed and adds a Suslin tree. In fact more is true, the generically added tree T has the additional property that for any Suslin tree S in the ground model $S \times T$ will be a Suslin tree in $V[G]$. This can be used to obtain a robust coding method (see also [9] for more applications)

Lemma 2.6. *Let* V *be a universe and let* $S \in V$ *be a Suslin tree. Let* $\mathbb{J} \in V$ *be Jech's forcing for adding a Suslin tree and let* G *be* $\mathbb{J}$*-generic over* V *and assume that* $T = \bigcup_{p \in G} p$ *is the generic tree. Then forcing with* $T \in V[G]$ *does preserve* S*, i.e. if* H *is* T*-generic over* $W[G]$ *we have that*

$$V[G][H] \models S \text{ is Suslin.}$$

Proof. Let $\dot{T}$ be the $\mathbb{J}$-name for the generic Suslin tree. We claim that $\mathbb{J} * \dot{T}$ has a dense subset which is σ-closed. As σ-closed forcings will always preserve ground model Suslin trees, this is sufficient. To see why the claim is true consider the following set:

$$\{(p, \check{q}) \, : \, p \in \mathbb{J} \land height(p) = \alpha + 1 \land \check{q} \text{ is a node of } p \text{ of level } \alpha\}.$$

It is easy to check that this set is dense and σ-closed in $\mathbb{J} * \dot{T}$.

□

A similar observation shows that a we can add an ω_1-sequence of such Suslin trees with a countably supported iteration.

Lemma 2.7. *Let S be a Suslin tree in V and let $\mathbb{P}$ be a countably supported product of length ω_1 of forcings $\mathbb{J}$ with G its generic filter. Then in $V[G]$ there is an ω_1-sequence of Suslin trees $\vec{T} = (T_\alpha : \alpha \in \omega_1)$ such that for any finite $e \subset \omega$ the tree $S \times \prod_{i \in e} T_i$ will be a Suslin tree in $V[G]$.*

These sequences of Suslin trees will be used for coding in our proof and get a name.

Definition 2.8. *Let $\vec{T} = (T_\alpha : \alpha < \kappa)$ be a sequence of Suslin trees. We say that the sequence is an independent family of Suslin trees if for every finite set $e = \{e_0, e_1, ..., e_n\} \subset \kappa$, the product $T_{e_0} \times T_{e_1} \times \cdots \times T_{e_n}$ is a Suslin tree again, provided the e_i's are pairwise different.*

We will use the following preservation result due to Miyamoto (see [17])

Theorem 2.9. *Let $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta)$ be a countable support iteration of proper forcings, let S be Suslin tree and assume that for every $\beta < \delta$, $\mathbb{P}_\beta \Vdash$ "$\mathbb{P}(\beta)$ preserves S as a Suslin tree." Then S remains a Suslin tree in the generic extension by $\mathbb{P}$.*

2.3 The ground model W of the iteration

We have to first create a suitable ground model W over which the actual iteration will take place. W will be a generic extension of L, satisfying CH and has the crucial property that in W there is an ω_1-sequence $\vec{S}$ of ω_1 trees which are an independent sequence of Suslin trees in the inner model $L[\vec{S}] \subset W$ and is $\Sigma_1(\omega_1)$-definable over $H(\omega_2)^W$. The sequence $\vec{S}$ will enable a coding method which is to some extent not depending on the surrounding universe, a feature we will exploit to a great extent in the upcoming.

In short, we will construct W in three steps. In the first step we generically add ω_1-many Suslin trees denoted by $\vec{S}$. In the second step we subsequently destroy all trees $\vec{S}$ via adding a cofinal ω_1-branch through every element of $\vec{S}$. In a third step we use a club adding forcing, which will make the sequence $\vec{S}$ $\Sigma_1(\omega_1)$-definable over the resulting universe. We will later use a coding forcing over W, which will code up some well-chosen ω_1-branches through $\vec{S}$ using almost disjoint coding forcing.

Turning to the detailed definition of W, we start with Gödels constructible universe L as our ground model. Recall that L comes equipped with a Σ_1-definable, global well-order $<_L$ of its elements. We first fix an appropriate sequence of stationary, co-stationary subsets of ω_1 using Jensen's $\diamondsuit$-sequence.

Fact 2.10. *In L there is a sequence $(a_\alpha : \alpha < \omega_1)$ of countable subsets of ω_1 such that any set $A \subset \omega_1$ is guessed stationarily often by the a_α's, i.e. $\{\alpha < \omega_1 : a_\alpha = A \cap \alpha\}$ is a stationary subset of ω_1. The sequence $(a_\alpha : \alpha < \omega_1)$ can be defined in a Σ_1 way over the structure L_{ω_1}.*

Proof. We shall only prove the claim about the Σ_1-definability and follow Jensen's original construction of the $\diamondsuit$-sequence. We define a sequence of pairs (a_α, c_α) by induction on α. If $\alpha = \beta + 1$, then $a_\alpha = c_\alpha = \alpha$. If α is a limit ordinal, then (a_α, c_α) is the $<_L$-least pair such that c_α is a closed and unbounded subset of α, $a_\alpha \subset \alpha$ and such that $a_\alpha \cap \eta \neq a_\eta$ for every $\eta \in c_\alpha$, provided such a pair exists. Otherwise let $a_\alpha = c_\alpha = \alpha$. It is well-known that the a_α's defined this way form a $\diamondsuit$-sequence. We let $\phi(\alpha, x)$ denote the statement: "x is the α-th entry of the $\diamondsuit$-sequence defined as above".

Now it is straightforward to check that L_{ω_1} is sufficient to correctly compute the sequence $((a_\alpha, c_\alpha) : \alpha < \omega_1)$ in a Σ_1-way. Indeed L_{ω_1} can correctly compute L_β, for $\beta < \omega_1$ with a Σ_1-formula. The latter structures, provided β is a limit ordinal, are able to define the $<_L$-wellorder up to their respective ordinal height. Thus if the countable L_β, β a limit ordinal, contains $((a_\alpha, c_\alpha : \alpha < \gamma)$, for some $\gamma < \beta$, then L_β will correctly compute (a_γ, c_γ) as $<_L$ and being closed and unbounded in some $\alpha < \beta$ are absolute notions between L_β and L. Consequentially, being a_α is $\Sigma_1(\alpha)$-definable over L_{ω_1}

$$x = a_\alpha \Leftrightarrow \exists \beta (\beta \text{ is a limit ordinal and } L_\beta \models \phi(\alpha, x)$$

and $x \in \{a_\alpha : \alpha < \omega_1\}$ if and only if $\exists \alpha (x = a_\alpha)$, which gives the claim. □

The $\diamondsuit$-sequence can be used to produce an easily definable sequence of L-stationary, co-stationary subsets of ω_1: we list the reals in L in an ω_1 sequence $(r_\alpha : \alpha < \omega_1)$, and let $\tilde{r}_\alpha \subset \omega_1$ be the unique element of 2^{ω_1} which copies r_α on its first ω-entries followed by ω_1-many 0's. Then, identifying $\tilde{r}_\alpha \in 2^{\omega_1}$ with the according subset of ω_1, we define for every $\beta < \omega_1$ a stationary, co-stationary set in the following way:

$$R'_\beta := \{\alpha < \omega_1 \, : \, a_\alpha = \tilde{r}_\beta \cap \alpha\}.$$

That each R'_β is stationary is clear by the definition of the $\diamondsuit$-sequence, it is also co-stationary as $\omega_1 \backslash R'_\beta$ necessarily must contain (modulo a set bounded in ω_1) the stationaryR'_γ, for $\gamma \neq \alpha$. It is clear that $\forall \alpha \neq \beta (R'_\alpha \cap R'_\beta \in \mathrm{NS}_{\omega_1})$ and we obtain a sequence of pairwise disjoint stationary sets as usual via setting for every $\beta < \omega_1$

$$R_\beta := R'_\beta \backslash \omega.$$

and let $\vec{R} = (R_\alpha : \alpha < \omega_1)$. We derive the following standard result

Lemma 2.11. *For any $\beta < \omega_1$, membership in R_β is uniformly Σ_1-definable over the model L_{ω_1}, i.e. there is a Σ_1-formula $\psi(v_0, v_1)$ such that for every $\beta < \omega_1$, $(\alpha \in R_\beta \Leftrightarrow L_{\omega_1} \models \psi(\alpha, \beta))$.*

Proof. First we note that there is a Σ_1-formula $\theta'(\eta, x)$ for which $L_{\omega_1} \models \theta'(\eta, x)$ is true if and only if "x is the η-th real in $<_L$, the canonical L-wellorder". It follows that there is a Σ_1-formula $\theta(\eta, \zeta, x)$ for which $L_{\omega_1} \models$

$\theta(\eta, \zeta, x)$ is true if and only if "x equals $\tilde{r}_\eta \cap \zeta$". Further recall that in the proof of the last lemma we found already a Σ_1-formula, let us denote it with $\varphi(\xi, y)$, such that $L_{\omega_1} \models \varphi(\xi, y)$ holds if and only if "y is the ξ-th element of the canonical $\diamondsuit$-sequence".

Then membership in R'_β can be expressed using the following formula:

$$\alpha \in R'_\beta \Leftrightarrow L_{\omega_1} \models \exists x(\varphi(\alpha, x) \wedge \theta(\beta, \alpha, x))$$

Note here that actually every countable L_γ, for γ a limit ordinal, which models (a sufficiently big fragment of) ZF^- and contains α and β is sufficient to witness membership of α in R'_β using the formula $\exists x(\varphi(\alpha, x) \wedge \theta(\beta, \alpha, x))$.

It follows that membership in R_β allows this representation:

$$\alpha \in R_\beta \Leftrightarrow L_{\omega_1} \models \exists x(\varphi(\alpha, x) \wedge \theta(\beta, \alpha, x)) \wedge \alpha \notin \omega$$

Note that the last formula is Σ_1, thus we found our desired $\psi(v_0, v_1)$. $\square$

We proceed with defining the universe W. Starting with L as the ground model we generically add $\aleph_1$-many Suslin trees using of Jech's Forcing $\mathbb{J} \in L$. We let

$$\mathbb{Q}^0 := \prod_{\beta \in \omega_1} \mathbb{J}$$

using countable support. This is a σ-closed, hence proper notion of forcing. In particular the stationarity of every $R_\alpha \in L$ is preserved. We denote the generic filter of $\mathbb{Q}^0$ with $\vec{S} = (S_\alpha : \alpha < \omega_1)$ and note that by Lemma 2.7 $\vec{S}$ is independent. We fix a definable bijection between $[\omega_1]^\omega$ and ω_1 and identify the trees in $(S_\alpha : \alpha < \omega_1)$ with their images under this bijection, so the trees will always be subsets of ω_1 from now on.

In a second step we destroy all the just added Suslin trees via adding cofinal branches through each $S \in \vec{S}$ using countable support again. That is, if we let S_β also denote the partial order when using the nodes of S_β as conditions, then we define

$$\mathbb{Q}^1 := \prod_{\beta \in \omega_1} S_\beta.$$

We note that we can rearrange the iteration $\mathbb{Q}^0 * \mathbb{Q}^1$ and write it as $\bigstar_{\beta < \omega_1}(\mathbb{J} * S_\beta) = \prod_{\beta < \omega_1}(\mathbb{J} * S_\beta)$, using countable support again. Now by the argument of the proof of Lemma 2.6, each factor $\mathbb{J} * S_\beta$ has a dense subset which is σ-closed. So the two step iteration $\mathbb{Q}^0 * \mathbb{Q}^1$ has itself a dense subset which is σ-closed. In particular $\mathbb{Q}^0 * \mathbb{Q}^1$ does not add any reals and is proper, hence preserves stationary subsets.

In a third step, working in $L[\mathbb{Q}^0][\mathbb{Q}^1]$, we code the trees from $\vec{S}$ into the sequence of L-stationary subsets $\vec{R}$ we produced earlier, using the method introduced in Lemma 2.3. It is important to note, that the forcing we are about to define does preserve Suslin trees, a fact we will show later. The forcing used in the third step will be denoted by $\mathbb{Q}^2$. Fix first a definable bijection $h \in L_{\omega_2}$ between $\omega_1 \times \omega_1$ and ω_1 and write $\vec{R}$ from now on in ordertype $\omega_1 \cdot \omega_1$ making implicit use of h, so we assume that $\vec{R} = (R_\alpha : \alpha < \omega_1 \cdot \omega_1)$.

The third forcing $\mathbb{Q}^2$ is defined over $L[\mathbb{Q}^0][\mathbb{Q}^1]$ as follows. We fix an arbitrary $\alpha < \omega_1$ and let $S_\alpha \subset \omega_1$ be the α-th Suslin tree in $\vec{S}$. Then we fix the α-th ω_1-block of $\vec{R}$ and let

$$\begin{aligned} E_\alpha := & \bigcup \{R_{\omega_1\alpha+2\beta+1} : \beta \text{ such that } S_\alpha(\beta) = 1\} \cup \\ & \bigcup \{R_{\omega_1\alpha+2\beta} : \beta \text{ such that } S_\alpha(\beta) = 0\}. \end{aligned}$$

Then we let

$$E := \bigcup_{\alpha<\omega_1} E_\alpha$$

and define

$$\mathbb{Q}^2 := \mathbb{P}_E$$

i.e. we shoot a club through $E \subset \omega_1$.

This way we can turn the generically added sequence of Suslin trees $\vec{S}$ into a definable sequence of Suslin trees using the ω-distributive forcing $\mathbb{Q}^2 = \mathbb{P}_E$. Indeed, if we work in $L[\vec{S} * \vec{b} * G]$, where $\vec{S} * \vec{b} * G$ is $\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2$-generic over L, then, as seen in Lemma 2.3

$$\begin{aligned} \forall \alpha, \gamma < \omega_1 (& \gamma \in S_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma} \text{ is not stationary and} \\ & \gamma \notin S_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma + 1} \text{ is not stationary}) \end{aligned}$$

Note here that the above formula can be used to make every S_α $\Sigma_1(\omega_1, \alpha)$ definable over $L[\vec{S} * G]$, which in turn yields the following lemma.

Lemma 2.12. *The sequence $\vec{S}$ is $\Sigma_1(\omega_1)$-definable over $L[\vec{S} * G]$.*

Proof. We claim that already $\aleph_1$-sized, transitive models of ZF^- which contain a club through the complement of exactly one element of every pair $\{(R_\alpha, R_{\alpha+1}) : \alpha < \omega_1\}$ are sufficient to compute correctly $\vec{S}$ via the following $\Sigma_1(\omega_1)$-formula:

$$\begin{aligned} \Psi(X, \omega_1) \equiv \exists M (& M \text{ transitive } \wedge M \models \mathsf{ZF}^- \wedge \omega_1 \in M \wedge \\ & M \models \forall \beta < \omega_1 \cdot \omega_1 (\text{either } R_{2\beta} \text{ or } R_{2\beta+1} \text{ is nonstationary}) \wedge \\ & M \models X \text{ is an } \omega_1 \cdot \omega_1\text{-sequence } (X_\alpha)_{\alpha<\omega_1\cdot\omega_1} \text{ of subsets of } \omega_1 \wedge \\ & M \models \forall \alpha, \gamma (\gamma \in X_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma} \text{ is not stationary and} \\ & \qquad \gamma \notin X_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma + 1} \text{ is not stationary}) \end{aligned}$$

We want to show that $X = \vec{S}$ if and only if $\Psi(X, \omega_1)$ is true in $L[\vec{S} * G]$. For the backwards direction, we assume that M is a model and $X \in M$ is a set, as on the right hand side of the above. We shall show that indeed $X = \vec{S}$. As M is transitive and a model of ZF^- it will compute every R_β, $\beta < \omega_1$ correctly by Lemma 2.11. As being nonstationary is a $\Sigma_1(\omega_1)$-statement, and hence upwards absolute, we conclude that if M believes to see a pattern written into (its versions of) the R_β's, this pattern is exactly the same as is seen by the real world $L[\vec{S} * G]$. But we know already that in $L[\vec{S} * G]$, the sequence $\vec{S}$ is written into the R_β's, thus $X = \vec{S}$ follows.

On the other hand, if $X = \vec{S}$, then

$$L[\vec{S} * G] \models \forall \beta < \omega_1 \cdot \omega_1 (\text{either } R_{2\beta} \text{ or } R_{2\beta+1} \text{ is nonstationary})$$
$$L[\vec{S} * G] \models X \text{ is an } \omega_1 \cdot \omega_1\text{-sequence } (X_\alpha)_{\alpha < \omega_1 \cdot \omega_1} \text{ of subsets of } \omega_1$$

and

$$\begin{aligned} L[\vec{S} * G] \models \forall \alpha, \gamma < \omega_1 (\gamma \in X_\alpha \Leftrightarrow & R_{\omega_1 \cdot \alpha + 2 \cdot \gamma} \text{ is not stationary and} \\ \gamma \notin X_\alpha \Leftrightarrow & R_{\omega_1 \cdot \alpha + 2 \cdot \gamma + 1} \text{ is not stationary}) \end{aligned}$$

By reflection, there is an $\aleph_1$-sized, transitive model M which models the assertions above, which gives the direction from left to right. $\square$

Let us set

$$W := L[\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2]$$

which will serve as our ground model for an iteration of length ω_1.

Our goal is to use $\vec{S}$ for coding again. For this it is essential, that the sequence remains independent in $L[\mathbb{Q}^0 * \mathbb{Q}^2]$ (note here that $\mathbb{Q}^1$, i.e. the forcing which destroys each element from $\vec{S}$ is missing here). First note that $\mathbb{Q}^0 * \mathbb{Q}^1 * \mathbb{Q}^2$ is in fact of the form $\mathbb{Q}^0 * (\mathbb{Q}^1 \times \mathbb{Q}^2)$, so considering $\mathbb{Q}^0 * \mathbb{Q}^2$ is reasonable.

To see the preservation of Suslin trees in $L[\mathbb{Q}^0 * \mathbb{Q}^2]$ we shall argue that forcing with $\mathbb{Q}^2$ over $L[\mathbb{Q}^0]$ preserves Suslin trees. The following line of reasoning is similar to arguments in [9]. Recall that for a forcing $\mathbb{P}$, θ sufficiently large and regular and $M \prec H(\theta)$, a condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic iff for every maximal antichain $A \subset \mathbb{P}$, $A \in M$, it is true that $A \cap M$ is predense below q. In the following we will write T_η to denote the η-th level of the tree T and $T \upharpoonright \eta$ to denote the set of nodes of T of height $< \eta$. The key fact is the following (see [17] for the case where $\mathbb{P}$ is proper)

Lemma 2.13. *Let T be a Suslin tree, $R \subset \omega_1$ stationary and $\mathbb{P}$ an R-proper poset. Let θ be a sufficiently large cardinal. Then the following are equivalent:*

1. *$\Vdash_{\mathbb{P}} T$ is Suslin*
2. *if $M \prec H_\theta$ is countable, $\eta = M \cap \omega_1 \in R$, and $\mathbb{P}$ and T are in M, further if $p \in \mathbb{P} \cap M$, then there is a condition $q < p$ such that for every condition $t \in T_\eta$, (q, t) is $(M, \mathbb{P} \times T)$-generic.*

Proof. For the direction from left to right note first that $\Vdash_{\mathbb{P}} T$ is Suslin implies $\Vdash_{\mathbb{P}} T$ is ccc, and in particular it is true that for any countable elementary submodel $N[\dot{G}_{\mathbb{P}}] \prec H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$, $\Vdash_{\mathbb{P}} \forall t \in T(t$ is $(N[\dot{G}_{\mathbb{P}}], T)$-generic). Now if $M \prec H(\theta)$ and $M \cap \omega_1 = \eta \in R$ and $\mathbb{P}, T \in M$ and $p \in \mathbb{P} \cap M$ then there is a $q < p$ such that q is $(M, \mathbb{P})$-generic. So $q \Vdash \forall t \in T(t$ is $(M[\dot{G}_{\mathbb{P}}], T)$-generic, and this in particular implies that (q, t) is $(M, \mathbb{P} \times T)$-generic for all $t \in T_\eta$.

For the direction from right to left assume that $\Vdash \dot{A} \subset T$ is a maximal antichain. Let $B = \{(x, s) \in \mathbb{P} \times T \,:\, x \Vdash_{\mathbb{P}} \check{s} \in \dot{A}\}$, then B is a predense subset in $\mathbb{P} \times T$. Let θ be a sufficiently large regular cardinal and let $M \prec H(\theta)$ be countable such that $M \cap \omega_1 = \eta \in R$ and $\mathbb{P}, B, p, T \in M$. By our assumption there is a $q <_{\mathbb{P}} p$ such that $\forall t \in T_\eta((q, t)$ is $(M, \mathbb{P} \times T)$-generic). So $B \cap M$ is predense below (q, t) for every $t \in T_\eta$, which yields that $q \Vdash_{\mathbb{P}} \forall t \in T_\eta \exists s <_T t(s \in \dot{A})$ and hence $q \Vdash \dot{A} \subset T \restriction \eta$, so $\Vdash_{\mathbb{P}} T$ is Suslin. $\square$

Lemma 2.14. *Let $R \subset \omega_1$ be stationary, co-stationary, then the club shooting forcing $\mathbb{P}_R$ preserves Suslin trees.*

Proof. Let T be an arbitrary Suslin tree from the ground model V. Because of Lemma 2.13, it is enough to show that for any regular and sufficiently large θ, every $M \prec H_\theta$ with $M \cap \omega_1 = \eta \in R$, and every $p \in \mathbb{P}_R \cap M$ there is a $q < p$ such that for every $t \in T_\eta$, (q, t) is $(M, (\mathbb{P}_R \times T))$-generic. Note first that, as T is Suslin, every node $t \in T_\eta$ is an (M, T)-generic condition. Further, as forcing with a Suslin tree is ω-distributive, $(H(\omega_1))^{M[G]} = (H(\omega_1))^M$ for every T-generic filter G over V. As $\mathbb{P}_R \subset H(\omega_1)$, we obtain that the set $M[G] \cap \mathbb{P}_R$ is independent of the choice of the generic filter G and equals $M \cap \mathbb{P}_R$. Likewise $M[G] \cap \omega_1 = M \cap \omega_1$, for every T-generic filter.

Next we note that for a countable M and a V-generic filter $G \subset T$, the model $M[G]$ is (up to isomorphism) uniquely determined by the $t \in T_\eta$, such that $t \in G$ and $\eta = M \cap \omega_1$. This is clear as we can transitively collapse $M[G]$ to obtain a structure of the form $\bar{M}[t]$, where $\bar{M}$ is the image of M under the collapse map and $t \in T_\eta$ is the unique node in T to which G is sent to by the collapse map. So for a countable $M \prec H(\theta)$, and $\eta = M \cap \omega_1$, we write $M[t]$ for the unique model of the form $M[G]$, for G T-generic over V and $t \in G \cap T_\eta$. With an argument almost identical to the one used in the proof of Lemma 2.2 it is not hard to see that if $M \prec H(\theta)$ is such that $M \cap \omega_1 \in R$ then an ω-length descending sequence of $\mathbb{P}_R$-conditions in M whose domains converge to $M \cap \omega_1$ has a lower bound as $M \cap \omega_1 \in R$.

We construct an ω-sequence of elements of $\mathbb{P}_R$ which has a lower bound which will be the desired condition q such that for every $t \in T_\eta$, (q, t) is $(M, \mathbb{P}_R \times T)$-generic. We list the nodes on T_η, $(t_i \,:\, i \in \omega)$ and consider the according generic extensions $M[t_i]$. In every $M[t_i]$ we list the $\mathbb{P}_R$-dense subsets of $M[t_i]$, $(D_n^{t_i} \,:\, n \in \omega)$, write the so listed dense subsets of $M[t_i]$

as an $\omega \times \omega$-matrix and enumerate this matrix in an ω-length sequence of dense sets $(D_i : i \in \omega)$. If $p = p_0 \in \mathbb{P}_R \cap M$ is arbitrary we can find, using the fact that $\forall i \, (\mathbb{P}_R \cap M[t_i] = M \cap \mathbb{P}_R)$, an ω-length, descending sequence of conditions below p_0 in $\mathbb{P}_R \cap M$, $(p_i : i \in \omega)$ such that $p_{i+1} \in M \cap \mathbb{P}_R$ is in D_i. By the usual density argument we can conclude that the domain of the conditions p_i converge to $M[t_i] \cap \omega_1 = M \cap \omega_1$. Then the p_i's have a lower bound $q = p_\omega \in \mathbb{P}_R$, namely $q = \bigcup_{i \in \omega} p_i \cup \{(\eta, \eta)\}$ and (t, q) is an $(M, T \times \mathbb{P}_R)$-generic condition for every $t \in T_\eta$ as any $t \in T_\eta$ is (M, T)-generic and every such t forces that q is $(M[T], \mathbb{P}_R)$-generic; moreover $q < p$ as desired. $\square$

We add a second proof of the last lemma, which is more straightforward at the cost of being less general.

Proof. Let T be a Suslin tree from the ground model V. We assume for a contradiction that there is a condition $p \in \mathbb{P}_R$ and a $\mathbb{P}_R$-name $\dot{A}$ such that

$$p \Vdash \text{“}\dot{A} \subset \check{T} \text{ is a maximal uncountable antichain”}$$

We let $M \prec H(\theta)$, $|M| = \aleph_0$, where θ is an arbitrary regular cardinal greater than $2^{\aleph_1}$. Additionally we demand that $\{\dot{A}, \mathbb{P}_R, p\} \subset M$ and, if we let $\delta := M \cap \omega_1$, we demand that $\delta \in R \subset \omega_1$. The latter is possible as $\{M \cap \omega_1 : M \prec H(\theta) \wedge \{\dot{A}, \mathbb{P}_R, p\} \subset M\}$ forms a club in ω_1, hence hits the stationary R. We know that

$$M \models p \Vdash \text{“}\dot{A} \subset \check{T} \text{ is a maximal antichain”}$$

hence, if we let $\bar{M}$ denote the transitive collapse of M and $\pi : M \to \bar{M}$ be the collapsing map,

$$\bar{M} \models p \Vdash \text{“}\pi(\dot{A}) \subset \pi(\check{T}) = \check{T} \cap \delta \text{ is a maximal antichain”}$$

Stepping outside of $\bar{M}$, we list the elements of $T \cap \delta$ as $(t_n : n \in \omega)$. Starting with $p =: p_0$ we recursively define a descending sequence of $\mathbb{P}_R$-conditions $(p_n : n \in \omega)$ such that for every $n \in \omega$, there is a $a_n \in T \cap \delta$ such that $p_{n+1} \Vdash$ “$\check{a}_n \in \dot{A}$ and $\check{a}_n$ and t_n are compatible in T”. The sequence $(p_n : n \in \omega)$ can be chosen such that $sup_{n \in \omega}(max\,(p_n)) = \delta \in R$. Hence there will be a lower bound $p_\omega \in \mathbb{P}_R$ for $(p_n : n \in \omega)$ in V. The lower bound p_ω will satisfy that there is a set $B \subset T$ in V, namely $B = \{a_n : n \in \omega\}$ such that

$$p_\omega \Vdash \text{“}\check{B} \text{ is a maximal antichain in } \check{T} \cap \delta\text{”}$$

and by absoluteness of the statement “$\check{B}$ is a maximal antichain in $\check{T} \cap \delta$”, we obtain that

$$V \models B \text{ is a maximal antichain in } T \cap \delta\text{”}.$$

As T is a Suslin tree, hence in particular a normal tree, we obtain that the δ-th level of T, denoted by T_δ, seals off B, i.e. for every $a \in B$ there is a $t_a \in T_\delta$ such that $a <_T t_\delta$. But this implies that B remains a maximal antichain in T, hence $p_\omega \Vdash$ "$\check{B} = \dot{A} \wedge |\check{B}| = \aleph_0$", which shows that $\mathbb{P}_R$ forces that every antichain of $\check{T}$ is countable, hence T remains Suslin after forcing with $\mathbb{P}_R$ as claimed.

□

If we let $A \subset \omega_1$ be an arbitrary subset in $L[\mathbb{Q}^0][\mathbb{Q}^2]$ and if we let $\mathbb{Q}^1_A := \prod_{\beta \in A} S_\beta$, and finally if $\alpha \notin A$, then by the above we know that S_α is still a Suslin tree in $L[\mathbb{Q}^0][\mathbb{Q}^2][\mathbb{Q}^1_A]$. Thus we can freely add ω_1-branches through some elements $\vec{S}$ whose index belongs to a set $A \subset \omega_1$, and add them to $L[\mathbb{Q}^0][\mathbb{Q}^2]$ without interfering with the Suslinity of all the other trees of $\vec{S}$ whose index is not in A. We summarize the last results to:

Theorem 2.15. *The universe $W = L[\mathbb{Q}^0][\mathbb{Q}^1][\mathbb{Q}^2]$ is an ω-distributive, $\aleph_2$-preserving generic extension of L and contains $\vec{S}$ which is an independent sequence of Suslin trees over $L[\mathbb{Q}^0 * \mathbb{Q}^2]$. However no tree from $\vec{S}$ is Suslin in W. Moreover $\vec{S}$ is $\Sigma_1(\omega_1)$-definable over W. If we let $A \subset \omega_1$ be an arbitrary set in $L[\mathbb{Q}^0 * \mathbb{Q}^2]$ and if $(b_\beta \subset S_\beta : \beta \in A)$ is a sequence of $\mathbb{Q}^1_A$-generic filters over $L[\mathbb{Q}^0][\mathbb{Q}^2]$, (i.e. generically added ω_1-branches) then for every $\alpha \notin A$, $L[\mathbb{Q}^0][(b_\beta \subset S_\beta : \beta \in A][\mathbb{Q}^2] \models$ "S_α is a Suslin tree".*

We end with a straightforward lemma which is used later in coding arguments.

Lemma 2.16. *Let T be a Suslin tree and let $\mathbb{A}_D(X)$ be the almost disjoint coding which codes a subset X of ω_1 into a real with the help of an almost disjoint family of reals D of size $\aleph_1$. Then*

$$\mathbb{A}_D(X) \Vdash T \text{ is Suslin}$$

holds.

Proof. This is clear as $\mathbb{A}_D(X)$ has the Knaster property, thus the product $\mathbb{A}_D(X) \times T$ is ccc and T must be Suslin in $V[\mathbb{A}_D(X)]$. □

3 Main Proof

3.1 Informal discussion of the idea

As the proof we aim for will be rather technical we want to discuss first some ideas which are used on an informal level. We shall concentrate on uniformizing one Π^1_3 set A_m. This is actually sufficient, as A_m could be the universal Π^1_3 set. If we fix a real x and consider its (assumed to be) non-empty x-section of A_m, denoted by $A_{m,x}$, then our goal is to single out exactly one real y such that (x, y) is the value of our uniformizing function $f(m, x)$. We shall aim to make the graph of $f(m, \cdot)$ Π^1_3-definable. This will be accomplished via coding every pair (x, y') which is not $(x, f(m, x))$ into the independent sequence of Suslin trees $\vec{S}$. We will see that "being coded into the $\vec{S}$"-sequence is a Σ^1_3-property, thus not being coded into $\vec{S}$ is Π^1_3 and if we can arrange that, for every x, $(x, f(m, x))$ is the unique pair of $A_{m,x}$ which is not coded into the $\vec{S}$-sequence, then indeed, we would have found a uniformizing function whose graph is Π^1_3, as desired.

The problem is of course, that coding reals into $\vec{S}$ means extending the universe, therefore the Π^1_3 set A_m will change, and the value $f(m, x)$ we chose, could end up not being an element of A_m anymore, while $A_{m,x}$ remains non-empty. In that situation, our attempt to create a Π^1_3 uniformizing function has failed. A closer inspection might lead to the impression that the task of determining for every real x a real y such that (x, y) will remain in A_m even after we coded every other pair into $\vec{S}$ is hopeless. Indeed it is e.g. easy to design a Π^1_3-set A_k, such that $A_{k,x}$ consists of exactly two points y_0 and y_1 and deciding to set $f(k, x) = y_0$, therefore coding up (x, y_1) will kick (x, y_0) out of A_k, while setting $f(k, x) = y_1$ and consequentially coding up (x, y_0) immediately kicks (x, y_1) out of A_k. This toy example can be extended to sets with infinite sections. It is also possible to construct two Π^1_3-sets A_k and A_l for which a setting a value for $f(k, x)$ will kick out the value $f(l, x)$ of A_l and so on.

The idea to solve these issues, is to turn the problem into a fixed point problem. We start with a base set of iterations, which we call allowable. If we consider a pair $(x, y) \in A_m$ for which we know that it can not be forced out of A_m with an allowable forcing, then it is safe to set $f(m, x) := y$, as long as we continue our iteration with an allowable forcing.

This reasoning yields a new set of rules for an iteration, and these new rules determine a subcollection of allowable forcings called 1-allowable. We can repeat this, via asking for a pair $(x, y) \in A_m$, whether there is an allowable $\mathbb{P}$ such that after using $\mathbb{P}$, (x, y) can not be kicked out of A_m with an 1-allowable forcing. These rules will form the 2-allowable forcings and so on.

These collections will be shrinking, but always non-empty, therefore they will stabilize, giving rise to a set we call ∞-allowable forcings. This is the right collection of forcings we want to use, and we start an iteration consisting entirely of ∞-allowable factors, where we set values $f(m, x) = y$ whenever a pair (x, y) can not be kicked out of A_m and $f(m, x)$ has not been defined yet; and otherwise use an ∞-allowable forcing which witnesses that (x, y) can be forced out of A_m with an ∞-allowable forcing. As ∞-allowable forcings are a fixed point under the derivation operator we roughly described above, this iteration will yield an ∞-allowable iteration again. So all the values we set for $f(m, \cdot)$ are safe, in that $(x, f(m, x))$ remains in A_m throughout the whole iteration. This ends a rough description of how the proof is set up.

3.2 ∞-allowable Forcings

We continue with the construction of the appropriate notions of forcing which we want to use in our proof. The goal is to iteratively shrink the set of notions of forcing we want to use until we reach a fixed point. All forcings will belong to a certain class, which we call allowable. These are just forcings which iteratively code reals into ω_1-many ω-blocks of Suslin trees from $\vec{S}$. To ensure some symmetry, we demand that the set of the ω_1-many ω-blocks is added by the usual ω_1-Cohen forcing, but computed as in L. This trick is inspired by the coding from [7], where they dub the places where the coding is happening as coding areas. Upshot of this coding method is to ensure, while being quite easy to define, that products of the coding are themselves a coding.

3.2.1 Coding reals in inner models of W

Our ground model shall be W. Let $x, y \in W$ be reals, let $m \in \omega$ and let $\gamma < \omega_1$ be an arbitrary ordinal. In the following we will write (x, y, m) for the real which recursively codes up x, y and m, using some fixed recursive coding. We will consider an inner model $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$ of W, which we assign to (x, y, m), which sees that the triple (x, y, m) is coded into the $\vec{S}$ at the γ-th ω-block, and moreover sees no other reals coded this way. We shall define $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$ now.

First we collect the ω-many ω_1-branches $(b_{\gamma+n} \subset S_{\gamma+n} : n \in \omega)$ to write the characteristic function of (x, y, m) into the γ-th ω block of $\vec{S}$. To be more specific, if b'_β denotes the $<$-least cofinal S_β-generic branch, which exists in W, then we let

$$b_{\gamma+n} := \begin{cases} b'_{\omega\gamma+2n} \text{ if } (x,y,m)(n) = 0 \\ b'_{\omega\gamma+2n+1} \text{ if } (x,y,m)(n) = 1 \end{cases}$$

This way, working over $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_{\gamma+n} : n \in \omega)]$, we can read off (x, y, m) via looking at the ω-block of $\vec{S}$-trees starting at γ and evaluate which tree in the ω-block has been destroyed.

Lemma 3.1. *Using the objects as defined in the discussion above. In the universe $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_{\gamma+n} : n \in \omega)]$ the real $(x, y, m) \in W$ can be defined using the following formula with one free variable v_0, $(*)_\gamma(v_0)$ which is, over $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_{\gamma+n} : n \in \omega)]$, equivalent to a $\Sigma_1(\gamma, \omega_1, v_0)$-formula.*

$$\begin{aligned}(*)_\gamma((x,y,m)) \Leftrightarrow\ & n \in (x,y,m) \textit{ if and only if } S_{\omega\cdot\gamma+2n+1} \textit{ has an } \omega_1\textit{-branch,} \\ & \textit{and } n \notin (x,y,m) \textit{ if and only if } S_{\omega\cdot\gamma+2n} \textit{ has an } \omega_1\textit{-branch.}\end{aligned}$$

Proof. Let us define first the forcing $\mathbb{P}_{(x,y,m,\gamma)}$ for which the sequence $(b_{\gamma+n} : n \in \omega)$ is a generic filter over $L[\mathbb{Q}^0][\mathbb{Q}^2]$. The forcing $\mathbb{P}_{(x,y,m,\gamma)}$ is defined over

$L[\mathbb{Q}^0][\mathbb{Q}^2]$ as a countably (i.e. fully) supported ω-length product which writes the characteristic function of (x, y, m) into the γ-th ω block of $\vec{S}$. To be more specific, the n-th factor of $\mathbb{P}_{(x,y,m,\gamma)}$ denoted by $\mathbb{P}_{(x,y,m,\gamma)}(n)$ is defined by

$$\mathbb{P}_{(x,y,m,\gamma)}(n) = \begin{cases} S_{\omega\gamma+2n} \text{ if } (x,y,m)(n) = 0 \\ S_{\omega\gamma+2n+1} \text{ if } (x,y,m)(n) = 1 \end{cases}$$

Note that $\mathbb{P}_{(x,y,m,\gamma)}$ is a regular subforcing of $\mathbb{Q}^1 \in L[\mathbb{Q}^0][\mathbb{Q}^2]$, which consisted of adding cofinal branches through every tree in $\vec{S}$. It is clear now that the sequence of the b_n's is generic for $\mathbb{P}_{(x,y,m,\gamma)}$ over $L[\mathbb{Q}^0][\mathbb{Q}^2]$.

We shall prove the Lemma now and work over $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$. Assume first that $n \in (x, y, m)$ i.e. $(x, y, m)(n) = 1$. Then, by definition, $\mathbb{P}_{(x,y,m,\gamma)}(n) = S_{\omega\gamma+2n+1}$, thus $S_{\omega\gamma+2n+1}$ adds generically an ω_1-branch through the tree $S_{\omega\gamma+2n+1}$. As $S_{\omega\gamma+2n+1}$ is a subforcing of $\mathbb{P}_{(x,y,m,\gamma)}$, and as the existence of an ω_1-branch through $S_{\omega\gamma+2n+1}$ is upwards absolute between universes of the same $\aleph_1$, we obtain that indeed, $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)] \models$ "$S_{\omega\cdot\gamma+2n+1}$ has an ω_1-branch". The proof for the case when $n \notin (x, y, m)$ is similar.

On the other hand, if $S_{\omega\gamma+2n+1}$ is not a Suslin tree in $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$, then we shall show that we must have used the forcing $S_{\omega\gamma+2n+1}$ at stage n in $\mathbb{P}_{(x,y,m,\gamma)}$. Indeed, we claim that the forcing $\mathbb{Q} := \prod_{m \neq 2n+1} S_{\omega\gamma+m}$ using countable support preserves the Suslin tree $S_{\omega\gamma+2n+1}$. This is sufficient, as $\mathbb{P}_{(x,y,m,\gamma)}$ is a subforcing of $\mathbb{Q}$, and if $S_{\omega\gamma+2n+1}$ remains Suslin in $L[\mathbb{Q}^0][\mathbb{Q}^2][\mathbb{Q}]$, it surely must be Suslin in $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$. To see that $S_{\omega\gamma+2n+1}$ is Suslin in $L[\mathbb{Q}^0][\mathbb{Q}^2][\mathbb{Q}]$, note that every factor of it preserves that $S_{\omega\gamma+2n+1}$ is Suslin and so the countable support must do so as well by theorem 2.9.

So, indeed if $S_{\omega\gamma+2n+1}$ is not a Suslin tree in $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$, we must have used $S_{\omega\gamma+2n+1}$ at stage n in $\mathbb{P}$, which means that $(x, y, m)(n) = 1$, as claimed. Again, the dual case when $S_{\omega\gamma+2n}$ has an ω_1-branch is similar.

We proceed to show that $(*)_\gamma(v_0)$ is, over $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$, equivalent to a $\Sigma_1(\gamma, \omega_1, v_0)$-formula. First note that, as just shown, $\mathbb{P}$ is a proper generic extension of $L[\mathbb{Q}^0][\mathbb{Q}^2]$, which in particular means that the pattern of stationary, non-stationary members of $(R_\alpha : \alpha < \omega_1 \cdot \omega_1)$ remains untouched when passing from $L[\mathbb{Q}^0][\mathbb{Q}^2]$ to $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$. Thus the sequence $\vec{S} \in L[\mathbb{Q}^0][\mathbb{Q}^2]$ is still definable over $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega)]$, using the same $\Sigma_1(\omega_1)$ formula $\Psi(X, \omega_1)$ from the proof of Lemma 2.12.

As a consequence $(*)_\gamma((x, y, m))$ is, over $L[\mathbb{Q}^0][\mathbb{Q}^2][(b_n : n \in \omega]$ equivalent to the following $\Sigma_1(\omega_1, \gamma, (x, y, m))$-formula:

$$\begin{aligned}\Phi(\omega_1, \gamma, (x, y, m)) \equiv \exists M, (b_n : n \in \omega), (&M \text{ transitive } \wedge M \models \mathsf{ZF}^- \wedge \\ &\{\omega_1, (b_n : n \in \omega)\} \subset M \wedge \\ &M \models \forall \beta \in [\omega_1 \cdot \omega\gamma, \omega_1 \cdot (\omega\gamma + \omega)) \\ &\qquad\qquad (\text{either } R_{2\beta} \text{ or } R_{2\beta+1} \text{ is nonstationary}) \wedge \\ &M \models \text{There is an } \omega\text{-sequence } (X_{\omega\gamma+k})_{k<\omega} \text{ of subsets of } \omega_1 \wedge \\ &M \models \forall k, \delta(\delta \in X_{\omega\gamma+k} \Leftrightarrow R_{\omega_1 \cdot (\omega\gamma+k)+2\cdot\delta} \text{ is not stationary and} \\ &\qquad \delta \notin X_{\omega\gamma+k} \Leftrightarrow R_{\omega_1 \cdot (\omega\gamma+k)+2\cdot\delta+1} \text{ is not stationary}) \wedge \\ &M \models \forall n \in \omega (n \in (x, y, m) \Leftrightarrow b_n \text{ is an} \\ &\qquad\qquad \omega_1\text{-branch through } X_{\omega\gamma+2n+1} \wedge \\ &\quad n \notin (x, y, m) \Leftrightarrow b_n \text{ is an } \omega_1\text{-branch through } X_{\omega\gamma+2n} \text{)"))}\end{aligned}$$

To verify the claimed equivalence, we shall only argue for the direction from right to left, as the other one is clear by reflection. We recall that by Lemma 2.11 and Lemma 2.12, if some transitive M is a model of ZF^- and contains ω_1, it will correctly compute the relevant elements from the $\vec{S}$-sequence. Last, if M is as claimed and $M \models$ "b is a cofinal branch through $X_{\omega\gamma+2n+1}$", then it must be true that $X_{\omega\gamma+2n+1}{=}S_{\omega\gamma+2n+1}$ and b really is a cofinal branch through $S_{\omega\gamma+2n+1}$ which gives the direction from right to left. $\square$

It is clear that the above coding is not tied to reals from W, that is reals from L. If we work over $\tilde{W}$ which is an arbitrary outer model of W by a proper forcing, then for any real $r \in \tilde{W}$, we can go to the according inner model of $\tilde{W}$ as described above, and the real r satisfies $\Phi(\omega_1, \gamma, r)$ in that inner model, and by upwards absoluteness in $\tilde{W}$ as well.

3.2.2 The Coding Forcing $\mathbb{P}_{(x,y,m)}$

We shall define the coding forcing we will use throughout this article. The forcing $\mathbb{P}_{(x,y,m)}$ is first defined over the universe W but its definition will work over generic extensions using iterated versions of the coding forcing $\mathbb{P}_{(x,y,m)}$ as well, which is what we are interested in most.

Let $(x, y, m) \in W$ be a real coding the triple consisting of $x, y \in \omega^\omega \cap W = \omega^\omega \cap L$ and $m \in \omega$. The coding forcing we are about to define will first add generically an ω_1-subset, whose coded initial segments will yield the set of starting points of ω-blocks of $\vec{S}$, where we code up the ω_1-branches through $\vec{S}$ in a way which will correspond to the real (x, y, m). To be more precise we define the coding forcing $\mathbb{P}_{(x,y,m)} \in W$ to be

$$\mathbb{P}_{(x,y,m)} := (\mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}_D(\dot{Y})$$

where $(\mathbb{C}(\omega_1))^L$ is ω_1-Cohen forcing as defined in L and $\dot{\mathbb{A}}_D(\dot{Y})$ is the (name of a) almost disjoint coding forcing which codes a certain set with $(\mathbb{C}(\omega_1))^L$-

name $\dot{Y}$ into a real. Note that, when working over W, the first forcing $(\mathbb{C}(\omega_1))^L$ equals just $\mathbb{C}(\omega_1)$ as defined in W. However we emphasize already now, that when iterating the coding forcing, we will stick to $(\mathbb{C}(\omega_1))^L$ in the definition of the coding forcing, even though $(\mathbb{C}(\omega_1))^L$ will lose σ-closure when working over universes with non-constructible reals. Thus when iterating these forcings we actually produce a hybrid of a product (the coordinates where $(\mathbb{C}(\omega_1))^L$ is used) and an iteration (the coordinates where we use almost disjoint coding). We shall see later that this is harmless, and the forcing will preserve ω_1 and CH when iterated.

We shall define the second forcing now, working in $W[g]$, for $g \subset (\mathbb{C}(\omega_1))^L$ be a generic filter over W. We fix a constructible bijection $\rho : [\omega_1]^\omega \to \omega_1$, and if $g \subset \omega_1$ is the generic subset of ω_1 added by $(\mathbb{C}(\omega_1))^L$ over W, we let $h := \{\rho(g \cap \alpha) \, : \, \alpha < \omega_1\}$. Note here that by σ-closure of $(\mathbb{C}(\omega_1))^L$, it will generically add a set whose initial segments are constructible, so ρ can be applied. To facilitate notation, we say that a set $C \subset \omega_1$ which satisfies $\forall \alpha < \omega_1 (C \cap \alpha \in L)$ is a *set coding a constructible sequence of ordinals*, if and only if there is a set $A \subset \omega_1$, $\forall \alpha < \omega_1 (A \cap \alpha \in L)$ and $\{\rho(A \cap \alpha) \, : \, \alpha < \omega_1\} = C$.

Then we list $h = (\alpha_i \, : \, i < \omega_1)$ and form the set $B \in W$ of branches through $\vec{S}$ which witness the pattern (x, y, m) on every ω-block of $\vec{S}$ with starting point in h. That is, for $\alpha_i \in h$ we let

$$B_{\alpha_i} = \{b_{\omega\alpha_i + 2n} \, : \, n \notin (x, y, m)\} \cup \{b_{\omega\alpha_i + 2n+1} \, : \, n \in (x, y, m)\}$$

We further collect all the club subsets we added to correctly define the elements of $\vec{S}$ which have an index corresponding to an index of a branch in $B := \bigcup_{i<\omega_1} B_{\alpha_i}$. More precisely:

- We let X_0 be the $<$-least (for some previously fixed wellorder of $H(\omega_2)$) set of the $\omega_1 \cdot \omega \cdot \omega_1$-many clubs which are necessary to correctly compute $S_{\omega\alpha_i + n}$ for every $n \in \omega$ and $\alpha_i \in h$ using the formula Ψ from Lemma 2.12.

- We let X_1 be $<$-least set of the ω_1-many ω_1-branches through elements of $(S_{\omega \cdot \alpha_i + n} \, : \, n \in \omega, \alpha_i \in h)$, so that the least ZF^--model of the form $L_\zeta[X_1]$ witnesses all the formulas $(*)_\gamma((x, y, m))$, $\gamma \in h$ from the last Lemma in the model $L[\mathbb{Q}^0][\mathbb{Q}^2][B] \subset W \subset W[g]$.

We fix a $\Sigma_1(\omega_1)$-definable bijection $\pi \in L$ between $(\omega_1 \cdot \omega) \cdot 2$ and ω_1, and use π to identify $X_0 \times X_1$ with its image under π which we denote with X. So $X \subset \omega_1$ codes in an easily definable way X_0 and X_1. It is clear that in $W[g] \supset L[\mathbb{Q}^0][\mathbb{Q}^2][B]$ any transitive model $M \in W[g]$ of a sufficiently big fragment of ZFC, which contains X as an element will also satisfy the following $\Sigma_1(\omega_1, X)$-formula with $N = L[X]^M$ being a witness:

$$
\begin{aligned}
\varphi((x,y,m)) \equiv & \exists N(N \text{ transitive }, |N| = \aleph_1, N \models \mathsf{ZF}^-, X \in N \wedge \\
& N \models \text{“}\exists h \subset \omega_1 (\forall \alpha < \omega_1 (h \cap \alpha \in L \wedge \\
& h \text{ is a set coding a contructible sequence of ordinals} \\
& \wedge (\forall \beta \in h \forall n \in \omega (n \in (x,y,m) \Rightarrow S_{\omega\beta+2n+1} \text{ has an } \omega_1\text{-branch} \wedge \\
& \qquad n \notin (x,y,m) \Rightarrow S_{\omega\beta+2n} \text{ has an } \omega_1\text{-branch}))))\text{”}
\end{aligned}
$$

Note here that in the above formula, we can actually demand that $n \in \omega \Leftrightarrow S_{\omega\beta+2n+1}$ has an ω_1-branch holds true, and likewise for $n \notin (x,y,m)$, but we will not need this strengthening. Note further that whenever we write $S_{\omega\beta+2n}$ has an ω_1-branch), we intend to actually use the $\Sigma_1(\omega_1)$-formula Ψ from the proof of lemma 2.12 to define the trees from $\vec{S}$.

Our goal is to reshape the set $X \subset \omega_1$ in such a way that the localized version of $\varphi((x,y,m))$ also works for suitable countable transitive models. The following argument takes place in $L[X] \subset L[\mathbb{Q}^0][\mathbb{Q}^2][B] \subset W$. First we fix an $\aleph_1$-sized ordinal β such that $X \in L_\beta[X]$ and $L_\beta[X] \models \mathsf{ZF}^- +$ “$\aleph_2$ exists”. Note then, that necessarily $L_\beta[X] \models \varphi((x,y,m))$. Then we pick the $<_{L[X]}$-least club $C \subset \omega_1, C \in L[X]$ and the $<_{L[X]}$-least sequence $(M_\alpha : \alpha \in C)$ of countable elementary submodels such that

$$\forall \alpha \in C (M_\alpha \prec L_\beta[X] \wedge M_\alpha \cap \omega_1 = \alpha)$$

Now let the set $Y \subset \omega_1$, $Y \in L[X]$ code the pair (C, X) in the following way. The odd entries of Y should code X and if $E(Y)$ denotes the set of even entries of Y and $\{c_\alpha : \alpha < \omega_1\}$ is the enumeration of C, then we demand that $E(Y)$ satisfies that

1. $E(Y) \cap \omega$ codes a well-ordering of type c_0.
2. $E(Y) \cap [\omega, c_0) = \emptyset$.
3. For all β, $E(Y) \cap [c_\beta, c_\beta + \omega)$ codes a well-ordering of type $c_{\beta+1}$.
4. For all β, $E(Y) \cap [c_\beta + \omega, c_{\beta+1}) = \emptyset$.

The upshot in forming this reshaped $Y \in L[X]$ is the following assertion, which shows that already countable transitive models of ZF^- which satisfy some mild additional assumptions, are already sufficient to see the branches corresponding to the characteristic function of (x,y,m).

Lemma 3.2. *Work in $\tilde{W}$ which should be an ω_1-preserving outer universe of $W[g]$. Let $X, C, Y \subset \omega_1$, $\gamma < \omega_1$ and $x, y \in W \cap \omega^\omega$ all be as defined above. For any countable transitive model $N \in \tilde{W}$ of $\mathsf{ZF}^- +$ “$\aleph_2$ exists”, such that $\omega_1^N = (\omega_1^L)^N$ and $Y \cap \omega_1^N \in N$, we have that*

$$N \models \varphi((x,y,m))$$

Proof. Let $N \in \tilde{W}$ be countable and transitive, and assume that $\omega_1^N = (\omega_1^L)^N$ and $Y \cap \omega_1^N \in N$. Then, $\omega_1^N \in C$, as otherwise there would be c_γ and $c_{\gamma+1}$ such that $\omega_1^N \in (c_\gamma, c_{\gamma+1})$. Item 3 in the definition of Y yields that N can see that $c_{\gamma+1}$ is countable, which contradicts $\omega_1^N < c_{\gamma+1}$.

We let $\bar{M}$ be the transitive collapse of $M_{\omega_1^N} \prec L_\beta[X]$, where $M_{\omega_1^N}$ belongs to the sequence of elementary submodels $(M_\alpha : \alpha < \omega_1)$ defined above. As $\omega_1^N \in C$, we can infer that $\bar{M}$ and N share the same ω_1, i.e. $\omega_1^N = \omega_1^{\bar{M}}$. Moreover

$$M_{\omega_1^N} \models \text{“The least } \mathsf{ZF}^-\text{-model } L_\zeta[X] \text{ witnesses that } \varphi((x,y,m)) \text{ is true”},$$

as $M_{\omega_1^N} \prec L_\beta[X]$ and as $L_\beta[X] \models$“The least ZF^--model $L_\zeta[X]$ witnesses $\varphi((x,y,m))$”. So

$$\bar{M} \models \text{“The least } \mathsf{ZF}^-\text{-model } L_{\bar{\zeta}}[X \cap \omega_1^{\bar{M}}] \text{ witnesses } \varphi((x,y,m)))\text{”}.$$

But N contains $Y \cap \omega_1^N$, so it contains $X \cap \omega_1^N$, and N can construct $L_{\bar{\zeta}}[X \cap \omega_1^N]$, so

$$N \models L_{\bar{\zeta}}[X \cap \omega_1^N] \text{ witnesses } \varphi((x,y,m)) \text{ holds true”},$$

and hence $N \models \varphi((x,y,m))$.

□

We shall use our just formed set $Y \subset \omega_1, Y \in L[X]$ to finally define the second forcing $\mathbb{A}_D(Y)$ of our two step iteration $\mathbb{P}_{(x,y,m)} = (\mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}(\dot{Y})$. We work in $W[g]$ as our ground model, (recall that g is our $(\mathbb{C}(\omega_1)^L$-Cohen generic subset) and we let the second forcing in the definition of $\mathbb{P}_{(x,y,m)}$ be the almost disjoint coding forcing $\mathbb{A}_D(Y)$ relative to our fixed almost disjoint family of reals $D = \{d_\alpha : \alpha < \omega_1\} \in L$ (D is defined right after Definition 2.4) to code the set $Y \in L[X] \subset W[g]$ into one real r. Conditions of $\mathbb{A}_D(Y)$ are pairs $(r, R) \in [\omega]^{<\omega} \times D^{<\omega}$ ordered by $(s, S) < (r, R)$ whenever it holds that

- $r \subset s$ and $R \subset S$.
- If $\alpha \in Y$ and $d_\alpha \in R$ then $r \cap d_\alpha = s \cap d_\alpha$.

In particular the definition of $\mathbb{A}_D(Y)$ only depends on the subset Y of ω_1 we code and $\mathbb{A}_D(Y)$ will be independent of the surrounding universe in which we define it, as long as it has the right ω_1 and contains the set Y. Moreover, we have shown already, that $\mathbb{A}_D(Y)$ preserves Suslin trees.

We let $G(1)$ be a $\mathbb{A}_D(Y)$-generic filter over $W[g]$, and let r_Y denote the generic real added by $G(1)$, which codes the set $Y \subset \omega_1$ in the following way:

$$\forall \alpha < \omega_1 (\alpha \in Y \Leftrightarrow r_Y \cap d_\alpha \text{ is finite}).$$

We note that the above equivalence holds for all ω_1-preserving outer models $W' \supset \tilde{W}[g][G(1)]$ as well (actually in all outer universes, though Y then might become countable, but we will not need that), by the absolute definition of $D \in L$. The real r_Y contains all the relevant information, such that arbitrary countable ZF^--models which contain r_Y and satisfy an additional mild technical assumption, suffice to witness that $\varphi((x, y, m))$ holds true.

Lemma 3.3. *Let $\tilde{W}$ be an outer universe of $W[g][G(1)]$ $\tilde{W} \models \mathsf{ZFC}$ and let (x, y, m) be our fixed real from the above. Working in $\tilde{W} \supset W[g][G(1)]$, the real $r_Y \in W[g][G(1)]$ has the following $\Pi^1_2((x, y, m))$-property there:*

$$\begin{aligned}(**)_{r_Y}(x, y, m) :\equiv & \textit{For any countable, transitive model } N \textit{ of } \mathsf{ZF}^- + \text{“}\aleph_2 \textit{ exists”} \\ & \textit{such that } \omega_1^N = (\omega_1^L)^N \textit{ and } r_Y \in N, \textit{ we have that} \\ & N \models \varphi((x, y, m))\end{aligned}$$

Proof. We assume first that $\tilde{W} = W[g][G(1)]$. As the assertion of $(**)_{r_Y}$ is a $\Pi^1_2(r_Y)$-statement, once we can show its truth in $W[g][G(1)]$, we know it will be true in all outer $\tilde{W} \supset W[g][G(1)]$ by Shoenfield absoluteness.

As $\omega_1^N = (\omega_1^L)^N$ and by the absoluteness of the decoding, we can infer that N will decode out of r_Y, using its own version of D (which is just $D \cap \omega_1^N$) the set $Y \cap \omega_1^N$, where Y is as in the previous lemma. So if $Y \cap \omega_1$ codes the set Z on its odd entries, then again by absoluteness of the decoding, $Z = X \cap \omega_1^N$, where X is again as in the previous lemma. Hence

$$N \models \text{“The least } \mathsf{ZF}^- \text{ model } L_\zeta[Z] \text{ witnesses } \varphi((x, y, m)) \text{ holds true”}$$

so $N \models \varphi((x, y, m))$ as asserted by the lemma. □

To summarize, for a given real $(x, y, m) \in W \cap \omega^\omega = L \cap \omega^\omega$ which in turn is the code for $x, y \in \omega^\omega$ and $m \in \omega$ the forcing $\mathbb{P}_{(x,y,m)} \in W$ is a proper forcing whose factors are of size $\aleph_1$ which generically adds a real r_Y such that the Π^1_2-property $(**)_{r_Y}((x, y, m))$ becomes true for (x, y, m). Speaking more generally, if $\tilde{W} \supset W$ is a generic extension of W and if there is a real $r \in \tilde{W}$ which witnesses $(**)_r((x, y, m))$ for a given real $(x, y, m) \in \tilde{W}$ then we say that r witnesses that the real (x, y, m) *is written into* $\vec{S}$, or that r witnesses that (x, y, m) *is coded into* $\vec{S}$. If $(x, y, m) \in \tilde{W}$ is such that there is a real r' such that (in $\tilde{W}$) r witnesses that (x, y, m) is coded into $\vec{S}$, then we just say that $\tilde{W}$ thinks that (x, y, m) is coded into $\vec{S}$ or that $\tilde{W}$ thinks that (x, y, m) is written into $\vec{S}$.

The statement “(x, y, m) is coded into $\vec{S}$” is a $\Sigma^1_3((x, y, m))$-formula. Indeed it is expressible using a formula of the form $\exists r \forall M(\Delta^1_2(r, M) \rightarrow \Delta^1_2(r, M, (x, y, m))$:

$$\begin{aligned}&\exists r \forall M (M \text{ is countable and transitive and } M \models \mathsf{ZF}^- + \text{“}\aleph_2 \text{ exists”} \\ &\qquad \text{and } \omega_1^M = (\omega_1^L)^M \text{ and } r, (x, y, m) \in M \rightarrow M \models \varphi((x, y, m)))\end{aligned}$$

As already seen in the above the truth of "(x, y, m) is coded into $\vec{S}$" is usually established via showing the slightly stronger formula which is $\Sigma^1_3((x, y, m))$ as well:

$$\exists r \forall M (M \text{ is countable and transitive and } M \models \mathsf{ZF}^- \text{"}\aleph_2 \text{ exists" and } (\omega_1^M = (\omega_1^L)^M \text{ and}$$
$$r, (x, y, m) \in M \rightarrow M \models \text{"}r \text{ codes a set } Y \text{ which in turn codes } X \subset \omega_1^M$$
$$\text{and for the least } \mathsf{ZF}^-\text{-model } L_\zeta[X]$$
$$L_\zeta[X] \models \exists h \subset (\omega_1^N)(h \text{ a set coding a constructible sequence of ordinals}$$
$$\wedge \forall n \in \omega \forall \xi \in h (n \in (x, y, m) \rightarrow S^{L[X]}_{\omega\xi+2n+1} \text{ has an } \omega_1\text{-branch}$$
$$n \notin (x, y, m) \rightarrow S^{L[X]}_{\omega\xi+2n} \text{ has an } \omega_1\text{-branch}))\text{")}.$$

The last Lemma has a converse. In particular, the projective and local statement $(**)_r((x, y, m))$ will determine how certain inner models of the surrounding universe will look like with respect to branches through $\vec{S}$.

Lemma 3.4. *Let $\tilde{W} \supset W$, $\tilde{W} \models \mathsf{ZFC}$ be an ω_1-preserving outer model. Let x, y be reals in $\tilde{W}$, let $m \in \omega$. Let $r \in \tilde{W}$ be a real such that $(**)_r((x, y, m))$ is true. Then also uncountable, transitive $M \in \tilde{W}$, $\{\omega_1, r\} \subset M$, $M \models \omega_1^M = \omega_1$ and $M \models \mathsf{ZF}^- +$ "$\aleph_2$ exists", will satisfy that $M \models \varphi((x, y, m))$ holds.*

Proof. Assume not, then there would be an uncountable, transitive M which is a counterexample to the assertion of the Lemma. By Löwenheim-Skolem, there would be a countable $N \prec M$, $r \in N$ which we can transitively collapse to obtain the transitive $\bar{N}$. But $\bar{N}$ would witness that $(**)_r((x, y, m))$ is not true for every countable, transitive model, which is a contradiction. □

Corollary 3.5. *Assume that $\tilde{W}$ is an outer universe of W with the same ω_1 and such that $\tilde{W}$ is stationary set preserving over W, in particular, stationary subsets of ω_1 in W remain stationary in $\tilde{W}$. Assume further that $r \in \tilde{W}$ is a real such that $\tilde{W} \models (**)_r((x, y, m))$ for a triple $(x, y, m) \in \tilde{W}$. Let $h \subset \omega_1$ be the set coding a constructible sequence whose existence is asserted by $\varphi((x, y, m))$ and which represents the set of ω-blocks of $\vec{S}$ where the pattern corresponding to $((x, y, m))$ is written. Assume that $\gamma \in h$. Then in $\tilde{W}$ we have that*

$$n \in (x, y, m) \Rightarrow L[r] \models \text{"}S_{\omega\gamma+2n+1} \textit{ has an } \omega_1\textit{-branch"}.$$

and

$$n \notin (x, y, m) \Rightarrow L[r] \models \text{"}S_{\omega\gamma+2n} \textit{ has an } \omega_1\textit{-branch"}.$$

Proof. Note first that by the last lemma,

$$L[r] \models \varphi((x, y, m))\text{"}$$

As $L[r]$ is an inner model of $\tilde{W}$ and the latter is a stationary set preserving outer model of W, we get that the pattern of stationary, not-stationary subsets of our distinguished sequence of L-stationary, co-stationary subsets $(R_\beta : \beta < \omega_1)$, which code up $\vec{S}$, is the same, no matter whether we compute it in $L[r]$, $\tilde{W}$ or W using our formula $\Psi(X, \omega_1)$ from the proof of lemma 2.12.

In particular, $L[r]$ computes $\vec{S}$ correctly. To finish the proof we just note that the statement of a of the existence of an ω_1-branch through some S_β is a $\Sigma_1(\omega_1)$-formula and hence upwards absolute, so the assertion follows immediately from the last lemma. $\square$

It is straightforward to see that the coding forcings can be iterated over W, coding more and more reals iteratively into $\vec{S}$, therefore filling up our distinguished Σ^1_3-set which consists of all reals coded into $\vec{S}$. Note that, as already mentioned above, we will stick however to $(\mathbb{C}(\omega_1))^L$ as our first factor of the coding forcing, even though, as soon as we are in a universe with non-constructible reals, which we will be in when iterating the coding forcings, $(\mathbb{C}(\omega_1))^L$ will not be the $\mathbb{C}(\omega_1)$-forcing as computed in the current universe. Consequentially an iteration of the coding forcings is in fact a hybrid of a product (the coordinates where we use $(\mathbb{C}(\omega_1))^L$) and an actual iteration (the coordinates where almost disjoint coding is used).

There are no issues with this however. If we let x, y be reals in W (that is x, y are in fact in L) and consider a two step iteration $\mathbb{P}_{(x,y,m)} * \mathbb{P}_{(z,v,k)}$, where z, v are $\mathbb{P}_{(x,y,m)}$-names for reals and k is a name for a natural number, then, by the commutativity of product forcing, we can rearrange

$$\mathbb{P}_{(x,y,m)} * \mathbb{P}_{(z,v,k)} = (\mathbb{C}(\omega_1) \times \mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}(\dot{Y_1}) * \dot{\mathbb{A}}(\dot{Y_2})$$

Note that the first factor is σ-closed, hence proper, and the second and third factor is ccc. The same holds true for transfinite iterations that is an iteration of coding forcings of infinite length. For technical reasons we shall use a mixed support when iterating the coding forcings, that is countable support for the product coordinates and finite support for the coordinates where we use almost disjoint coding forcing. This mixed support iteration can always be re-arranged such that we start with a countably supported product of $(\mathbb{C}(\omega_1))^L$'s followed by an iteration of almost disjoint coding forcings. Thus using countable support immediately gives us that a countable support iteration of our coding forcings results in an α_1-preserving itself.

As an alternative, and equivalent way of formulating the coding forcing, we could have replaced our ground model W with $W[G]$, where G is a generic for a countably supported product of $\aleph_1$-many copies of $\mathbb{C}(\omega_1)$-Cohen forcing. Note that $W[G]$ has again the same reals as W which has only constructible reals. With this new ground model, the new coding forcing $\mathbb{P}_{(x,y,m)}$ would be to pick one of the $\aleph_1$-many $\mathbb{C}(\omega_1)$-generics, dub it g, then form h as usual with the help of g and code up the reshaped set $Y \subset \omega_1$ which codes the relevant information into a real, in a way entirely analogue to the one we defined in this section. We believe that our present, and equivalent approach is a bit more intuitive, this is why we defined the coding forcing the way we did it.

The definition of $\mathbb{P}_{(x,y,m)}$ has a certain degree of absoluteness. A fact we will exploit heavily.

Lemma 3.6. *Let $(x, y, m) \in W$ and let $\tilde{W} \supset W$, $\tilde{W} \models \mathsf{ZFC}$, $\omega_1^{\tilde{W}} = \omega_1$. Then $\mathbb{P}_{(x,y,m)}$ as defined in $\tilde{W}$ contains a dense subset A which is an element of W. For $r_1, r_2 \in A$ it holds that*

$$W \models r_1 <_{\mathbb{P}_{(x,y,m)}} r_2 \Leftrightarrow \tilde{W} \models r_1 <_{\mathbb{P}_{(x,y,m)}} r_2.$$

In particular if $\forall i \in \{1,2\}$ $x_i, y_i \in \tilde{W} \cap 2^\omega$, $m_i \in \omega$, then

$$\tilde{W} \models \mathbb{P}_{(x_1,y_1,m_1)} * \dot{\mathbb{P}}_{(x_2,y_2,m_2)} = \mathbb{P}_{(x_1,y_1,m_1)} \times \mathbb{P}_{(x_2,y_2,m_2)}$$

Proof. The dense subset A of $\mathbb{P}_{(x,y,m)} = (\mathbb{C}(\omega_1))^L * \mathbb{A}_D(\dot{Y})$ is just $A := \{(p, \check{q}) : p \in P(0)$ and a $q \in [\omega]^{<\omega} \times D^{<\omega}\}$, and this dense set is computed in an absolute way in every universe which contains $\vec{S}$.

To show that also the order $<$ on $\mathbb{P}_{(x,y,m)}$ does not depend on the surrounding universe $\tilde{W}$, it suffices to remark that $<$ only depends on the first coordinate $p \in (\mathbb{C}(\omega_1))^L$, the forcing being of course absolute. Indeed, by the definition of $\mathbb{P}_{(x,y,m)}$ all further manipulations of p use absolute computations performed in $L[\vec{S}][p]$ (see the steps in the definition of $\mathbb{P}_{(x,y,m)}$ which define the reshaped set $Y \subset \omega_1$ in $L[X]$), so the absoluteness of $<$ of $\mathbb{P}_{(x,y,m)}$ is shown. $\square$

3.2.3 Allowable Forcings

Next we define the set of forcings which we will use in our proof. They belong to a well-defined set, we call allowable forcings:

Definition 3.7. *Let W be our ground model. Let $\alpha < \omega_1$ and let $F \in W$, $F : \alpha \to W$ be a bookkeeping function. A mixed support iteration $\mathbb{P} = (\mathbb{P}_\beta : \beta < \alpha)$ is called allowable (relative to the bookkeeping function F) if the function $F : \alpha \to W$ determines $\mathbb{P}$ inductively as follows:*

- *$\mathbb{P}_0$ is the trivial forcing.*
- *We assume that $\beta > 0$ and $\mathbb{P}_\beta$ is defined. We let G_β be a $\mathbb{P}_\beta$-generic filter over W and assume that $F(\beta) = (\dot{x}, \dot{y}, \dot{m})$, for a triple of $\mathbb{P}_\beta$-names. We assume that $\dot{x}^{G_\beta} =: x$, $\dot{y}^{G_\beta} =: y$ are reals, $\dot{m}^{G_\beta} =: m$ is a natural number. Then we let the forcing $\mathbb{P}(\beta)$ we want to use at stage β be the coding forcing $\mathbb{P}_{(x,y,m)}$.*

  Otherwise, $\mathbb{P}(\beta)$ is the trivial forcing. We use mixed support that is full support on the ccordinates which use $(\mathbb{C}(\omega_1))^L$ and finite support on the coordinates which use almost disjoint coding forcing.

If $\mathbb{P} \in W$ is a forcing such that there is an $\alpha < \omega_1$ and an $F \in W$, $F : \alpha \to W$ such that $\mathbb{P}$ is allowable with respect to F, then we often just drop the F and simply say that $\mathbb{P} \in W$ is allowable.

As allowable forcings form the base set of an inductively defined shrinking process, they are sometimes also denoted by 0-allowable to emphasize this fact. Intuitively for an allowable forcing, the bookkeeping F hands us at every step reals of the form (x, y, m) and we add a $\mathbb{C}(\omega_1)$-set which gives us the places where we code up the relevant branches to compute (x, y, m) using the coding mechanism described in the previous section.

Lemma 3.8. *1. If $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta) \in W$ is allowable then for every $\beta < \delta$, $\mathbb{P}_\beta \Vdash |\mathbb{P}(\beta)| = \aleph_1$, thus every factor of $\mathbb{P}$ is forced to have size $\aleph_1$.*

2. Every allowable forcing over W is $\aleph_1$ and CH preserving.

3. The product of two allowable forcings is allowable again.

Proof. The first assertion follows immediately from the definition.

The second one was dealt with already at the end of the previous section. Indeed, every allowable $\mathbb{P} = \bigstar_{\beta<\delta} P(\beta) = \bigstar_{\beta<\delta}(((\mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}(\dot{Y}_\beta))$ can be rewritten as $(\prod_{\beta<\delta}(\mathbb{C}(\omega_1))^L) * \bigstar_{\beta<\delta}\dot{\mathbb{A}}_D(\dot{Y}_\beta)$ (again with mixed support). The latter representation is easily seen to be of the form $\mathbb{P} * \bigstar_{\beta<\delta}\dot{\mathbb{A}}_D(\dot{Y}_\beta)$, where $\mathbb{P}$ is σ-closed and the second part is a finite support iteration of ccc forcings, hence $\aleph_1$ is preserved. That CH is preserved as well is standard.

To see that the third item is true, we invoke lemma 3.6 to immediately see that a two step iteration $\mathbb{P}_1 * \mathbb{P}_2$ of two allowable $\mathbb{P}_1, \mathbb{P}_2 \in W$ is in fact a product. Note that this tacitly uses the well-known fact that countable sets of ordinals in a proper generic extension can be covered by countable sets of ordinals from the ground model. As the iteration of two allowable forcings (in fact the iteration of countably many allowable forcings) is allowable as well, the proof is done. $\square$

The second assertion of the last lemma immediately gives us the following:

Corollary 3.9. *Let $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta) \in W$ be an allowable forcing over W. Then $W[\mathbb{P}] \models \mathsf{CH}$. Further, if $\mathbb{P} = (\mathbb{P}(\alpha) : \alpha < \omega_1) \in W$ is an ω_1-length iteration such that each initial segment of the iteration is allowable over W, then $W[\mathbb{P}] \models \mathsf{CH}$.*

The set of triples of (names of) reals which are enumerated by the bookkeeping function $F \in W$ which comes along with an allowable $\mathbb{P} = (\mathbb{P}(\beta) : \beta < \delta)$, we call the set of reals coded by $\mathbb{P}$. That is, if

$$\mathbb{P}(\beta) = (\mathbb{C}(\omega_1))^L * \dot{\mathbb{A}}_D(\dot{Y}_{(\dot{x}_\beta, \dot{y}_\beta, \dot{m}_\beta)})$$

and $G \subset \mathbb{P}$ is a generic filter and if we let for every $\beta < \delta$, $\dot{x}_\beta^G =: x_\beta$, $\dot{y}_\beta^G =: y_\beta$, $\dot{m}_\beta^G =: m_\beta$, then $\{(x_\beta, y_\beta, m_\beta) : \beta < \alpha\}$ is the set of reals coded by $\mathbb{P}$ and G (though we will suppress the G). Next we show, that iterations of 0-allowable forcings will not add unwanted witnesses to our distinguished Σ^1_3-formula $\psi((x, y, m))$, where

$$\psi((x, y, m)) \equiv \exists r \forall M (M \text{ is countable and transitive and } M \models \mathsf{ZF}^- + \text{“}\aleph_2 \text{ exists”} \\ \text{and } \omega_1^M = (\omega_1^L)^M \text{ and } r, (x, y, m) \in M \rightarrow M \models \varphi((x, y, m)))$$

Lemma 3.10. *If $\mathbb{P} \in W$ is allowable, $\mathbb{P} = (\mathbb{P}_\beta : \beta < \delta)$, $G \subset \mathbb{P}$ is generic over W and $\{(x_\beta, y_\beta, m_\beta) : \beta < \delta\}$ is the set of (triples of) reals which is coded as we use $\mathbb{P}$. Let $\psi(v_0)$ be the distinguished formula from above. Then in $W[G]$, the set of reals which satisfy $\psi(v_0)$ is exactly $\{(x_\beta, y_\beta, m_\beta) : \beta < \delta\}$, that is, we do not code any unwanted information accidentally.*

Proof. Let G be $\mathbb{P}$ generic over W. Let $g = (g_\beta : \beta < \delta)$ be the set of the δ many ω_1 subsets added by the $(\mathbb{C}(\omega_1))^L$-part of the factors of $\mathbb{P}$. We let $\rho : ([\omega_1]^\omega)^L \rightarrow \omega_1$ be our fixed, constructible bijection and let $h_\beta = \{\rho(g_\beta \cap \alpha) : \alpha < \omega_1\}$. Note that the family $\{h_\beta : \beta < \delta\}$ forms an almost disjoint family of subsets of ω_1. Thus there is $\alpha < \omega_1$ such that $\alpha > h_{\beta_1} \cap h_{\beta_2}$ for $\beta_1 \neq \beta_2 < \delta$ and additionally, α is an index not used by the iterated coding forcing $\mathbb{P}$, where we say that an index i of $\vec{S}$ is used by $\mathbb{P}$ whenever an ω_1-branch through S_i is coded by a factor of $\mathbb{P}$.

We fix such an α and $S_\alpha \in \vec{S}$. We claim that there is no real in $W[G]$ such that $W[G] \models L[r] \models$ “S_α has an ω_1-branch”. We show this by pulling out the forcing S_α out of $\mathbb{P}$. Indeed if we consider $W[\mathbb{P}] = L[\mathbb{Q}^0][\mathbb{Q}^1][\mathbb{Q}^2][\mathbb{P}]$, and if S_α is as described already, we can rearrange this to $W[\mathbb{P}] = L[\mathbb{Q}^0][\mathbb{Q}'^1 \times S_\alpha][\mathbb{Q}^2][\mathbb{P}] = W[\mathbb{P}'][S_\alpha]$, where $\mathbb{Q}'^1$ is $\prod_{\beta \neq \alpha} S_\beta$ and $\mathbb{P}'$ is $\mathbb{Q}^0 * \mathbb{Q}'^1 * \mathbb{Q}^2 * \mathbb{P}$.

Note now that, as S_α is ω-distributive, $2^\omega \cap W[\mathbb{P}] = 2^\omega \cap W[\mathbb{P}']$, as S_α is still a Suslin tree in $W[\mathbb{P}']$ by the fact that $\vec{S}$ is independent, and no factor of $\mathbb{P}'$ besides the trees from $\vec{S}$ used in $\mathbb{P}'$ destroys Suslin trees. But this implies that

$$W[\mathbb{P}'] \models \neg \exists r L[r] \models \text{“}S_\alpha \text{ has an } \omega_1\text{-branch”}$$

as the existence of an ω_1-branch through S_α in the inner model $L[r]$ would imply the existence of such a branch in $W[\mathbb{P}']$. Further and as no new reals appear when passing to $W[\mathbb{P}]$ we also get

$$W[\mathbb{P}] \models \neg \exists r L[r] \models \text{“}S_\alpha \text{ has an } \omega_1\text{-branch”}.$$

On the other hand any unwanted information, i.e. any $(x, y, m) \notin \{(x_\beta, y_\beta, m_\beta) : \beta < \delta\}$ such that $W[G] \models \psi((x, y, m))$ will satisfy that there is a real r such that

$$n \in (x, y, m) \rightarrow L[r] \models \text{“}S_{\omega\gamma + 2n+1} \text{ has an } \omega_1\text{-branch”}$$

and

$$n \notin (x,y,m) \to L[r] \models \text{“}S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch”}.$$

by corollary 3.5, for ω_1-many γ's.

But by the argument above, only trees which we used in one of the factors of $\mathbb{P}$ have this property, so there can not be unwanted codes.

$\square$

Let $\mathbb{P} = (\mathbb{P}_\beta : \beta < \delta) \in W$ be allowable. Let $A \subset \beta$, $A \in W$ be such that the forcing $\mathbb{P}_A := \bigstar_{\eta \in A} \mathbb{P}(\eta)$ (i.e. the iteration which uses the factors of $\mathbb{P}_\delta$ whose indices are in A using mixed support) is a forcing in W (which is automatically a subforcing of $\mathbb{P}$) and let $i_{A\delta}$ be the canonical embedding which maps $\mathbb{P}_A$ to $\mathbb{P}_\delta$ via

$$i_{A\delta}(p) = p' \text{ where } p' \text{ is such that } p'(\eta) = \begin{cases} p(\eta) & \text{if } \eta \in A \\ 1 & \text{else} \end{cases}.$$

Hence there are $\mathbb{P}_\delta$-names $\dot{x}$ which can, in a canonical way, be identified with a $\mathbb{P}_A$-name namely as long as all the $\mathbb{P}_\delta$-conditions of $\dot{x}$ are in fact $\mathbb{P}_A$-conditions, using the identification $i_{A\delta}$. For the rest of this article we will identify $\mathbb{P}_A$-names with their corresponding $\mathbb{P}_\delta$-names, which will simplify the language. In particular, in the definition of allowable forcings, if at stage β, $F(\beta)$ is a $\mathbb{P}_\beta$-name for a real, which is also a $\mathbb{P}_A$-name under the just described identification, then we will treat the name as if it was a $\mathbb{P}_A$-name, using i_A.

3.2.4 1-allowability

Given the notion of allowable, we can form a first approximation to the set of forcings we eventually want to use in our proof. We call these forcings 1-allowable. To motivate this notion, recall our strategy to force a model where the Π^1_3-uniformization property holds. We list the Π^1_3-formulas $(\varphi_m)_{m\in\omega}$ with two free variables in some recursive way, and let $A_m = \{(x,y) : \varphi_m(x,y)\}$ be the according sets. We let f_m denote the uniformizing function for A_m and write $f(m,x)$ for f_m's values at x. The goal is to pick for every $m \in \omega$ and every real x for which the x-section of A_m is non-empty, a value $f(m,x)$ such that $(x, f(m,x)) \in A_m$ and such that for every $y' \neq f(m,x)$ with $(x,y') \in A_m$, the triple (x,y',m) is coded somewhere in the $\vec{S}$, sequence. As being coded into $\vec{S}$ is a Σ^1_3-property, the unique $(x,y) \in A_m$ which is not coded into $\vec{S}$, is a Π^1_3-property. This way, the graph of f_m becomes a Π^1_3-definable set.

The underlying idea of forming 1-allowable forcings is the following line of reasoning. We will restrict ourselves to a simplified toy example first which we will describe now. Work in W. Assume that $x \in W$ is a real, $A_m \subset 2^\omega \times 2^\omega$ is a Π^1_3-set such that the x-section $A_{m,x}$ of A_m has exactly two elements y_1 and y_2. Assume further that no allowable $\mathbb{P} \in W$ will add new elements to the x-section of A_m. Our modest goal is to find a good value for $f(m, x)$ only, thus we will leave out the question of uniformizing all other A_k's and all other x-sections of A_m and their interferences among each other, as these make any easy attempt of a solution immediately extremely complicated.

Now, when we want to implement the above ansatz, we have to decide which one of the two reals y_1 or y_2 should become the value of our uniformizing function. This also means that we have to code up the other real somewhere into $\vec{S}$. Suppose we randomly decide to let y_1 be the $f(m, x)$-value, and code (x, y_2, m) into $\vec{S}$ and call this forcing $\mathbb{R}$. If we follow up with an arbitrary allowable $\mathbb{P} \in W[\mathbb{R}]$, how would the choice of $f(m, x)$ become the wrong one? Well, it could be that $\mathbb{R} * \mathbb{P}$ eventually adds a real which witnesses that (x, y_1) is not an element of A_m anymore. This however is not a real problem, as we could still be in the situation that there is a further allowable $\mathbb{Q} \in W[\mathbb{R} * \mathbb{P}]$ which forces (x, y_2) out of A_m as well, in which situation the problem of finding a value for $f(m, x)$ has disappeared, as $A_{m,x}$ has become empty in $W[\mathbb{R} * \mathbb{P} * \mathbb{Q}]$ and will remain empty by our assumptions.

So the actual problematic or pathological situation is the following: after we used $\mathbb{P}$ over $W[\mathbb{R}]$, (x, y_1) is not an element of A_m anymore, yet $(x, y_2) \in A_m$ and there is no additional $\mathbb{Q} \in W[\mathbb{R} * \mathbb{P}]$ which forces (x, y_2) out of A_m. This pathological situation means that our attempt to carry out our ansatz has failed, and we would have to start all over again.

The main idea is now to exploit this dead end to make progress in finding values for uniformizing functions. Indeed the above pathological situation can be exploited to fully settle the problem of defining $f(m, x)$ in a satisfying way. Note that the fact that

$$(*) \qquad W[\mathbb{R} * \mathbb{P}] \models \text{“There is no allowable } \mathbb{Q} \text{ such that } \mathbb{Q} \Vdash (x, y_2) \notin A_m\text{”}$$

also means that in particular no allowable $\mathbb{Q} \in W$ can kick (x, y_2) out of A_m. Indeed, assume for a contradiction that $\mathbb{Q} \in W$ is allowable, yet there is a $q \in \mathbb{Q}$ such that $q \Vdash (x, y_2) \notin A_m$, then $\mathbb{Q}_{\leq q} = (p \in \mathbb{Q} : p \leq q\}$ is such that $\mathbb{R} * \mathbb{P} \Vdash$ “$\mathbb{Q}_{\leq q}$ is allowable”. But, using Shoenfield absoluteness, we have that $W[\mathbb{R} * \mathbb{P}] \models$ “$\mathbb{Q}_{\leq q}$ is allowable and $\mathbb{Q}_{\leq q} \Vdash (x, y_2) \notin A_m$”, which is a contradiction to $(*)$. So, for the toy example, we found a value for $f(m, x)$, namely y_2, which will remain in A_m for all future allowable forcing extensions of W, thus we are safe in coding (x, y_1, m) into $\vec{S}$ using $\mathbb{P}_{(x, y_1, m)}$.

Before we start to define the notion of 1-allowable which is a refined and iterated version of the ideas above, we add the following definition which is standard and will be useful:

Definition 3.11. *Let $\mathbb{P} \in W$ be an allowable forcing and let $\dot{r} \in W$ be a $\mathbb{P}$-name of a real, i.e. $\mathbb{P} \Vdash \dot{r} \in 2^\omega$. Then we say that $\dot{r}$ is a nice ($\mathbb{P}$)-name of* *a real, whenever is has the following form*

$$\dot{r} = \{((n, m_p^n), p) \,:\, p \in A_n(\dot{r})\},$$

where for every $n \in \omega$, $A_n(\dot{r})$ is a maximal, (necessarily) countable antichain in $\mathbb{P}$, and for every $n \in \omega$ and every $p \in A_n(\dot{r})$, $m_p^n \in \omega$ and for every $p \in A_n(\dot{r})$,

$$p \Vdash \dot{r}(n) = m_p^n.$$

Note that such a nice $\mathbb{P}$-name is always an element of $H(\omega_2)^W$.

There is an analogue notion of nice name of an ordinal, and it is immediate that if $\mathbb{P} \in W$ is allowable and $\tau \in W$ is a $\mathbb{P}$-name of a countable ordinal which is a nice $\mathbb{P}$-name, then τ is an element of $H(\omega_2)^W$ as well. We will often tacitly assume that names are in fact nice names to make notation a bit easier.

We let $<$ denote some fixed wellorder of $H(\omega_2)^W$, not necessarily definable, which helps us to define the iteration. We demand that $<$ has the property that if $\mathbb{P} = (\mathbb{P}_\beta \,:\, \beta < \delta) \in W$ is allowable over W and $\beta_1 < \beta_2 < \delta$, then every $\dot{x} \in W$ which is a $\mathbb{P}_{\beta_1}$-name is $<$ than $\dot{y}$ for every $\mathbb{P}_{\beta_2}$-name $\dot{y}$. Moreover $<$ should satisfy that whenever $\mathbb{P}^1 = (\mathbb{P}^1_\beta \,:\, \beta < \delta_1)$ and $\mathbb{P}^2 = (\mathbb{P}^2_\beta \,:\, \beta < \delta_2)$ are two allowable forcings over W and $\delta_1 < \delta_2$ and $\dot{x}$ is a nice $\mathbb{P}^1$-name of a real and $\dot{y}$ is a nice $\mathbb{P}^2$ name for a real then $\dot{x} < \dot{y}$.

Now we define the notion of 1-allowability via induction. We work over W as our ground model. We let $\eta < \omega_1$, and let $F : \eta \to W^3$ be a bookkeeping function. The values $F(\beta)$ are triples and are written as $(F(\beta)_0, F(\beta)_1, F(\beta)_2)$. With the help of F we will define two objects inductively.

Assume we are at stage $\beta < \eta$ of our iteration and that we have already created the following list of objects:

- The forcing iteration $\mathbb{P}_\beta$ up to stage β which is an allowable forcing over W and G_β a $\mathbb{P}_\beta$-generic filter over W. For $\beta = 0$ we let $\mathbb{P}_\beta$ be the trivial forcing.

- The set $I_\beta = \dot{I}_\beta^{G_\beta} = \{(\dot{x}^{G_\beta}, \dot{y}^{G_\beta}, \dot{m}^{G_\beta}, \dot{\gamma}^{G_\beta}) \,:\, \dot{m}$ is a $\mathbb{P}_\beta$-name for a natural number, $\dot{x}, \dot{y}$ are $\mathbb{P}_\beta$-names of reals, $\dot{\gamma}$ is a name for an ordinal$\}$ of possible preliminary values of f. If $(x, y, m, \gamma) \in I_\beta$, we say that the potential $f(m, x)$-value y has rank γ, or just that (x, y, m) has rank γ. The concept of ranked f-values will become clear as we proceed in the proof. We let $I_0 = \emptyset$.

To make things intelligible, we argue in $W[G_\beta]$, that is semantically. The definitions to come will be uniformly working for all possible G_β, so it is straightforward to translate things back into forcing language using names.

We assume that $F(\beta)_0 = (\dot{x}, \dot{y}, \dot{m})$ and assume that there is an $A \subset \beta$, $A \in W$, such that $\dot{x}, \dot{y}, \dot{m}$ are in fact $\mathbb{P}_A$-names where $\mathbb{P}_A = \bigstar_{\beta \in A} \mathbb{P}(\beta)$, and $\mathbb{P}_A \in W$. We let $x = \dot{x}^{G_\beta}$, $y = \dot{y}^{G_\beta}$, $m = \dot{m}^{G_\beta}$. We define the next forcing $\mathbb{P}(\beta)$, and a new $f(m,x)$-value which will determine the new $I_{\beta+1}$ according to these rules:

(a) Let $G_A := G_\beta \upharpoonright A$.

- We collect all $\mathbb{P}_\beta$-names for reals $\dot{a}$ such that $\dot{a}^{G_\beta}$ is in fact an element of $W[G_A]$. For every such $\mathbb{P}_\beta$-name $\dot{a}$ we pick the $<$-least, nice name $\dot{b} \in W^{\mathbb{P}_\beta}$ such that $\dot{a}^{G_\beta} = \dot{b}^{G_\beta}$ and collect the these names $\dot{b}$ into a set called C. We assume that there is a $<$-least, nice $\mathbb{P}_A$-name $\dot{y_0}$ in C such that $\dot{y_0}^{G_A} = y_0$,

$$W[G_\beta] \models (x, y_0) \in A_m$$

and for which there is no allowable forcing $\mathbb{R} \in W[G_\beta]$ such that

$$W[G_\beta] \models \text{“}\mathbb{R} \Vdash (x, y_0) \notin A_m\text{”}.$$

If this is the case, then we define $\mathbb{P}(\beta)$ in $W[G_\beta]$ and, letting $G_{\beta+1}$ be a $\mathbb{P}_\beta * \mathbb{P}(\beta)$-generic filter, we define $f(m,x)$ in $W[G_{\beta+1}]$ as follows:

- We assume first that $F(\beta)_1 = \dot{E}$ where $\dot{E}$ is a $\mathbb{P}_A$-name of an infinite, countable set of reals. We set $E = \dot{E}^{G_A}$ and let

$$\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$$

where the latter product uses mixed support.
Else we just pick the $<$-least $\mathbb{P}_A$-name for a an infinite, countable set dubbed $\dot{E}$. Letting $E = \dot{E}^{G_A}$ we define

$$\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$$

where the latter product uses mixed support. We also let $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{P}(\beta)$ and let $G_{\beta+1} = G_\beta * G(\beta)$ be its generic filter.

- We set a new f value, i.e. we set $f(m,x) := y_0$ and assign in $W[G_{\beta+1}]$ the rank 0 to the value (x, y_0, m). We update $I_{\beta+1}^{G_{\beta+1}} := I_\beta^{G_\beta} \cup \{(x, y_0, m, 0)\}$.

(b) We assume that case (a) is not true. In that situation we let the bookkeeping F fully guess what to force with. We assume that $F(\beta)_1$ is a nice $\mathbb{P}_A$ name for a pair of reals of the form $(\dot{x}', \dot{y}_0)$ such that $\dot{x}'^{G_A} = x$, $\dot{y}_0^{G_A} = y_0$, together with a name for an ordinal $\dot{\xi}$ such that

$\mathbb{P}_A \Vdash \dot{\xi} > 0$. We assume that $F(\beta)_2 = \dot{E}$ is a $\mathbb{P}_A$-name of an infinite, countable set of reals. We set $E = \dot{E}^{G_A}$ and let

$$\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$$

$\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{P}(\beta)$, and let $G(\beta)$ be a $\mathbb{Q}_0$-generic filter over $W[G_\beta]$ and $G_{\beta+1} = G_\beta * G(\beta)$.

Further we update our set I_β of preliminary values for f to

$$I_{\beta+1} := I_\beta \cup \{(x, y_0, m, \xi)\}.$$

Otherwise, i.e. when $F(\beta)_1$ and $F(\beta)_2$ do not have the desired form we pick the $<$-least pair of $\mathbb{P}_A$-names of reals, $(\dot{x}', \dot{y}_0)$ such that $\dot{x'}^{G_A} = x$, $\dot{y}_0^{G_A} = y_0$, pick the least $\mathbb{P}_A$-name of a infinite, countable set of reals $\dot{E}$, and, working in $W[G_A]$, define

$$\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$$

Also we let $G(\beta)$ be a $\mathbb{P}(\beta)$-generic filter over $W[G_\beta]$ and set $W[G_{\beta+1}] = W[G_\beta * G(\beta)]$.

Then, working in $W[G_{\beta+1}]$, we update $I_{\beta+1} := I_\beta \cup \{(x, y_0, m, 1)$.

This ends the definition of 1-allowability in the successor stages.

If we arrive at a limit stage β in our iteration, we take the inverse limit of the initial segments, i.e.

$$\mathbb{P}_\beta := inv\, lim(\mathbb{P}_\nu \,:\, \nu < \beta).$$

For an arbitrary $\mathbb{P}_\beta$-generic filter G_β we let

$$I_\beta^{G_\beta} := \bigcup_{\xi < \beta} I_\xi^{G_\xi} = \{(m, x, y, \zeta) \,:\, \exists \xi < \beta((m, x, y, \zeta) \in I_\xi^{G_\xi})\}.$$

Definition 3.12. *Work in W. Let $\eta < \omega_1$ and assume that $F : \eta \to W^3$ is a bookkeeping function. If $\mathbb{P} = (\mathbb{P}_\beta \,:\, \beta < \eta)$ is an allowable forcing and $I = I_\eta$ such that $\mathbb{P}$, I are the result of applying the rules (a) and (b) together with F over W, then we say that $(\mathbb{P}, I)$ is 1-allowable with respect to F (over W). If I is clear from the context we often just say $\mathbb{P}$ is 1-allowable with respect to F. We say $\mathbb{P}$ is 1-allowable if there is an F such that $\mathbb{P}$ is 1-allowable with respect to F. Note here that, similar to the way we write the iteration $(\mathbb{P}_\beta \,:\, \beta < \eta)$, the set I is in fact a $\mathbb{P}_\eta$-name for these objects, even though we write them as if they were not.*

Before continuing proving some properties of 1-allowable forcings we want to add a couple of remarks concerning its definition.

- Note that there are necessarily Π^1_3-formulas φ_m, where case (a) must apply whenever $m \in \omega$ is considered by the bookkeeping, e.g if φ_m is logically equivalent to a true Σ^1_2-formula. In that case we can not alter its truth value by any additional forcing. As a result, the notion of 1-allowable is different from 0-allowable and the set of 1-allowable forcings is a proper subset of the set of 0-allowable forcings.
- In the definition of case (a), we refrain from considering all pairs of reals (x, y) from $W[G_\beta]$, but instead just scan through all pairs which are in the inner model $W[G_A]$ with x as the first coordinate. This stratification has technical advantages which shall become clear in the process of the arguments later. The upshot of this choice is that it enables a strategy to pick potential f_m-values in such a way that they will line up in a nice way as we go along in our 1-allowable iteration. The idea to not just pick a promising f_m-values once, and keep it for the rest of the iteration, but instead add potential $f_m(x)$-values in every step of the iteration ensures that we will not run into problems when dealing with products of allowable forcings. (If we would pick one fixed $f(m, x)$-value at a certain stage of the iteration and would want to keep it, throughout the iteration we run into problems when trying to keep 1-allowable forcings closed under products.)
- The set of potential f-values I does not have an influence on how the 1-allowable forcing $\mathbb{P}$ is defined at every step. Indeed, the definition of $\mathbb{P}(\beta)$ does only depend on F which also determines $\mathbb{P}_\beta$. We use I to make some arguments more transparent.

3.2.5 Definition of 1-allowable over arbitrary allowable extensions

We note that the definition of 1-allowable works over arbitrary allowable generic extensions $W[\mathbb{P}]$ of W as well. Given an allowable $\mathbb{P} \in W$ and a bookkeeping $F \in W[\mathbb{P}]$ $F : \delta \to W[\mathbb{P}]^3$ we can compute a 1-allowable iteration $(\mathbb{P}_\beta : \beta < \delta)$ and $(I_\beta : \beta < \delta)$ over $W[\mathbb{P}]$ using F with the new ground model $W[\mathbb{P}]$. In that situation we say that $(\mathbb{P}_\beta : \beta < \delta)$ and $(I_\beta : \beta < \delta)$ are 1-allowable over $W[\mathbb{P}]$ with respect to F and initial value $(\mathbb{P}, \emptyset)$. A fully analogue definition is possible when we additionally decide to use a different starting value $I := I_0 \in W$, even though it will not influence the actual definition of the 1-allowable forcing determined by F.

It follows from the definition that if $\mathbb{P}^0 := (\mathbb{P}^0_\beta : \beta < \delta^0)$ and $I^0 := (I^0_\beta : \beta < \delta^0)$ are 1-allowable with respect to some F^0 over W and if $(\mathbb{P}^1_\beta : \beta < \delta^1)$ and $(I^1_\beta : \beta < \delta^1)$ are 1-allowable with respect to some F^1 over $W[\mathbb{P}^0]$ with initial values $(\mathbb{P}^0, I^0)$, then the two step iteration of $\mathbb{P}^0$ and then $\mathbb{P}^1$ together with I_0 and I^1 is 1-allowable over W whose witness is the concatenation of the bookkeeping function F followed by (the accordingly slightly reformulated version of) F'. In particular, given a 1-allowable iteration $(\mathbb{P}_\beta : \beta < \delta)$, $(\mathbb{R}_\beta : \beta < \delta)$ and $(I_\beta : \beta < \delta)$ over W of length $\delta > 1$ then whenever we let $0 < \eta < \delta$ and split the iteration into two parts $(\mathbb{P}_\beta : \beta < \eta)$ and $(I_\beta : \beta < \eta)$ and the tail $(\mathbb{P}_\beta : \eta < \beta < \delta)$ and $(I_\beta : \eta < \beta < \delta)$ then the second part of that split will always be a 1-allowable iteration over $W[\mathbb{P}_\eta]$ with starting values $(\mathbb{P}_\eta, I_\eta)$.

3.2.6 Properties of 1-allowable forcings

We will derive some consequences from the definition of 1-allowability. The first thing we note is that 1-allowable forcings are closed under taking products.

Lemma 3.13. *Let $F_1 : \delta_1 \to W^3$ and $F_2 : \delta_2 \to W^3$ be two bookkeeping functions in W, let $\mathbb{P}_1 \in W$ be the 1-allowable forcing with respect to F_1 and let $\mathbb{P}_2 \in W$ be the 1-allowable forcing with respect to F_2. Then $\mathbb{P}_1 \times \mathbb{P}_2$ is a 1-allowable forcing relative to a bookkeeping function F' which is definable from F_1 and F_2.*

Proof. We shall define a bookkeeping function F' such that $\mathbb{P}_1 \times \mathbb{P}_2$ is 1-allowable relative to F'. For ordinals $\beta < \delta_1$ we let $F'(\beta) = F_1(\beta)$. Then the 1-allowable forcing which will be produced on the first δ_1-many stages is $\mathbb{P}_1$.

For $\beta > \delta_1$, we let $F'(\beta) = F_2(\beta - \delta_1)$. Then we claim that $F' \restriction [\delta_1, \delta_1 + \delta_2)$ using the rules of 1-allowability will produce $\mathbb{P}_2$.

First we prove by induction on $\beta \in [\delta_1, \delta_1 + \delta_2)$ that if β is a stage such that for $\beta - \delta_1$, case (a) applies when building $\mathbb{P}_2$ over W using F_2, then case (a) also must apply at stage β when building the forcing using F' over W and vice versa. That is, there is no difference in which case applies when forming $\mathbb{P}_2$ over W using F_2 and $\mathbb{P}_2$ over $W[\mathbb{P}_1]$ using F'.

Indeed, assume that $\beta < \delta_2$ and we have already established that $F' \restriction \delta_1 + \beta$ produces the 1-allowable $\mathbb{P}^1 \times \mathbb{P}^2_\beta$. Now assume that β is such that case (a) applies when working over W with F_2. This means that there is a (x, y_0) which can not be kicked out of A_m by a further 0-allowable forcing over $W[G_\beta]$.

Now we assume for a contradiction, that if we work over $W[\mathbb{P}_1][\mathbb{P}^2_\beta]$ and consider $F'(\delta_1 + \beta) = F_2(\beta)$, we are not in case (a). But the model $W[G_A]$ considered by $F'(\delta_1 + \beta)$ is the same as the model $W[G_A]$ considered by $F_2(\beta)$ when working over W by definition of 1-allowability. As we are not in case (a), there is an allowable forcing $\mathbb{P} \in W[\mathbb{P}^1][\mathbb{P}^2_\beta]$ such that $\mathbb{P} \Vdash (x, y_0) \notin A_m$. But then $\mathbb{P}^1 \times \mathbb{P} \in W[\mathbb{P}^2_\beta]$ is allowable and witnesses that we are not in case (a) when working with F_2 over W to define $\mathbb{P}_2$, which is a contradiction.

For the other direction, we assume that we arrived at stage $\beta \in [\delta_1, \delta_1 + \delta_2)$ when forming $\mathbb{P}_1 \times \mathbb{P}_2$ using F' and we are in case (a) there with the pair of reals (x, y_0) witnessing this. We shall show that we are also in case (a) at stage $\beta - \delta_1$ when forming $\mathbb{P}_2$ using F_2 over W.

Assume not, then working in $W[\mathbb{P}^2_\beta]$, for every $(x, y) \in A_m$, $(x, y) \in W[G_A]$, there is is an allowable $\mathbb{P}_{x,y} \in W[\mathbb{P}^2_\beta]$ which forces $\mathbb{P}_{x,y} \Vdash (x, y) \notin A_m$. But then there is also an allowable $\mathbb{P}_{x,y_0} \in W[\mathbb{P}^2_\beta]$ such that $\mathbb{P}_{x,y_0} \Vdash (x, y_0) \notin A_m$. But $\mathbb{P}_{x,y_0}$ is also in $W[\mathbb{P}_1][\mathbb{P}^2_\beta]$ and is also allowable there, which is a contradiction again.

So we must be in the same cases at stage β when defining $\mathbb{P}_2(\beta)$ over $W[\mathbb{P}_1]$ and when defining $\mathbb{P}_2(\beta)$ using F_2 over W. But then we let $F'(\beta)$ be such that it does exactly what $F_2(\beta)$ does. This implies that $\mathbb{P}_1 \times \mathbb{P}_2 \restriction \beta + 1$ is 1-allowable with respect to $F' \restriction \beta + 1$ and the induction step is proven.

For β being limit there is noting to show as the β-th forcing is uniquely determined by $\mathbb{P}_{\beta'}, \beta' < \beta$.

Thus F' witnesses that $\mathbb{P}_1 \times \mathbb{P}_2$ is 1-allowable.

□

The second thing we note is that a 1-allowable forcing $\mathbb{P}^0$ can always be further extended by another 1-allowable forcing such that $f(m, x)$-values of rank 0 which are defined along $\mathbb{P}^0$ satisfy the desired form of uniqueness in terms of which potential $f(m, x)$ values of $A_{m,x}$ have not been coded into $\vec{S}$.

In the following we will simply write $\mathbb{P}^0 \Vdash$ "$\mathbb{P}^1$ is 1-allowable" as a short way of saying that the (over W) 1-allowable $\mathbb{P}^0 := (\mathbb{P}^0_\beta : \beta < \delta^0)$ and $I^0 := (I^0_\beta : \beta < \delta^0)$ forces that $(\mathbb{P}^1_\beta : \beta < \delta^1)$ and $(I^1_\beta : \beta < \delta^1)$ are 1-allowable with respect to some F^1 over $W[\mathbb{P}^0]$ with initial values $(\mathbb{P}^0, I^0)$.

Lemma 3.14. *Let $\gamma < \omega_1$, and let $(\mathbb{P}^0_\beta : \beta < \gamma) = \mathbb{P}$ and $I^0_\gamma = I$ be such that $(\mathbb{P}^0, I^0)$ is a 1-allowable forcing over W with respect to some $F : \gamma \to W^3$. Let $x \in W[\mathbb{P}]$ and $m \in \omega$.*

If there is a real $y \in W[\mathbb{P}^0]$ such that $(x, y, m, 0) \in I^0$, then for every $\mathbb{P}^1 \in W[\mathbb{P}^0]$ which satisfies that $\mathbb{P}^0 \Vdash$ "$\mathbb{P}^1$ is 1-allowable", there is a further $\mathbb{P}^2 \in W[\mathbb{P}^0 * \mathbb{P}^1]$ such that $\mathbb{P}^0 * \mathbb{P}^1 \Vdash$ "$\mathbb{P}^2$ is 1-allowable" and such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$ there is a pair of reals x, y_0 such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$, if I^2 is the unique set of potential $f(\cdot, \cdot)$-values one obtains when forming the 1-allowable $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2 * \mathbb{P}^3$, then $(x, y_0, m, 0)$ is the unique quadruple in I^2 containing m and x which is not coded into $\vec{S}$.

Also there is no $\mathbb{P}^3 \in W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$ such that

$$W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2] \models \text{"}\mathbb{P}^3 \textit{ is 1-allowable"}$$

and such that

$$W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2 * \mathbb{P}^3] \Vdash (x, y_0) \notin A_m.$$

Proof. We argue towards the first assertion. We let $G^0 = G^0_\gamma \subset \mathbb{P}^0$ be a $\mathbb{P}^0$-generic filter over W, fix $x \in W[G^0] \cap 2^\omega$ and $m \in \omega$ for which there is a y with $(x, y, m, 0) \in I^0$. As (x, y, m) has rank 0, there must be a least stage $\beta < \gamma$ in our iteration for which there is an $A \subset \beta$, $A \in W$ and $\mathbb{P}^0_A$-names $\dot{x}$, $\dot{y}'$ such that $F(\beta)_0 = (\dot{x}, \dot{y}', \dot{m})$, $\dot{x}^{G^0_A} = x$, $\dot{y}'^{G^0_A} = y'$ and $\dot{m}^{G^0_A} = m$ and case (a) applies at stage β with (x, y) as our new f_m value of rank 0.

We claim, that if $\eta > \beta$ and $F(\eta)_0 = (\dot{x}', \dot{z}, \dot{m}')$, such that $\dot{x}'^{G^0_\eta} = x$, $\dot{z}^{G^0_\eta} = z$ and $\dot{m}'^{G^0_\eta} = m$, and the $B \subset \eta$, $B \in W$ is such that $\dot{x}$ and $\dot{z}$ are $\mathbb{P}_B$-names and $(x, z, m) \in W[G_B]$ is such that $A \subset B$, then again case (a) must apply.

Indeed if $\eta > \beta$ is such a stage, then our (x, y) is such that $W[G_\eta] \models (x, y) \in A_m$, as the intermediate iteration between stage β and stage η is a 0-allowable forcing. Moreover we have that

$$W[G^0_\eta] \models \text{“}\forall \mathbb{P}'(\mathbb{P}' \text{ is 0-allowable”} \rightarrow \mathbb{P}' \Vdash (x, y) \in A_m),$$

as otherwise, by the closure of 0-allowable forcings under products, it would witness that we were not in case (a) at stage β, which is a contradiction to our assumption. Hence (x, y) still witnesses that we are in case (a) at stage η. Consequentially whenever $\mathbb{P}^1 * \mathbb{P}^2 \in W$ is such that $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is 1-allowable over W, and $F^2 \in W$ is a bookkeeping function witnessing this, then whenever η and $B \supset A$ are such that $F^2(\eta)_0 = ((\dot{u}, \dot{z}, \dot{n})$, the latter being $\mathbb{P}_B$-names which satisfy $\dot{u}^{G_\eta} = x$, $\dot{z}^{G_\eta} = z$ and $\dot{n}^{G_\eta} = m$, for a $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\eta$-generic filter G_η, then again case (a) must apply. Thus the claim is proved.

Let $\mathbb{P}^1 \in W$ be such that $W \models$ “$\mathbb{P}^0 * \mathbb{P}^1$ is 1-allowable” and let $F^1 \in W$, $F^1 : \gamma^1 \rightarrow W$, be a bookkeeping which witnesses this. In order to prove the lemma we shall define a bookkeping $F^2 \in W$, $F^2 : \omega_1 \rightarrow W$ (note that the domain of F^2 is ω_1) such that if $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is the outcome of the applications of the rules (a) and (b) for 1-allowability guided by F^2, then, there is a stage $\gamma_2 < \omega_1$ such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\gamma]$ there is (x, y_0) such that $(x, y_0, m, 0)$ is the unique quadruple in $I^2_{\gamma_2}$ which is not coded into $\vec{S}$.

We let $F^2 \upharpoonright \gamma^1 = F^1 \upharpoonright \gamma^1$, and for $\xi \geq \gamma^1$, $\xi < \omega_1$ we define $F^2(\xi) = (F^2(\xi)_0, F^2(\xi)_1, F^2(\xi)_2)$ in the following way, which will ensure that $\mathbb{P}^2$ is as desired. First let

$$F^2(\xi)_0 := (\dot{x}, \dot{z}, \check{m})$$

where $\dot{x}$ is the $<$-least nice $\mathbb{P}^0$-name for x and $\dot{z}$ is the $<$-least nice $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\xi$-name for a real such that there is no proper subset $B \subsetneq \gamma_0 + \gamma_1 + \xi$ such that $\dot{z}$ is a $\mathbb{P}_B$-name. Note that such a name $\dot{z}$ always exists, e.g. we can pick the $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\xi$-name of a real $\dot{z}$ to code all names for generic filters for all almost disjoint coding forcings used so far in $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\xi$.

Further we let $F^2(\xi)_1$ be such that it picks the $<$-least, nice $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\xi$-name for a countable, infinite set $E \in W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\xi]$ which contains the set

$I^0 \cup I^1 \cup I^2_\xi$. Note that such an E must exist as we use iterations which have countable length, hence each I^i must be countable. Then we define

$$F(\xi)_2 := \dot{E}.$$

As usual we form $\mathbb{P}^2(\xi)$ accordingly.

We claim now that there is a $\gamma_2 < \omega_1$, $\gamma_2 \geq \gamma_1$, such that $F^2 \upharpoonright \gamma_2$ witnesses that $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_{\gamma_2}$ is as desired.

Indeed, by the choice of the $\dot{z}$ from $F^2(\xi)_0 = (\dot{x}, \dot{z}, \dot{m})$, there must be a stage $\eta < \omega_1$, which is such that from η on, we will always put some fixed quadruple $(\dot{x}, \dot{y}_0, \dot{m}, 0)$ to $I_{\eta'}$, where $\eta' > \eta$, as otherwise we would obtain an infinite $<$-decreasing chain of forcing names, which is nonsense. Now, by the above claim, we must be in case (a) at both stages η and $\eta + 1$. This in particular means that $\mathbb{P}^2(\eta)$ is such that (x, y_0) can not be kicked out of A_m anymore with additional 0-allowable forcings.

As a consequence, the set E which $F^2(\eta + 1)_2$ picks, will contain I^2_η, so we will code up all elements of I^2_η except (x, y_0, m) into $\vec{S}$. Thus $\gamma_2 = \eta + 2$ and $\mathbb{P}^2_{\eta+2}$ is as desired.

The second assertion of the lemma is clear and is actually proved already above for (x, y), as if η is minimal such that $(x, y_0, m, 0)$ is put into I^2_η, then no 0-allowable forcing can kick (x, y_0) out of A_m. But the tail of the iteration $\mathbb{P}_{[\eta, \gamma_2)}$ is such that

$$W[G_{\eta+1}] \models \text{“}\mathbb{P}_{[\eta,\gamma_2)} \text{ is 0-allowable”}.$$

Hence (x, y_0) must be in A_m throughout the iteration as claimed.

$\square$

In particular, the last Lemma tells us that for those A_m where we found f-values of rank 0 in a 1-allowable iteration, these f-values are valid ones as they will stay in A_m throughout the 1-allowable iteration and we can always use additional 1-allowable forcings such that there is exactly one value for every x and m with non-empty A_m section at x which is not coded into $\vec{S}$.

So 1-allowable forcings already provide a first step in finding reasonable candidates for the f_m-values. Nevertheless there are still issues, stemming from the usual “moving target” problem. Indeed, when defining 1-allowable we ask at every stage if we can find a pair (x, y) for A_m such that (x, y) will remain in A_m for all additional 0-allowable $\mathbb{P}'$. But when moving on in our 1-allowable iteration we will not just produce a 0-allowable iteration, we will in fact produce a 1-allowable iteration, so we should additionally ask at every stage whether we can find (x, y) such that (x, y) can not be kicked out of A_m by a further 1-allowable forcing. After all, these new pairs (x, y) would be good candidates for our uniformizing f_m as well, as long as we continue to force with allowable forcings which are also 1-allowable which is exactly what we do when forcing with a 1-allowable iteration. This additional question we add at every stage will yield the notion of 2-allowable, and this reasoning can now be iterated transfinitely often.

3.2.7 $\alpha + 1$-allowability

We define next a derivative acting on the set of allowable forcings over W. Inductively we assume that for an ordinal α and any bookkeeping function $F \in W$, we have already defined the notion of η-allowable with respect to F for every $\eta \leq \alpha$. We also assume that the definitions of η-allowable work in a uniform way for any allowable extension of W and all starting values $(\mathbb{P}_0, I_0) \in W$. This is in line with the already observed behaviour of 0-allowable and 1-allowable forcings. In particular this means that for an arbitrary allowable generic extension W' of W, and every $\eta \leq \alpha$, we have defined already a set of rules which, in combination with a bookkeeping $F \in W'$ will produce over W':

- An allowable forcing $\mathbb{P} = \mathbb{P}_\delta = (\mathbb{P}_\beta : \beta < \delta) \in W'$, the actual forcing which is used in the iteration. We let G_δ denote a $\mathbb{P}_\delta$-generic filter over W'.
- A set $I = \dot{I}_\delta^{G_\delta} = \{(\dot{x}^{G_\delta}, \dot{y}^{G_\delta}, \dot{m}^{G_\delta}, \dot{\gamma}^{G_\delta}) : m \in \omega, \dot{x}, \dot{y}, \dot{\gamma}$ are $\mathbb{P}$-names of elements of ω, 2^ω and ω_1 respectively$\}$. The set $I \in W'[G_\beta]$ is the set of potential values for the uniformizing function f, we want to define. We note that there can be several values $(x, y_1, m, \xi_1), ..., (x, y_n, m, \xi_n)$ for one x and one m. We say that (x, y, m) has rank ξ if $(x, y, m, \xi) \in I$. Again a (x, y, m) can have several ranks. The idea here is to use the (x, y, m)'s whose rank is minimal for our eventual values of $f_m(x)$, and for which we will prove a uniqueness result, similar to Lemma 3.14, so the choice is well-defined.

Similar to our already established jargon, if the result of applying the rules for η-allowable over the model W and $F \in W$ is the pair $(\mathbb{P}, I) \in W$ then we say that $\mathbb{P}$ is η-allowable with respect F (over W), or often just $\mathbb{P}$ is η-allowable if there is an F, I such that $\mathbb{P}$ is η-allowable with respect to F. Likewise we will say that some forcing $\mathbb{P} \in W$ is η-allowable with starting values $(\mathbb{P}_0, I_0) \in W$ over the model W.

Given that we know everything above we aim to define the derivation of the α-allowable forcings over W (but the definition works in a uniform way for any allowable extensions W' of W) which we call $\alpha + 1$-allowable (again over W). The definition is a uniform extension of 1-allowability. A $\delta < \omega_1$-length iteration $\mathbb{P} = (\mathbb{P}_\beta : \beta < \delta) \in W$ is called $\alpha + 1$-allowable over W (or relative to W) if it is recursively constructed using two ingredients. First a bookkeeping function $F \in W$, $F : \delta \to W^3$, where for every $\beta < \delta$, we write $F(\beta) = ((F(\beta)_0, (F(\beta))_1, (F(\beta))_2)$ for the according values of the coordinates. Second two cases which are similar to the ones for 1-allowability, which add to the cases of α-allowable one additional rule, and which determine along with F how the iteration $\mathbb{P}$ and the set of f-values I are constructed from arbitrary starting values $(\mathbb{P}_0, I_0) \in W$.

The two cases shall be defined now. We fix a bookkeeping function $F \in W$, $F\delta \to W^3$ for $\delta < \omega_1$. We assume that we are at stage β of our construction and we assume inductively that we already created the following list of objects:

- The forcing $\mathbb{P}_\beta \in W$ up to stage β, along with a $\mathbb{P}_\beta$-generic filter G_β over W. We let $\mathbb{P}_0 = \emptyset$.
- The set $I_\beta = \dot{I}_\beta^{G_\beta} = \{(\dot{x}^{G_\beta}, \dot{y}^{G_\beta}, \dot{m}^{G_\beta}, \dot{\zeta}^{G_\beta}) : \dot{m}, \dot{x}, \dot{y}, \dot{\zeta}$ are $\mathbb{P}_\beta$-names of elements of ω, 2^ω and ω_1 respectively$\}$ of already defined, potential values for the uniformizing function $\dot{f}^{G_\beta}(m, \cdot)$. We let $I_0 = \emptyset$.

We emphasize that the set of possible f-values will change along the iteration. The iteration is defined in a way, that values of f must be added if we encounter a new and possible value of $\dot{f}^{G_\beta}(m, x)$ of lesser rank. Working in $W[G_\beta]$ we shall now define the next forcing of our iteration $\mathbb{P}(\beta)$ together with a possibly updated set of possible values for the uniformizing function $f(m, x)$. We assume that $F(\beta)_0 = (\dot{x}, \dot{y}, \dot{m})$ and let $A \subset \beta$, $A \in W$ be such that $\dot{x}, \dot{y}, \dot{m}$ are $\mathbb{P}_A = \bigstar_{\eta \in A} \mathbb{P}(\eta)$-names, where we demand that $\mathbb{P}_A \in W$ and a subforcing of $\mathbb{P}_\beta$. We let $x = \dot{x}^{G_\beta}$, $y = \dot{y}^{G_\beta}$ and $\dot{m}^{G_\beta}$ and split into cases:

(a) Let $G_A := G_\beta \upharpoonright A$.

   - There is an ordinal $\zeta < \alpha + 1$, which is chosen to be minimal for which the following holds:
   - First we collect all $\mathbb{P}_\beta$-names for reals $\dot{a}$ such that $\dot{a}^{G_\beta}$ is in fact an element of $W[G_A]$. For every such $\mathbb{P}_\beta$-name $\dot{a}$ we pick the $<$-least, nice name $\dot{b} \in W^{\mathbb{P}_\beta}$ such that $\dot{a}^{G_\beta} = \dot{b}^{G_\beta}$ and collect the these names $\dot{b}$ into a set called C. We assume that there is a $<$-least, nice $\mathbb{P}_A$-name $\dot{y_0}$ in C such that $\dot{y_0}^{G_A} = y_0$,

     $$W[G_\beta] \models (x, y_0) \in A_m$$

     and for which there is no further ζ-allowable forcing $\mathbb{R} \in W[G_\beta]$ such that $W[G_\beta] \models$ "$\mathbb{R} \Vdash (x, y_0) \notin A_m$. If this is the case, then we set the following:

   - We assume first that $F(\beta)_1 = \dot{E}$ where $\dot{E}$ is a $\mathbb{P}_A$-name of a countable, infinite set of reals. If we let $E = \dot{E}^{G_A}$ then we define, using mixed support

     $$\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}.$$

Else we just pick the $<$-least $\mathbb{P}_A$-name for a set $\dot{E}$, where $\mathbb{P}_A \Vdash \dot{E} \in [2^\omega]^\omega \wedge |\dot{E}| = \aleph_0$. We define $\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$.
We also let $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{P}(\beta)$ and let $G_{\beta+1} = G_\beta * G(\beta)$ be its generic filter.

- We set a new f value, i.e. we set $f(m,x) := y_0$ and assign in $W[G_{\beta+1}]$ the rank ζ to the value (x, y_0, m). We update $I_{\beta+1}^{G_{\beta+1}} := I_\beta^{G_\beta} \cup \{(x, y_0, m, \zeta)\}$.

Note that for every forcing $\mathbb{P} \in W[G_{\beta+1}]$ such that $W[G_{\beta+1}] \models \mathbb{P}$ is ζ-allowable, we have that $W[G_{\beta+1}] \models \mathbb{P} \Vdash (x, y_0) \in A_m$, by construction.

(b) We assume that case (a) is not true. In that situation we again let the bookkeeping F fully guess what to force with. We assume that $F(\beta)_1$ is a nice $\mathbb{P}_A$ name for a pair of reals of the form $(\dot{x'}, \dot{y}_0)$ such that $\dot{x'}^{G_A} = x$ together with a name for an ordinal $\dot{\xi}$ such that $\mathbb{P}_A \Vdash \dot{\xi} > \alpha$. We assume that $F(\beta)_2$ is a $\mathbb{P}_A$-name of an infinite, countable set of reals $\dot{E}$, then letting $\dot{E}^{G_A} = E$ we define $\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$. We let $G(\beta)$ be a $\mathbb{P}(\beta)$-generic filter over $W[G_\beta]$ and $G_{\beta+1} = G_\beta * G(\beta)$.

Further we update our set I_β of preliminary values for f to

$$I_{\beta+1} := I_\beta \cup \{(x, y_0, m, \xi)\}.$$

Otherwise, i.e. when $F(\beta)_1, F(\beta)_2$ do not have the desired form we pick the $<$-least pair of $\mathbb{P}_A$-names of reals, $(\dot{x'}, \dot{y}_0)$ such that $\dot{x'}^{G_A} = x$, pick the least $\mathbb{P}_A$-name $\dot{E}$ of an ω-sized set of reals and, working in $W[G_A]$, let

$$\mathbb{P}(\beta) := \prod_{s \in E \wedge s \neq y_0} \mathbb{P}_{(x,s,m)}$$

Also we let $G(\beta)$ be a $\mathbb{P}(\beta)$-generic filter over $W[G_\beta]$ and set $W[G_{\beta+1}] = W[G_\beta * G(\beta)]$.

Then, working in $W[G_{\beta+1}]$, we update $I_{\beta+1} := I_\beta \cup \{(x, y_0, m, \alpha + 1)$.

At limit stages η of $\alpha + 1$-allowable forcings we take the inverse limit of the initial segments, i.e.

$$\mathbb{P}_\eta := inv\, lim(\mathbb{P}_\nu \, : \, \nu < \eta).$$

Finally we let

$$I_\eta^{G_\eta} := \{(m, x, y, \zeta) \, : \, \exists \xi < \eta((m, x, y, \zeta) \in I_\xi^{G_\xi})\}.$$

This ends the definition of the rules for $\alpha + 1$-allowability over the ground model W. To summarize:

Definition 3.15. *Assume that $F \in W$, $F : \eta \to W^3$ is a bookkeeping function and that $\mathbb{P} = (\mathbb{P}_\beta : \beta < \eta)$ and $I = (I_\beta : \beta < \eta)$ is the result of applying the above defined rules together with F over W. Then we say that $(\mathbb{P}, I)$ is $\alpha + 1$-allowable with respect to F (over W). Often, I is clear from context, and we will just say $\mathbb{P}$ is $\alpha + 1$-allowable with respect to F. We also say that $\mathbb{P}$ is $\alpha + 1$-allowable over W if there is an F such that $\mathbb{P}$ is $\alpha + 1$-allowable with respect to F.*

We add a couple of remarks concerning the definition of $\alpha + 1$-allowable:

- The case (a) is the iterated version of case (a) in the definition of 1-allowable. Note that we minimize on the rank ζ of the potential $f(m, x)$-value. The reason for this is that this makes it easier to show that the notion of α-allowable becomes stronger and stronger as we increase α, as we will prove later.
- The definition of $\alpha + 1$-allowable adds one more constraint to the definition of α-allowable in case (a) in that it considers not only forcings which are β-allowable for $\beta < \alpha$, but also considers α-allowable forcings as well. So it is intuitively clear, and will be proved in Lemma 3.18 below, that the set of α-allowable forcings is shrinking as α increases. This in effect yields that there are more and more pairs of reals $(x, y) \in A_m$ which can not be kicked out of A_m any more by additional α-allowable forcings, as α grows. Which in turn yields more cases where (a) must apply, so more constraints in the definition of α allowable as α rises. So the shrinking process of α-allowable forcings, as α increases, reinforces itself due to the choice of the definitions.
- If $(\mathbb{P}_\beta : \beta < \delta)$ is $\alpha + 1$-allowable over W (with trivial initial values), and if $\beta_0 < \delta$ is an intermediate stage where case (a) of the definition of $\alpha + 1$-allowable applies. Then the tail forcing $\mathbb{P}_{[\beta_0, \delta)}$ is such that $\mathbb{P}_\beta \Vdash$ "$\mathbb{P}_{[\beta_0, \delta)}$ is $\alpha + 1$-allowable with starting values $(\mathbb{P}_\beta, I_\beta)$".

3.2.8 Definition of $\alpha+1$-allowability over arbitrary ground models

We note that the definition of $\alpha + 1$-allowability carries over to arbitrary, 0-allowable generic extensions of $W[\mathbb{Q}]$. This is fully analogous to the 1-allowable case.

Definition 3.16. *Let $\mathbb{P}^0 \in W$ be an allowable forcing. If $F \in W[\mathbb{P}^0]$, $F : \eta \to W[\mathbb{P}^0]^3$ is a bookkeeping function and if $\mathbb{P}$ and I are the result of applying the two cases for $\alpha+1$-allowability together with F and initial values $\mathbb{P}^0 \neq \emptyset$, and $I^0 \neq \emptyset$, then we say that $\mathbb{P}$ is $\alpha + 1$-allowable with respect to F and starting values $(\mathbb{P}^0, I^0)$ over $W[\mathbb{P}^0]$.*

As before, it follows readily from the above definition that if $\mathbb{P}^0 := (\mathbb{P}^0_\beta : \beta < \delta^0)$ and $I^0 := (I^0_\beta : \beta < \delta^0)$ are $\alpha + 1$-allowable with respect to some F^0 over W and if $(\mathbb{P}^1_\beta : \beta < \delta^1)$ and $(I^1_\beta : \beta < \delta^1)$ are $\alpha + 1$-allowable with respect to some F^1 over $W[\mathbb{P}^0]$ with initial values $(\mathbb{P}^0, I^0)$, then the two step iteration of $\mathbb{P}^0$ and then $\mathbb{P}^1$ together with I^1 is $\alpha+1$-allowable over W whose witness is the concatenation of the bookkeeping function F followed by (the accordingly slightly reformulated version of) F'.

The other direction is also true. In particular, given an $\alpha + 1$-allowable iteration $(\mathbb{P}_\beta : \beta < \delta)$ and $(I_\beta : \beta < \delta)$ over W of length $\delta > 1$ then whenever we let $0 < \eta < \delta$ and split the iteration into two parts $(\mathbb{P}_\beta : \beta < \eta)$, $(I_\beta : \beta < \eta)$ and the tail $(\mathbb{P}_\beta : \eta < \beta < \delta)$, $(\mathbb{R}_\beta : \eta < \beta < \delta)$ and $(I_\beta : \eta < \beta < \delta)$ then the second part of that split will always be a $\alpha + 1$-allowable iteration over $W[\mathbb{P}_\eta]$ with starting values $(\mathbb{P}_\eta, I_\eta)$.

3.2.9 Definition of α-allowable for limit α

Next we want to define the notion α-allowable, when α is a limit ordinal. First we assume inductively that we know already, what β-allowable means, for any $W[G]$, where G is generic for some allowable forcing. Note that this assumption is perfectly sound by the discussion above. Then α-allowable will be defined using the very same rules as above, using some bookkeeping F in the background. Thus, at every stage β of an α-allowable forcing, if (x, y') is handed to us by the bookkeeping (together with the generic filter G_β) and , we ask whether there exists for $\zeta = 0$ a pair (x, y), a forcing $\mathbb{P}$ and a γ such that the conclusion of (a) becomes true. If not then we ask the same question for $\zeta = 1$, and so on. If (a) never applies for any $\zeta < \alpha$ we pass to (b). To summarize:

Definition 3.17. *Let $\eta < \omega_1$, $F \in W$ and $F : \eta \to W^3$. Let α be a limit ordinal and assume that $\mathbb{P} = (\mathbb{P}_\beta : \beta < \eta)$ and $I_\beta = (I_\eta : \eta < \beta)$ is the outcome of applying the rules (a) and (b). together with F over W. Then we say that the pair $(\mathbb{P}, I)$ is α-allowable with respect to F over W. The notion of α-allowable with respect to F and starting values $(\mathbb{P}_0, I_0)$ is defined in the now well-known, analogous way.*

3.2.10 Properties of $\alpha + 1$-allowable forcings

Lemma 3.18. *Work in W. If $\mathbb{P}$ is β-allowable over W and $\alpha < \beta$, then $\mathbb{P}$ is α-allowable over W. Thus the sequence of α-allowable forcings (over W) is decreasing with respect to the $\subset$-relation.*

Proof. Let $\alpha < \beta$, let $\mathbb{P}$ be a β-allowable forcing and let $F \in W$ be the bookkeeping function which, together with the rules from above determine $\mathbb{P} \in W$. We will show that there is a bookkeeping function $F' \in W$ such that $\mathbb{P}$ can be seen as an α-allowable forcing determined by F'. The idea is to let the new bookkeeping function F' be such that it simulates the reasoning we would do for a β-allowable forcing at every stage, even though it is an α-allowable forcing.

We start with setting $F'(\eta) = F(\eta)$ until we hit a stage where a difference in what case applies occurs for the first time. Let γ be the least stage such that F together with the rule applied at γ, when considering $\mathbb{P}$ as an α-allowable forcing yields a different case than when considering $\mathbb{P}$ as a β-allowable forcing. By the minimality of γ, $\mathbb{P}_\gamma$ and I_γ must coincide when considering $\mathbb{P}_\gamma$ as an α and a β-allowable forcing respectively. It is clear from the definitions, that at stage γ, when working with the β-allowable rules case (a) must apply whereas case (b) applies when working with the rules for α-allowable.

So $F(\gamma)_0 = (\dot{x}, \dot{y}, m)$, and as usual we let G_γ be the generic filter for the forcing and let $x = \dot{x}^{G_\gamma}$ and $y = \dot{y}^{G_\gamma}$, and there is a potential $f(m,x)$-value of rank $\leq \beta$ in the universe $W[G_A]$, where $A \subset \gamma$ is such that $\dot{x}$ and $\dot{y}$ are in fact $\mathbb{P}_A$-names, $(x,y) \in W[G_A]$; on the other hand there is no potential $f(m,x)$-value of rank $\leq \alpha$. To be more precise, at stage γ when working with the rules for β-allowable we obtain:

- A quadruple $(x, y_0, m, \xi) \in I_{\gamma+1}$, where $\xi \in (\alpha, \beta]$ and $(x, y_0) \in W[G_A]$, where $A \subset \gamma$ is such that $\dot{x}$, $\dot{y}_0$ are in fact $\mathbb{P}_A$-names.
- A countably infinite set $E = \dot{E}^{G_A}$ of reals.
- A forcing $\mathbb{P}(\gamma) = \prod_{s \in E s \neq y_0} \mathbb{P}_{(x,s,m)}$.

We define $F'(\gamma)$ as follows: $F'(\gamma) = (F'(\gamma)_0, F'(\gamma)_1, F'(\gamma)_2)$ such that $F'(\gamma)_0 = F(\gamma)_0$ and such that $(x, y, m, \zeta) \in I_{\gamma+1}$ and $\mathbb{P}(\gamma)$ are guessed correctly by $F'(\gamma)$. To be more precise we let $F'(\gamma)_i$ be $\mathbb{P}_\gamma$-names such that uniformly, for any $\mathbb{P}_\gamma$-generic filter G_γ:

- $F'(\gamma)_1^{G_\gamma} := (x, y_0, \xi)$,
- $F'(\gamma)_2^{G_\gamma} := E$.

Note that this definition of F' is entirely in W, the use of G_γ in its definition is uniform and can, as always be removed in the common way.

The upshot is that when applying the rules for α-allowable at stage γ using F', the result is exactly the same as when applying the rules for β-allowable at γ using F.

To summarize, if γ is the least stage such that we find ourselves in different cases when following the rules for α and β-allowable using F, then there is an F' such that the β-allowable iteration $\mathbb{P}_{\gamma+1}$ using F is also an α-allowable iteration using F'. But this line of argumentation can be iterated. Indeed, after we dealt with (m, x, y) at γ, we can go to the least stage $\gamma' > \gamma$ where the rules for β-allowable using F yield a different case than the rules for α-allowable using F'. We apply the same arguments from above to see that we can pretend that we are in an α-allowable iteration as we proceed in $\mathbb{P}$. Until we hit a new triple for which again we use the just described argument and so on. Thus there is a bookkeeping F' such that $\mathbb{P}$ is α-allowable with respect to F'.

□

The above proof immediately generalizes.

Corollary 3.19. *Let $\mathbb{P}^0 \in W$ be allowable and let I^0 be an arbitrary set. If $\mathbb{P}$ is β-allowable over $W[\mathbb{P}^0]$ with starting values $(\mathbb{P}^0, I^0)$ and $\alpha < \beta$, then $\mathbb{P}$ is α-allowable over $W[\mathbb{P}^0]$ with starting values $(\mathbb{P}^0, I^0)$.*

Lemma 3.20. *Let $F_1 : \delta_1 \to W^3$ be a bookkeeping function which determines an α-allowable forcing $\mathbb{P}^1 = (\mathbb{P}^1_\beta : \beta < \delta_1)$. Likewise let $F_2 : \delta_2 \to W^3$ be a bookkeeping function which determines an α-allowable forcing $\mathbb{P}^2 = (\mathbb{P}^2_\beta : \beta < \delta_2)$. Then the product $\mathbb{P}_1 \times \mathbb{P}_2$ is α-allowable relative to a bookkeeping function $F \in W$ which is definable from F_1 and F_2.*

Proof. The proof is very similar to the already established lemma 3.13. We will prove it by induction on α. For $\alpha = 0$ and $\alpha = 1$ the lemma is true.

Now assume that the lemma is true for α. We shall argue that it is also true for $\alpha + 1$. We define $F : \delta_1 + \delta_2 \to W^3$ as follows. For $\beta < \delta_1$, we let $F(\beta) = F_1(\beta)$. Note then that the outcome of producing an $\alpha + 1$-allowable forcing using $F \upharpoonright \delta_1$ is trivially $\mathbb{P}^1$. For $\beta \geq \delta_1$ we set $F(\beta) = F_2(\beta - \delta_1)$.

Claim. *If we build the $\alpha + 1$-allowable forcing using $F \upharpoonright [\delta_1, \delta_1 + \delta_2)$ over $W[\mathbb{P}^1]$, then the outcome is $\mathbb{P}^2$.*

Proof of the Claim. We show first, that if $\beta < \delta_2$ is a stage and we define $\mathbb{P}^2$ over W using F_2 and case (a) in the definition of $\alpha + 1$-allowable applies, then case (a) also applies at stage $\beta + \delta_1$ when defining the $\alpha + 1$-allowable forcing using $F \upharpoonright [\delta_1, \delta_1 + \delta_2)$ over $W[\mathbb{P}^1]$ and vice versa.

This is again shown by induction, this time on $\beta \in [0, \delta_2)$. Assume that the claim is true for a $\beta \in [0, \delta_2)$, we shall show that at stage $\beta + 1$, if we are in case (a) when constructing an $\alpha + 1$-allowable $\mathbb{P}^2$ over W using F^2, then we must be in case (a) at stage $\beta + 1 + \delta_1$ when using F over $W[\mathbb{P}^1]$ and vice versa.

So, we are at stage $\beta + 1$, and case (a) applies for $F^2(\beta + 1)$ when working over W. This means that there is a $\xi < \alpha + 1$ and $F^2(\beta + 1)_0 = (\dot{x}, \dot{y}, \dot{m})$ is a pair of $\mathbb{P}_A \subset \mathbb{P}^2_\beta$-names of reals, and in $W[G_A]$, there is a pair (x, y) which will remain in A_m for all future ξ-allowable forcings over $W[\mathbb{P}^2_\beta]$.

Note now that $\mathbb{P}^1 * \mathbb{P}^2_\beta$ is $\alpha + 1$-allowable, by induction hypothesis, thus it is also ξ-allowable by the last lemma. Assume now that we are in case (b) at stage $\beta + 1 + \delta_1$ when working over $W[\mathbb{P}^1]$ using F. We know that $F(\beta + 1 + \delta_1) = F_2(\beta + 1)$, thus $F(\beta + 1 + \delta_1)_0 = (\dot{x}, \dot{y}, \dot{m})$ and following the rules of $\alpha+1$-allowability, we scan through the very same $W[G_A]$ for a potential $f_m(x)$-value of rank ξ. Assume for a contradiction that we are in case (b) at stage $\beta+1+\delta_1$. Thus for $(x,y) \in W[G_A]$ there is a ξ-allowable forcing $\mathbb{Q} \in W[\mathbb{P}^1 * \mathbb{P}^2_\beta]$ such that $\mathbb{Q} \Vdash (x,y) \notin A_m$. But then $\mathbb{P}_1 \times \mathbb{Q} \in W[\mathbb{P}^2_\beta]$ is ξ-allowable, by induction hypothesis as $\xi < \alpha+1$ and by Shoenfield absoluteness $\mathbb{P}_1 \times \mathbb{Q} \Vdash (x,y) \notin A_m$. But this means that at stage $\beta+1$, when working over W using F^2, there is a ξ-allowable forcing which kicks (x,y) out of A_m, thus we were not in case (a) at stage $\beta+1$ when using F^2 and working over W to define $\mathbb{P}^2$, which is a contradiction.

If on the other hand, $\beta+1+\delta_1$ is such that we are in case (a) when using F over W, and we consider stage $\beta+1$ using F_2 over W, we shall show that we are in case (a) there as well. Thus, if $F(\beta+1+\delta_1)_0 = (\dot{x},\dot{y},\dot{m})$ and the latter is a $\mathbb{P}_A$-name for $A \subset \beta+1$, we find a $\zeta < \alpha+1$ and $(x,y) \in W[G_A]$ which can not be kicked out of A_m by a further $\mathbb{R} \in W[\mathbb{P}^1][\mathbb{P}^2_\beta]$ which is ζ-allowable over $W[\mathbb{P}^1][\mathbb{P}^2_\beta]$.

By the definition of F, $F^2(\beta+1)_0 = F(\beta+1+\delta_1) = (\dot{x},\dot{y},\dot{m})$ and we have to scan through the very same $W[G_A]$ for a potential $f(m,x)$ value, when working over W using F_2 at stage $\beta+1$ as well. Assume we are not in case (a) there, then for every potential pair of reals $(x,z) \in W[G_A]$, there is an α-allowable forcing $\mathbb{R}(x,z) \in W[\mathbb{P}^2_\beta]$ such that $\mathbb{R}(x,z) \Vdash (x,z) \notin A_m$. This holds in particular for the (x,y) from above. But then $\mathbb{R}(x,y)$ is α-allowable in $W[\mathbb{P}^2_\beta]$ hence $\mathbb{R}(x,y) \times \mathbb{P}^1$ is ξ-allowable in $W[\mathbb{P}^2_\beta]$, and $\mathbb{R}(x,y) \times \mathbb{P}^1 \Vdash (x,y) \notin A_m$. Hence, we can not be in case (a) when at stage $\beta+1+\delta_1$ and working over W using F but this is what we assumed. This contradiction ends the proof of the claim. $\square$

Knowing that the claim is true we obtain that in particular $\mathbb{P}^1 * \mathbb{P}^2$ is $\alpha+1$-allowable and as $\mathbb{P}^1 * \mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^2$ the latter is $\alpha+1$-allowable. This ends the proof for $\alpha+1$-allowability.

If α is a limit ordinal, the proof is completely analogue. All one has to do is replace in the above proof every instance of $\zeta < \alpha+1$ with $\zeta < \alpha$. $\square$

As the tail of an α-allowable forcing is always seen to be α-allowable by the intermediate model with the according ground model, it follows that once we encounter a potential $f_m(x)$-value y of rank $< \alpha$, in an α-allowable iteration, that (x,y) will remain in A_m for the rest of the α-allowable iteration. To be more precise:

Lemma 3.21. *Let $(\mathbb{P}, I)$ be α-allowable over W with respect to F of length $\eta < \omega_1$, let G be $\mathbb{P}$-generic and let $(x,y,m,\xi) \in I$ for some $\xi < \alpha$. Then in $W[G]$, $(x,y) \in A_m$ and for every $\mathbb{Q} \in W[G]$ such that $\mathbb{P} \Vdash$ "$\mathbb{Q}$ is ξ-allowable", it holds that $\mathbb{Q} \Vdash (x,y) \in A_m$.*

Proof. Let β be the least stage in $\mathbb{P}$ such that (x,y,m,ξ) is added to I_β. Then, as $\xi < \alpha$, we must be in case (a) at stage β. This means that we found a pair (x,y) which can not be kicked out of A_m with an additional ξ-allowable forcing, for a $\xi < \alpha$ as in the lemma. The tail however is an α-allowable forcing over $W[G_\beta]$, hence also ξ-allowable and thus $(x,y) \in A_m$ throughout the tail of the iteration.

Adding an additional $\mathbb{Q}$ which is ξ-allowable to the tail of $\mathbb{P}$ does not alter the argument, which proves the second assertion of the lemma. $\square$

Lemma 3.22. *Let $\gamma < \omega_1$, and let $F : \gamma \to W^3$ be a bookkeeping function in W. Assume that $\mathbb{P}^0 = (\mathbb{P}^0_\beta : \beta < \gamma)$ and $I^0 = I^0_\gamma$ be such that $(\mathbb{P}^0, I^0)$ is an α-allowable forcing over W with respect to F. Let $x \in W[\mathbb{P}^0]$ and $m \in \omega$. If there is a real $y \in W[\mathbb{P}^0]$ and a $\xi < \alpha$ such that $(x,y,m,\xi) \in I^0$, then for every $\mathbb{P}^1 \in W$ which satisfies that $\mathbb{P}^0 * \mathbb{P}^1$ is α-allowable over W, there is a further $\mathbb{P}^2 \in W$ such that $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is α-allowable and such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$ there is a $\zeta \le \xi$ and a pair of reals x, y_0 such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$, if I^2 is the unique set of potential $f(\cdot,\cdot)$-values one obtains when forming the α-allowable $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, then (x, y_0, m, ζ) is the unique quadruple in I^2 containing m and x which is not coded into $\vec{S}$.*

Proof. This is very similar to the proof of Lemma 3.14. We let $G^0 = G^0_\gamma \subset \mathbb{P}^0$ be a $\mathbb{P}^0$-generic filter over W, fix $x \in W[G^0] \cap 2^\omega$ and $m \in \omega$ for which there is a y with $(x,y,m,\xi) \in I^0$. As (x,y,m) has rank $\xi < \alpha$, there must be a least stage $\beta < \gamma$ in our iteration for which there is an $A \subset \beta$, $A \in W$ and $\mathbb{P}^0_A$-names $\dot{x}$, $\dot{y}'$ such that $F(\beta)_0 = (\dot{x}, \dot{y}', \dot{m})$, $\dot{x}^{G^0_A} = x$, $\dot{y}'^{G^0_A} = y'$ and $\dot{m}^{G^0_A} = m$ and case (a) applies at stage β with (x,y) as our new f_m value of rank ξ.

We claim, that if $\eta > \beta$, $\mathbb{P}_\eta$ is an arbitrary α- allowable forcing over $W[\mathbb{P}_\beta]$, $G_\eta \subset \mathbb{P}_\eta$ a generic filter over W and $B \subset \eta$, $A \subset B$, $B \in W$, is such that $\mathbb{P}_B := \bigstar_{\beta \in B} \mathbb{P}(\beta)$ is an allowable forcing over W and a regular subforcing of $\mathbb{P}_\eta$, and such that $F(\eta)_0 = (\dot{x}', \dot{z}, \dot{m}')$ is a triple of $\mathbb{P}_B$-names with $\dot{x}'^{G_\eta} = x$, $\dot{z}^{G_\eta} = z$ and $\dot{m}'^{G_\eta} = m$, then again case (a) must apply.

Indeed if $\eta > \beta$ is such a stage, then

$$W[G_\eta] \models \forall \mathbb{P}' (\mathbb{P}' \text{is } \xi\text{-allowable} \to \mathbb{P}' \Vdash (x,y) \in A_m).$$

This is true, as otherwise, we let $\mathbb{P}'$ be a ξ-allowable forcing over $W[G_\eta]$ for which $\mathbb{P}' \Vdash (x,y) \notin A_m$ holds true. At stage β of our iteration $\mathbb{P}$ however, we were in case (a), hence there is no ξ-allowable forcing $\mathbb{R} \in W[G_\beta]$ such that $\mathbb{R} \Vdash (x,y) \notin A_m$. But the intermediate forcing $\mathbb{P}_{\beta,\eta}$ is α-allowable for $\alpha > \xi$, and $\mathbb{P}'$ is ξ-allowable over $W[G_\beta]$, so $\mathbb{P}_{\beta,\eta} * \mathbb{P}'$ witnesses that there is a ξ-allowable forcing which kicks (x,y) out of A_m, which is nonsense and the claim is proved.

Consequentially whenever $\mathbb{P}^1 * \mathbb{P}^2 \in W$ is such that $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ξ-allowable over W, and $F^2 \in W$ is a bookkeeping function witnessing this, then whenever η is such that $F^2(\eta)_0 = (\dot{u}, \dot{z}, \dot{n})$, where $\dot{u}, \dot{z}$ and $\dot{n}$ are $\mathbb{P}_B$-names for a regular subforcing $\mathbb{P}_B$ and a set $B \supset A$ and additionally $\dot{u}^{G^0_\eta} = x$, $\dot{z}^{G^0_\eta} = z$ and $\dot{n}^{G^0_\eta} = m$, then again case (a) must apply.

Let $\mathbb{P}^1 \in W$ be such that $W \models$ "$\mathbb{P}^0 * \mathbb{P}^1$ is α-allowable" and let $F^1 \in W$, $F^1 : \gamma^1 \to W$, be a bookkeeping which witnesses this. We shall define a bookkeping $F^2 \in W$, $F^2 : \omega_1 \to W$ such that if $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is the outcome of the applications of the rules (a) and (b) for α-allowability guided by F^2, then, in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$ there is a $\zeta \leq \xi$ and (x, y_0) such that (x, y_0, m, ζ) is the unique quadruple in I^2 which is not coded into $\vec{S}$.

We let $F^2 \upharpoonright \gamma^1 = F^1 \upharpoonright \gamma^1$, and for $\eta \geq \gamma^1$, $\eta < \omega_1$ we define

$$F^2(\eta)_0 := (\dot{x}, \dot{z}, \check{m})$$

where $\dot{x}$ is the $<$-least nice $\mathbb{P}^0$-name for x and $\dot{z}$ is the $<$-least nice $\mathbb{P}^2_\eta$-name for a real such that $\dot{z}$ codes up the ω-many reals coded so far by the forcing $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\eta$. (This has the effect that whenever β is such that $F(\beta)_0 = (\dot{x}, \dot{z}, m)$ then the model we pick in order to scan through potential $f(m, x)$-values is always the full $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\beta]$.

Further we let $F^2(\eta)_1$ be such that it picks the $<$-least $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\eta$-name for a set $E \in W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_\eta]$ which contains the set I^2_η. As a result, the forcing η-th forcing we use will be $\mathbb{P}^2(\eta) := \prod_{y \in E} \mathbb{P}_{(x,y,m)}$.

We claim now that there is a $\gamma_2 < \omega_1$, $\gamma_2 \geq \gamma_1$, such that $F^2 \upharpoonright \gamma_2$ witnesses that $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_{\gamma_2}$ is as desired.

Indeed, by the choice of the $\dot{z}$ from $F^2(\xi)_0 = (\dot{x}, \dot{z}, \dot{m})$, there must be a stage $\eta < \omega_1$, which is such that from η on, we will always put some fixed quadruple $(\dot{x}, \dot{y}_0, \dot{m}, \xi)$ to $I_{\eta'}$, where $\eta' > \eta$, as otherwise we would either obtain an infinite $<$-decreasing sequence of forcing names for reals, which is nonsense. Now we must be in case (a) at both stages η and $\eta + 1$. This in particular means that $\mathbb{P}^2(\eta)$ is such that there is a forcing $\mathbb{R}^2(\eta) \in W[G^2_\eta][\mathbb{P}^2(\eta)]$ such that (x, y_0) can not be kicked out of A_m anymore with additional ζ-allowable forcings.

As a consequence, the set E which $F(\eta + 1)_2$ picks, will contain I^2_η and the forcing at stage $\eta + 1$ will code up all elements of I^2_η except (x, y_0, m) into $\vec{S}$. Thus $\mathbb{P}^2_{\eta+2}$ is as desired, as soon as we can show that (x, y_0, m) is not coded into $\vec{S}$ in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$, which we shall prove now.

By the definition above, η is the least stage such that (x, y_0, m, ξ) has been added to I^2_η. The rules of the definition of α-allowable then tell us that we must be in case (a) at stage η and in case (a) at $\eta+1$ as well, and (x, y, m) is not coded into $\vec{S}$ in both stages.

Thus we have to argue that (x, y_0, m) has not been coded into $\vec{S}$ at an earlier stage $\eta' < \eta$. Assume for a contradiction that there is such an $\eta' < \eta$ such that at stage η', (x, y_0, m) got coded into $\vec{S}$. We must have been in case (a) at η'.

Therefore, at stage η' there must have been a triple (x, z, m, ξ_0) which we added to $I^2_{\eta'}$ instead of (x, y_0, m, ξ). Note that (x, z) will stay in A_m for any further α-allowable iteration again by the lemma 3.21. So either $\xi_0 < \xi$, in which case we must not add (x, y_0, m, ξ) to I_η at stage η, which is a contradiction. Or $\xi = \xi_0$ and the $<$-least $\mathbb{P}_{\eta'}$-name for (x, z) is $<$-below the $<$-least $\mathbb{P}_{\eta'}$-name for (x, y_0). But then, again, we must not add (x, y_0, m, ξ) to I_η as (x, z) would be the better candidate, which is a contradiction. So, indeed, (x, y_0, m) is not coded into $\vec{S}$ in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2_{\gamma_2}]$, which finishes the proof of the lemma.

$\square$

The last lemma has the following straightforward generalization:

Corollary 3.23. *Let $\gamma < \omega_1$, and let $F : \gamma \to W^4$ be a bookkeeping function in W. Assume that $\mathbb{P}^0 = (\mathbb{P}^0_\beta : \beta < \gamma)$ and $I^0 = I^0_\gamma$ be such that $(\mathbb{P}^0, I^0)$ is an α-allowable forcing over W with respect to F. Let $x \in W[\mathbb{P}^0]$ and $m \in \omega$. If there is a real $y \in W[\mathbb{P}^0]$ and a $\xi < \alpha$ such that $(x, y, m, \xi) \in I^0$, then for every $\mathbb{P}^1 \in W$ which satisfies that $\mathbb{P}^0 * \mathbb{P}^1$ is α-allowable over W, and every countable set $C \subset 2^\omega \cap W[\mathbb{P}^0 * \mathbb{P}^1]$ there is a further $\mathbb{P}^2 \in W$ such that $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is α-allowable and such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$ there is a pair of reals x, y_0 such that in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$, if I^2 is the unique set of potential $f(\cdot, \cdot)$-values one obtains when forming the α-allowable $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, then (x, y_0, m) is the unique triple such that there is a $\xi < \alpha$ such that $(x, y_0, m, \xi) \in I^2$ which is not coded into $\vec{S}$. Additionally all reals of the form (x, z, m), where $z \in C, z \neq y_0$ are coded into $\vec{S}$ as well in $W[\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2]$.*

Proof. The proof is almost identical to the one of Lemma 3.22, but we define $\dot{E}$ from $F(\eta)_2 = \dot{E}$ to not only contain I^2_η but also $\check{C}$. With this minor modification the proof works as above. $\square$

Lemma 3.24. *Work in W. For any α, the set of α-allowable forcings is nonempty.*

Proof. By induction on α. If there are α-allowable forcings over W, then every bookkeeping function $F \in W$, $F : \delta \to W^4$ together with the rules (a) and (b) will create a nontrivial $\alpha + 1$-allowable forcing just by the way we chose to define $\alpha + 1$-allowability. For limit ordinals, the same reasoning applies. $\square$

As a direct consequence we obtain that there must be an ordinal α such that for every $\beta > \alpha$, the set of α-allowable forcings over W must equal the set of β-allowable forcings over W. Indeed every allowable forcing is an $\aleph_1$-sized partial order in W, thus there are only set-many of them (modulo isomorphism), and the decreasing sequence of α-allowable forcings must eventually stabilize at a set which also must be non-empty.

Definition 3.25. *Let α_0 be the least ordinal such that for every $\beta > \alpha_0$, the set of α_0-allowable forcings over W is equal to the set of β-allowable forcings over W.*

The set of ∞-allowable forcings can also be described in the following way. A $\delta < \omega_1$-length iteration $\mathbb{P} = (\mathbb{P}_\alpha : \alpha < \delta)$ is ∞-allowable if it is recursively constructed following a bookkeeping function F and a modified version of the two rules from above: we ask in (a) whether there exists an ordinal ζ at all for which the antecedens of (a) is true. If there is such an ordinal ζ we proceed as described in (a) if not we use (b). Note that for $m \in \omega$ and a real x we will have several potential y's such that $(x, y, m, \xi) \in I$ as we go along in an ∞-allowable iteration. The ranks of the potential values form a decreasing sequence of ordinals, thus, once we set a value $f(m, x)$, we can be sure that eventually there will be a value for $f(m, x)$ which will not change any more in rank.

3.3 Definition of the universe in which the Π^1_3 uniformization property holds

The notion of ∞-allowable will be used now to define the universe in which the Π^1_3-uniformization property is true. We let W be our ground model and start an ω_1-length iteration whose initial segments are all ∞-allowable with respect to W. We are using the following rules in combination with some bookkeeping $F \in W$. The actual properties of F are not really relevant, F should however satisfy that

- $F : \omega_1 \to H(\omega_1)$ is surjective.
- For every $x \in H(\omega_1)$, the set $F^{-1}(x)$ should be unbounded in ω_1.

Inductively we assume that we are at stage $\beta < \omega_1$ of our iteration and the allowable forcings $\mathbb{P}_\beta$, $\mathbb{R}_\beta$ have been defined already. We assume additionally that the value $F(\beta) = (F(\beta)_0, F(\beta)_1)$ is in fact a pair of elements in $H(\omega_1)$ and $F(\beta)_0 = (\eta_1, \eta_2, m)$ where $\eta_1 \le \beta$ and η_2 are ordinals and $m \in \omega$. We let $(\dot{x}, \dot{y})$ be the η_2-th nice $\mathbb{P}_{\eta_1}$ name of a pair of reals relative to our wellorder $<$ of $H(\omega_2)^W$. If we set $\dot{x}^{G_\beta} = x$, $\dot{y}^{G_\beta} = y$ then we further assume that $W[G_\beta] \models (x, y) \in A_m$. Recall that α_0 is the least ordinal such that the notion of α-allowable stabilizes. We split into two main cases, following the definition of α_0-allowable.

(a) We work in $W[G_{\eta_1}]$ and assume the following.

  - There is an ordinal $\zeta \le \alpha_0$, which is chosen to be minimal for which
  - there is a $<$-least pair of nice $\mathbb{P}_{\eta_1}$-names $(\dot{x}', \dot{y}')$ such that $\dot{x}'^{G_{\eta_1}} = x$ and $\dot{y}'^{G_{\eta_1}} = y'$ and $W[G_\beta] \models (x, y') \in A_m$ and for every ζ-allowable $\mathbb{R} \in W[G_\beta]$,

$$\mathbb{R} \Vdash (x, y') \in A_m.$$

If this is the case, then we set the following:

- We pick the $<$-least non-empty set of reals $E \in W[G_{\eta_1}]$ which does not contain y'. We let $\mathbb{P}'(\beta) := \mathbb{Q}_E = \prod_{s \in E} \mathbb{P}_{(x,y,m)}$. We also let $\mathbb{P}'_{\beta+1} = \mathbb{P}_\beta * \mathbb{P}'(\beta)$ and let $G'_{\beta+1}$ be its generic filter.
- We set a new f value, i.e. we set $f(m,x) := y'$ and assign the rank ξ to the value. We update $I_{\beta+1}^{G'_{\beta+1}} := I_\beta^{G'_\beta} \cup \{(x,y',m,\xi)\}$.

Then we follow up with a second α_0-allowable forcing. Let $F(\beta)_1 = \eta_3$ for $\eta_3 < \omega_1$, and assume that the η_3-th $\mathbb{P}_{\eta_1}$-name of a countable set of reals is $\dot{C} \in W$. Then, over $W[G'_{\beta+1}]$, use the $<$-least, α_0-allowable forcing $(\mathbb{P}^2, I^2)$ such that $\mathbb{P}'_{\beta+1} * \mathbb{P}^2$ is α_0-allowable, whose existence is granted by corollary 3.23, and which codes all elements of the form (x,a,m) where $a \in C = \dot{C}^{G'_{\beta+1}}$ into $\vec{S}$, except for maybe one element. We finally set $\mathbb{P}(\beta) := \mathbb{P}'(\beta) * \mathbb{P}^2$, let $\mathbb{P}_{\beta+1} := \mathbb{P}_\beta * \mathbb{P}(\beta)$ and let $G_{\beta+1}$ be a $\mathbb{P}_{\beta+1}$-generic filter over W. Then we let $I_{\beta+1} = I^2$.

(b) We assume that case (a) is not true. So for every $\xi \leq \alpha_0$, in particular for $\xi = \alpha_0$, every pair $(x,z) \in W[G_{\eta_1}]$ (so in particular for the pair (x,y)) and every $E \in W[G_{\eta_1}] \cap [2^\omega]^\omega$ and function $h \in W[G_{\eta_1}]$, there is a further $(\mathbb{P}^{(x,y)},, I^{(x,y,h)} \in W[G_\beta]$ such that

$$\text{“}(\mathbb{P}^{(x,y)}, I^{(x,y)})\text{is } \alpha_0\text{-allowable over the ground model } W[G_\beta]\text{”}$$

and

$$\mathbb{P}^{(x,y)} \Vdash (x,y) \notin A_m$$

We pick the $<$-least such α_0-allowable forcing $\mathbb{P}^{(x,y)}$ and use it at stage β, to force

$$\mathbb{P}(\beta) \Vdash (x,y) \notin A_m.$$

Then we update $I_{\beta+1}$ to be $I_\beta \cup I^{x,y}$. In other words, we use an additional α_0-allowable forcing over $W[G_\beta]$ to kick (x,y) out of A_m.

As always we use mixed support. This ends the definition of our iteration $((\mathbb{P}_\beta, I_\beta) : \beta < \omega_1)$. We set $\mathbb{P}_{\omega_1}$ to be the direct limit of $(\mathbb{P}_\beta : \beta < \omega_1)$, and $I_{\omega_1} = \bigcup_{\beta < \omega_1} I_\beta$.

We next derive its basic properties. First we note that the iteration is such that there is an $F' \in W$, $F' : \omega_1 \to W$, and such that for every $\delta < \omega_1$, $((\mathbb{P}_\beta, I_\beta) : \beta < \delta)$ is α_0-allowable over W with respect to $F' \upharpoonright \delta$. Indeed the bookkeeping F can be used to readily derive such an $F' \in W$.

Fact 3.26. *The just defined iteration $(\mathbb{P}_{\omega_1}, I_{\omega_1}) \in W$ is such that every initial segment is α_0-allowable over W relative to a fixed $F' \in W$.*

As a consequence, the f_m-values of rank $< \alpha_0$ we define as we go along the iteration are such that they will certainly belong to A_m in the final model by Lemma 3.21. We let G_{ω_1} be $\mathbb{P}_{\omega_1}$-generic over W. What is left, is to show that in $W[G_{\omega_1}]$, for every $m \in \omega$ and every real x such that $A_{m,x} \neq \emptyset$, we do have exactly one pair of reals $(x, y) \in A_m$ such that (x, y, m) is not coded into $\vec{S}$. The next lemma does exactly that, and is the main step in proving that the Π^1_3-uniformization property holds true in $W[G_{\omega_1}]$.

Lemma 3.27. *In $W[G_{\omega_1}]$ the following dichotomy holds true:*

1. *Either (x, m) is such that there is a real y and $\xi < \alpha_0$ such that $(x, y, m, \xi) \in I$. Then there is a unique real y_0 such that*

   $W[G_{\omega_1}] \models$ "$(x, y_0) \in A_m \wedge (x, y_0, m)$ *is not coded somewhere into* $\vec{S}$".

2. *Or (x, m) is such that for every real y, if $(x, y, m, \xi) \in I$, then $\xi \geq \alpha_0$, in which case*

$$W[G_{\omega_1}] \models \text{"The } x\text{-section of } A_m \text{ is empty"}$$

Proof. We assume first that the assumptions of case 1 are true, i.e. there is a y and $\xi < \alpha_0$ such that $(x, y, m, \xi) \in I$. Then there is a real $y_0 \in W[G_{\omega_1}]$ (and an attached ordinal $\xi_0 < \alpha_0$) whose $\mathbb{P}_{\omega_1}$-name is $<$-minimal among all such names. We let β be the least stage where we add (x, y_0, m, ξ_0) to I_β.

Claim. $W[G_{\omega_1}] \models (x, y_0) \in A_m$.

Proof of the first Claim. This follows immediately from the lemma 3.21. □

Claim.

$W[G_{\omega_1}] \models$ "y_0 *is the unique real such that*

(x, y_0, m) *is not coded somewhere in the $\vec{S}$-sequence.*"

Proof of the second Claim. We shall prove the second claim. First we show that (x, y_0, m) is not coded somewhere into the $\vec{S}$-sequence. It is clear by the argument from the proof of lemma 3.22, that from stage β on, we will not code (x, y_0, m) into $\vec{S}$. So the only possibility that we coded up (x, y_0, m) is that there is a stage $\eta < \beta$ of our iteration $\mathbb{P}_{\omega_1}$ where we coded (x, y_0, m) into $\vec{S}$. At stage η, we can not be in case 2, as (x, y_0) and the intermediate forcing $\mathbb{P}_{[\eta,\beta)}$ witness that we must be in case 1 at η. So we must be in case 1, but we add a different (x, y', m, ξ_0) to I_η. But then, also at stage β, we will add (x, y', m, ξ_0) to I_β by the argument from the proof of Lemma 3.22. This is a contradiction so indeed (x, y_0, m) is not coded into $\vec{S}$.

In order to show that it is the unique real of the form (x, y, m) which is not coded, it is sufficient to note that for every other $y \neq y_0$, if η is such that $F(\eta)_1$ will output $\dot{C}$ and $y \in \dot{C}^{G_\eta}$, then, at stage η, we will code up (x, y, m) as the coding will be a factor of the iteration $\mathbb{Q}_h$ we use at η. By the choice of F, such stages are in fact unbounded in ω_1.

Thus Claim 2 is proved, which also finishes the proof of the Lemma under the assumptions of the first case of our Lemma. □

We shall prove now that under the assumptions of the second case of our Lemma, its conclusion does hold, i.e. we need to show that if (x, m) is such that for every real y, if $(x, y, m, \xi) \in I$, then $\xi \geq \alpha_0$, then $W[G_{\omega_1}] \models$ "The x-section of A_m is empty".

But under these assumptions, whenever we are at a stage β such that there is a y such that $F(\beta) = (x, y, m)$, then case 2 of the definition of $\mathbb{P}_{\omega_1}$ must apply. But for every such y, at stage β, we ensure with an α_0-allowable forcing that $W[G_{\beta+1}] \models (x, y) \notin A_m$. By upwards absoluteness of Σ^1_3-formulas we obtain in the end

$$W[G_{\omega_1}] \models \neg\exists y((x, y) \in A_m).$$

This finishes the poof of the Lemma.

□

Corollary 3.28. *In $W[G_{\omega_1}]$ the Π^1_3-uniformization property is true. For A_m an arbitrary Π^1_3-set, we get that*

$y = f(m, x)$

if and only if

$(x, y) \in A_m$ *and* $\neg\exists r(\forall M(M$ *is countable and transitive and* $M \models \mathsf{ZF}^- +$ "$\aleph_2$ *exists*" *and* $\omega_1^M = (\omega_1^L)^M$ *and* $r, (x, y, m) \in M \rightarrow M \models \varphi((x, y, m)))$.

Proof. It suffices to note that the formula on the right and side is indeed Π^1_3. This is clear as it is of the form $\Pi^1_3 \wedge \neg\Sigma^1_3$. □

4 Forcing over canonical inner models with Woodin cardinals

4.1 Coding over M_1

As stated in the beginning, we can apply this proof in the context of canonical inner models with Woodin cardinals. Recall that under the axiom of projective determinacy PD, the odd levels of the projective hierarchy will satisfy the uniformization property. Our construction will yield a universe in which the Π^1_4-uniformization property is true, thus producing a model for the "wrong" side for the first time. The complexities in its proof may serve as another example of empiric evidence, that the regularity properties implied by PD are natural, and violating them needs considerable effort. The proof which we present should, modulo some technicalities lift to higher levels of the projective hierarchy. The theorem could also be proved using $L[U]$ as our ground model, or even weaker, working over $L^{\#}$, the minimal transitive class-sized model which is closed under sharps for sets, reducing the large cardinal assumption, but at the cost of not being liftable, this is why we settle to prove it using M_1 as the ground model.

Theorem 4.1. *Assume that the canonical inner model with one Woodin cardinal, M_1, exists. Then there is a generic extension of M_1, in which the Π^1_4-uniformization property is true.*

The proof of the theorem is closely modeled after the L case. We will first introduce some of the properties of M_1 which are crucial for our needs, but assume from this point on that the reader is familar with the basic notions of inner model theory. Recall that M_1 is a proper class premouse containing a Woodin cardinal (see [22], pp. 81 for a definition of M_1). Every initial segment $\mathcal{J}^{M_1}_\beta$ is ω-sound and 1-small, where we say that a premouse $\mathcal{M}$ is 1-small iff whenever κ is the critical point of an extender on the $\mathcal{M}$-sequence then

$$\mathcal{J}^{\mathcal{M}}_\kappa \models \neg\exists\delta(\delta \text{ is Woodin}).$$

The reals of M_1 admit a Σ^1_3-definable wellorder (see [22], Theorem 4.5), the definition of the wellorder makes crucial use of a weakened notion of iterability, the so-called Π^1_2-iterability which we shall introduce.

Let $\mathcal{M}$ be a premouse, $\mathcal{T}$ be an ω-maximal iteration tree b a branch through $\mathcal{T}$ and α an ordinal. Then b is α-good if, whenever $\mathcal{N} = \mathcal{M}^{\mathcal{T}}_b$ or $\mathcal{N}$ is the α-th iterate of some initial segment $\mathcal{P} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$ using a single extender E (and its images under the iteration map) on the $\mathcal{P}$-sequence, then α is in the wellfounded part of $\mathcal{N}$. Then we say that a premouse $\mathcal{M}$ is Π^1_2-iterable, if player II has a winning strategy in the game $\mathcal{G}'_\omega(\mathcal{M}, 1)$, where $\mathcal{G}'_\omega(\mathcal{M}, 1)$, is defined just as the ordinary weak two player game $W\mathcal{G}_\omega(\mathcal{M}, 1)$ (see e.g. [23] pp. 65 for a definition), with the exception that player I not only plays an ω-maximal, countable putative iteration tree $\mathcal{T}$ but additionally has to play a countable ordinal $\alpha < \aleph^{M_1}_1$. Then player II does not have to play a wellfounded branch through $\mathcal{T}$ (as it would be the case for iterability), but instead can play a cofinal branch b through $\mathcal{T}$ such that b is α-good in order to win.

The winning strategy for II for $\mathcal{G}'_\omega(\mathcal{M}, 1)$ guarantees that $\mathcal{M}$ can be compared to any countable premouse which is an initial segment of M_1.

Lemma 4.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be ω-sound premice which both project to ω. Assume that $\mathcal{M}$ is an initial segment of M_1 and $\mathcal{N}$ is Π^1_2-iterable, and let*

Σ *denote the winning strategy for player II in* $\mathcal{G}_\omega(\mathcal{M}, \omega_1 + 1)$. *Then we can successfully compare* $\mathcal{M}$ *and* $\mathcal{N}$ *and consequentially* $\mathcal{M} \lhd \mathcal{N}$ *or* $\mathcal{N} \unlhd \mathcal{M}$.

It is relatively straightforward to check that the set of reals which code Π^1_2-iterable, countable premice is itself a Π^1_2-definable set in the codes (see [22], Lemma 1.7). Modulo the last lemma, this implies that there is a nice definition of a cofinal set of countable initial segments of M_1 in ω_1-preserving forcing extensions $M_1[G]$ of M_1, (in fact this definiton holds in all outer models of M_1 with the same ω_1):

Lemma 4.3. *Let* $M_1[G]$ *be an* ω_1*-preserving forcing extension of* M_1. *Then in* $M_1[G]$ *there is* Π^1_2*-definable set* $\mathcal{I}$ *of premice which are of the form* $\mathcal{J}^{M_1}_\eta$ *for some* $\eta < \omega_1$. $\mathcal{I}$ *is defined as*

$$\mathcal{I} := \{\mathcal{M} \text{ ctbl premouse} : \mathcal{M} \text{ is } \Pi^1_2\text{-iterable, } \omega\text{-sound and projects to } \omega\},$$

and the set

$$\{\eta < \omega_1 : \exists \mathcal{N} \in \mathcal{I}(\mathcal{N} = \mathcal{J}^{M_1}_\eta)\}$$

is cofinal in ω_1.

In particular $M_1|\omega_1$ is $\Sigma_1(\omega_1)$-definable in ω_1-preserving generic extensions of M_1, as $x \in M_1|\omega_1$ if and only if there is a transitive $U \models \mathsf{ZF}^-$, $\omega_1 \subset U$, $\aleph^U_1 = \aleph_1$ such that $U \models \exists \mathcal{M} \in \mathcal{I} \wedge x \in \mathcal{M}$, which suffices using Shoenfield absoluteness. A similar argument also shows that $\{M_1|\omega_1\}$ is $\Sigma_1(\omega_1)$ definable, as we can successfully compute it in transitive ω_1-containing models, via the following $\Sigma_1(\omega_1)$-formula:

$$\begin{aligned}(*) \quad X = M_1|\omega_1 \Leftrightarrow \exists U (&U \text{ is a transitive model of } \mathsf{ZF}^- \wedge \omega_1 \subset U \wedge \\ &U \models \forall \alpha < \omega_1 \exists r \in \mathcal{I}(\alpha \in (r \cap Ord)) \wedge \\ &X \text{ is transitive and } X \cap Ord = \omega_1 \wedge \\ &\forall x \in \mathcal{I}(x \subset X) \wedge \forall y \in X \exists x \in \mathcal{I}(y \in x))\end{aligned}$$

Indeed, if the left hand side of $(*)$ is true, then any transitive U which contains $M_1|\omega_1$ as an element and which models ZF^- will witness the truth of the right hand side, which is an immediate consequence of Shoenfield absoluteness.

If the right hand side is true, then, using the fact that Σ^1_3-statements are upwards absolute between U and the real world, U will contain an ω_1-height, transitive structure X which contains all countable initial segments of M_1, and such that every $y \in X$ is included in some element of $M_1|\omega_1$, in other words X must equal $M_1|\omega_1$.

We shall argue now, that the coding forcings, we defined earlier over the constructible universe, can be adapted to M_1. The first thing to note is that $M_1|\omega_1$ can define a $\diamondsuit$-sequence in the same way as L_{ω_1} can. Indeed, as M_1 has a Δ^1_3-definable wellorder of the reals whose definition relativizes to

$M_1|\omega_1$ we can repeat Jensen's original proof in M_1 to construct a candidate for the $\diamondsuit$-sequence, via picking at every limit stage $\alpha < \omega_1$ the $<_{M_1}$-least pair $(a_\alpha, c_\alpha) \in P(\alpha) \times P(\alpha)$ which witnesses that the sequence we have created so far is not a $\diamondsuit$-sequence. The proof that this defines already a witness for $\diamondsuit$ is finished as usual with a condensation argument. Hence we shall show that if $\mathcal{J}_\beta^{M_1}$ is least such that $(a_\alpha : \alpha < \omega_1)$ and $(A, C) \in \mathcal{J}_\beta^{M_1}$, where (A, C) is the $<_{M_1}$-least witness for $(a_\alpha)_{\alpha<\omega_1}$ not being a $\diamondsuit$-sequence, then there is an countable $N \prec \mathcal{J}_\beta^{M_1}$ such that the transitive collapse $\bar{N}$ is an initial segment of M_1.

To see that in fact every such N collapses to an initial of M_1, recall the condensation result as in [23], Theorem 5.1, which we can state in our situation as follows:

Theorem 4.4. *Let $\mathcal{M}$ be an initial segment of M_1. Suppose that $\pi : \bar{N} \to \mathcal{M}$ is the inverse of the transitive collapse and $crit(\pi) = \rho_\omega^{\bar{N}}$, then either*

1. *$\bar{N}$ is a proper initial segment of $\mathcal{M}$, or*
2. *there is an extender E on the $\mathcal{M}$-sequence such that $lh(E) = \rho_\omega^{\bar{N}}$, and $\bar{N}$ is a proper initial segment of $Ult_0(\mathcal{M}, E)$.*

We shall argue, that in our situation, the second case is ruled out, hence every $N \prec \mathcal{J}_\beta^{M_1}$ collapses to an initial segment of M_1. Indeed, due to the ω-soundness of $\mathcal{J}_\beta^{M_1}$, every $N \prec \mathcal{J}_\beta^{M_1}$ will satisfy that

$$\rho_\omega^N = \rho_\omega^{\mathcal{J}_\beta^{M_1}} = \omega_1^{\mathcal{J}_\beta^{M_1}},$$

hence $crit(\pi) = \omega_1^{\bar{N}} = \rho_\omega^{\bar{N}}$ by elementarity of π.

But $\bar{N}|\omega_1^{\bar{N}} = N|\omega_1^{\bar{N}}$, and as $\bar{N}|\omega_1^{\bar{N}}$ thinks that ω is its largest cardinal, $N|\omega_1^{\bar{N}}$ must believe this as well. But then there can not be an extender on the N-sequence which is indexed at $\omega_1^{\bar{N}}$, as otherwise $N|\omega_1^{\bar{N}}$ would think that $\omega_1^{\bar{N}}$ is inaccessible, which is a contradiction. Hence, the condition $lh(E) = \rho_\omega^{\bar{N}}$ is impossible and all that is left is case 1, so $\bar{N}$ is an initial segment of M_1.

This shows that Jensen's construction of a $\diamondsuit$-sequence succeeds when applied to M_1. It is straightforward to verify that the recursive construction can be carried out in $M_1|\omega_1$ by absoluteness. Consequentially the $\diamondsuit$-sequence is a Σ_1-definable class over $M_1|\omega_1$.

We can use the $\diamondsuit$-sequence to construct an ω_1-length sequence of M_1-subsets of ω_1 which are stationary, co-stationary just as in L. We let R_α be $\{\beta < \omega_1 \mid a_\beta = r_\alpha \cap \beta\}$, where r_α is the α-th M_1 real in its canonical wellorder. The sequence $(R_\alpha : \alpha < \omega_1)$ is $\Sigma_1(\omega_1)$-definable, which works for all ω_1-preserving generic extensions of M_1, by our discussion above. Indeed in order to find R_α in some $M_1[G]$, where G is a generic filter for an ω_1-preserving forcing, then the formula $(*)$ will define $\{M_1|\omega_1\}$ in a $\Sigma_1(\omega_1)$-way, and the latter can internally define R_α.

Hence, we can reproduce the stationary kill forcings we used to obtain $W = L[\mathbb{Q}^0][\mathbb{Q}^1][\mathbb{Q}^2]$ from L in exactly the same way over M_1 and obtain an ω_1-preserving, ω-distributive generic extension W^* over M_1, in which there is a $\Sigma_1(\omega_1)$-definable sequence of independent ω_1 trees $\vec{S}$, which are Suslin in the inner model $M_1[\mathbb{Q}^0][\mathbb{Q}^2]$.

We shall work in W^* from now on, and reproduce the coding forcings we defined in W. Given an arbitrary real coding a r triple (x, y, m) we can define the coding forcing $\mathbb{P}_{(x,y,m)}$ in almost the same way as we did over W, the only exception is that we use $\mathbb{C}(\omega_1)^{M_1}$, i.e. ω_1-Cohen forcing as evaluated in M_1 as the first factor. If $g \subset \omega_1$ we let h be the set one obtains when applying the $<_{M_1}$-least bijection $\rho : \omega_1^\omega \to \omega_1$, $\rho \in M_1$ pointwise to g, i.e. $h = \rho``\{g \cap \alpha \, : \, \alpha < \omega_1\}$. As before, the set $h$ determines which $\omega$-blocks of $\vec{S}$ should have written the $(x, y, m)$-pattern into it. To emulate the previous jargon, we say that $h$ *codes an $M_1$-sequence of ordinals*, if there is a $g$ such that $h = \rho``\{g \cap \alpha \, : \, \alpha < \omega_1\}$. We collect the set $M_1|\omega_1$, the relevant clubs through M_1-stationary sets, and the branches through the Suslin trees which create the pattern which codes up $w = (x, y, m)$, the set $h \subset \omega_1$ and write everything into one set $X \subset \omega_1$. Note that if $\gamma \in h$ is arbitrary, if $L_\zeta[X]$ is the least ZF^--model which contains $X \subset \omega_1$, we obtain that

$$\begin{aligned} L_\zeta[X] \models & n \in (x, y, m) \to S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch and} \\ & n \notin (x, y, m) \to S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch} \end{aligned}$$

Our next goal is to rewrite the set X, such that already suitable countable models can read off w. Here we our argument has to diverge from the W-case, as M_1's definition is more complicated.

We first note that any transitive, $\aleph_1$-sized ZF^- model M which contains X will satisfy

$$\begin{aligned} (M, \in, \mathcal{J}_{\omega_1}^{M_1}) \models & \text{“Decoding X yields a model } m \text{ and } m = \mathcal{J}_{\omega_1}^{M_1} = \bigcup_{\mathcal{J}_\eta^{M_1} \in \mathcal{I}} \mathcal{J}_\eta^{M_1}, \\ & \text{some clubs } \vec{c} \text{ through elements of } m \text{ which code Suslin trees } \vec{s} \\ & \text{some branches } \vec{b} \text{ through } \vec{s}, \\ & \text{a set } h \subset \omega_1 \text{ which codes an } M_1\text{-sequence of ordinals such that} \\ & \text{ for the least } \mathsf{ZF}^- \text{ model of the form } L_\zeta[X] \text{ we have that} \\ & L_\zeta[X] \models \forall \gamma \in h (n \in (x, y, m) \to S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch} \\ & \qquad n \notin (x, y, m) \to S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch})) \end{aligned}$$

In particular, this will be true for a ZF^- + “$\aleph_2$ exists” model of the form $(L_\xi[X], \in, \mathcal{J}_{\omega_1}^{M_1})$, $\xi < \aleph_2$. If we consider the club

$$C := \{\eta < \omega_1 \, : \, \exists (M, \in, P) \prec (L_\xi[X], \in, \mathcal{J}_{\omega_1}^{M_1})(|M| = \aleph_0 \wedge \eta = \omega_1 \cap M)\}$$

then if $(N, \in)$ is an arbitrary countable transitive model of ZF^- such that $X \cap \omega_1^N \in N$ and $\omega_1^N \in C$, then N will decode out of $X \cap \omega_1^N$ exactly what $(\bar{M}, \in, \mathcal{J}_\eta^{M_1})$ decodes, where the latter is the transitive collapse of $(M, \in, P) \prec (L_\xi[X], \in, \mathcal{J}_{\omega_1}^{M_1})$, where $X \in M, M \cap \omega_1 = \omega_1^N$. In particular, if we denote the Δ_1-definable decoding functions with dec_1, dec_2 and dec_3 respectively, then we obtain

$$
\begin{aligned}
N \models \exists m_1 \exists \vec{c} \exists \vec{b} (& dec_1(X \cap \omega_1^N) = m_1 \wedge dec_2(X \cap \omega_1^N) = \vec{c} \\
& \wedge dec_3(X \cap \omega_1^N) = \vec{b} \wedge dec_4(X \cap \omega_1^N) = h \cap \omega_1^N \\
& \text{and for the least } \mathsf{ZF}^- \text{ model of the form } L_\zeta[X \cap \omega_1^N] \text{ we have that} \\
& L_\zeta[X \cap \omega_1^N] \models \forall \gamma \in h (n \in (x, y, m) \rightarrow S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch} \\
& \qquad\qquad n \notin (x, y, m) \rightarrow S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch}))).
\end{aligned}
$$

Further, as $dec_1(X \cap \omega_1^N) = m_1 = \mathcal{J}_\eta^{M_1}$, we get that

$$dec_1(X \cap \omega_1^N) \in \mathcal{I}.$$

Now let the set $Y \subset \omega_1$ code the pair (C, X) such that the odd entries of Y should code X and if $Y_0 := E(Y)$ where the latter is the set of even entries of Y and $\{c_\alpha : \alpha < \omega_1\}$ is the enumeration of C then

1. $E(Y) \cap \omega$ codes a well-ordering of type c_0.
2. $E(Y) \cap [\omega, c_0) = \emptyset$.
3. For all β, $E(Y) \cap [c_\beta, c_\beta + \omega)$ codes a well-ordering of type $c_{\beta+1}$.
4. For all β, $E(Y) \cap [c_\beta + \omega, c_{\beta+1}) = \emptyset$.

We obtain a version of the which works for suitable countable transitive models:

> Let M be an arbitrary countable transitive model of $\mathsf{ZF}^- +$ "$\aleph_2$ exists" for which there is a $\mathcal{J}_\eta^{M_1} \in \mathcal{I}$ such that $\omega_1^M = \omega_1^{\mathcal{J}_\eta^{M_1}}$ and $\mathcal{J}_\eta^{M_1} \in M$. Assume that $Y \cap \omega_1^M \in M$ then M can decode out of $Y \cap \omega_1$,
>
> – a model m,
> – some clubs $\vec{c}$ through m-stationary sets $\vec{s}$, (such that of every consecutive pair in $\vec{s}$ *exactly* one of the pair is not stationary anymore as witnessed by an element of $\vec{c}$), which in turn yield a sequence $\vec{s}$ of m-Suslin trees;
> – a set $h \subset \omega_1^M$ such that $\forall \alpha < \omega_1^M (h \cap \alpha \in m)$ and which codes an M_1-sequence of ordinals

– and some branches $\vec{b}$ through elements of $\vec{s}$ such that for the least $\mathsf{ZF}^- +$ "$\aleph_2$ exists"-model of the form $L_\zeta[m, \vec{s}, \vec{b}]$:

$$L_\zeta[m, \vec{s}, \vec{b}] \models \forall \gamma \in h(n \in (x, y, m) \to S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch}$$
$$n \notin (x, y, m) \to S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch}))).$$

Moreover m is an M_1 initial segment as seen from the outside, i.e. $m = \mathcal{J}_\eta^{M_1} \in \mathcal{I}$.

Thus we have a local version of the property $(*)$. In the last step, we use almost disjoint coding forcing again, to obtain a real r_Y which codes our set $Y \subset \omega_1$ relative to the $\mathcal{J}_{\omega_1}^{M_1}$-definable almost disjoint family of reals. Thus we obtain the following formula $\psi((x, y, m), r_Y)$ holds, where $\psi((x, y, m), r_Y)$ is defined to be:

For M an arbitrary countable transitive model of $\mathsf{ZF}^- +$ "$\aleph_2$ exists", and $r_Y \in M$ and for which there is a $\mathcal{J}_\eta^{M_1} \in \mathcal{I}$ such that $\omega_1^M = \omega_1^{\mathcal{J}_\eta^{M_1}}$ and $\mathcal{J}_\eta^{M_1} \in M$. Assume that $r_Y \in M$ then M, relative to the a.d. family of reals from $\mathcal{J}_\eta^{M_1}$, can decode out of r_Y the following

– a model m,
– some clubs $\vec{c}$ through m-stationary sets $\vec{s}$, (such that of every consecutive pair in $\vec{s}$ *exactly* one of the pair is not stationary anymore as witnessed by an element of $\vec{c}$) which in turn yield a sequence $\vec{s}$ of m-Suslin trees;
– a set $h \subset \omega_1^M$ such that $\forall \alpha < \omega_1^M (h \cap \alpha \in m)$ and which codes an M_1-sequence of ordinals
– and some branches $\vec{b}$ through elements of $\vec{s}$, whose indices live on ω-blocks with starting values in h such that the least ZF^-+"$\aleph_2$ exists" model of the form $L_\zeta[m, \vec{c}, \vec{b}]$

$$L_\zeta[m, \vec{c}, \vec{b}] \models \models \forall \gamma \in h(n \in (x, y, m) \to S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch}$$
$$n \notin (x, y, m) \to S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch})).$$

Moreover m is an M_1 initial segment as seen from the outside, i.e. $m = \mathcal{J}_\eta^{M_1} \in \mathcal{I}$.

A straightforward calculation shows that the statement $\psi((x, y, m), r_Y)$ is of the form $(\Sigma_3^1 \to \Pi_3^1)$, thus it is a Π_3^1-formula, and stating the existence of such a real r_Y is Σ_4^1.

The existence of a real r witnessing $\psi((x, y, m), r)$ is sufficient to conclude that $L[r]$ contains branches through $\aleph_1$-many trees from $\vec{S}$.

Lemma 4.5. *Let w be a real which codes $(m,x,y) \in (\omega \times 2^\omega \times 2^\omega)$ and let r be such that $\psi((x,y,m),r)$ is true. Then, working inside $L[r]$, there is a set $h \subset \omega_1$ such that $\forall \alpha < \omega_1 (h \cap \alpha \in M_1|\omega_1)$, such that h codes an M_1-sequence of ordinals and such that*

$$\begin{aligned} \forall \gamma \in h(n \in (x,y,m) &\rightarrow S_{\omega\gamma+2n+1} \textit{ has an } \omega_1\textit{-branch} \\ n \notin (x,y,m) &\rightarrow S_{\omega\gamma+2n} \textit{ has an } \omega_1\textit{-branch} \end{aligned}$$

Proof. We note first that $\psi((x,y,m),r)$ must also be true (ignoring its statements involving $\mathcal{I}$) for models of uncountable size where we replace $\mathcal{J}^{M_1}_\eta$ with $\mathcal{J}^{M_1}_{\omega_1}$. Indeed, if M would be an uncountable, transitive model containing r and $\mathcal{J}^{M_1}_{\omega_1}$ for which $\psi((x,y,m),r)$ is wrong, then we let $\bar{N}$ be the transitive collapse of $N \prec M$, $r, \mathcal{J}^{M_1}_\eta \in N$ and $\bar{N}$ would reject $\psi((x,y,m),r)$ as well, even though $\bar{N}$ is of the right form, which gives us a contradiction.

But if $\psi((x,y,m),r)$ holds for arbitrarily large models M, it must be true in the universe $L[r]$. Indeed if some $\aleph_1$-sized ZF^--model of the form $L_\zeta[M,\vec{C},\vec{B}]$, where $M,\vec{C},\vec{B}$ are just the unions of the computations of $m,\vec{c}$ and $\vec{b}$ in suitable countable transitive models of increasing (with limit ω_1) ordinal height, then first note that $M = M_1|\omega_1$ and $L_\zeta[M,\vec{C},\vec{B}]$ sees that there is a set $h \subset \omega_1$ such that $\forall \alpha < \omega_1 (h \cap \alpha \in M_1|\omega_1)$ such that

$$\begin{aligned} L_\zeta[M,\vec{C},\vec{B}] \models \models \forall \gamma \in h(n \in (x,y,m) &\rightarrow S_{\omega\gamma+2n+1} \text{ has an } \omega_1\text{-branch} \\ n \notin (x,y,m) &\rightarrow S_{\omega\gamma+2n} \text{ has an } \omega_1\text{-branch})). \end{aligned}$$

and the computation of $\vec{S}$ must be correct. As the existence of an ω_1-branch through S_α is upwards absolute to $L[r]$ we obtain that indeed, in $L[r]$, there is a set h of desired form such that $w = (x,y,m)$ is coded at every γ-th ω-block of $\vec{S}$ for $\gamma \in h$. $\square$

So to summarize our discussion so far, if we let W^* be our ground model, which is defined as reproducing the move from L to W with M_1 as starting point, then there is a way of coding arbitrary reals x into the $\vec{S}$-sequence, and the statement "x is coded into $\vec{S}$" is $\Sigma^1_4(x)$.

Consequentially we can reproduce the proof of the Π^1_3-uniformization property over W^*. We list all the Π^1_4-formulas, form the set of ∞-allowable forcings over M_1 and eventually define an ω_1-lenght iteration of ∞-allowable forcings just as before. The only changes are that the coding argument has to be altered as described above, and the use of the two-step Σ^1_3-generic absoluteness of M_1 instead of Shoenfield absoluteness, which makes it possible to uniformize Π^1_4-formulas. The generic two-step Σ^1_3 absoluteness of M_1 follows from M_1 being closed under sharps and the well-known result of Martin-Solovay and Woodin (see [4], Theorem 3). This ends the hopefully sufficiently detailed sketch of the proof of Theorem 4.1.

4.2 Forcing over M_n

This section shall outline how to make the adjustments when applying our forcing to the canonical inner models with n-many Woodin cardinals, denoted as usual with M_n. For every such M_n, there exists a notion of Π^1_{n+1}-iterability, which is sufficient to characterize countable initial segments of M_n, even in our ccc generic extensions of M_n.

Fact 4.6. *Let $M_n[G]$ be an ω_1-preserving forcing extension of M_n. Then in $M_n[G]$ there is Π^1_{n+1}-definable set $\mathcal{I}_n$ of premice which are of the form $\mathcal{J}_\eta^{M_n}$ for some $\eta < \omega_1$. $\mathcal{I}_n$ is defined as*

$$\mathcal{I}_n := \{\mathcal{M} \text{ ctbl premouse} : \mathcal{M} \text{ is } \Pi^1_{n+1}\text{-iterable, } \omega\text{-sound and projects to } \omega\},$$

and the set

$$\{\eta < \omega_1 : \exists \mathcal{N} \in \mathcal{I}(\mathcal{N} = \mathcal{J}_\eta^{M_n})\}$$

is cofinal in ω_1.

The sets $\mathcal{I}_n$ will be used to run a coding argument just as described for M_1, with the obvious replacements. The second fact we need concerns generic (two-step) absoluteness of the M_n's. This is true because of a generalization of the Martin-Solovay result due to, most likely Steel and Woodin (see [21], Lemma 3.7), and the fact that M_n is closed under the $x \mapsto M_k^\#(x)$ operation for $k < n$ and every real $x \in M_n$.

Fact 4.7. *For every $n \in \omega$, M_n is Σ^1_{n+2}-generic absolute for forcings of size the second largest Woodin cardinal.*

These two results suffice to run the proof of the Π^1_{n+}-uniformization property as follows: we start with M_n as our ground model and pass first to W^* which contains a $\Sigma_1(\omega_1)$-definable sequence of independent Suslin trees. Then, working in W^*, we list all the Π^1_{n+3}-formulas and repeat the construction of ∞-allowable forcings over W^*. The role of Shoenfield absoluteness is replaced by taking advantage of the generic absoluteness result from above. We use the Π^1_{n+1}-definable set of M_n initial segments to form with the coding forcings Σ^1_{n+3}-predicates for "being coded into $\vec{S}$", similar to the M_1-case. This will obtain:

Theorem 4.8. *For any $n \in \omega$, if the canonical inner model with n Woodin cardinals exists, there is a universe in which the Π^1_{n+3}-uniformization property holds.*

We believe that the above can be improved, indeed we conjecture that for every $n \in \omega$, the Π^1_n-uniformization property can be forced over L.

5 Further possible applications and open problems

In this last section we want to sketch a second application of the proof method we just presented, and introduce some natural follow-up questions which are likely very old and have been asked already somewhere else. First we want to point out that the we expect the proof to be applicable to the generalized Baire space $\kappa^{<\kappa}$. In particular, the Π^1_1-uniformization problem in $\kappa^{<\kappa}$ should (consistently) have a positive solution.

If $\kappa = \omega_1$, then we can first force a universe W over L, which contains an ω_2-sequence of independent Suslin trees $\vec{S}$. Every initial segment of $\vec{S}$ of length $\gamma < \omega_2$, is $\Sigma_1(\omega_1, \gamma)$-definable, and we can repeat the reasoning for ∞-allowable forcings, but this time the forcings should have length $< \omega_2$ and we do not need almost disjoint coding forcings, as we can read off ω_1-length patterns written into $\vec{S}$ using $\aleph_1$-sized transitive models of ZF^-. This readily yields

Theorem 5.1. *The Π^1_1-uniformization property for the generalized Baire space $\omega_1^{<\omega_1}$ can be forced over L.*

The general case for κ a regular cardinal is possibly solved in a similar way, but we will not investigate it here. We end with some questions:

Question 1. *Can the Π^1_n-uniformization property be forced over L for $n > 3$?*

Question 2. *Is it possible to separate the Π^1_3 reduction property from the Π^1_3 uniformization property, i.e. is there a universe in which Π^1_3-reduction holds but Π^1_3-uniformization is false? What about the case $n > 3$?*

The method we introduced is limited so far to local effects. It would be interesting to force a less local or even global behaviour:

Question 3. *Given a pair $n, m \in \omega$ such that $n \neq m, n \neq m+1$. Can one force a universe in which the Π^1_n and the Π^1_m uniformization property does hold simultaneously?*

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