

Exercise Sheet 1

1. Consider the just-identified model $y_i = b_i$ we have discussed in the lecture. The LASSO is the minimiser of the objective function:

$$\sum_{i=1}^n (y_i - b_i)^2 + \lambda \sum_{i=1}^n |b_i|$$

Show that the estimator takes the form:

$$\hat{b}_i = \begin{cases} y_i - \lambda/2 & \text{if } y_i > \lambda/2 \\ y_i + \lambda/2 & \text{if } y_i < -\lambda/2 \\ 0 & \text{if } -\lambda/2 \leq y_i \leq \lambda/2 \end{cases}$$

Hint: Use the fact that $|b| = b - 2 \cdot b \cdot \mathbf{1}_{\{b < 0\}}$ and $|0| = 0$. And derive the first order conditions for the cases $b < 0$, $b > 0$, and $b = 0$.

2. Prove Bayes Theorem. Hint: use the definition of $P(H|E)$ and $P(E|H)$, respectively, and solve.
3. Let us assume we are “nature”, i.e. we know the true distributions and relationships between the binary dependent variable y , the continuous predictors x_1 and x_2 , and the continuous errors ε . They can be summarised as follows:

- $y^*(x) = 0.5 \cdot x_1 + 0.5 \cdot x_2 + \varepsilon$
- $x_1, x_2 \sim U[0, 1]$ (uniformly distributed between 0 and 1)
- $\varepsilon \sim \mathcal{N}(0, 0.1^2)$
- $\delta(x) = \mathbf{1}_{\{y^*(x) > 0.5\}}$

- (a) Create a dataset of $n = 200$ observations: iid draws of $(y, x_1, x_2, \varepsilon)$ satisfying the conditions above.
- (b) Plot them in a 2d plot with different colours for different cases $y = 0$ and $y = 1$.
- (c) What is the Bayes classifier for this problem?
- (d) Calculate the Bayes decision boundary: i.e. the curve at which the Bayes classifier would (on average!) be indifferent between the two cases.
- (e) Plot the Bayes decision boundary in the above graph.
- (f) What is its shape?
4. Repeat Exercise 3 with
- $y^*(x) = x_1 + 2 \cdot x_2 + x_2^2 + \varepsilon$
 - $x_1 \sim U[1, 2]$ and $x_2 \sim U[-1, 0]$.

What is the shape of the Bayes decision boundary now?

5. Consider the function

$$f(x) = (a - x)^2 + (b - x^2)^2$$

Show that it's first and second derivative are:

$$\begin{aligned} f'(x) &= -2(a - x) + 2(b - x^2)(-2x) \\ f''(x) &= 2 - 4(b - x^2) + 8x^2. \end{aligned}$$