Consumer surplus

Let $\Phi_i = \max_{j \in \mathcal{J}} u_{ij}$ be the optimal utility of individual i by choosing option j from the set of options $\mathcal{J}$. For simplicity we let $\mathcal{J} = \{0, 1\}$, for binary outcomes. We assume individuals have utlity $u_{ij} = \delta_j + \varepsilon_{ij}$ where ε_{ij} are unobserved individual-specific tastes for good j with standard Gumbel distribution (mean zero and "scale" one). Further, we assume that observations (y_i, x_i) are iid copies of a random vector and drop the subscript i from now on.

Since the $\max_j u_j \leq \phi$ if and only if $u_j \leq \phi$ for all $j \in \mathcal{J}$ we get

$$
\begin{aligned}
F_\Phi(\phi) = P(\Phi \leq \phi) &= P(\{\delta_0 + \varepsilon_0 \leq \phi\} \cap \{\delta_1 + \varepsilon_1 \leq \phi\}) && (1)\\
&= P(\{\delta_0 + \varepsilon_0 \leq \phi\}) P(\{\delta_1 + \varepsilon_1 \leq \phi\}) && (2)\\
&= P(\{\varepsilon_0 \leq \phi - \delta_0\}) P(\{\varepsilon_1 \leq \phi - \delta_1\}) && (3)\\
&= F_\varepsilon(\phi - \delta_0) F_\varepsilon(\phi - \delta_1) && (4)
\end{aligned}
$$

where the second row follows from independence of ε and the last row from the identical distributions of ε_i. Plugging in the cdf of the Gumbel distribution F_ε we get

$$
\begin{aligned}
F_\Phi(\phi) &= e^{-e^{-(\phi-\delta_0)}} e^{-e^{-(\phi-\delta_1)}} && (5)\\
&= e^{-e^{-\phi}(e^{\delta_0} + e^{\delta_1})} && (6)\\
&= e^{-e^{-\phi} L} && (7)
\end{aligned}
$$

where $L = e^{e^{\delta_0} + e^{\delta_1}}$. To get the density we differentiate with respect to ϕ using the chain rule:

$$f_\Phi(\phi) = F'_\Phi(\phi) = e^{-e^{-\phi} L}(-e^{-\phi} L)(-1) = F_\Phi(\phi) L e^{-\phi}.$$

Finally, we have an expression for the expected consumer surplus

$$
\begin{aligned}
\mathbb{E}[\Phi] &= \int_{-\infty}^{\infty} \phi f_\phi(\phi) d\phi && (8)\\
&= \int_{-\infty}^{\infty} \phi e^{-e^{-\phi} L} e^{-\phi} L d\phi && (9)
\end{aligned}
$$

Now, we change variables to $t = e^{-\phi}$ such that $-\log(t) = \phi$ and get:

$$
\begin{aligned}
\mathbb{E}[\Phi] &= \int_{e^{-(-\infty)}}^{e^{-\infty}} -\log(t) e^{-tL} L t \frac{d\phi}{dt} dt && (10)\\
&= \int_{+\infty}^{0} -\log(t) e^{-tL} L t \left(-\frac{1}{t}\right) dt && (11)\\
&= L \int_{0}^{\infty} \log(t) e^{-tL} dt = \log L + \gamma && (12)
\end{aligned}
$$

where the last integral is well known and evaluates to $\frac{-\log L - \gamma}{L}$ where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.