Exercise Sheet 1

1. Consider the just-identified model $y_i = b_i$ we have discussed in the lecture. The LASSO is the minimiser of the objective function:

$$\sum_{i=1}^{n} (y_i - b_i)^2 + \lambda \sum_{i=1}^{n} |b_i|$$

Show that the estimator takes the form:

$$\widehat{b}_i = \begin{cases} y_i - \lambda/2 & \text{if } y_i > \lambda/2 \\ y_i + \lambda/2 & \text{if } y_i > \lambda/2 \\ 0 & \text{if } -\lambda/2 \le y_i \le -\lambda/2 \end{cases}$$

Hint: Use the fact that $|b| = b - 2 \cdot b \cdot \mathbf{1}_{\{b < 0\}}$ and |0| = 0. And derive the first order conditions for the cases b < 0, b > 0, and b = 0.

- 2. Prove Bayes Theorem. Hint: use the definition of P(H|E) and P(E|H), respectively, and solve.
- 3. Let us assume we are "nature", i.e. we know the true distributions and relationships between the binary dependent variable y, the continuous predictors x_1 and x_2 , and the continuous errors ε . They can be summarised as follows:
 - $y^*(x) = 0.5 \cdot x_1 + 0.5 \cdot x_2 + \varepsilon$
 - $x_1, x_2 \sim U[0,1]$ (uniformly distributed between 0 and 1)
 - $\varepsilon \sim \mathcal{N}(0, 0.1^2)$
 - $\delta(x) = \mathbf{1}_{\{y^*(x) > 0.5\}}$
 - (a) Create a dataset of n = 200 observations: iid draws of $(y, x_1, x_2, \varepsilon)$ satisfying the conditions above.
 - (b) Plot them in a 2d plot with different colours for different cases y = 0 and y = 1.
 - (c) What is the Bayes classifier for this problem?
 - (d) Calculate the Bayes decision boundary: i.e. the curve at which the Bayes classifier would (on average!) be indifferent between the two cases.
 - (e) Plot the Bayes decision boundary in the above graph.
 - (f) What is its shape?
- 4. Repeat Exercise 3 with
 - $y^*(x) = x_1 + 2 \cdot x_2 + x_2^2 + \varepsilon$
 - $x_1 \sim U[1,2]$ and $x_2 \sim U[-1,0]$.

What is the shape of the Bayes decision boundary now?

5. Consider the function

$$f(x) = (a - x)^2 + (b - x^2)^2$$

Show that it's first and second derivative are:

$$f'(x) = -2(a-x) + 2(b-x^2)(-2x)$$

$$f''(x) = 2 - 4(b-x^2) + 8x^2.$$