Multivariable Calculus Self-Learning Module

Exercises

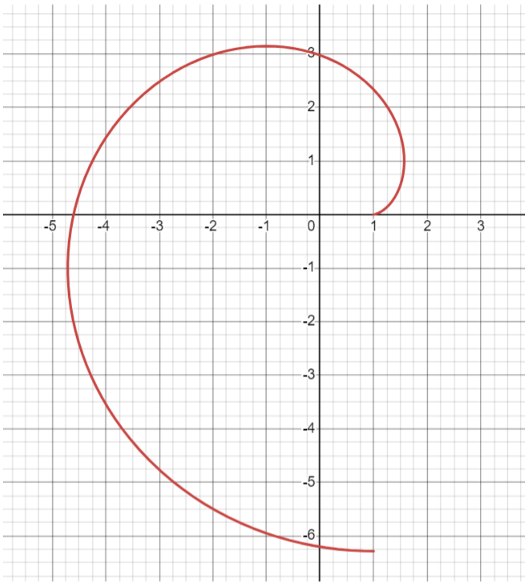
Contents

1 Exercise 1

Calculate the length of curve C, where C is the curve:

$$t \mapsto (\cos(t) + t\sin(t),\, \sin(t) - t\cos(t))$$

for $0 \leq t \leq 2\pi$.



1.1 Hint 1

There is a formula for calculating the length of the curve. You must know this one! The formula is of the form:

$$\int_{\dots}^{\dots} |\dots|\, dt$$

1.2 Hint 2

The formula for the length is:

$$\int_0^{2\pi} |f'(t)|\, dt$$

1.3 Hint 3

$$\begin{aligned} f'(t) &= \frac{d}{dt}(\cos(t) + t\sin(t),\, \sin(t) - t\cos(t)) \\ &= (-\sin(t) + \sin(t) + t\cos(t),\, \cos(t) - \cos(t) + t\sin(t)) \\ &= (t\cos(t),\, t\sin(t)) \end{aligned}$$

So

$$\begin{aligned} |f'(t)| &= \sqrt{t^2\cos^2(t) + t^2\sin^2(t)} \\ &= \sqrt{t^2} \\ &= |t| \end{aligned}$$

1.4 Solution

Since $t \geq 0$, we have $|t| = t$, so:

$$\begin{aligned}
\int_0^{2\pi} |f'(t)|\, dt &= \int_0^{2\pi} t\, dt \\
&= \left[\frac{t^2}{2}\right]_0^{2\pi} \\
&= 2\pi^2
\end{aligned}$$

2 Exercise 2

Consider the curve C given by $f(x, y) = 0$, where

$$f(x, y) = (x - y)^2 + 4(x + y) - 4$$

Determine the point on C at which $x + y$ is maximal.

2.1 Hint 1

We want to maximize $g(x, y) = x + y$ subject to the constraint $f(x, y) = 0$. Use Lagrange.

2.2 Hint 2

Solve the equation $\nabla g = \lambda \nabla f$.

2.3 Solution

$$\nabla g(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\nabla f(x, y) = \begin{pmatrix} 2(x - y) + 4 \\ -2(x - y) + 4 \end{pmatrix}$$

So

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2(x - y) + 4 \\ -2(x - y) + 4 \end{pmatrix}$$

Note that $\lambda = 0$ gives

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is a contradiction, so $\lambda \neq 0$ and we get the system:

$$\begin{cases} 2(x - y) + 4 = \dfrac{1}{\lambda} \\ -2(x - y) + 4 = \dfrac{1}{\lambda} \end{cases}$$

Subtracting the two equations gives:

$$2(x - y) + 4 + 2(x - y) - 4 = \frac{1}{\lambda} - \frac{1}{\lambda} \Leftrightarrow 4(x - y) = 0 \Leftrightarrow x = y$$

Since we are looking for the point that maximizes $x + y$ on C, we substitute $y = x$ in $f(x, y) = 0$:

$$(x - x)^2 + 4(x + x) - 4 = 0 \Leftrightarrow 8x = 4 \Leftrightarrow x = \frac{1}{2}$$

So

$$(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and the maximum is

$$g\left(\frac{1}{2}, \frac{1}{2}\right) = 1.$$

3 Exercise 3

Consider the function

$$f(x,y,z) = \frac{1}{x} + \frac{1}{8y} + \frac{1}{27z}.$$

Find the point on the unit sphere (i.e. the sphere centered at $(0,0,0)$ of radius 1) at which f is maximal and the point at which it is minimal. Calculate also these maximum and minimum values.

3.1 Hint 1

The surface of a sphere centered at (x_0, y_0, z_0) of radius R has the formula

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

3.2 Hint 2

Let

$$g(x,y,z) = x^2 + y^2 + z^2 - 1.$$

We want to maximize $f(x,y,z)$ given the constraint $g(x,y,z) = 0$. Use Lagrange.

3.3 Hint 3

Solve the equation $\nabla f = \lambda \nabla g$.

3.4 Hint 4

$$\nabla f(x,y,z) = \begin{pmatrix} -\dfrac{1}{x^2} \\ \\ -\dfrac{1}{8y^2} \\ \\ -\dfrac{1}{27z^2} \end{pmatrix}$$

$$\nabla g(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

So

$$\begin{pmatrix} -\dfrac{1}{x^2} \\ \\ -\dfrac{1}{8y^2} \\ \\ -\dfrac{1}{27z^2} \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Note that the left-hand side is always non-zero, so $\lambda \neq 0$. Hence:

$$\begin{cases} -\dfrac{1}{x^2} = 2\lambda x \\ \\ -\dfrac{1}{8y^2} = 2\lambda y \\ \\ -\dfrac{1}{27z^2} = 2\lambda z \end{cases} \Leftrightarrow \begin{cases} x^3 = -\dfrac{1}{2\lambda} \\ \\ y^3 = -\dfrac{1}{8 \cdot 2\lambda} \\ \\ z^3 = -\dfrac{1}{27 \cdot 2\lambda} \end{cases} \Leftrightarrow \begin{cases} x = -\sqrt[3]{\dfrac{1}{2\lambda}} \\ \\ y = -\dfrac{1}{2}\sqrt[3]{\dfrac{1}{2\lambda}} \\ \\ z = -\dfrac{1}{3}\sqrt[3]{\dfrac{1}{2\lambda}} \end{cases} \Leftrightarrow \begin{cases} x = -\sqrt[3]{\dfrac{1}{2\lambda}} \\ \\ y = -\dfrac{1}{2} \cdot x \\ \\ z = -\dfrac{1}{3} \cdot x \end{cases}$$

3.5 Hint 5

The points lie on the unit sphere, so $x^2 + y^2 + z^2 = 1$ must also hold.

3.6 Solution

$$\begin{aligned}
x^2 + \left(-\frac{1}{2}\cdot x\right)^2 + \left(-\frac{1}{3}\cdot x\right)^2 &= 1\\
\Leftrightarrow x^2 \cdot \left(1 + \frac{1}{4} + \frac{1}{9}\right) &= 1\\
\Leftrightarrow x^2 \cdot \frac{49}{36} &= 1\\
\Leftrightarrow x^2 &= \frac{36}{49}\\
\Leftrightarrow x &= \pm\frac{6}{7}
\end{aligned}$$

We get two solutions:

$$\begin{aligned}
x &= \frac{6}{7}\\
y &= -\frac{1}{2}\cdot x = -\frac{3}{7}\\
z &= -\frac{1}{3}\cdot x = -\frac{2}{7}
\end{aligned}$$

and

$$\begin{aligned}
x &= -\frac{6}{7}\\
y &= -\frac{1}{2}\cdot x = \frac{3}{7}\\
z &= -\frac{1}{3}\cdot x = \frac{2}{7}
\end{aligned}$$

In order to determine which one gives the minimum and which one the maximum, we substitute in f:

$$\begin{aligned}
f\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right) &= \frac{1}{-6/7} + \frac{1}{8\cdot(-3/7)} + \frac{1}{27\cdot(-2/7)}\\
&= -\frac{7}{6} - \frac{7}{24} - \frac{7}{54}\\
&= -\frac{373}{216}
\end{aligned}$$

and

$$\begin{aligned}
f\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right) &= \frac{1}{6/7} + \frac{1}{8\cdot 3/7} + \frac{1}{27\cdot 2/7}\\
&= \frac{7}{6} + \frac{7}{24} + \frac{7}{54}\\
&= \frac{373}{216}
\end{aligned}$$

So f achieves a maximum value of $373/216$ at

$$\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

and a minimum value of $-373/216$ at

$$\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right).$$

4 Exercise 4

Consider the function

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

defined on the set

$$K = \{(x, y) : 0 < x \leq 1, 0 < y \leq 1\}.$$

Determine where f assumes its minimum, and what that minimum value is.

4.1 Hint 1

Calculate ∇f.

4.2 Hint 2

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left(\frac{xy \cdot 2x - (x^2 + y^2) \cdot y}{x^2 y^2}\right) \\ &= \frac{x^2 y - y^3}{x^2 y^2} \\ &= \frac{x^2 - y^2}{x^2 y}\end{aligned}$$

By symmetry, we have:

$$\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{xy^2}$$

So

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x^2 - y^2}{x^2 y} \\ \\ \frac{y^2 - x^2}{xy^2} \end{pmatrix}$$

4.3 Hint 3

Solve $\nabla f = 0$.

4.4 Hint 4

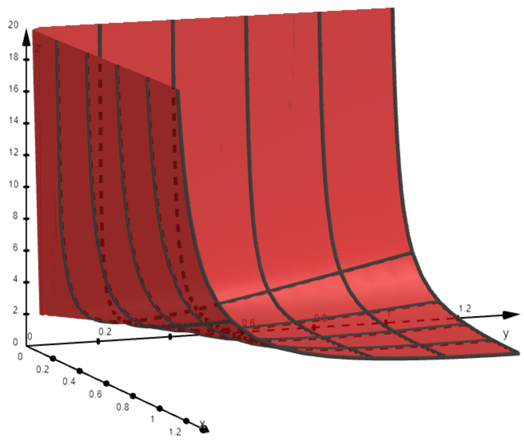
$$\begin{pmatrix} \frac{x^2 - y^2}{x^2 y} \\ \\ \frac{y^2 - x^2}{xy^2} \end{pmatrix} = 0 \Leftrightarrow x^2 - y^2 = 0 \Leftrightarrow x = \pm y$$

4.5 Solution

Note that since $x, y > 0$ on K, we cannot have $x = -y$. Substituting $y = x$ in f we get:

$$f(x, x) = \frac{x^2 + x^2}{x^2} = 2$$

So f achieves its minimum value 2 on the entire line segment $x = y$, $0 < x \leq 1, 0 < y \leq 1$.


z
y
x
0
0.2
0.4
0.6
0.8
1
1.2
2
4
6
8
10
12
14
16
18
20

5 Exercise 5

Consider the surface S defined by $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + y^2 + z^2 + 3xy - z - 11$$

1. Check that $A = (1, 1, 3)$ lies on S.
2. Give an equation of the form $\alpha x + \beta y + \gamma z = \delta$ describing the tangent plane to S at point A.

5.1 Hint 1

1. Substitute $x = 1$, $y = 1$, $z = 3$ in f to get:

$$1^2 + 1^2 + 3^2 + 3 \cdot 1 \cdot 1 - 3 - 11 = 0.$$

5.2 Hint 2

2. There are several ways to go about this, but in most (if not all) you need to compute ∇f at the point A.

5.3 Hint 3

$$\nabla f(x, y, z) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \\ \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x + 2y \\ 2z - 1 \end{pmatrix}$$

So

$$\nabla f(1, 1, 3) = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 3 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

5.4 Hint 4

The gradient of f at A is a vector orthogonal to the tangent plane to S at point A. How do you find an equation of a plane if you know a vector orthogonal to it and a point on it?

5.5 Solution

The tangent plane is perpendicular to the vector $\nabla f(1, 1, 3) = (5, 5, 5)$ and passes through the point $(1, 1, 3)$, so its equation is:

$$5 \cdot (x - 1) + 5 \cdot (y - 1) + 5 \cdot (z - 3) = 0 \Leftrightarrow 5x + 5y + 5z = 25$$

Remark: *There is also a formula given in the book as the linearization of f at $A = (x_0, y_0, z_0)$:*

$$L(x, y, z) = f(A) + f_x(A) \cdot (x - x_0) + f_y(A) \cdot (y - y_0) + f_z(A) \cdot (z - z_0)$$

(this is essentially a first order Taylor expansion). The formula will yield the exact same answer. However, applying formulas without understanding is like eating your food without chewing!