THIRD ORDER FIBONACCI SEQUENCES

STEFAN MAXIM

ABSTRACT: I begin by introducing a special class of third-order Fibonacci sequences. I then investigate some connections with the standard Fibonacci sequences.

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1. Introduction

By a Fibonacci sequence, we mean any sequence in which a certain term is the sum of the two preceding ones. So, if (F_n) is such a sequence, then its (second order) recurrence formula is written as follows:

(FS-2) $F_{n+2} = F_{n+1} + F_n$

(or, equivalently: $F_{n+2} - F_{n+1} - F_n = 0$), for all $n \in N := \{0, 1, ...\}$.

To obtain the general solution of this recurrence, one starts with the condition that $(F_n = \lambda^n; n \in N)$ should fulfill (FS). This yields that

$\lambda^{n+2} - \lambda^{n+1} - \lambda^n = 0$, $\forall n \in N$; or, equivalently, $\lambda^2 - \lambda - 1 = 0$.

Let λ_1 and λ_2 be the roots of this equations; then,

$\lambda_1 = (1/2)(1 + \sqrt{5}) \approx 1.632$, $\lambda_2 = (1/2)(1 - \sqrt{5}) = 1 - \lambda_1 \approx -0.632$.

Note that, as a direct consequence of this,

$0 < |\lambda_2| < 1 < |\lambda_1|$; hence, $0 < |\lambda_2/\lambda_1| < 1$.

Furthermore, note that the relations above can be obtained without any approximation technique through the original Viète relations,

(V-rela) $\lambda_1 + \lambda_2 = 1$, $\lambda_1\lambda_2 = -1$,

and are deductible through its direct factorization:

$\lambda^2 - \lambda - 1 = (\lambda - \lambda_1)(\lambda - \lambda_2)$;

Using these facts, one can obtain the general solution for the recurrence (FS-2):

(FS-2-sol) $F_n = C_1\lambda_1^n + C_2\lambda_1^n$, $n \in N$,

where C_1 and C_2 are constants. To determine their values, one can fix the values of F_0 and F_1 and solve. To ensure consistency with its recursive property, the natural choices for F_0 and F_1 would be

(FS-01) $F_0 = 1$, $F_1 = 1$.

By substituting back into (FS-2), one obtains the system

(FC-2-sys) $C_1 + C_2 = 1$, $C_1\lambda_1 + C_2\lambda_2 = 1$;

The determinant of this system is

$\Delta = \det((1, 1); (\lambda_1, \lambda_2))^\top = \lambda_2 - \lambda_1 \neq 0$.

(Where $\top$ signifies the transpose operation). Using the Cramer rule, we obtain a unique solution (D_1, D_2) of (FC-2-sys), expressed as

$D_1 = (\lambda_2 - 1)/(\lambda_2 - \lambda_1)$, $D_2 = (1 - \lambda_1)/(\lambda_2 - \lambda_1)$.

Consequentially, the solution of (FS-2)+(FS-01) has the form

$$F_n = D_1\lambda_1^n + D_2\lambda_2^n,\ n \in N;$$

we then say that (F_n) is a *Fibonacci sequence of order two*; in short: a *2-order Fibonacci sequence*. Note that, by its very definition, all elements of the sequence (F_n) are natural numbers. For example, the first 10 elements of this sequence are

$$F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,$$
$$F_5 = 8,\ F_6 = 13,\ F_7 = 21,\ F_8 = 34,\ F_9 = 55.$$

Now that we have established how we define the Fibonacci sequence, we can begin to discuss the asymptotic behavior of the sequence. However, before we begin, we must note some preliminary facts.

Let (U_n) and (V_n) both be sequences over $R_+^0 :=]0,\infty[$. We can say that

(ad-1) (U_n) is *asymptotically equivalent* to (V_n), if $\lim_n(U_n/V_n) = \gamma$, for some $\gamma \in R_+^0$

(ad-2) (U_n) is *asymptotic inferior* to (V_n), if $\lim_n(U_n/V_n) = 0$

(ad-3) (U_n) is *asymptotic superior* to (V_n), if $\lim_n(U_n/V_n) = \infty$.

Proposition 1.1. *Let (U_n) be a geometric sequence with a common ratio of λ_1 over R_+^0:*

$$U_n = U_0\lambda_1^n,\ n \in N,\ \text{where } U_0 > 0.$$

Then,

(11-1) *the Fibonacci sequence (F_n) is asymptotic equivalent with the progression (U_n), since*

$$\lim_n(F_n/U_n) = D_1/U_0\ (> 0)$$

(11-2) *As a consequence,*

$\lim_n(F_{n+1}/F_n) = \lambda_1$*; so that, at least asymptotically, (F_n) behaves like a geometric series with a common ratio of λ_1.*

Proof. **(I)** According to our established definition,

$$\lim_n(F_n/U_n) = \lim_n(D_1\lambda_1^n + D_2\lambda_2^n)/(U_0\lambda_1^n) =$$
$$\lim_n[(D_1/U_0) + (D_2/U_0)(\lambda_2/\lambda_1)^n] = D_1/U_0.$$

From here, if we denote $\xi = \lambda_2/\lambda_1$, we have (see above)

$0 < |\xi| < 1$; from which we observe, $\lim_n \xi^n = 0$.

Thus, the first part is proved. Note that, as a direct consequence,

$$\lim_n(U_n/F_n) = 1/\lim_n(F_n/U_n) = 1/(D_1/U_0) = U_0/D_1.$$

(II) From the previous stage, one has (under our notations)

$$\lim_n(F_{n+1}/F_n) = \lim_n[(F_{n+1}/U_{n+1})(U_{n+1}/U_n)(U_n/F_n)] =$$
$$(D_1/U_0)(\lambda_1)(U_0/D_1) = \lambda_1;$$

thus proving the second part. □

2. Statement of the problem

To depict the function of 2-order Fibonacci sequence (F_n), we can use it as the abstract model of numbering the pairs of rabbits concerning a sequence of time units (months, for example). In what follows, we will begin by making a certain modification to the aforementioned process. Specifically, we will denote

G_n=the number of rabbit pairs at the n-th month, $n \in N$.

The conditions below are acceptable, for each $n \in N$:

(cond-1) in the time interval $]n, n+(3/2)]$, each rabbit pair (from the class of all G_n pairs) mates, resulting in a single rabbit
(cond-2) in the time interval $]n+(3/2), n+3]$, each pair (from the class of all G_n pairs) mates (again), resulting in another single rabbit
(cond-3) in the time interval $]n, n+3]$, each pair (from the class of all G_n pairs) mates with a new rabbit pair.

As a consequence of this, the abstract formula for our problem now becomes

(GS-3) $G_{n+3} = G_{n+2} + G_n$
(or, equivalently: $G_{n+3} - G_{n+2} - G_n = 0$), $n \in N$.

Where the initial conditions imposed are (in accordance to our previous convention)

(GS-012) $G_0 = 1$, $G_1 = 1$, $G_2 = 1$.

The sequence (G_n) fulfilling (GS-3) and (GS-012) will be referred to as a *Fibonacci sequence of order three*; in short: a *3-order Fibonacci sequence.*

3. Solution of (GS-3)+(GS-012)

To get the general solution for the recurrence (GS-3), one starts with the condition that $(G_n = \mu^n; n \in N)$ should fulfill (GS-3). This results in

(CE-3) $\mu^{n+3} - \mu^{n+2} - \mu^n = 0$, $\forall n \in N$; or, equivalently, $\mu^3 - \mu^2 - 1 = 0$.

Through an analysis of the associated function $g : R \to R$ defined as

$g(\mu) = \mu^3 - \mu^2 - 1$, $\mu \in R$,

it follows that

(r-sol) its characteristic equation (CE-3) has only a single real solution μ_1, which is approximated as $\mu_1 = 3/2$
(c-sol) the characteristic equation (CE-3) has
two (conjugate) complex solutions $\mu_2 = \alpha + i\beta$, $\mu_3 = \alpha - i\beta$.

As for μ_1's actual value, we observe that

$g(1.46) = -0.019$, $g(1.47) = 0.015$;

so, a more precise value for the real solution would be $\mu_1 = 1.47$. That being said, for the sake of simplicity in our calculation, we will continue to refer to μ_1's previous value $\mu_1 = 1.5 = 3/2$.

As for the real constants (α, β) appearing in our equations for (μ_2, μ_3), they can be obtained using the Viète relations. To this end, let us begin with the factorization

$\mu^3 - \mu^2 - 1 = (\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)$.

A direct identification of these polynomials returns the relations

$\mu_1 + \mu_2 + \mu_3 = 1$, $\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = 0$, $\mu_1\mu_2\mu_3 = 1$.

From the first and third equations, we can conclude that

$\mu_2 + \mu_3 = 1 - \mu_1 = 1 - 3/2 = -1/2$, $\mu_2\mu_3 = 1/\mu_1 = 1/(3/2) = 2/3$;

that is (using the representation consistent with how we defined μ_2 and μ_3)

$2\alpha = -1/2$, $\alpha^2 + \beta^2 = 2/3$.

The solution of this system is

(solu) $\alpha = -1/4$, $\beta = \sqrt{2/3 - 1/16} = (1/4)\sqrt{29/3}$.

This, finally, gives us the general solution of our recurrence (GS-3)

(GS-3-sol) $G_n = K_1\mu_1^n + K_2\mu_2^n + K_3\mu_3^n$, $n \in N$.

where, for now, (K_1, K_2, K_3) are complex numbers. From here, the particular solution of the recurrence that fulfills (GS-012) can be obtained by passing our imposed conditions on this equation as follows:

(GS-3-sys) $K_1\mu_1^0 + K_2\mu_2^0 + K_3\mu_3^0 = 1$,
$K_1\mu_1^1 + K_2\mu_2^1 + K_3\mu_3^1 = 1$, $K_1\mu_1^2 + K_2\mu_2^2 + K_3\mu_3^2 = 1$.

The determinant of this system is

$\Delta = \det((1, 1, 1), (\mu_1, \mu_2, \mu_3), (\mu_1^2, \mu_2^2, \mu_3^2))^\top =$
$(\mu_2 - \mu_1)(\mu_3 - \mu_1)(\mu_3 - \mu_2) \neq 0$.

By the Cramer rule, we therefore get a unique (complex) solution (K_1^*, K_2^*, K_3^*) of (GS-3-sys), expressed as

$K_1^* = [(\alpha - 1)^2 + \beta^2]/[(\alpha - \mu_1)^2 + \beta^2]$,
$K_2^* = [(1 - \mu_1)(\alpha - 1 - i\beta)]/[(\alpha - \mu_1 + i\beta)(-2i\beta)] = S + iT$,
$K_3^* = [(1 - \mu_1)(\alpha - 1 + i\beta)]/[(\alpha - \mu_1 - i\beta)(2i\beta)] = S - iT$,

where the real pair (S, T) depends on this equations. As a consequence, the solution of the problem (GS-3)+(GS-012) has the form

(GS-3-sol) $G_n = K_1^*\mu_1^n + K_2^*\mu_2^n + K_3^*\mu_3^n$, $n \in N$;

where (K_1^*, K_2^*, K_3^*) is the triple of complex numbers given above. However, to provide a more appropriate solution, we have to transform our complex solutions (μ_2, μ_3) of (EC-3) into polar form. To this end, remember that for every complex number, we have the polar representation

$\alpha + i\beta = \rho(cos\theta + i\sin\theta)$, where
$\rho = \sqrt{\alpha^2 + \beta^2}$, $\cos\theta = \alpha/\rho$, $\sin\theta = \beta/\rho$,

From this, we will derive the well-known Moivre formula (see, for instance, Brown et al [1, Ch 14, Sect 14-5])

$(\alpha + i\beta)^n = \rho^n(\cos n\theta + i\sin n\theta)$, $n \in N$.

Passing to the complex solutions of (CE-3), we have

$-1/4 + i(1/4)\sqrt{29/3} = \rho(\cos\theta + i\sin\theta)$,
$-1/4 - i(1/4)\sqrt{29/3} = \rho(\cos\theta - i\sin\theta)$, where
$\rho = \sqrt{2/3}$, $\cos\theta = (-1/4)\sqrt{3/2}$, $\sin\theta = (1/4)\sqrt{29/2}$;
hence, $\theta = \arccos{(-1/4)}\sqrt{3/2}$;

and this gives the expression of their powers

$\mu_2^n = (\alpha + i\beta)^n = \rho^n(\cos n\theta + i\sin n\theta)$,
$\mu_3^n = (\alpha - i\beta)^n = \rho^n(\cos n\theta - i\sin n\theta)$, $n \in N$.

By substituting this all back into the expression (GS-3-sol) of the particular solution, we obtain, after some small calculations

(GS-3-sol-real) $G_n = H_1\mu_1^n + H_2\rho^n\cos n\theta + H_3\rho^n\sin n\theta$, $n \in N$;

where (H_1, H_2, H_3) is a triple of real numbers. Note that, unfortunately, this representation is not exact since the values of (μ_1, μ_2, μ_3) we used are only loose approximations of the exact solutions of the characteristic equation (CE-3). That being said, by using some techniques of solving polynomial equations of degree 3, we can write the exact representation of these solutions; namely

$\mu_1 = (1/3)(1 + \sqrt[3]{A} + \sqrt[3]{B}]$,
$\mu_2 = (1/3)[1 - P\sqrt[3]{A} - Q\sqrt[3]{B}]$, $\mu_3 = (1/3)[1 - Q\sqrt[3]{A} - P\sqrt[3]{B}]$,

where, for simplicity, we denoted

$A = (1/2)(29 - 3\sqrt{93})$, $B = (1/2)(29 + 3\sqrt{93})$,
$P = (1/2)(1 - i\sqrt{3})$, $Q = (1/2)(1 + i\sqrt{3})$.

As a result, an "exact" representation of the solutions, related to the one in (GS-3-sol-real) are available. However, from a practical perspective, these expressions hold little value since they cannot be used in their current crude form. Thus, an approximation of these (as is shown in GS-3-sol-real) is preferable.

Finally, note that despite the complicated expression for our solution (littered with algebraic and trigonometric functions) all elements of the sequence (G_n) are still natural numbers. For example, the first 10 elements of this sequence are

$G_0 = 1$, $G_1 = 1$, $G_2 = 1$, $G_3 = 2$, $G_4 = 3$,
$G_5 = 4$, $G_6 = 6$, $G_7 = 9$, $G_8 = 13$, $F_9 = 19$.

Lastly, an algebraic theory for these numbers is still waiting to be discovered. Hopefully, future developments in this field will yield some results that will change that.

4. Asymptotic properties

In the following, we will discuss an asymptotic behavior of the solution (GS-3-sol-real) for the third order Fibonacci sequence (G_n)

Let us first return to the Fibonacci sequence of order two, (F_n). Its characteristic equation is written as

$f(\lambda) := \lambda^2 - \lambda - 1 = 0$; with the roots:
$\lambda_1 = (1/2)(1 + \sqrt{5}) \approx 1.632$, $\lambda_2 = (1/2)(1 - \sqrt{5}) \approx -0,632$.

On the other hand, for the Fibonacci sequence of order three (G_n), its characteristic equation is written as

$g(\mu) := \mu^3 - \mu - 1 = 0$; with the roots:
$\mu_1 \approx 1.5$, $\mu_2 = \alpha + i\beta$, $\mu_3 = \alpha - i\beta$;

where (α, β) are given by the precise relations above. Note that, according to some classical results expressed in Stewart et al [3, Ch 4, Sect 4-1]

$g'(\mu) = 3\mu^2 - 2\mu$, $\mu \in R$;
thus, g is increasing on the interval $[2/3, \infty[$.

This, along with

$g(1) = -1$, $g(\lambda_1) = \lambda_1^3 - \lambda_1^2 - 1 = \lambda_1^2(\lambda_1 - 1) - 1 =$
$(\lambda_1 + 1)(\lambda_1 - 1) - 1 = \lambda_1^2 - 2 = \lambda_1 - 1 > 0$

tells us that

$1 < \mu_1 < \lambda_1$; thus $0 < \rho = \sqrt{2/3} < 1 < \mu_1 < \lambda_1$.

Proposition 4.1. *Let (V_n) be a geometric series with a common ratio of μ_1 over R_+^0:*

$V_n = V_0\mu_1^n$, $n \in N$, *where* $V_0 > 0$.

Then,

(41-1) *the 3-order Fibonacci sequence (G_n) is asymptotic equivalent with the progression (V_n), since, as established in (ad-1),*

$\lim_n(G_n/V_n) = H_1/V_0$ (> 0)

(41-2) *As a consequence, we obtain that*

$\lim_n(G_{n+1}/G_n) = \mu_1$; *so that, asymptotically, (G_n) acts as a geometric series with a common ratio of μ_1*

(41-3) *the 3-order Fibonacci sequence* (G_n) *is asymptotic inferior with respect to the 2-order Fibonacci sequence* (F_n)*, in the sense that:* $\lim_n(G_n/F_n) = 0$.

Proof. **(I)** Denote, for simplicity,

$\sigma = \rho/\mu_1$; hence, $0 < \sigma < 1$, $\lim_n \sigma^n = 0$.

According to this definition, we get that

$$\lim_n(G_n/V_n) = \lim_n(H_1\mu_1^n + H_2\rho^n \cos n\theta + H_3\rho^n \sin n\theta)/(V_0\mu_1^n)] =$$
$$\lim_n[(H_1/V_0) + (H_2/V_0)\sigma^n \cos n\theta + (H_3/V_0)\sigma^n \sin n\theta] = H_1/V_0,$$

where we imply that

(imp-1) $|\sigma^n \cos n\theta| \le \sigma^n$, $\forall n$, and $\lim_n \sigma^n = 0$ imply $\lim_n \sigma^n \cos n\theta = 0$,
(imp-2) $|\sigma^n \sin n\theta| \le \sigma^n$, $\forall n$, and $\lim_n \sigma^n = 0$ imply $\lim_n \sigma^n \sin n\theta = 0$,

Thereby proving the first part. As a result,

$$\lim_n(V_n/G_n) = 1/\lim_n(G_n/V_n) = 1/(H_1/V_0) = V_0/H_1.$$

(II) By a previous observation, all elements in the sequence (G_n) are natural numbers; Specifically, positive numbers. In this case, we can conclude that

$$\lim_n(G_{n+1}/G_n) = \lim_n[(G_{n+1}/V_{n+1})(V_{n+1}/V_n)(V_n/G_n)] =$$
$$(H_1/V_0)(\mu_1)(V_0/H_1) = \mu_1;$$

thereby proving the second part as well.

(II) Remember that we introduced the geometric series

$U_n = U_0\lambda_1^n$, $n \in N$.

Furthermore, we also showed that

$\lim_n(F_n/U_n) = D_1/U_0(> 0)$.

Finally, through a previous comparison relation, we showed that

$0 < \mu_1 < \lambda_1$; hence, $0 < \eta := \mu_1/\lambda_1 < 1$.

Combining all of these, we can conclude that

$$\lim_n(G_n/F_n) = \lim_n(G_n/V_n)(V_n/U_n)(U_n/F_n) =$$
$$(H_0/V_0)(V_0/U_0)(\eta^n)(U_0/D_1) = 0;$$

□

As for the import of our results, it may prove useful when we wish to study our sequence (G_n) for large values of n. It should also be useful when evaluating the quotient sequence (G_n/F_n) for the same values of n. As a consequence of this last property, the sequence (G_n) may better describe the growth progression in question. Further aspects may be found in Mitchell et al [2], Waldschmidt [4], and the references therein.

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Email address: `stefan.teodor.maxim@gmail.com`