1 Exact solution derivation for wave equation

We consider the linear wave equation restricted to the surface of a sphere of radius R. The equation is given by

$$u_{tt}(\theta,\phi,t) = c^2 \Delta_{S^2} u(\theta,\phi,t) \tag{1}$$

where Δ_{S^2} is the Laplace–Beltrami operator on the sphere of radius R. In spherical coordinates, the Laplace–Beltrami operator is given explicitly by

$$\Delta_{S^2} = \frac{1}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

It is a classical result that the spherical harmonics Y_ℓ^m are eigenfunctions of the Laplace-Beltrami operator on the sphere with eigen values $-\ell(\ell+1)$, that is

$$\Delta_{S^2} Y_\ell^m(\theta,\phi) = -\frac{\ell(\ell+1)}{R^2} Y_\ell^m(\theta,\phi) \quad -\ell \le m \le \ell \tag{2}$$

The problem has initial conditions in the form

$$u(\theta,\phi,0) = f(\theta,\phi) \quad \frac{\partial u}{\partial t}(\theta,\phi,0) = g(\theta,\phi)$$

where f and g represent the initial displacement and initial velocity on the sphere respectively.

We can write the function $u(\theta,\phi)$ on a sphere in terms of

$$u(\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_{\ell m} Y_\ell^m(\theta,\phi) \tag{3}$$

where $Y_\ell^m(\theta,\phi)$ is the set of spherical harmonics given by

$$Y_n^m(\theta,\phi) = N_{\ell m} P_n^m(\cos\theta) e^{im\phi} \tag{4}$$

with normalization constant

$$N_{\ell m} = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}$$

Substituting (3) into the wave equation and differentiating we get

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \ddot{u}_{\ell m}(t)\, Y_\ell^m(\theta,\phi) = c^2 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_{\ell m}(t)\, \Delta_{S^2} Y_\ell^m(\theta,\phi).$$

Using equation (2) we obtain

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \ddot{u}_{\ell m}(t)\, Y_\ell^m(\theta,\phi) = -\frac{c^2}{R^2} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \ell(\ell+1)\, u_{\ell m}(t)\, Y_\ell^m(\theta,\phi)$$

This reduces the PDE to a family of uncoupled ordinary differential equations for each mode

$$\ddot{u}_{\ell m}(t) + \omega_\ell^2 u_{\ell m}(t) = 0, \quad \omega_\ell = \frac{c}{R}\sqrt{\ell(\ell+1)}$$

where each coefficient $u_{\ell m}(t)$ evolves independently. Therefore, the solution to the above differential equation is given by

$$u_{\ell m}(t) = A_{\ell m}\cos(\omega_\ell t) + B_{\ell m}\sin(\omega_\ell t) \implies \dot{u}_{\ell m}(t) = -A_{\ell m}\omega_\ell \sin(\omega_\ell t) + B_{\ell m}\omega_\ell \cos(\omega_\ell t) \quad (5)$$

Using the initial conditions value we get

$$u_{\ell m}(0) = A_{\ell m}(1) + 0 = f_{\ell m} \implies A_{\ell m} = f_{\ell m}$$

and for the velocity

$$\dot{u}_{\ell m}(0) = -0 + B_{\ell m}\omega_\ell(1) = g_{\ell m} \implies B_{\ell m} = \frac{g_{\ell m}}{\omega_\ell}$$

Therefore the solution from (5) can be written as

$$u_{\ell m}(t) = f_{\ell m}\cos(\omega_\ell t) + \frac{g_{\ell m}}{\omega_\ell}\sin(\omega_\ell t)$$

Here $f_{\ell m}$ and $g_{\ell m}$ are the spherical harmonic coefficients of the initial displacement $f(\theta, \phi)$ and initial velocity $g(\theta, \phi)$ respectively.

For f to be defined we first choose a point C on the sphere where the Gaussian hill will be centered. The point is given by

$$C = (\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)R$$

where the colatitude is $\theta_0 =$ and $\phi_0 =$. For any other point (x, y, z) on the sphere, the distance from the center point C is measured by the angle at the sphere's center between the two points. This angle is found using the dot formula

$$\alpha = \cos^{-1}\left(\frac{(x, y, z)\cdot C}{R^2}\right)$$

The angle α represents the shortest path along the surface on the sphere between the point (x, y, z) and the chosen center point C.

Using this, the initial displacement is defined as a Gaussian bump centered at C which is given by

$$f(x, y, z) = A\exp\left(-\frac{\alpha^2}{2\sigma^2}\right)$$

Here $A = 1$ is the peak amplitude of the wave and $\sigma = 0.1482$ determines the width of the Gaussian bump. The initial velocity is chosen to be zero

$$g(x, y, z) = 0$$

We need to expand the functions f and g to use them. The coefficients are obtained by projection

$$f_{\ell m} = \int f(\theta, \phi) Y_\ell^m(\theta, \phi)^* d\Omega, \quad g_{\ell m} = \int g(\theta, \phi) Y_\ell^m(\theta, \phi)^* d\Omega$$

where $d\Omega = \sin\theta d\theta d\phi$ denoting the surface element on the sphere. These coefficients provide the initial data for the modal amplitudes $u_{\ell m}(t)$ which evolve according to the exact solution formula. As seen earlier, the spherical harmonics are defined as

$$Y_\ell^m(\theta, \phi) = N_{\ell m}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

where $N_{\ell m}$ is the normalization constant and P_ℓ^m is the associated Legendre polynomial. Taking the complex conjugate gives

$$Y_\ell^m(\theta, \phi)^* = N_{\ell m}\, P_\ell^m(\cos\theta)\, e^{-im\phi}$$

We can therefore show the conjugation symmetry as

$$Y_\ell^m(\theta, \phi) = (-1)^m\, Y_\ell^{-m}(\theta, \phi)^*$$