

Summary of Exact Solution Method: 1D Wave Equation (Neumann BCs)

1. Problem Definition

We consider the homogeneous wave equation on a finite domain $\Omega = [-1, 1]$ with homogeneous Neumann boundary conditions (zero slope at ends) and a specific initial condition.

Governing Equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in (-1, 1), t > 0$$

Boundary Conditions (Neumann):

$$\frac{\partial u}{\partial x}(-1, t) = 0, \quad \frac{\partial u}{\partial x}(1, t) = 0$$

Physically, this represents "free ends" or insulated boundaries where the wave reflects perfectly without inversion.

Initial Conditions:

$$u(x, 0) = \psi(x) = \exp \left[- \left(\frac{x - x_0}{\sigma} \right)^2 \right] \quad (\text{Gaussian Pulse})$$
$$\frac{\partial u}{\partial t}(x, 0) = 0 \quad (\text{Zero Initial Velocity})$$

2. Solution via Separation of Variables

The standard analytical approach assumes the solution can be factored into spatial and temporal components: $u(x, t) = X(x)T(t)$.

Step A: Spatial Eigenvalue Problem

Substituting the ansatz into the PDE leads to the spatial ODE:

$$X''(x) + \lambda^2 X(x) = 0$$

Subject to boundary conditions $X'(-1) = X'(1) = 0$.

- The eigenvalues are $\lambda_n = n\pi$ (for domain length 2).
- The eigenfunctions are **Cosines**: $X_n(x) = \cos(n\pi x)$.
- Note: If the boundary conditions were Dirichlet ($u = 0$), the eigenfunctions would be Sines.

Step B: Temporal Problem

The temporal ODE is $T''(t) + c^2 \lambda_n^2 T(t) = 0$. The general solution is $T_n(t) = A_n \cos(c\lambda_n t) + B_n \sin(c\lambda_n t)$.

- Since the initial velocity is zero ($u_t(x, 0) = 0$), the sine terms vanish ($B_n = 0$).

Step C: General Series Solution

By superposition, the full solution is the sum of all modes:

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi c t) \cos(n\pi x)$$

Step D: Determining Coefficients (A_n)

The coefficients A_n are determined by projecting the initial condition $\psi(x)$ onto the basis functions (Fourier Cosine Transform):

$$A_n = \int_{-1}^1 \psi(x) \cos(n\pi x) dx$$

In the provided Python script, this integral is computed numerically using `scipy.integrate.quad` because the Gaussian pulse does not have a simple closed-form finite Fourier series.

3. Reference (BibTeX)

Use the following reference for the derivation of the wave equation on finite domains using separation of variables.

```
@misc{ward_wave_eqn,
    author      = {Ward, Michael J.},
    title       = {{Lecture 23: The wave equation on finite domains -- solution by separation of variables}},
    howpublished = {University of British Columbia, Math 316 Course Notes},
    year        = {n.d.},
    url         = {[https://personal.math.ubc.ca/~ward/teaching/m316/lecture23.pdf]}(http://personal.math.ubc.ca/~ward/teaching/m316/lecture23.pdf),
    note        = {Accessed: 2025-11-21}
}
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