Exercise Sheet 3

1. Let Ω be our sample space, and define two independent random variables on it: Y_1 and Y_2. Let $Y_i \sim \text{Bernoulli}(\theta)$ for $i = 1, 2$. Note that Bernoulli a distributed r.v. Y_i takes values 0 or 1 with the parameter θ corresponding to $P(Y_i = 1)$. We consider only two parameters $\theta \in \Theta = \{\frac{1}{3}, \frac{2}{3}\}$.

   - What possible realisations of (Y_1, Y_2) could you get?
   - For each of them, write down the likelihood for both parameter values.
   - What is the maximum likelihood estimator for each sample?
   - Interpret.
   - Once you have obtained $\widehat{\beta}$ how would you define your Bayes classifier?

2. For the logit model we had

$$P(Y_i = y | X_i = x; \beta) = \frac{1}{1 + e^{-x_i^T \beta}}$$

   and log-likelihood

$$l(\beta) = \sum_{i=1}^{n} \left[y_i \log P(Y_i = 1 | X_i = x; \beta) + (1 - y_i) \log(1 - P(Y_i = 1 | X_i = x; \beta)) \right].$$

   - Explain the principle of MLE. How do we find β?
   - For our numerical procedure we require the $l'(\beta) = \frac{dl(\beta)}{d\beta}$. Derive it.
   - Does the solution look somewhat familiar? Explain.

3. We are trying to predict individual `defaults` (y) using account `balance` (x_1) and `income` (x_2) as inputs/predictors. You can use generative AI as guidance for the following exercises but make sure to always double-check!

   - Install and load the R library `ISLR` and use the data `Default`.
   - Split the dataset into a training and a validation set.
   - Plot `defaults` as a function of `balance`.
   - Add a logistic regression line to this plot.
   - Now estimate the logistic regression for the model with both x_1 and x_2 as predictors.
   - Plot the defaults (y) with different colours in (x_1, x_2) space.
   - Derive and add the Bayes decision boundary to this plot.

4. *Advanced:* Implement the Maximum Likelihood estimator for $\widehat{\beta}$ based on the likelihood (and its gradient) in Question 1. *Hint:* use the function `optim(b, fn = ll, gr = grad)` where `b` is the start value, `ll` is the likelihood function (as a function of `b`) and grad is the gradient of `ll` also as a function of `b`.