

Exercise

Calculate the length of the curve C ,

where C is the curve $\vec{r}(t) = (\cos(t), \sin(t))$
when $0 \leq t \leq 2\pi$.

[Picture]

Hint 1: there is a formula for calculating
the length of the curve. You need to know
this one!

It is of the form $\int_0^{2\pi} |f(t)| dt$.

Hint 2

$$\int_0^{2\pi} |f(t)| dt$$

$$\begin{aligned} \int_0^{2\pi} |f(t)| dt &= \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= \int_0^{2\pi} \sqrt{t^2} dt \end{aligned}$$

Now be careful: $\sqrt{t^2} = |t|$,
but since t is from 0 to 2π , $t \geq 0$.

$$\text{So: } \int_0^{2\pi} t dt = \left[\frac{t^2}{2} \right]_0^{2\pi} = 2\pi^2$$

(1)

Exercise: Consider the curve given
by the equation

$$0 = f(x, y) = \cancel{x^2} \cdot (x-y)^2 + 4(x+y) - 4$$

Determine where on this curve $x+y$ is the largest.

→ Hint: write $g(x, y) = x+y$ and use Lagrange

→ Hint: calculate $\nabla f = \lambda \nabla g$

→ Hint: $\begin{pmatrix} 2(x-y) + 4 \\ -2(x-y) + 4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $2(x-y) + 4 = \lambda$
 $-2(x-y) + 4 = \lambda$

several ways to solve this.

One way: add the equations
& subtract the equations.

→ add: $8 = 2\lambda \Rightarrow \lambda = 4$ (you don't need this though)

subtract: $4(x-y) = 0$

$$\Rightarrow x = y$$

→ You are on the equation 2
 $(x-y)^2 + 4(x+y) - 4 = 0$

So substitute ~~$x \neq y$~~ . $y=x$

→ $(x-x)^2 + 4(x+x) - 4 = 0 \Rightarrow$
 $8x = 4 \Rightarrow x = \frac{1}{2}$

so $y = x = \frac{1}{2}$

and the point is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

The function is here

~~$f\left(\frac{1}{2}, \frac{1}{2}\right) = (1-1)^2 + 4\left(\frac{1}{2} + \frac{1}{2}\right) - 4$~~

$g\left(\frac{1}{2}, \frac{1}{2}\right) = 1$

Exercise: On the surface of a sphere of radius 1, centred at $(0, 0, 0)$,

where is the function $f(x, y, z) = \frac{2x^2 + 2y^2}{x} + \frac{1}{8y} + \frac{1}{2z^2}$

maximal and where minimal?

And what is this minimum and maximum?

Hint: the surface of a sphere has the formula

$$x^2 + y^2 + z^2 = R^2, \text{ where } R \text{ is the radius.}$$

Hint: You need to use Lagrange's method

Let $g(x, y, z) = x^2 + y^2 + z^2 - 1.$

we need to find where $\nabla f = \lambda \cdot \nabla g$ for some $\lambda \neq 0.$

Hint:

$$\begin{pmatrix} 2x & -\frac{1}{x^2} \\ -\frac{1}{8y^2} & 2y \\ -\frac{1}{2z^2} & 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

(2)

$$\text{So } x^3 = \frac{-1}{2\lambda} \quad . \quad x = \sqrt[3]{\frac{-1}{2\lambda}}$$

$$y^3 = \frac{-1}{8\cdot 2\lambda} \Rightarrow y = \sqrt[3]{\frac{-1}{8\cdot 2\lambda}}$$

$$z^3 = \frac{-1}{27\cdot 2\lambda} \quad z = \sqrt[3]{\frac{-1}{27\cdot 2\lambda}}$$

What now?

→ Hint: the points lie on the surface of the sphere

→ Hint so $x^2 + y^2 + z^2 = 1$ must hold

Hint: $x = \sqrt[3]{\frac{-1}{2\lambda}}$

$$y = \sqrt[3]{\frac{-1}{2\lambda}} \cdot \frac{1}{2}$$

$$z = \sqrt[3]{\frac{-1}{2\lambda}} \cdot \frac{1}{3}$$

$$\Rightarrow \left(\sqrt[3]{\frac{-1}{2\lambda}}\right)^2 + \frac{1}{4} \left(\sqrt[3]{\frac{-1}{2\lambda}}\right)^2 + \frac{1}{9} \left(\sqrt[3]{\frac{-1}{2\lambda}}\right)^2 = 1$$

(continued)
but $\Rightarrow \left(\sqrt[3]{\frac{-1}{2\lambda}}\right)^2 \left(1 + \frac{1}{4} + \frac{1}{9}\right) = 1$

$$\left(\sqrt[3]{\frac{-1}{2\lambda}}\right)^2 \left(\frac{36}{36} + \frac{9}{36} + \frac{4}{36}\right) = 1$$

$$\Rightarrow \left(\sqrt[3]{\frac{-1}{2\lambda}}\right)^2 = \frac{36}{49} \Rightarrow \sqrt[3]{\frac{-1}{2\lambda}} = \pm \frac{6}{7}$$

(3)

two solutions:

$$x = -\frac{6}{7}$$

$$y = \frac{1}{2} \cdot -\frac{6}{7} = -\frac{3}{7}$$

$$z = \frac{1}{3} \cdot -\frac{6}{7} = -\frac{2}{7}$$

and

$$x = \frac{6}{7}$$

$$y = \frac{3}{7}$$

$$z = \frac{2}{7}$$

(notice indeed:

$$\left(-\frac{6}{7} \right)^2 + \left(-\frac{3}{7} \right)^2 + \left(-\frac{2}{7} \right)^2 = 1$$

which is minimum which max?
And what are the values?

Simple: substitute in f.

$$f\left(-\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}\right) = -\frac{7}{6} + \frac{1}{8} \cdot \frac{7}{3} + \frac{1}{27} \cdot \frac{7}{2}$$

$$= -\frac{7}{6} - \frac{7}{24} - \frac{7}{54}$$

$$= -\frac{343}{216} = -\frac{7^3}{2^3 \cdot 3^3} = \left(-\frac{7}{6}\right)^3$$

$$f\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right) = \left(\frac{7}{6}\right)^3 \quad \text{minimum}$$

and

maximum.



Exercise:
Consider the function

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

on the shape $k = h(x, y) \mid \begin{cases} 0 < x \leq 1 \\ 0 < y \leq 1 \end{cases}$

(a square, tri)

Determine where f assumes its minimum,
and what this minimum is.

Let's calculate ∇f .

Then ...

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left(xy \cdot (2x) - (x^2 + y^2) \cdot y \right) \\ &= \frac{x^2y - y^3}{x^2y^2} = \frac{x^2 - y^2}{x^2y} \end{aligned}$$

$$\frac{\partial f}{\partial y} = \frac{x^2 - y^2}{x^2y^2}$$

$$\cdot \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \dots$$

Now solve $\nabla f = 0$:

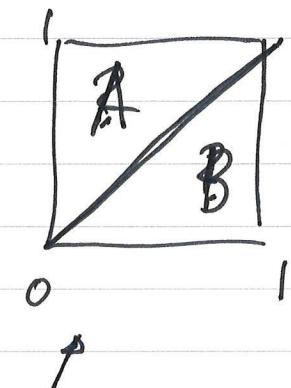
$\frac{x^2 - y^2}{x^2y} = 0$ since $x, y > 0$ this means

$$\begin{aligned} x^2 - y^2 &= 0 \\ \rightarrow x^2 = y^2 &\Rightarrow x = y \text{ or } x = -y \\ &\text{(but this is impossible since } x > 0, y > 0) \end{aligned}$$

Hmt: At $y=x$

$$f(x,y) = \frac{x^2+y^2}{(x+1)(y+1)} = \frac{2x^2}{x^2+2x+1} = 2$$

This is a whole line where the value 2 is assumed.



geogebra
picture?

Note that f goes to ∞ if x or y approach 0.

Or you can

$$\text{calculate } f\left(\frac{1}{2}, 1\right)$$

$$= \frac{\frac{1}{4} + 1}{\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{1}{2}} = \frac{5}{2}$$

and see that the A values are above the minimum at A .



Solution:

$$\nabla f(A) = (5, 5, 5) \text{ so the equation}$$

is

$$5X + 5Y + 5Z = 5.$$

Since $A = (1, 1, 3)$ is on this plane,

$$5 \cdot 1 + 5 \cdot 1 + 5 \cdot 3 = 25 \\ 25 = 25$$

so:

$$5X + 5Y + 5Z = 25$$

is the answer.

NOTE: there is also a formula given
in the book as: the linearization of f at A :

$$L(x, y, z) = f(A) + f_x(A)(x - x_0) + f_y(A)(y - y_0) \\ + f_z(A)(z - z_0)$$

$$\text{where } A = (x_0, y_0, z_0)$$

which will yield exactly this answer.

However, applying formulas without understanding
is like eating your food without chewing!

Exercise: Consider the surface $f(x, y, z) = 0$ (1)
 where $f(x, y, z) = z^2 + x^2 + 3xy + y^2 - z - 11$

a) Check that $A = (1, 1, 3)$ is on the surface.

b) Give an equation $\alpha X + \beta Y + \gamma Z = \delta$
 describing the tangent plane to $f = 0$ at A .

Solution:

Substitute $x=1, y=1, z=3$ into
 $z^2 + x^2 + 3xy + y^2 - z - 11$ and see that
 it becomes 0.

Hint: there are several ways to go,
 but in most, if not all you need

to compute ∇f at the point A .

$\nabla f(A) : \nabla f = \begin{pmatrix} df/dx \\ df/dy \\ df/dz \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 3x+2y \\ 2z-1 \end{pmatrix}$

$\nabla f(A) = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 2 \cdot 3 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$

How to use $\nabla f(A)$?

The gradient is a vector
 orthogonal to the plane at that
 point.

How do you find an equation of
 a plane, if you have a vector
 orthogonal to it?

(turn over)

Exercise: Consider the plane \mathcal{P} which contains the points $A = (1, 1, 1)$, $B = (2, 3, 9)$, $C = (3, 5, 4)$

Try to think of as many ways as possible to determine the equation $\alpha X + \beta Y + \gamma Z = S$ of this plane.

Hint:

Method 1: just substitute the points in $\alpha X + \beta Y + \gamma Z = S$ and solve for α, β, γ, S

Method 2: Make the vectors \vec{AB}, \vec{AC} and compute $\vec{AB} \times \vec{AC}$. Then ...

Method 3: find a vector (α, β, γ) which is orthogonal to \vec{AB}, \vec{AC} .

So calculate $\vec{AB} \cdot (\alpha, \beta, \gamma)$
 $\vec{AC} \cdot (\alpha, \beta, \gamma)$
 and then ...

→ Sol 1: $\begin{cases} \alpha + \beta + \gamma = S \\ 2\alpha + 3\beta + 9\gamma = S \\ 3\alpha + 5\beta + 4\gamma = S \end{cases} \quad \left. \begin{array}{l} R_1: R_2 - 2R_1 \\ R_3: R_3 - 3R_1 \end{array} \right. \quad \left. \begin{array}{l} \alpha + \beta + \gamma = S \\ \beta + 7\gamma = S \\ 2\beta + \gamma = S \end{array} \right\} \dots$

→ Sol 2: $\vec{AB} \times \vec{AC} = \begin{vmatrix} 1 & 2 & 8 \\ 2 & 4 & 3 \end{vmatrix} = (-26, 13, 0) \dots$
extra steps plz

$$(1, 2, 8) \cdot (\alpha, \beta, \gamma) = \alpha + 2\beta + 8\gamma = 0$$

$$(2, 4, 3) \cdot (\alpha, \beta, \gamma) = 2\alpha + 4\beta + 3\gamma = 0$$

$$\Rightarrow R_2: R_2 - 2R_1, \quad \left. \begin{array}{l} \alpha + 2\beta + 8\gamma \\ -13\gamma = 0 \end{array} \right\} \Rightarrow \gamma = 0, \quad \alpha + 2\beta = 0$$

so pick $\beta = 1, \alpha = -2, \gamma = 0$

$$\text{so } -2X + Y + 0Z = S$$

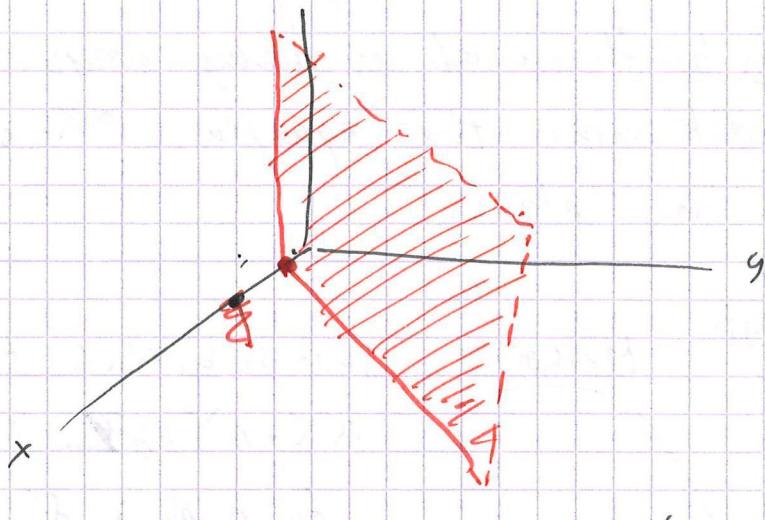
Substitute $(1, 1, 1)$: $-2 + 1 = S \Rightarrow S = -1$

so $-2X + Y = -1$ (see other side)

Note: in \mathbb{R}^3 , the equation $-2x + y = -1$ is
a plane and not a line. \underline{z}

(geogebra picture:

(geogebra on
wolfram
alpha.)
ask me for
help if
it takes long!)



16

85

$$\begin{array}{rcl} \cancel{50} & \cancel{= 1500} \\ \cancel{14} & & \end{array}$$
$$\begin{array}{rcl} \cancel{7} & = 14.15 \\ \cancel{50} & = 1500 \end{array}$$

A function f

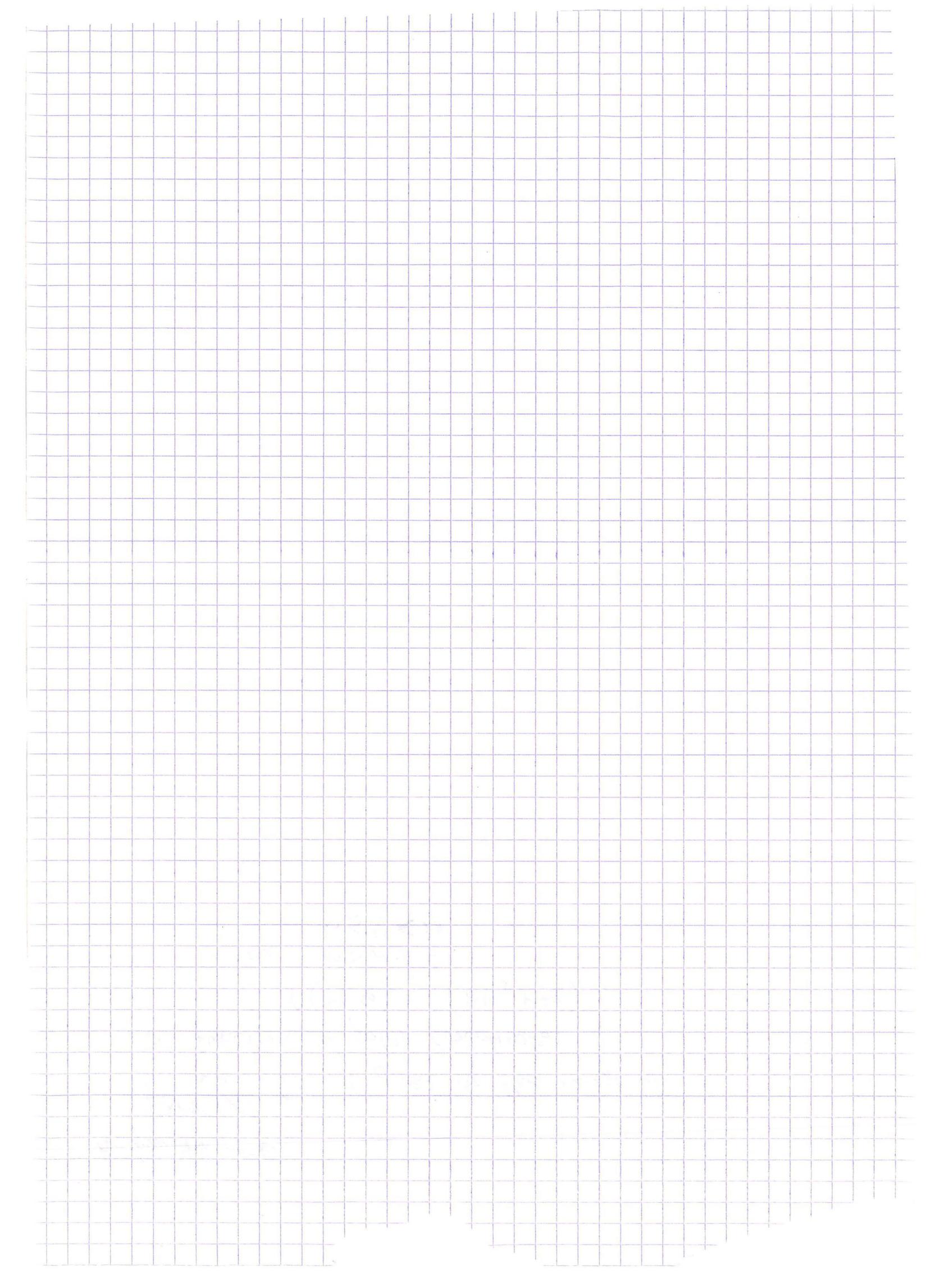
Exercise:

A curve C is given as the image of
a function in polar coordinates:

$$f(r, \varphi) = (r\varphi, r+\varphi)$$

$$\text{where } 0 \leq r \leq 4$$

$$0 \leq \varphi \leq 2\pi$$



Give the point

①

$$A = (2, 4, 0), B = (0, 3, 3), C = (4, 0, 4),$$

find all point(s) which are equidistant to A, B, C.

Hint: take say $P = (x, y, z)$ is such a point.

Make equations giving the distances being equal

Hint: $|PA| = \sqrt{(x-2)^2 + (y-4)^2 + (z-0)^2}$ etc.

Hint: $|PA|^2 = |PB|^2 = |PC|^2$

Hint: $(x-2)^2$

$$\begin{aligned} & x^2 - 4x + 4 + y^2 + 16y + 4 + z^2 \\ &= \cancel{x^2} - \cancel{6x} + \cancel{9} = x^2 + y^2 - 6y + 9 + z^2 - 6z + 9 \\ &= x^2 - 8x + 16 + y^2 + z^2 - 8y - 16 \end{aligned}$$

Hint subtract $x^2 + y^2 + z^2$ from these and simplify:

$$-4x - 4y + 8 = -6y - 6z + 18 = -8x - 8z + 32$$

subtract 8:

$$-4x - 4y = -6y - 6z + 10 = -8x - 8z + 24$$

divide by 2:

$$2x + 2y = 3y + 3z - 10 = 4x + 4z - 12$$

(2)

Hint: Note that you in fact just have 2 equations!

~~exes~~
$$2x+2y = 3y+3z = 10$$

$$2x+2y = 4x+4z = 12$$

$$\text{(or } 3y+3z-10 = 4x+4z-12\text{)}$$

Hint: solving this you get

$$2x-y-3z = -10$$

$$-2x+2y-4z = -12$$

This becomes linear algebra! (The solution is intersection of 2 planes a line thus!)

~~geogebra~~

~~exes~~
$$\text{Hint: } \left(\begin{array}{ccc|c} 2 & -1 & -3 & -10 \\ -2 & 2 & -4 & -12 \end{array} \right) \xrightarrow{R_2: R_2 + R_1}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & -3 & -10 \\ 0 & 1 & -7 & -22 \end{array} \right) \xrightarrow{R_1: R_1 + R_2}$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -10 & -32 \\ 0 & 1 & -7 & -22 \end{array} \right) \xrightarrow{R_1: \frac{1}{2}R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -16 \\ 0 & 1 & -7 & -22 \end{array} \right)$$

$$\text{So } x-5z = -16 \Rightarrow x = 5z-16 \\ y-7z = -22 \Rightarrow y = 7z-22$$

$z = \text{free}$

$$\text{so: } (x, y, z) = (5z-16, 7z-22, z)$$

for any $z \in \mathbb{R}$. A line of points, thus.

Given are 5 points.

$$A = (0, 0, 0)$$

$$B = (2, 0, 0)$$

$$C = (4, 4, 0)$$

$$D = (4, 0, 4)$$

$$E = (0, 0, 4)$$

Find the point(s) which are equidistant to A, B, C, D, E (i.e. have the same distance to A, B, C, D, E).

Hint: Write $P = (x, y, z)$ and make equations of the distance

$$|PA|, |PB|, |PC|, |PD|, |PE|$$

Hint 1 $|PB| = |(x-2, y, z)| = \sqrt{(x-2)^2 + y^2 + z^2}$

Hint: $|PA|^2 = |PB|^2 \neq |PC|^2 = |PD|^2 = |PE|^2$

Hint: $\cancel{(x-2)^2} + y^2 + z^2 =$
 $x^2 + y^2 + z^2 = (x-2)^2 + y^2 + z^2 = (x-y)^2 + (y-4)^2 + z^2$
 $= (x-y)^2 + (y-4)^2 = (y-4)^2 + (z-4)^2$

Hint: $0 = -4x + 4$

$$0 = -8x + 16 - 8y + 16$$

$$0 = -8x + 16 - 8y + 16$$

$$0 = -8y + 16 - 8z + 16$$

Hint: $x = 1$

$$-8y = -8x + 32 = -8 \cdot 1 + 32 = 24 \Rightarrow y = 3$$

$$-8z = -8x + 32 = 24 \Rightarrow z = 3$$

$$-8y = -8z + 32 \Rightarrow -8 \cdot 3 = -8 \cdot 3 + 32$$

$$\Rightarrow 24 = 0 \text{ contradiction:}$$

There is no such point!

Exercise:

$$\frac{2}{3}\sqrt{2} + \sqrt{t}$$

(Peter)

A rocket is flying through space.

At time t it is at location $f(t) = (\cos t, \sin t, \frac{2}{3}\sqrt{2} + \sqrt{t})$ for $t \geq 0$.

At some point the rocket reaches speed 66.

~~At this moment the rocket loses a piece of its hull. This piece keeps moving at the same speed and direction as the rocket has at that moment. When is this?~~

~~What is the location of the piece at time $t=100$?~~

↳ Hint: compute a formula for the speed of the rocket at time t

Hint: $f'(t) = (\cos t - t \sin t, \sin t + t \cos t, \frac{2}{3}\sqrt{2}) \sqrt{t}$

we need $|f'(t)| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 2t^3}$

~~to calculate~~

simplify this!

→ Hint: $|f'(t)| = \sqrt{\cos^2 t - 2t \cos t \sin t + t^2 \sin^2 t +$

~~Solutions~~ $+ \sin^2 t + 2t \cos t \sin t + t^2 \cos^2 t + 2t^3$

$$= \sqrt{1 + t^2 + 2t} = \text{since } \sqrt{(1+t)^2} = |1+t|$$

Since $t \geq 0$: $= 1+t$

Continue from here.

$$1+t = 66 \Rightarrow t = 65$$

~~At this moment, the location of the rocket is ...~~

~~the speed of the rocket is ...~~

~~thus, the speed of the rocket hull piece is ...~~

~~So, a formula for the location of the hull piece is:~~



~~(See other side)~~

At $t=66$, the location of the rocket is $(66 \cos(66), 66 \sin(66), \frac{3}{2} V_2 (66)^{\frac{3}{2}})$
the velocity is

Exercise: Consider the surface $z = y^3 - 6xy + x^2$ where we restrict ourselves to $x \geq y \geq 0$. (1) Figure out where z is maximal/minimal on this area.

Hint: "I have no idea how to start"

→ You need to calculate the extremal values, and you need to check if the edges of the area ~~are~~ perhaps have some max/min.

Hint:

It's a good idea to draw the area.

Can you reason something about the maximum?

Hint / partial solution:

(Geogebra applet I made)

The area is unbounded, and if you go to the right, z gets lower and larger. Easier to reason this is:

If $y = 0$, then $z = x^2$.

x can be anything (as $x \geq y$)

and this makes z as large as you want.

So there is no maximum.

Hint: computation of extremal values

$$0 = \frac{dz}{dx} = -6y + 2x \quad \left. \begin{array}{l} x = 3y \\ 3y^2 = 6x \end{array} \right\} \Rightarrow 3y^2 = 18y$$

$$0 = \frac{dz}{dy} = 3y^2 - 6x \quad \left. \begin{array}{l} x = 3y \\ 3y^2 = 6x \end{array} \right\}$$

$$\Rightarrow y = 0 \text{ or } y = 6 \Rightarrow \text{points } (0,0) = A \\ (18,6) = B$$

Now figure out if these are local max/min or saddle points!

Hint: calculate local max/min

(2)

$$\begin{aligned} H &= \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 \\ &= \frac{\partial (-6y+2x)}{\partial x} \cdot \frac{\partial (3y^2-6x)}{\partial y} - \left(\frac{\partial (3y^2-6x)}{\partial x} \right)^2 \\ &= 2 \cdot 6y - (-6)^2 \\ &= 12y - 36 \end{aligned}$$

$$(0,0) : H = -36 < 0 \text{ saddle point}$$

$$(18,6) : H = 12 \cdot 6 - 36 = 36 > 0$$

~~if~~ $\frac{\partial^2 z}{\partial x^2} = 2 > 0$ at $(18,6)$, so
this is a local minimum.

Try to finish the exercise, with solid argumentation on the edges of the area

Hint: You need to consider the lines

$$y=x \quad \text{and} \quad y=0$$

and analyze the function there

separately (only minimum values interest us)

The Hint

$$y=0 : z = x^2 \text{ . minimum at } x=0 \\ (0,0)$$

$$y=x : z = x^3 - 6x \cdot x + x^2 = x^3 - 5x^2$$

minimum?

$$0 = \frac{d(x^3 - 5x^2)}{dx} = 3x^2 - 10x$$

$$\Rightarrow x=0 \text{ (had that already)} \quad \text{or} \quad x = \frac{10}{3}$$

$$\frac{\partial^2 z}{\partial x^2} \frac{d^2(x^3 - 5x^2)}{dx^2} = 6x - 10 \quad (3)$$

so at $x = \frac{10}{3}$ we get $6 \cdot \frac{10}{3} - 10 = 10 > 0$

so at $(\frac{10}{3}, \frac{10}{3})$ we have a local min.

Try to wrap it up!

Hints/Solutions:

Only candidates are

$$(0,0) \rightarrow z=0$$

$$(18, 6) \rightarrow z = 6^3 - 6 \cdot 18 \cdot 6 + 18^2 = -108$$

$$(\frac{10}{3}, \frac{10}{3}) \rightarrow z = \left(\frac{10}{3}\right)^3 - 6 \cdot \left(\frac{10}{3}\right)^2 + \left(\frac{10}{3}\right)^2 = -\frac{5200}{27} \approx -18.5$$

So minimum is at $(18, 6)$.

Additional ideas/shortcuts:

which is a local minimum

If you would only have one extremal value then that ~~value~~ point is also a global ~~minimum~~ minimum. However, if you also have a saddle point, then that messes up everything.

For example, here our function

$z = y^3 - 6xy + x^2$ has unbounded from below values when $x=0, y \rightarrow -\infty$ (then $z = y^3 \rightarrow -\infty$).

But, in our particular exercise, this saddle point is on the EPGE. This means that automatically the local minimum will be a global minimum - but this is quite advanced reasoning, which is indeed quicker but not something I expect anyone to come up with.

1. *Phragmites* -
2. *Scirpus* -
3. *Cyperus* -
4. *Schoenoplectus* -
5. *Equisetum* -
6. *Lemna* -
7. *Utricularia* -
8. *Hydrocharis* -
9. *Elodea* -
10. *Myriophyllum* -
11. *Sparganium* -
12. *Phalaris* -
13. *Agrostis* -
14. *Phragmites* -
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16. *Cyperus* -
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II

Consider

~~Determine where~~ on the square

$$K := \{ (x, y) \mid 0 \leq x \leq 2\pi, 0 \leq y \leq \pi \}$$

the function $f(x, y) = (x+1) \sin(xy)$.

Determine the maximal and ~~where~~ minimal values it assumes on K .

Hint: You CAN use some ad-hoc methods, but let's do this structurally, in a way which always works.)

Calculate ∇f - what can you do with this?

(And - is determining extremal values enough?)

Hint: $\nabla f = \begin{pmatrix} \sin(xy) + y \cdot (x+1) \cos(xy) \\ x \cdot (x+1) \cos(xy) \end{pmatrix}$

$\nabla f = 0$ will yield the extremal values.

$$\nabla f = 0 \Rightarrow \begin{cases} \sin(xy) + y(x+1) \cos(xy) = 0 & (1) \\ x(x+1) \cos(xy) = 0 & (2) \end{cases}$$

$$(2) : x=0 \vee x+1=0 \vee \cos(xy)=0$$

$$x=-1 \Rightarrow xy = \frac{\pi}{2}$$

$$(\text{out of domain}) \quad \text{or } xy = \frac{3\pi}{2}$$

~~case $x=0$:~~ Now substitute these cases in ①

[2]

$$x=0: \sin(0 \cdot y) + y(0+1) \cos(0 \cdot y) = 0$$

$$0 + y = 0$$

$$\Rightarrow y=0$$

$$A=(0,0)$$

$$xy = \frac{\pi}{2}, \quad \sin\left(\frac{\pi}{2}\right) + y(x+1) \cos\left(\frac{\pi}{2}\right) = 0$$

$$1 + y(x+1) \circ = 0$$

$$1 = 0$$

No solution.

$$xy = \frac{3\pi}{2} \quad \sin\left(\frac{3\pi}{2}\right) + y(x+1) \cos\left(\frac{3\pi}{2}\right) = 0$$

$$-1 = 0$$

No solution

A is the only extremal value.

Is A now the only maximum or minimum?

→ No - you have a bounded domain.
So you need to check the edges of
the square K!

How do you do that?

The edges are the lines

$$x=0$$

$$x=2\pi$$

$$y=0$$

$$y=2\pi$$



so let's look at $f(x,y)$ on these lines:

(B)

$$x=0: f(0,y) = \sin(y) = 0$$

$$y=2\pi: \underline{\lim} f(2\pi,y) = (2\pi+1) \cdot \sin(2\pi y) \Leftarrow$$

This is maximal if $\sin(2\pi y)$ is;
 $\underset{\text{max}}{\sin}$

so the highest value will be
when this is 1, $\sin(2\pi) = 1$

lowest if $\sin(2\pi) = -1$

so: $2\pi+1$ and $-2\pi-1$.

$$y=0: \underline{\lim} f(x,0) = \sin(x+1) \sin(0) = 0$$

$$x=2\pi: \underline{\lim} f(x,2\pi) = (x+1) \sin(2\pi x)$$

This has the same max/min

Note: it is not asked where, so
the final answer is:

maximum $2\pi+1$

minimum $-2\pi-1$

(If you want to

know: the maximum is assumed

~~at several points~~: at many values

~~$\sin(2\pi x) =$~~

$(t_1, 2\pi), \left(\frac{3}{4}, 2\pi\right), \left(\frac{5}{4}, 2\pi\right), \dots$

$(2\pi, \frac{1}{4}), \left(2\pi, \frac{3}{4}\right), \left(2\pi, \frac{5}{4}\right), \dots$

(similar for the minimum)

