Exercise Sheet 1

1. Consider the just-identified model $y_i = b_i$ we have discussed in the lecture. The LASSO is the minimiser of the objective function:

$$\sum_{i=1}^{n}(y_i - b_i)^2 + \lambda \sum_{i=1}^{n}|b_i|$$

Show that the estimator takes the form:

$$\widehat{b_i} = \begin{cases} y_i - \lambda/2 & \text{if } \ y_i > \lambda/2 \\ y_i + \lambda/2 & \text{if } \ y_i > \lambda/2 \\ 0 & \text{if } \ -\lambda/2 \leq y_i \leq \lambda/2 \end{cases}$$

Hint: Use the fact that $|b| = b - 2 \cdot b \cdot \mathbf{1}_{\{b<0\}}$ and $|0| = 0$. And derive the first order conditions for the cases $b < 0$, $b > 0$, and $b = 0$.

2. Prove Bayes Theorem. Hint: use the definition of $P(H|E)$ and $P(E|H)$, respectively, and solve.

3. Let us assume we are "nature", i.e. we know the true distributions and relationships between the binary dependent variable y, the continuous predictors x_1 and x_2, and the continuous errors ε. They can be summarised as follows:

   - $y^*(x) = 0.5 \cdot x_1 + 0.5 \cdot x_2 + \varepsilon$
   - $x_1, x_2 \sim U[0, 1]$ (uniformly distributed between 0 and 1)
   - $\varepsilon \sim \mathcal{N}(0, 0.1^2)$
   - $\delta(x) = \mathbf{1}_{\{y^*(x)>0.5\}}$

   (a) Create a dataset of $n = 200$ observations: iid draws of $(y, x_1, x_2, \varepsilon)$ satisfying the conditions above.
   (b) Plot them in a 2d plot with different colours for different cases $y = 0$ and $y = 1$.
   (c) What is the Bayes classifier for this problem?
   (d) Calculate the Bayes decision boundary: i.e. the curve at which the Bayes classifier would (on average!) be indifferent between the two cases.
   (e) Plot the Bayes decision boundary in the above graph.
   (f) What is its shape?

4. Repeat Exercise 3 with

   - $y^*(x) = x_1 + 2 \cdot x_2 + x_2^2 + \varepsilon$
   - $x_1 \sim U[1, 2]$ and $x_2 \sim U[-1, 0]$.

   What is the shape of the Bayes decision boundary now?

5. Consider the function

$$f(x) = (a - x)^2 + (b - x^2)^2$$

Show that it's first and second derivative are:

$$f'(x) = -2(a - x) + 2(b - x^2)(-2x)$$
$$f''(x) = 2 - 4(b - x^2) + 8x^2.$$