# Multivariable Calculus Self-Learning Module

Exercises

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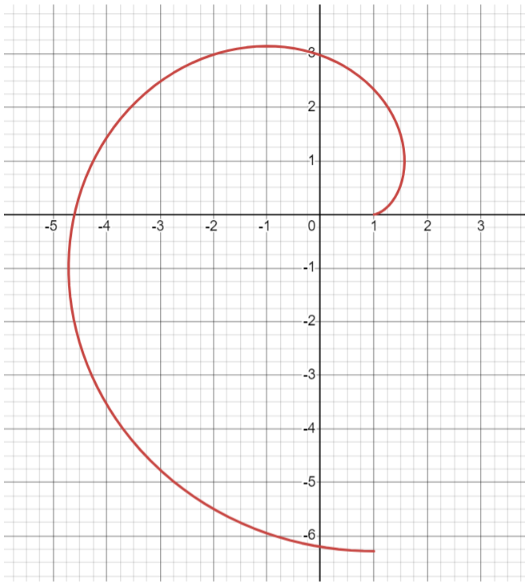
# 1 Exercise 1

Calculate the length of curve $C$, where $C$ is the curve:

$$t \mapsto (\cos(t) + t\sin(t),\ \sin(t) - t\cos(t))$$

for $0 \le t \le 2\pi$.




## 1.1 Hint

There is a formula for calculating the length of the curve. You must know this one! The formula is of the form:

$$\int_{\cdots}^{\cdots} |\ldots|\, dt$$

## 1.2 Hint

The formula for the length is:

$$\int_0^{2\pi} |f'(t)|\, dt$$

## 1.3 Hint

$$\begin{aligned} f'(t) &= \frac{d}{dt}(\cos(t) + t\sin(t),\ \sin(t) - t\cos(t)) \\ &= (-\sin(t) + \sin(t) + t\cos(t),\ \cos(t) - \cos(t) + t\sin(t)) \\ &= (t\cos(t),\ t\sin(t)) \end{aligned}$$

So

$$\begin{aligned} |f'(t)| &= \sqrt{t^2\cos^2(t) + t^2\sin^2(t)} \\ &= \sqrt{t^2} \\ &= |t| \end{aligned}$$

## 1.4 Solution

Since $t \geq 0$, we have $|t| = t$, so:

$$\begin{aligned}
\int_0^{2\pi} |f'(t)|\, dt &= \int_0^{2\pi} t\, dt \\
&= \left[\frac{t^2}{2}\right]_0^{2\pi} \\
&= 2\pi^2
\end{aligned}$$

## 2 Exercise 2

Consider the curve $C$ given by $f(x,y)=0$, where

$$f(x,y)=(x-y)^2+4(x+y)-4$$

Determine the point on $C$ at which $x+y$ is maximal.

### 2.1 Hint

We want to maximize $g(x,y)=x+y$ subject to the constraint $f(x,y)=0$. Use Lagrange.

### 2.2 Hint

Solve the equation $\nabla g=\lambda\nabla f$.

### 2.3 Solution

$$\nabla g(x,y)=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\nabla f(x,y)=\begin{pmatrix}2(x-y)+4\\-2(x-y)+4\end{pmatrix}$$

So

$$\begin{pmatrix}1\\1\end{pmatrix}=\lambda\begin{pmatrix}2(x-y)+4\\-2(x-y)+4\end{pmatrix}$$

Note that $\lambda=0$ gives

$$\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

which is a contradiction, so $\lambda\neq 0$ and we get the system:

$$\begin{cases}2(x-y)+4=\dfrac{1}{\lambda}\\[2ex]-2(x-y)+4=\dfrac{1}{\lambda}\end{cases}$$

Subtracting the two equations gives:

$$2(x-y)+4+2(x-y)-4=\frac{1}{\lambda}-\frac{1}{\lambda}\iff 4(x-y)=0\iff x=y$$

Since we are looking for the point that maximizes $x+y$ on $C$, we substitute $y=x$ in $f(x,y)=0$:

$$(x-x)^2+4(x+x)-4=0\iff 8x=4\iff x=\frac{1}{2}$$

So

$$(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$$

and the maximum is

$$g\left(\frac{1}{2},\frac{1}{2}\right)=1.$$

# 3 Exercise 3

Consider the function

$$f(x,y,z) = \frac{1}{x} + \frac{1}{8y} + \frac{1}{27z}.$$

Find the point on the unit sphere (i.e. the sphere centered at $(0,0,0)$ of radius 1) at which $f$ is maximal and the point at which it is minimal. Calculate also these maximum and minimum values.

## 3.1 Hint

The surface of a sphere centered at $(x_0, y_0, z_0)$ of radius $R$ has the formula

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

## 3.2 Hint

Let

$$g(x,y,z) = x^2 + y^2 + z^2 - 1.$$

We want to maximize $f(x,y,z)$ given the constraint $g(x,y,z) = 0$. Use Lagrange.

## 3.3 Hint

Solve the equation $\nabla f = \lambda \nabla g$.

## 3.4 Hint

$$\nabla f(x,y,z) = \begin{pmatrix} -\dfrac{1}{x^2} \\[2ex] -\dfrac{1}{8y^2} \\[2ex] -\dfrac{1}{27z^2} \end{pmatrix}$$

$$\nabla g(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

So

$$\begin{pmatrix} -\dfrac{1}{x^2} \\[2ex] -\dfrac{1}{8y^2} \\[2ex] -\dfrac{1}{27z^2} \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Note that the left-hand side is always non-zero, so $\lambda \neq 0$. Hence:

$$\begin{cases} -\dfrac{1}{x^2} = 2\lambda x \\[2ex] -\dfrac{1}{8y^2} = 2\lambda y \\[2ex] -\dfrac{1}{27z^2} = 2\lambda z \end{cases} \iff \begin{cases} x^3 = -\dfrac{1}{2\lambda} \\[2ex] y^3 = -\dfrac{1}{8 \cdot 2\lambda} \\[2ex] z^3 = -\dfrac{1}{27 \cdot 2\lambda} \end{cases} \iff \begin{cases} x = -\sqrt[3]{\dfrac{1}{2\lambda}} \\[2ex] y = -\dfrac{1}{2}\sqrt[3]{\dfrac{1}{2\lambda}} \\[2ex] z = -\dfrac{1}{3}\sqrt[3]{\dfrac{1}{2\lambda}} \end{cases} \iff \begin{cases} x = -\sqrt[3]{\dfrac{1}{2\lambda}} \\[2ex] y = -\dfrac{1}{2} \cdot x \\[2ex] z = -\dfrac{1}{3} \cdot x \end{cases}$$

## 3.5 Hint

The points lie on the unit sphere, so $x^2 + y^2 + z^2 = 1$ must also hold.

### 3.6 Solution

$$
\begin{aligned}
x^2 + \left(-\frac{1}{2}\cdot x\right)^2 + \left(-\frac{1}{3}\cdot x\right)^2 &= 1 \\
\iff x^2 \cdot \left(1 + \frac{1}{4} + \frac{1}{9}\right) &= 1 \\
\iff x^2 \cdot \frac{49}{36} &= 1 \\
\iff x^2 &= \frac{36}{49} \\
\iff x &= \pm\frac{6}{7}
\end{aligned}
$$

We get two solutions:

$$
\begin{aligned}
x &= \frac{6}{7} \\
y &= -\frac{1}{2}\cdot x = -\frac{3}{7} \\
z &= -\frac{1}{3}\cdot x = -\frac{2}{7}
\end{aligned}
$$

and

$$
\begin{aligned}
x &= -\frac{6}{7} \\
y &= -\frac{1}{2}\cdot x = \frac{3}{7} \\
z &= -\frac{1}{3}\cdot x = \frac{2}{7}
\end{aligned}
$$

In order to determine which one gives the minimum and which one the maximum, we substitute in $f$:

$$
\begin{aligned}
f\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right) &= \frac{1}{-6/7} + \frac{1}{8\cdot(-3/7)} + \frac{1}{27\cdot(-2/7)} \\
&= -\frac{7}{6} - \frac{7}{24} - \frac{7}{54} \\
&= -\frac{373}{216}
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right) &= \frac{1}{6/7} + \frac{1}{8\cdot 3/7} + \frac{1}{27\cdot 2/7} \\
&= \frac{7}{6} + \frac{7}{24} + \frac{7}{54} \\
&= \frac{373}{216}
\end{aligned}
$$

So $f$ achieves a maximum value of $373/216$ at

$$
\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)
$$

and a minimum value of $-373/216$ at

$$
\left(\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right).
$$

# 4 Exercise 4

Consider the function

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

defined on the set

$$K = \{(x, y) : 0 < x \leq 1, 0 < y \leq 1\}.$$

Determine where $f$ assumes its minimum, and what that minimum value is.

## 4.1 Hint

Calculate $\nabla f$.

## 4.2 Hint

$$\begin{aligned}\frac{\partial f}{\partial x} &= \left(\frac{xy \cdot 2x - (x^2 + y^2) \cdot y}{x^2 y^2}\right) \\ &= \frac{x^2 y - y^3}{x^2 y^2} \\ &= \frac{x^2 - y^2}{x^2 y}\end{aligned}$$

By symmetry, we have:

$$\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{xy^2}$$

So

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{x^2 - y^2}{x^2 y} \\ \frac{y^2 - x^2}{xy^2} \end{pmatrix}$$

## 4.3 Hint

Solve $\nabla f = 0$.

## 4.4 Hint

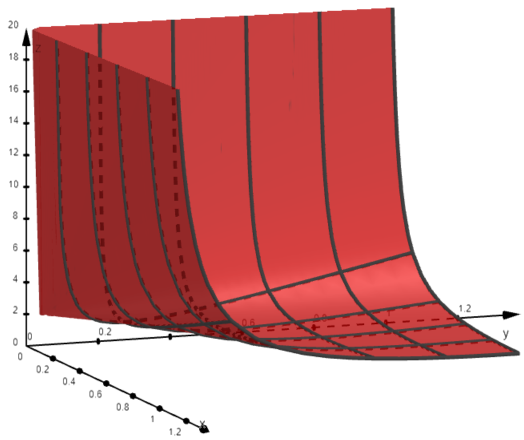
$$\begin{pmatrix} \frac{x^2 - y^2}{x^2 y} \\ \frac{y^2 - x^2}{xy^2} \end{pmatrix} = 0 \iff x^2 - y^2 = 0 \iff x = \pm y$$

## 4.5 Solution

Note that since $x, y > 0$ on $K$, we cannot have $x = -y$. Substituting $y = x$ in $f$ we get:

$$f(x, x) = \frac{x^2 + x^2}{x^2} = 2$$

So $f$ achieves its minimum value 2 on the entire line segment $x = y$, $0 < x \leq 1, 0 < y \leq 1$.


20
18
16
14
12
10
8
6
4
2
0
z
0
0.2
1.2
y
0
0.2
0.4
0.6
0.8
1
1.2
x

# 5 Exercise 5

Consider the surface $S$ defined by $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + y^2 + z^2 + 3xy - z - 11$$

1. Check that $A = (1, 1, 3)$ lies on $S$.
2. Give an equation of the form $\alpha x + \beta y + \gamma z = \delta$ describing the tangent plane to $S$ at point $A$.

## 5.1 Hint

1. Substitute $x = 1$, $y = 1$, $z = 3$ in $f$ to get:

$$1^2 + 1^2 + 3^2 + 3 \cdot 1 \cdot 1 - 3 - 11 = 0.$$

## 5.2 Hint

2. There are several ways to go about this, but in most (if not all) you need to compute $\nabla f$ at the point $A$.

## 5.3 Hint

$$\nabla f(x, y, z) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \\ \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 3x + 2y \\ 2z - 1 \end{pmatrix}$$

So

$$\nabla f(1, 1, 3) = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 3 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

## 5.4 Hint

The gradient of $f$ at $A$ is a vector orthogonal to the tangent plane to $S$ at point $A$. How do you find an equation of a plane if you know a vector orthogonal to it and a point on it?

## 5.5 Solution

The tangent plane is perpendicular to the vector $\nabla f(1, 1, 3) = (5, 5, 5)$ and passes through the point $(1, 1, 3)$, so its equation is:

$$5 \cdot (x - 1) + 5 \cdot (y - 1) + 5 \cdot (z - 3) = 0 \iff 5x + 5y + 5z = 25$$

***Remark:*** *There is also a formula given in the book as the linearization of $f$ at $A = (x_0, y_0, z_0)$:*

$$L(x, y, z) = f(A) + f_x(A) \cdot (x - x_0) + f_y(A) \cdot (y - y_0) + f_z(A) \cdot (z - z_0)$$

*(this is essentially a first order Taylor expansion). The formula will yield the exact same answer. However, applying formulas without understanding is like eating your food without chewing!*

# 6 Exercise 6

Consider the plane $P$ which contains the points $A = (1, 1, 1)$, $B = (2, 3, 9)$, $C = (3, 5, 4)$. Try to think of as many was as possible to determine the equation $\alpha x + \beta y + \gamma z = \delta$ of this plane.

## 6.1 Method 1

Just substitute the coordinates of the 3 points in $\alpha x + \beta y + \gamma z = \delta$ and solve for $\alpha$, $\beta$, $\gamma$, $\delta$.

## 6.2 Solution 1

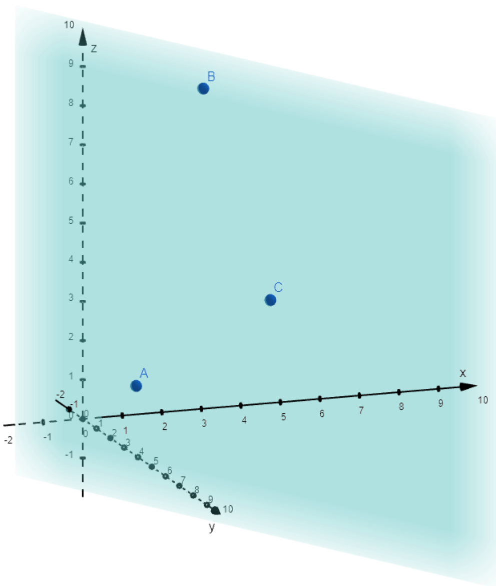
$$
\begin{aligned}
&\begin{cases} \alpha + \beta + \gamma = \delta \\ 2\alpha + 3\beta + 9\gamma = \delta \quad | \quad R_2 \leftarrow R_2 - 2R_1 \\ 3\alpha + 5\beta + 4\gamma = \delta \quad | \quad R_3 \leftarrow R_3 - 3R_1 \end{cases} \\
\iff &\begin{cases} \alpha + \beta + \gamma = \delta \\ \beta + 7\gamma = -\delta \\ 2\beta + \gamma = -2\delta \quad | \quad R_3 \leftarrow R_2 - \frac{1}{2}R_3 \end{cases} \\
\iff &\begin{cases} \alpha + \beta + \gamma = \delta \\ \beta + 7\gamma = -\delta \\ \frac{13}{2}\gamma = 0 \end{cases} \\
\iff &\begin{cases} \alpha = 2\delta \\ \beta = -\delta \\ \gamma = 0 \end{cases}
\end{aligned}
$$

Note that $\delta$ is a free variable here, and

$$2\delta \cdot x - \delta \cdot y + 0 \cdot z = \delta$$

defines the same plane for any $\delta \neq 0$ (for $\delta = 0$ we just get $0 = 0$ which is satisfied by any $x, y, z$, i.e. we don't get a plane but the entire $\mathbb{R}^3$). So we can choose for example $\delta = 1$ to get the following equation for $P$:

$$2x - y = 1$$




## 6.3 Method 2

Make the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and compute the vector $\overrightarrow{AB} \times \overrightarrow{AC}$. Then...

## 6.4 Solution 2

$$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (2,3,9) - (1,1,1) = (1,2,8)$$
$$\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (3,5,4) - (1,1,1) = (2,4,3)$$

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} 1 & 2 & 8 \\ 2 & 4 & 3 \\ e_x & e_y & e_z \end{vmatrix} \\ &= \begin{vmatrix} 2 & 8 \\ 4 & 3 \end{vmatrix} \cdot e_x - \begin{vmatrix} 1 & 8 \\ 2 & 3 \end{vmatrix} \cdot e_y + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \cdot e_z \\ &= -26 \cdot e_x + 13 \cdot e_y + 0 \cdot e_z \\ &= (-26, 13, 0)\end{aligned}$$

Now $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to $P$ and $A$ lies on $P$, so we get the following equation for $P$:

$$-26 \cdot (x-1) + 13 \cdot (y-1) + 0 \cdot (z-1) = 0 \iff -26x + 13y = -13$$

## 6.5 Method 3

Find a vector $(\alpha, \beta, \gamma)$ that is orthogonal to $\overrightarrow{AB}$, $\overrightarrow{AC}$ by solving the system

$$\overrightarrow{AB} \cdot (\alpha, \beta, \gamma) = 0$$
$$\overrightarrow{AC} \cdot (\alpha, \beta, \gamma) = 0$$

Then...

## 6.6 Solution 3

$$\begin{aligned} &\begin{cases} (1,2,8) \cdot (\alpha, \beta, \gamma) = 0 \\ (2,4,3) \cdot (\alpha, \beta, \gamma) = 0 \end{cases} \\ \iff &\begin{cases} \alpha + 2\beta + 8\gamma = 0 \\ 2\alpha + 4\beta + 3\gamma = 0 \quad | \quad R_2 \leftarrow R_2 - 2R_1 \end{cases} \\ \iff &\begin{cases} \alpha + 2\beta + 8\gamma = 0 \\ -13\gamma = 0 \end{cases} \\ \iff &\begin{cases} \alpha = -2\beta \\ \gamma = 0 \end{cases} \end{aligned}$$

Choose $\beta = 1$, so $(\alpha, \beta, \gamma) = (-2, 1, 0)$, and the equation for $P$ becomes:

$$-2 \cdot x + 1 \cdot y + 0 \cdot z = \delta$$

Substitute $(1,1,1)$ in the equation to find $\delta$:

$$\delta = -2 + 1 = -1$$

So the equation for $P$ is:

$$-2x + y = -1$$

# 7 Exercise 7

Find all points that are equidistant to the 3 points $A = (2, 2, 0)$, $B = (0, 3, 3)$, $C = (4, 0, 4)$.

## 7.1 Hint

Let $P = (x, y, z)$ be such a point. Write down equations for the distances of $P$ from $A, B, C$ to be equal, i.e.:

$$\left|\overrightarrow{PA}\right|^2 = \left|\overrightarrow{PB}\right|^2 = \left|\overrightarrow{PC}\right|^2$$

## 7.2 Hint

$$\left|\overrightarrow{PA}\right|^2 = (x-2)^2 + (y-2)^2 + (z-0)^2,$$

etc.

## 7.3 Hint

$$\begin{aligned}
&(x-2)^2 + (y-2)^2 + (z-0)^2 \\
&= (x-0)^2 + (y-3)^2 + (z-3)^2 \\
&= (x-4)^2 + (y-0)^2 + (z-4)^2 \\
\iff& \\
&x^2 + y^2 + z^2 - 4x - 4y + 8 \\
&= x^2 + y^2 + z^2 - 6y - 6z + 18 \\
&= x^2 + y^2 + z^2 - 8x - 8z + 32 \\
\iff& \\
&= -4x - 4y \\
&= -6y - 6z + 10 \\
&= -8x - 8z + 24 \\
\iff& \\
&= 2x + 2y \\
&= 3y + 3z - 10 \\
&= 4x + 4z - 12
\end{aligned}$$

## 7.4 Hint

Note that in fact we have 2 equations:

$$\begin{cases} 2x + 2y = 3y + 3z - 10 \\ 2x + 2y = 4x + 4z - 12 \end{cases} \iff \begin{cases} 2x - y - 3z = -10 \\ -2x + 2y - 4z = -12 \end{cases}$$

So the solution is the intersection of 2 planes, i.e. a line! How can we find the equation of the line in parametric form?

## 7.5 Hint

This is now a linear algebra problem.

$$
\begin{aligned}
&\left(\begin{array}{ccc|c} 2 & -1 & -3 & -10 \\ -2 & 2 & -4 & -12 \end{array}\right) \quad | \quad R_2 \leftarrow R_2 + R_1 \\
\rightarrow &\left(\begin{array}{ccc|c} 2 & -1 & -3 & -10 \\ 0 & 1 & -7 & -22 \end{array}\right) \quad | \quad R_1 \leftarrow R_1 + R_2 \\
\rightarrow &\left(\begin{array}{ccc|c} 2 & 0 & -10 & -32 \\ 0 & 1 & -7 & -22 \end{array}\right) \quad | \quad R_1 \leftarrow \frac{1}{2} R_1 \\
\rightarrow &\left(\begin{array}{ccc|c} 1 & 0 & -5 & -16 \\ 0 & 1 & -7 & -22 \end{array}\right)
\end{aligned}
$$

So

$$
\begin{cases} x - 5z = -16 \\ y - 7z = -22 \end{cases} \iff \begin{cases} x = 5z - 16 \\ y = 7z - 22 \end{cases},
$$

where $z$ is free. So there is a line of points equidistant from $A, B, C$ given by the equation:

$$
(x, y, z) = (5\lambda - 16, 7\lambda - 22, \lambda)
$$

for $\lambda \in \mathbb{R}$.