

Multivariable Calculus Self-Learning Module

Exercises

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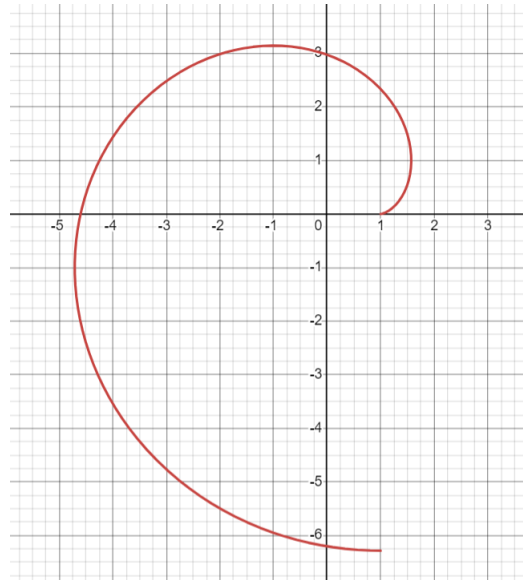
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1 Exercise 1

Calculate the length of curve C , where C is the curve:

$$t \mapsto (\cos(t) + t \sin(t), \sin(t) - t \cos(t))$$

for $0 \leq t \leq 2\pi$.



1.1 Hint

There is a formula for calculating the length of the curve. You must know this one! The formula is of the form:

$$\int_{\dots}^{\dots} |\dots| dt$$

1.2 Hint

The formula for the length is:

$$\int_0^{2\pi} |f'(t)| dt$$

1.3 Hint

$$\begin{aligned} f'(t) &= \frac{d}{dt}(\cos(t) + t \sin(t), \sin(t) - t \cos(t)) \\ &= (-\sin(t) + \sin(t) + t \cos(t), \cos(t) - \cos(t) + t \sin(t)) \\ &= (t \cos(t), t \sin(t)) \end{aligned}$$

So

$$\begin{aligned} |f'(t)| &= \sqrt{t^2 \cos^2(t) + t^2 \sin^2(t)} \\ &= \sqrt{t^2} \\ &= |t| \end{aligned}$$

1.4 Solution

Since $t \geq 0$, we have $|t| = t$, so:

$$\begin{aligned}\int_0^{2\pi} |f'(t)| \, dt &= \int_0^{2\pi} t \, dt \\ &= \left[\frac{t^2}{2} \right]_0^{2\pi} \\ &= 2\pi^2\end{aligned}$$

2 Exercise 2

Consider the curve C given by $f(x, y) = 0$, where

$$f(x, y) = (x - y)^2 + 4(x + y) - 4$$

Determine the point on C at which $x + y$ is maximal.

2.1 Hint

We want to maximize $g(x, y) = x + y$ subject to the constraint $f(x, y) = 0$. Use Lagrange.

2.2 Hint

Solve the equation $\nabla g = \lambda \nabla f$.

2.3 Solution

$$\begin{aligned}\nabla g(x, y) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \nabla f(x, y) &= \begin{pmatrix} 2(x - y) + 4 \\ -2(x - y) + 4 \end{pmatrix}\end{aligned}$$

So

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2(x - y) + 4 \\ -2(x - y) + 4 \end{pmatrix}$$

Note that $\lambda = 0$ gives

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is a contradiction, so $\lambda \neq 0$ and we get the system:

$$\begin{cases} 2(x - y) + 4 = \frac{1}{\lambda} \\ -2(x - y) + 4 = \frac{1}{\lambda} \end{cases}$$

Subtracting the two equations gives:

$$2(x - y) + 4 + 2(x - y) - 4 = \frac{1}{\lambda} - \frac{1}{\lambda} \iff 4(x - y) = 0 \iff x = y$$

Since we are looking for the point that maximizes $x + y$ on C , we substitute $y = x$ in $f(x, y) = 0$:

$$(x - x)^2 + 4(x + x) - 4 = 0 \iff 8x = 4 \iff x = \frac{1}{2}$$

So

$$(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and the maximum is

$$g\left(\frac{1}{2}, \frac{1}{2}\right) = 1.$$

3 Exercise 3

Consider the function

$$f(x, y, z) = \frac{1}{x} + \frac{1}{8y} + \frac{1}{27z}.$$

Find the point on the unit sphere (i.e. the sphere centered at $(0, 0, 0)$ of radius 1) at which f is maximal and the point at which it is minimal. Calculate also these maximum and minimum values.

3.1 Hint

The surface of a sphere centered at (x_0, y_0, z_0) of radius R has the formula

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

3.2 Hint

Let

$$g(x, y, z) = x^2 + y^2 + z^2 - 1.$$

We want to maximize $f(x, y, z)$ given the constraint $g(x, y, z) = 0$. Use Lagrange.

3.3 Hint

Solve the equation $\nabla f = \lambda \nabla g$.

3.4 Hint

$$\nabla f(x, y, z) = \begin{pmatrix} -\frac{1}{x^2} \\ -\frac{1}{8y^2} \\ -\frac{1}{27z^2} \end{pmatrix}$$

$$\nabla g(x, y, z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

So

$$\begin{pmatrix} -\frac{1}{x^2} \\ -\frac{1}{8y^2} \\ -\frac{1}{27z^2} \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Note that the left-hand side is always non-zero, so $\lambda \neq 0$. Hence:

$$\begin{cases} -\frac{1}{x^2} = 2\lambda x \\ -\frac{1}{8y^2} = 2\lambda y \\ -\frac{1}{27z^2} = 2\lambda z \end{cases} \iff \begin{cases} x^3 = -\frac{1}{2\lambda} \\ y^3 = -\frac{1}{8 \cdot 2\lambda} \\ z^3 = -\frac{1}{27 \cdot 2\lambda} \end{cases} \iff \begin{cases} x = -\sqrt[3]{\frac{1}{2\lambda}} \\ y = -\frac{1}{2} \sqrt[3]{\frac{1}{2\lambda}} \\ z = -\frac{1}{3} \sqrt[3]{\frac{1}{2\lambda}} \end{cases} \iff \begin{cases} x = -\sqrt[3]{\frac{1}{2\lambda}} \\ y = -\frac{1}{2} \cdot x \\ z = -\frac{1}{3} \cdot x \end{cases}$$

3.5 Hint

The points lie on the unit sphere, so $x^2 + y^2 + z^2 = 1$ must also hold.

3.6 Solution

$$\begin{aligned}x^2 + \left(-\frac{1}{2} \cdot x\right)^2 + \left(-\frac{1}{3} \cdot x\right)^2 &= 1 \\ \Leftrightarrow x^2 \cdot \left(1 + \frac{1}{4} + \frac{1}{9}\right) &= 1 \\ \Leftrightarrow x^2 \cdot \frac{49}{36} &= 1 \\ \Leftrightarrow x^2 &= \frac{36}{49} \\ \Leftrightarrow x &= \pm \frac{6}{7}\end{aligned}$$

We get two solutions:

$$\begin{aligned}x &= \frac{6}{7} \\ y &= -\frac{1}{2} \cdot x = -\frac{3}{7} \\ z &= -\frac{1}{3} \cdot x = -\frac{2}{7}\end{aligned}$$

and

$$\begin{aligned}x &= -\frac{6}{7} \\ y &= -\frac{1}{2} \cdot x = \frac{3}{7} \\ z &= -\frac{1}{3} \cdot x = \frac{2}{7}\end{aligned}$$

In order to determine which one gives the minimum and which one the maximum, we substitute in f :

$$\begin{aligned}f\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right) &= \frac{1}{-6/7} + \frac{1}{8 \cdot (-3/7)} + \frac{1}{27 \cdot (-2/7)} \\ &= -\frac{7}{6} - \frac{7}{24} - \frac{7}{54} \\ &= -\frac{373}{216}\end{aligned}$$

and

$$\begin{aligned}f\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right) &= \frac{1}{6/7} + \frac{1}{8 \cdot 3/7} + \frac{1}{27 \cdot 2/7} \\ &= \frac{7}{6} + \frac{7}{24} + \frac{7}{54} \\ &= \frac{373}{216}\end{aligned}$$

So f achieves a maximum value of $373/216$ at

$$\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

and a minimum value of $-373/216$ at

$$\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right).$$

4 Exercise 4

Consider the function

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

defined on the set

$$K = \{(x, y) : 0 < x \leq 1, 0 < y \leq 1\}.$$

Determine where f assumes its minimum, and what that minimum value is.

4.1 Hint

Calculate ∇f .

4.2 Hint

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left(\frac{xy \cdot 2x - (x^2 + y^2) \cdot y}{x^2 y^2} \right) \\ &= \frac{x^2 y - y^3}{x^2 y^2} \\ &= \frac{x^2 - y^2}{x^2 y}\end{aligned}$$

By symmetry, we have:

$$\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{xy^2}$$

So

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x^2 - y^2}{x^2 y} \\ \frac{y^2 - x^2}{xy^2} \end{pmatrix}$$

4.3 Hint

Solve $\nabla f = 0$.

4.4 Hint

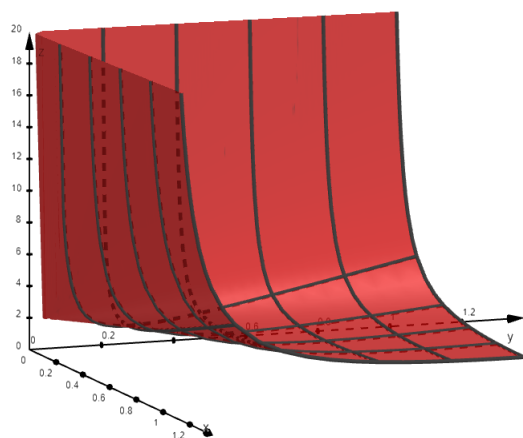
$$\begin{pmatrix} \frac{x^2 - y^2}{x^2 y} \\ \frac{y^2 - x^2}{xy^2} \end{pmatrix} = 0 \iff x^2 - y^2 = 0 \iff x = \pm y$$

4.5 Solution

Note that since $x, y > 0$ on K , we cannot have $x = -y$. Substituting $y = x$ in f we get:

$$f(x, x) = \frac{x^2 + x^2}{x^2} = 2$$

So f achieves its minimum value 2 on the entire line segment $x = y$, $0 < x \leq 1, 0 < y \leq 1$.



5 Exercise 5

Consider the surface S defined by $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + y^2 + z^2 + 3xy - z - 11$$

1. Check that $A = (1, 1, 3)$ lies on S .
2. Give an equation of the form $\alpha x + \beta y + \gamma z = \delta$ describing the tangent plane to S at point A .

5.1 Hint

1. Substitute $x = 1$, $y = 1$, $z = 3$ in f to get:

$$1^2 + 1^2 + 3^2 + 3 \cdot 1 \cdot 1 - 3 - 11 = 0.$$

5.2 Hint

2. There are several ways to go about this, but in most (if not all) you need to compute ∇f at the point A .

5.3 Hint

$$\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x + 2y \\ 2z - 1 \end{pmatrix}$$

So

$$\nabla f(1, 1, 3) = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 3 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

5.4 Hint

The gradient of f at A is a vector orthogonal to the tangent plane to S at point A . How do you find an equation of a plane if you know a vector orthogonal to it and a point on it?

5.5 Solution

The tangent plane is perpendicular to the vector $\nabla f(1, 1, 3) = (5, 5, 5)$ and passes through the point $(1, 1, 3)$, so its equation is:

$$5 \cdot (x - 1) + 5 \cdot (y - 1) + 5 \cdot (z - 3) = 0 \iff 5x + 5y + 5z = 25$$

Remark: There is also a formula given in the book as the linearization of f at $A = (x_0, y_0, z_0)$:

$$L(x, y, z) = f(A) + f_x(A) \cdot (x - x_0) + f_y(A) \cdot (y - y_0) + f_z(A) \cdot (z - z_0)$$

(this is essentially a first order Taylor expansion). The formula will yield the exact same answer. However, applying formulas without understanding is like eating your food without chewing!

6 Exercise 6

Consider the plane P which contains the points $A = (1, 1, 1)$, $B = (2, 3, 9)$, $C = (3, 5, 4)$. Try to think of as many ways as possible to determine the equation $\alpha x + \beta y + \gamma z = \delta$ of this plane.

6.1 Method 1

Just substitute the coordinates of the 3 points in $\alpha x + \beta y + \gamma z = \delta$ and solve for α , β , γ , δ .

6.2 Solution 1

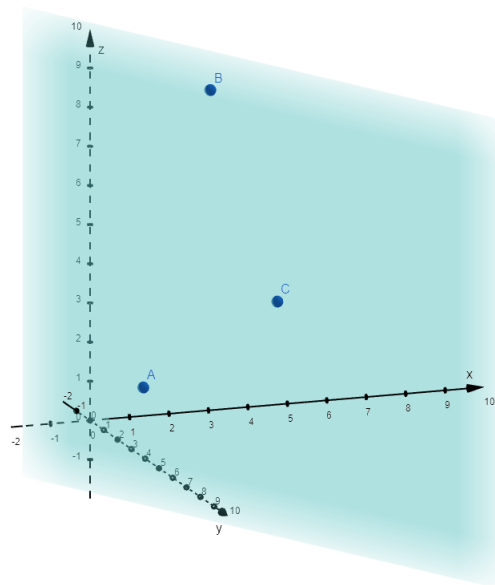
$$\begin{aligned} & \begin{cases} \alpha + \beta + \gamma = \delta \\ 2\alpha + 3\beta + 9\gamma = \delta \\ 3\alpha + 5\beta + 4\gamma = \delta \end{cases} \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array} \\ \Leftrightarrow & \begin{cases} \alpha + \beta + \gamma = \delta \\ \beta + 7\gamma = -\delta \\ 2\beta + \gamma = -2\delta \end{cases} \quad R_3 \leftarrow R_2 - \frac{1}{2}R_3 \\ \Leftrightarrow & \begin{cases} \alpha + \beta + \gamma = \delta \\ \beta + 7\gamma = -\delta \\ \frac{13}{2}\gamma = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \alpha = 2\delta \\ \beta = -\delta \\ \gamma = 0 \end{cases} \end{aligned}$$

Note that δ is a free variable here, and

$$2\delta \cdot x - \delta \cdot y + 0 \cdot z = \delta$$

defines the same plane for any $\delta \neq 0$ (for $\delta = 0$ we just get $0 = 0$ which is satisfied by any x, y, z , i.e. we don't get a plane but the entire \mathbb{R}^3). So we can choose for example $\delta = 1$ to get the following equation for P :

$$2x - y = 1$$



6.3 Method 2

Make the vectors \overrightarrow{AB} , \overrightarrow{AC} and compute the vector $\overrightarrow{AB} \times \overrightarrow{AC}$. Then...

6.4 Solution 2

$$\overrightarrow{AB} = \vec{B} - \vec{A} = (2, 3, 9) - (1, 1, 1) = (1, 2, 8)$$

$$\overrightarrow{AC} = \vec{C} - \vec{A} = (3, 5, 4) - (1, 1, 1) = (2, 4, 3)$$

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} 1 & 2 & 8 \\ 2 & 4 & 3 \\ e_x & e_y & e_z \end{vmatrix} \\ &= \begin{vmatrix} 2 & 8 \\ 4 & 3 \end{vmatrix} \cdot e_x - \begin{vmatrix} 1 & 8 \\ 2 & 3 \end{vmatrix} \cdot e_y + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \cdot e_z \\ &= -26 \cdot e_x + 13 \cdot e_y + 0 \cdot e_z \\ &= (-26, 13, 0)\end{aligned}$$

Now $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to P and A lies on P , so we get the following equation for P :

$$-26 \cdot (x - 1) + 13 \cdot (y - 1) + 0 \cdot (z - 1) = 0 \iff -26x + 13y = -13$$

6.5 Method 3

Find a vector (α, β, γ) that is orthogonal to \overrightarrow{AB} , \overrightarrow{AC} by solving the system

$$\overrightarrow{AB} \cdot (\alpha, \beta, \gamma) = 0$$

$$\overrightarrow{AC} \cdot (\alpha, \beta, \gamma) = 0$$

Then...

6.6 Solution 3

$$\begin{aligned}&\begin{cases} (1, 2, 8) \cdot (\alpha, \beta, \gamma) = 0 \\ (2, 4, 3) \cdot (\alpha, \beta, \gamma) = 0 \end{cases} \\ \iff &\begin{cases} \alpha + 2\beta + 8\gamma = 0 \\ 2\alpha + 4\beta + 3\gamma = 0 \end{cases} \quad | \quad R_2 \leftarrow R_2 - 2R_1 \\ \iff &\begin{cases} \alpha + 2\beta + 8\gamma = 0 \\ -13\gamma = 0 \end{cases} \\ \iff &\begin{cases} \alpha = -2\beta \\ \gamma = 0 \end{cases}\end{aligned}$$

Choose $\beta = 1$, so $(\alpha, \beta, \gamma) = (-2, 1, 0)$, and the equation for P becomes:

$$-2 \cdot x + 1 \cdot y + 0 \cdot z = \delta$$

Substitute $(1, 1, 1)$ in the equation to find δ :

$$\delta = -2 + 1 = -1$$

So the equation for P is:

$$-2x + y = -1$$