Multivariable Calculus Self-Learning Module

Exercises

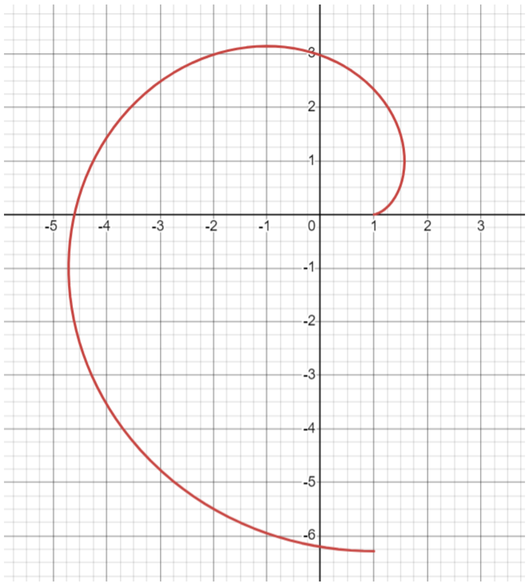
Contents

1 Exercise

Calculate the length of curve C, where C is the curve:

$$t \mapsto (\cos(t) + t\sin(t),\, \sin(t) - t\cos(t))$$

for $0 \leq t \leq 2\pi$.



1.1 Hint

There is a formula for calculating the length of the curve. You must know this one! The formula is of the form:

$$\int_{\dots}^{\dots} |\dots|\, dt$$

1.2 Hint

The formula for the length is:

$$\int_0^{2\pi} |f'(t)|\, dt$$

1.3 Hint

$$\begin{aligned} f'(t) &= \frac{d}{dt}(\cos(t) + t\sin(t),\, \sin(t) - t\cos(t)) \\ &= (-\sin(t) + \sin(t) + t\cos(t),\, \cos(t) - \cos(t) + t\sin(t)) \\ &= (t\cos(t),\, t\sin(t)) \end{aligned}$$

So

$$\begin{aligned} |f'(t)| &= \sqrt{t^2\cos^2(t) + t^2\sin^2(t)} \\ &= \sqrt{t^2} \\ &= |t| \end{aligned}$$

1.4 Solution

Since $t \geq 0$, we have $|t| = t$, so:

$$\begin{aligned} \int_0^{2\pi} |f'(t)|\, dt &= \int_0^{2\pi} t\, dt \\ &= \left[\frac{t^2}{2} \right]_0^{2\pi} \\ &= 2\pi^2 \end{aligned}$$

2 Exercise

Consider the curve C given by $f(x, y) = 0$, where

$$f(x, y) = (x - y)^2 + 4(x + y) - 4$$

Determine the point on C at which $x + y$ is maximal.

2.1 Hint

We want to maximize $g(x, y) = x + y$ subject to the constraint $f(x, y) = 0$. Use Lagrange.

2.2 Hint

Solve the equation $\nabla g = \lambda \nabla f$.

2.3 Solution

$$\nabla g(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\nabla f(x, y) = \begin{pmatrix} 2(x - y) + 4 \\ -2(x - y) + 4 \end{pmatrix}$$

So

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2(x - y) + 4 \\ -2(x - y) + 4 \end{pmatrix}$$

Note that $\lambda = 0$ gives

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is a contradiction, so $\lambda \neq 0$ and we get the system:

$$\begin{cases} 2(x - y) + 4 = \dfrac{1}{\lambda} \\ \\ -2(x - y) + 4 = \dfrac{1}{\lambda} \end{cases}$$

Subtracting the two equations gives:

$$2(x - y) + 4 + 2(x - y) - 4 = \frac{1}{\lambda} - \frac{1}{\lambda} \implies 4(x - y) = 0 \implies x = y$$

Since we are looking for the point that maximizes $x + y$ on C, we substitute $y = x$ in $f(x, y) = 0$:

$$(x - x)^2 + 4(x + x) - 4 = 0 \implies 8x = 4 \implies x = \frac{1}{2}$$

So

$$(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and the maximum is

$$g\left(\frac{1}{2}, \frac{1}{2}\right) = 1.$$

3 Exercise

Consider the function

$$f(x,y,z) = \frac{1}{x} + \frac{1}{8y} + \frac{1}{27z}.$$

Find the point on the unit sphere (i.e. the sphere centered at $(0,0,0)$ of radius 1) at which f is maximal and the point at which it is minimal. Calculate also these maximum and minimum values.

3.1 Hint

The surface of a sphere centered at (x_0, y_0, z_0) of radius R has the formula

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

3.2 Hint

Let

$$g(x,y,z) = x^2 + y^2 + z^2 - 1.$$

We want to maximize $f(x,y,z)$ given the constraint $g(x,y,z) = 0$. Use Lagrange.

3.3 Hint

Solve the equation $\nabla f = \lambda \nabla g$.

3.4 Hint

$$\nabla f(x,y,z) = \begin{pmatrix} -\dfrac{1}{x^2} \\[2ex] -\dfrac{1}{8y^2} \\[2ex] -\dfrac{1}{27z^2} \end{pmatrix}$$

$$\nabla g(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

So

$$\begin{pmatrix} -\dfrac{1}{x^2} \\[2ex] -\dfrac{1}{8y^2} \\[2ex] -\dfrac{1}{27z^2} \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Note that the left-hand side is always non-zero, so $\lambda \neq 0$. Hence:

$$\begin{cases} -\dfrac{1}{x^2} = 2\lambda x \\[2ex] -\dfrac{1}{8y^2} = 2\lambda y \\[2ex] -\dfrac{1}{27z^2} = 2\lambda z \end{cases} \implies \begin{cases} x^3 = -\dfrac{1}{2\lambda} \\[2ex] y^3 = -\dfrac{1}{8 \cdot 2\lambda} \\[2ex] z^3 = -\dfrac{1}{27 \cdot 2\lambda} \end{cases} \implies \begin{cases} x = -\sqrt[3]{\dfrac{1}{2\lambda}} \\[2ex] y = -\dfrac{1}{2}\sqrt[3]{\dfrac{1}{2\lambda}} \\[2ex] z = -\dfrac{1}{3}\sqrt[3]{\dfrac{1}{2\lambda}} \end{cases} \implies \begin{cases} x = -\sqrt[3]{\dfrac{1}{2\lambda}} \\[2ex] y = -\dfrac{1}{2} \cdot x \\[2ex] z = -\dfrac{1}{3} \cdot x \end{cases}$$

3.5 Hint

The points lie on the unit sphere, so $x^2 + y^2 + z^2 = 1$ must also hold.

3.6 Solution

$$\begin{aligned} x^2 + \left(-\frac{1}{2}\cdot x\right)^2 + \left(-\frac{1}{3}\cdot x\right)^2 &= 1 \\ \implies x^2 \cdot \left(1 + \frac{1}{4} + \frac{1}{9}\right) &= 1 \\ \implies x^2 \cdot \frac{49}{36} &= 1 \\ \implies x^2 &= \frac{36}{49} \\ \implies x &= \pm\frac{6}{7} \end{aligned}$$

We get two solutions:

$$\begin{aligned} x &= \frac{6}{7} \\ y &= -\frac{1}{2}\cdot x = -\frac{3}{7} \\ z &= -\frac{1}{3}\cdot x = -\frac{2}{7} \end{aligned}$$

and

$$\begin{aligned} x &= -\frac{6}{7} \\ y &= -\frac{1}{2}\cdot x = \frac{3}{7} \\ z &= -\frac{1}{3}\cdot x = \frac{2}{7} \end{aligned}$$

In order to determine which one gives the minimum and which one the maximum, we substitute in f:

$$\begin{aligned} f\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right) &= \frac{1}{-6/7} + \frac{1}{8\cdot(-3/7)} + \frac{1}{27\cdot(-2/7)} \\ &= -\frac{7}{6} - \frac{7}{24} - \frac{7}{54} \\ &= -\frac{373}{216} \end{aligned}$$

and

$$\begin{aligned} f\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right) &= \frac{1}{6/7} + \frac{1}{8\cdot 3/7} + \frac{1}{27\cdot 2/7} \\ &= \frac{7}{6} + \frac{7}{24} + \frac{7}{54} \\ &= \frac{373}{216} \end{aligned}$$

So f achieves a maximum value of $373/216$ at

$$\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

and a minimum value of $-373/216$ at

$$\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right).$$

4 Exercise

Consider the function

$$f(x,y) = \frac{x^2+y^2}{xy}$$

defined on the set

$$K = \{(x,y) : 0 < x \le 1, 0 < y \le 1\}.$$

Determine where f assumes its minimum, and what that minimum value is.

4.1 Hint

Calculate ∇f.

4.2 Hint

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left(\frac{xy \cdot 2x - (x^2+y^2) \cdot y}{x^2y^2}\right) \\ &= \frac{x^2y - y^3}{x^2y^2} \\ &= \frac{x^2-y^2}{x^2y}\end{aligned}$$

By symmetry, we have:

$$\frac{\partial f}{\partial y} = \frac{y^2-x^2}{xy^2}$$

So

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x^2-y^2}{x^2y} \\ \frac{y^2-x^2}{xy^2} \end{pmatrix}$$

4.3 Hint

Solve $\nabla f = 0$.

4.4 Hint

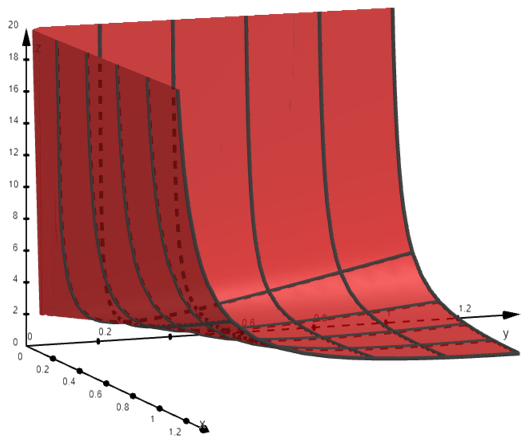
$$\begin{pmatrix} \frac{x^2-y^2}{x^2y} \\ \frac{y^2-x^2}{xy^2} \end{pmatrix} = 0 \implies x^2 - y^2 = 0 \implies x = \pm y$$

4.5 Solution

Note that since $x, y > 0$ on K, we cannot have $x = -y$. Substituting $y = x$ in f we get:

$$f(x,x) = \frac{x^2+x^2}{x^2} = 2$$

So f achieves its minimum value 2 on the entire line segment $x = y$, $0 < x \le 1, 0 < y \le 1$.


20
18
16
14
12
10
8
6
4
2
0
z
0
0.2
0.4
0.6
0.8
1
1.2
x
0.2
1.2
y

5 Exercise

Consider the surface S defined by $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + y^2 + z^2 + 3xy - z - 11$$

1. Check that $A = (1, 1, 3)$ lies on S.
2. Give an equation of the form $\alpha x + \beta y + \gamma z = \delta$ describing the tangent plane to S at point A.

5.1 Hint

1. Substitute $x = 1$, $y = 1$, $z = 3$ in f to get:

$$1^2 + 1^2 + 3^2 + 3 \cdot 1 \cdot 1 - 3 - 11 = 0.$$

5.2 Hint

2. There are several ways to go about this, but in most (if not all) you need to compute ∇f at the point A.

5.3 Hint

$$\nabla f(x, y, z) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \\ \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x + 2y \\ 2z - 1 \end{pmatrix}$$

So

$$\nabla f(1, 1, 3) = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 3 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

5.4 Hint

The gradient of f at A is a vector orthogonal to the tangent plane to S at point A. How do you find an equation of a plane if you know a vector orthogonal to it and a point on it?

5.5 Solution

The tangent plane is perpendicular to the vector $\nabla f(1, 1, 3) = (5, 5, 5)$ and passes through the point $(1, 1, 3)$, so its equation is:

$$5 \cdot (x - 1) + 5 \cdot (y - 1) + 5 \cdot (z - 3) = 0 \implies 5x + 5y + 5z = 25$$

Remark: *There is also a formula given in the book as the linearization of f at $A = (x_0, y_0, z_0)$:*

$$L(x, y, z) = f(A) + f_x(A) \cdot (x - x_0) + f_y(A) \cdot (y - y_0) + f_z(A) \cdot (z - z_0)$$

(this is essentially a first order Taylor expansion). The formula will yield the exact same answer. However, applying formulas without understanding is like eating your food without chewing!

6 Exercise

Consider the plane P which contains the points $A = (1, 1, 1)$, $B = (2, 3, 9)$, $C = (3, 5, 4)$. Try to think of as many was as possible to determine the equation $\alpha x + \beta y + \gamma z = \delta$ of this plane.

6.1 Method 1

Just substitute the coordinates of the 3 points in $\alpha x + \beta y + \gamma z = \delta$ and solve for α, β, γ, δ.

6.2 Solution 1

$$\begin{cases} \alpha + \beta + \gamma = \delta \\ 2\alpha + 3\beta + 9\gamma = \delta \quad | \quad R_2 \leftarrow R_2 - 2R_1 \\ 3\alpha + 5\beta + 4\gamma = \delta \quad | \quad R_3 \leftarrow R_3 - 3R_1 \end{cases}$$

$$\implies \begin{cases} \alpha + \beta + \gamma = \delta \\ \beta + 7\gamma = -\delta \\ 2\beta + \gamma = -2\delta \quad | \quad R_3 \leftarrow R_2 - \frac{1}{2}R_3 \end{cases}$$

$$\implies \begin{cases} \alpha + \beta + \gamma = \delta \\ \beta + 7\gamma = -\delta \\ \frac{13}{2}\gamma = 0 \end{cases}$$

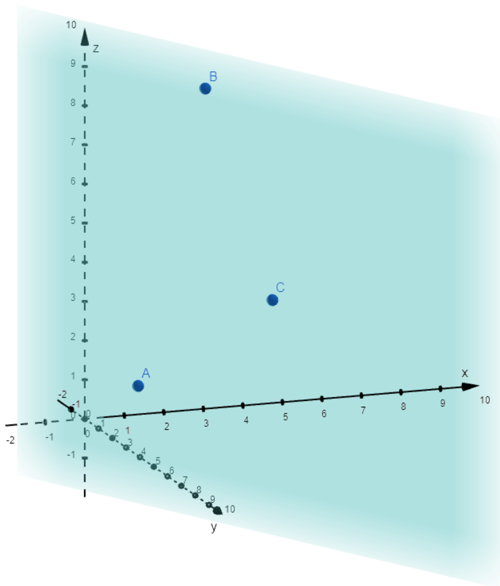
$$\implies \begin{cases} \alpha = 2\delta \\ \beta = -\delta \\ \gamma = 0 \end{cases}$$

Note that δ is a free variable here, and

$$2\delta \cdot x - \delta \cdot y + 0 \cdot z = \delta$$

defines the same plane for any $\delta \neq 0$ (for $\delta = 0$ we just get $0 = 0$ which is satisfied by any x, y, z, i.e. we don't get a plane but the entire $\mathbb{R}^3$). So we can choose for example $\delta = 1$ to get the following equation for P:

$$2x - y = 1$$



6.3 Method 2

Make the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and compute the vector $\overrightarrow{AB} \times \overrightarrow{AC}$. Then...

6.4 Solution 2

$$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (2,3,9) - (1,1,1) = (1,2,8)$$
$$\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (3,5,4) - (1,1,1) = (2,4,3)$$

$$\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} 1 & 2 & 8 \\ 2 & 4 & 3 \\ e_x & e_y & e_z \end{vmatrix} \\
&= \begin{vmatrix} 2 & 8 \\ 4 & 3 \end{vmatrix} \cdot e_x - \begin{vmatrix} 1 & 8 \\ 2 & 3 \end{vmatrix} \cdot e_y + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \cdot e_z \\
&= -26 \cdot e_x + 13 \cdot e_y + 0 \cdot e_z \\
&= (-26, 13, 0)
\end{aligned}$$

Now $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to P and A lies on P, so we get the following equation for P:

$$-26 \cdot (x-1) + 13 \cdot (y-1) + 0 \cdot (z-1) = 0 \implies -26x + 13y = -13$$

6.5 Method 3

Find a vector (α, β, γ) that is orthogonal to $\overrightarrow{AB}$, $\overrightarrow{AC}$ by solving the system

$$\overrightarrow{AB} \cdot (\alpha, \beta, \gamma) = 0$$
$$\overrightarrow{AC} \cdot (\alpha, \beta, \gamma) = 0$$

Then...

6.6 Solution 3

$$\begin{aligned}
&\begin{cases} (1,2,8) \cdot (\alpha, \beta, \gamma) = 0 \\ (2,4,3) \cdot (\alpha, \beta, \gamma) = 0 \end{cases} \\
\implies &\begin{cases} \alpha + 2\beta + 8\gamma = 0 \\ 2\alpha + 4\beta + 3\gamma = 0 \quad | \quad R_2 \leftarrow R_2 - 2R_1 \end{cases} \\
\implies &\begin{cases} \alpha + 2\beta + 8\gamma = 0 \\ -13\gamma = 0 \end{cases} \\
\implies &\begin{cases} \alpha = -2\beta \\ \gamma = 0 \end{cases}
\end{aligned}$$

Choose $\beta = 1$, so $(\alpha, \beta, \gamma) = (-2, 1, 0)$, and the equation for P becomes:

$$-2 \cdot x + 1 \cdot y + 0 \cdot z = \delta$$

Substitute $(1,1,1)$ in the equation to find δ:

$$\delta = -2 + 1 = -1$$

So the equation for P is:

$$-2x + y = -1$$

7 Exercise

Find all points that are equidistant to the 3 points $A = (2, 2, 0)$, $B = (0, 3, 3)$, $C = (4, 0, 4)$.

7.1 Hint

Let $P = (x, y, z)$ be such a point. Write down equations for the distances of P from A, B, C to be equal, i.e.:

$$\left|\overrightarrow{PA}\right|^2 = \left|\overrightarrow{PB}\right|^2 = \left|\overrightarrow{PC}\right|^2$$

7.2 Hint

$$\left|\overrightarrow{PA}\right|^2 = (x-2)^2 + (y-2)^2 + (z-0)^2,$$

etc.

7.3 Hint

$$\begin{aligned}
(x-2)^2 + (y-2)^2 + (z-0)^2 &= (x-0)^2 + (y-3)^2 + (z-3)^2 \\
&= (x-4)^2 + (y-0)^2 + (z-4)^2 \\
&\Longrightarrow \\
x^2 + y^2 + z^2 - 4x - 4y + 8 &= x^2 + y^2 + z^2 - 6y - 6z + 18 \\
&= x^2 + y^2 + z^2 - 8x - 8z + 32 \\
&\Longrightarrow \\
-4x - 4y &= -6y - 6z + 10 \\
&= -8x - 8z + 24 \\
&\Longrightarrow \\
2x + 2y &= 3y + 3z - 10 \\
&= 4x + 4z - 12
\end{aligned}$$

7.4 Hint

Note that in fact we have 2 equations:

$$\begin{cases} 2x + 2y = 3y + 3z - 10 \\ 2x + 2y = 4x + 4z - 12 \end{cases} \Longrightarrow \begin{cases} 2x - y - 3z = -10 \\ -2x + 2y - 4z = -12 \end{cases}$$

So the solution is the intersection of 2 planes, i.e. a line! How can we find the equation of the line in parametric form?

7.5 Hint

This is now a linear algebra problem.

$$\begin{aligned}
&\left(\begin{array}{ccc|c} 2 & -1 & -3 & -10 \\ -2 & 2 & -4 & -12 \end{array}\right) \quad | \quad R_2 \leftarrow R_2 + R_1 \\
\rightarrow &\left(\begin{array}{ccc|c} 2 & -1 & -3 & -10 \\ 0 & 1 & -7 & -22 \end{array}\right) \quad | \quad R_1 \leftarrow R_1 + R_2 \\
\rightarrow &\left(\begin{array}{ccc|c} 2 & 0 & -10 & -32 \\ 0 & 1 & -7 & -22 \end{array}\right) \quad | \quad R_1 \leftarrow \frac{1}{2} R_1 \\
\rightarrow &\left(\begin{array}{ccc|c} 1 & 0 & -5 & -16 \\ 0 & 1 & -7 & -22 \end{array}\right)
\end{aligned}$$

So

$$\begin{cases} x - 5z = -16 \\ y - 7z = -22 \end{cases} \implies \begin{cases} x = 5z - 16 \\ y = 7z - 22 \end{cases},$$

where z is free. So there is a line of points equidistant from A, B, C given by the equation:

$$(x, y, z) = (5\lambda - 16, 7\lambda - 22, \lambda)$$

for $\lambda \in \mathbb{R}$.

8 Exercise

Consider the following 5 points:

$$A = (0,0,0), \quad B = (2,0,0), \quad C = (4,4,0), \quad D = (4,0,4), \quad E = (0,0,4)$$

Find all the points that are equidistant from A, B, C, D, E.

8.1 Hint

Let $P = (x, y, z)$ be such a point. Write down equations for the distances of P from A, B, C to be equal, i.e.:

$$\left|\overrightarrow{PA}\right|^2 = \left|\overrightarrow{PB}\right|^2 = \left|\overrightarrow{PC}\right|^2 = \left|\overrightarrow{PD}\right|^2 = \left|\overrightarrow{PE}\right|^2$$

8.2 Hint

$$\left|\overrightarrow{PB}\right|^2 = (x-2)^2 + (y-0)^2 + (z-0)^2,$$

etc.

8.3 Hint

$$\begin{aligned}
(x-0)^2 + (y-0)^2 + (z-0)^2 &= (x-2)^2 + (y-0)^2 + (z-0)^2 \\
&= (x-4)^2 + (y-4)^2 + (z-0)^2 \\
&= (x-4)^2 + (y-0)^2 + (z-4)^2 \\
&= (x-0)^2 + (y-0)^2 + (z-4)^2 \\
&\Longrightarrow \\
x^2 + y^2 + z^2 &= x^2 + y^2 + z^2 - 4x + 4 \\
&= x^2 + y^2 + z^2 - 8x - 8y + 32 \\
&= x^2 + y^2 + z^2 - 8x - 8z + 32 \\
&= x^2 + y^2 + z^2 - 8z + 16 \\
&\Longrightarrow \\
0 &= -4x + 4 \\
&= -8x - 8y + 32 \\
&= -8x - 8z + 32 \\
&= -8z + 16
\end{aligned}$$

8.4 Solution

We have:

$$0 = -4x + 4 \implies x = 1$$

For $x = 1$:

$$0 = -8x - 8y + 32 = -8y + 24 \implies y = 3$$

For $x = 1, y = 3$:

$$0 = -8x - 8z + 32 \implies z = 3$$

But now for $x = 1$, $y = 3$, $z = 3$:

$$-8z + 16 = 0 \implies -8 = 0,$$

contradiction! So there are no points equidistant from A, B, C, D, E.

9 Exercise

A rocket is flying through space. At time t, for $t \geq 0$, it is at location

$$f(t) = \left(t\cos(t),\ t\sin(t),\ \frac{2\sqrt{2}}{3}\, t\sqrt{t} \right).$$

At which point (x, y, z) is the rocket's speed equal to 37?

9.1 Hint

Compute a formula for the speed of the rocket at time t.

9.2 Hint

The velocity of the rocket at time t is given by:

$$f'(t) = \left(\cos(t) - t\sin(t),\ \sin(t) + t\cos(t),\ \sqrt{2t} \right)$$

The speed is given by $|f'(t)|$. Calculate (and simplify) this!

9.3 Hint

$$\begin{aligned}
|f'(t)| &= \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2 + \left(\sqrt{2t}\right)^2} \\
&= \sqrt{\cos(t)^2 - 2\cos(t)\, t\sin(t) + t^2\sin(t)^2 + \sin(t)^2 + 2\sin(t)\, t\cos(t) + t^2\cos(t)^2 + 2t} \\
&= \sqrt{t^2\left(\sin(t)^2 + \cos(t)^2\right) + 2t + \left(\sin(t)^2 + \cos(t)^2\right)} \\
&= \sqrt{t^2 + 2t + 1} \\
&= \sqrt{(t+1)^2} \\
&= |t+1| \\
&= t + 1,
\end{aligned}$$

since $t \geq 0$.

9.4 Solution

$$1 + t = 37 \implies t = 36$$

At $t = 36$ the rocket is at point:

$$\begin{aligned}
f(36) &= \left(36\cos(36),\ 36\sin(36),\ \frac{2\sqrt{2}}{3} \cdot 36\sqrt{36} \right) \\
&= \left(36\cos(36),\ 36\sin(36),\ 144\sqrt{2} \right)
\end{aligned}$$

10 Exercise

Consider the surface

$$f(x, y) = y^3 - 6xy + x^2$$

restricted to the region

$$A = \{(x, y) : x \geq y \geq 0\}.$$

Find where f is maximal/minimal on A.

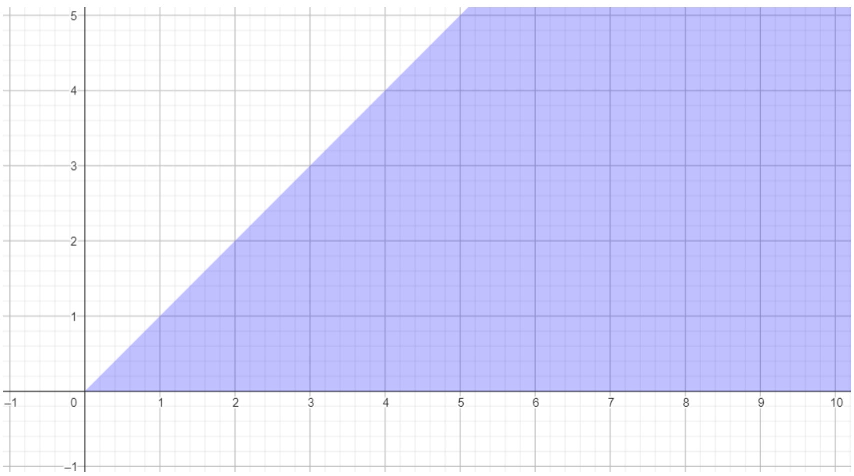
10.1 Hint

Find any critical points inside A, and check also if the boundary of A contains any extreme values.

10.2 Hint

It is a good idea to draw A. Can you reason something about the maximum?

10.3 Hint



The region is unbounded, and f can be made arbitrarily large by going further into the right. Easiest way to prove this is as follows: let $y = 0$, then $f(x, y) = x^2$. Since x can be any non-negative number, f can be arbitrarily large. Hence, f has no maximum on A.

10.4 Hint

Let's compute the critical points on the inside of A.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$$

$$\implies \begin{cases} -6y + 2x = 0 \\ 3y^2 - 6x = 0 \end{cases}$$

$$\implies \begin{cases} x = 3y \\ y^2 = 6y \end{cases}$$

So $y = 0$ and $x = 0$ or $y = 6$ and $x = 18$. Now figure out if these are local minima or saddle points!

10.5 Hint

Let's calculate the Hessian for f.

$$\begin{aligned}
H(x,y) &= \frac{\partial^2 f}{\partial^2 x} \cdot \frac{\partial^2 f}{\partial^2 y} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\
&= \left(\frac{\partial}{\partial x}(-6y+2x)\right) \cdot \left(\frac{\partial}{\partial y}(3y^2-6x)\right) - \left(\frac{\partial}{\partial y}(-6y+2x)\right)^2 \\
&= 2 \cdot 6y - (-6)^2 \\
&= 12y - 36
\end{aligned}$$

- $H(0,0) = -36 < 0$, so $(0,0)$ is a saddle point.
- $H(18,6) = 12 \cdot 6 - 36 = 36 > 0$ and $\frac{\partial^2 f}{\partial^2 x} = 2 > 0$, so f has a local minimum at $(18,6)$.

Try to finish this exercise with solid argumentation about the edges of A. You need to consider the lines $y = x$ and $y = 0$ and analyze the function on each line separately (only minimum values interest us).

10.6 Hint

- $y = 0$: $f(x,y) = x^2$, minimum at $x = 0$ (i.e. at point $(0,0)$).
- $y = x$: $f(x,y) = x^3 - 5x^2$. Where is the minimum?

$$\frac{d}{dx}(x^3 - 5x^2) = 0 \implies 3x^2 - 10x = 0 \implies x = 0 \quad \text{or} \quad x = \frac{10}{3}$$

$$\frac{d^2}{dx^2}(x^3 - 5x^2) = 6x - 10$$

At $x = \frac{10}{3}$ we get $6 \cdot \frac{10}{3} - 10 > 0$, so at $\left(\frac{10}{3}, \frac{10}{3}\right)$ we have a local minimum.

Try to wrap it up!

10.7 Solution

We determined that f has no maxima on A. Now regarding the minima, the only candidates are:

- $(0,0)$: $f(0,0) = 0$
- $(18,6)$: $f(18,6) = 6^3 - 6 \cdot 18 \cdot 6 + 18^2 = -108$
- $\left(\frac{10}{3}, \frac{10}{3}\right)$: $f\left(\frac{10}{3}, \frac{10}{3}\right) = \left(\frac{10}{3}\right)^3 - 6 \cdot \left(\frac{10}{3}\right)^2 + \left(\frac{10}{3}\right)^2 = -\frac{500}{27} \approx -18.5$

So the minimum is at $(18,6)$.

Additional ideas/shortcuts: *If you only had one extremal value which is a local minimum, then that point would necessarily be a global minimum. However, saddle points mess up everything! For example, for $x = 0$ we have*

$$f(x,y) = y^3 - 6xy + x^2 = y^3$$

which is unbounded from below as $y \to -\infty$, i.e. the function has no global minimum on the real plane. In our particular exercise however, the saddle point is on the edge of the domain. This automatically implies that the local minimum is a global minimum. However, this is quite advanced reasoning, and while it is quicker, it is not something I expect anyone to come up with.

11 Exercise

Consider the function

$$f(x,y) = (x+1)\sin(xy)$$

defined on the square

$$K = \{(x,y) : 0 \le x \le 2\pi, 0 \le y \le 2\pi\}.$$

Determine any maxima/minima of f on K.

11.1 Hint

While you can use some ad-hoc methods, try to do this question structurally in a way that always works. First, calculate ∇f. What can you do with this? Is determining the critical points of f enough?

11.2 Hint

$$\nabla f(x,y) = \begin{pmatrix} \sin(xy) + y \cdot (x+1)\cos(xy) \\ x(x+1)\cos(xy) \end{pmatrix}$$

Setting $\nabla f(x,y) = 0$ will yield the critical points.

$$x(x+1)\cos(xy) = 0 \implies \begin{cases} x = 0, \quad \text{or} \\ x+1 = 0, \quad \text{or} \\ \cos(xy) = 0 \end{cases} \implies \begin{cases} x = 0, \quad \text{or} \\ x = -1, \quad \text{or} \\ xy = \frac{k\pi}{2}, \quad k = 1, 3, 5, \dots \end{cases}$$

Since (x,y) must be in K we only have the cases:

- $x = 0$
- $xy = \frac{\pi}{2}$
- $xy = \frac{3\pi}{2}$

Now substitute these values in $\sin(xy) + y \cdot (x+1)\cos(xy) = 0$.

- $x = 0$: $\sin(0) + y \cdot 1 \cdot \cos(0) = 0 \implies 0 + y = 0 \implies y = 0$
- $xy = \frac{\pi}{2}$: $\sin(\frac{\pi}{2}) + y \cdot (x+1)\cos(\frac{\pi}{2}) = 0 \implies 1 + 0 = 0$, contradiction!
- $xy = \frac{3\pi}{2}$: $\sin(\frac{3\pi}{2}) + y \cdot (x+1)\cos(\frac{3\pi}{2}) = 0 \implies -1 + 0 = 0$, contradiction!

So we get a critical point at $(0,0)$, at which $f(0,0) = 0$. Are we done?

11.3 Hint

No! Since the function is defined on the bounded domain K, we also need to check the edges of K, i.e. the lines: $x = 0$, $x = 2\pi$, $y = 0$, $y = 2\pi$.

11.4 Hint

On $x = 0$: $f(0,y) = \sin(0) = 0$.

11.5 Hint

On $x = 2\pi$: $f(2\pi, y) = (2\pi + 1)\sin(2\pi y)$. This is maximal/minimal whenever $\sin(2\pi y)$ is maximal/minimal. So the max value is for $2\pi + 1$ (achieved e.g. for $y = \frac{1}{4}$) and the min value is $-2\pi - 1$ (achieved e.g. for $y = \frac{3}{4}$).

11.6 Hint

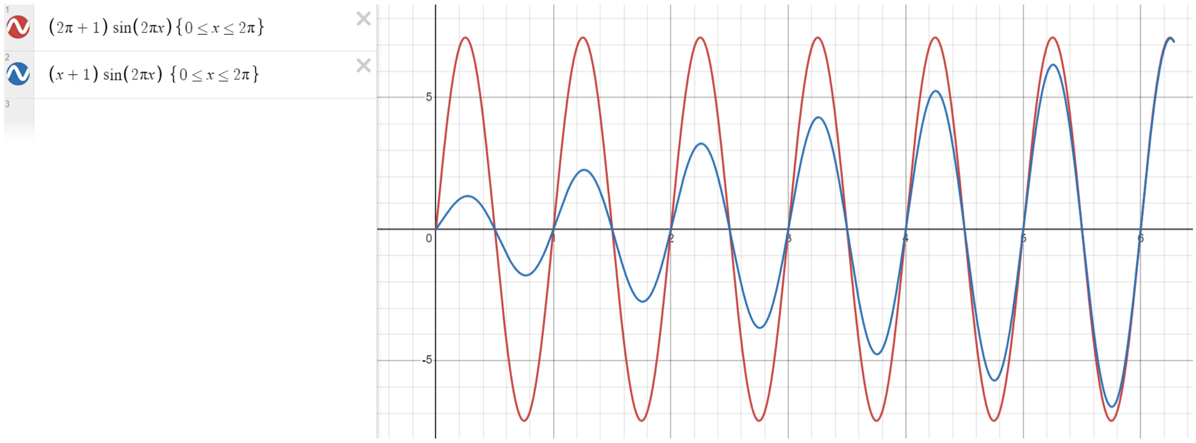
On $y = 0$: $f(x,0) = (x+1)\sin(0) = 0$.

11.7 Solution

On $y = 2\pi$: $f(x, 2\pi) = (x+1)\sin(2\pi x)$. While we could try and find the maxima/minima of this function, we can save a lot of time by noticing that for $0 \leq x \leq 2\pi$ we have

$$|(x+1)\sin(2\pi x)| \leq |(2\pi+1)\sin(2\pi x)|,$$

so this case cannot produce any new extreme values compared to the case $x = 2\pi$.



Since the question only asks for the maximum/minimum values themselves and not where they assumed, the final answer is:

- maximum: $2\pi + 1$
- minimum: $-2\pi - 1$