Second set of **Exercises** for the tutorials on logic in CS 3511
(Except where indicated, informal proofs will suffice.)

1. Find a counterexample, if possible, to these statements, where the u.d. consists of all (negative and non-negative) integers:

a. $\forall x \forall y (x^2 = y^2 \rightarrow x = y)$
Answer: $x = 3, y = -3$
b. $\forall x \exists y (y^2 = x)$
Answer: $x = -3$**; also, x=5**
c. $\forall x \forall y (xy \geq x)$
Let x **be any integer and** y **any negative integer**

2. An advert in an Aberdeen supermarket says: "All frozen foods reduced by up to 70%".
a) Is the sentence false if all frozen foods have been reduced by 1%?
On my readings (see below), it is true. This indicates that there is something about the sentence that is not captured by these formulas – Perhaps the suggestion that 'quite a few' frozen foods are reduced by something 'close to' 70%**.**
b) Propose one or more logical translations of this sentence
Answer: Sentences like this are quite unclear, so it's not as if there is exactly one correct answer. One possibility is $\forall x(F(x) \rightarrow \neg(RM(x))$**, where** $RM(x)$ **stands for 'x is reduced by more than 70%'. A perhaps more plausible option is the conjunction of the above with** $\forall x(F(x) \rightarrow \neg(RE(x))$**, where** $RE(x)$ **stands for 'x is reduced (by some percentage)'. A disadvantage of this answer is that it doesn't use logic to formalise the relation between** RE **and** RM**. A better answer might make use of the predicate** $R(x,y)$ **meaning 'x is reduced by a percentage of y'. In this case the answer suggested above becomes** $(1)\&(2)$**, where** $(1) = \forall x(F(x) \rightarrow \neg\exists y(y > 70\&R(x,y)))$**, and** $(2) = \forall x(F(x) \rightarrow \exists y(y > 0\&R(x,y)))$**.**
One might also want to add that some articles must be reduced by as much as 70%: $\exists x(F(x) \wedge R(x,70))$**.**

3. Use quantifiers to express the associative law for multiplication of real numbers.

Answer: $\forall xyz((x.y).z) = (x.(y.z))$

4. Show, using a sequence of equivalences, that the following two statements must have the same truth value:

a) $\neg\exists x \forall y P(x,y)$
b) $\forall x \exists y \neg P(x,y)$

Answer: $\neg\exists x \forall y P(x,y) \Leftrightarrow \forall x \neg\forall y P(x,y) \Leftrightarrow \forall x \exists y \neg P(x,y)$

5. A number is called an upperbound (UB) of a set S of numbers iff it is greater than or equal to every member of S. The number x is called the *least upper bound* (LUB) of a set S of real numbers iff x is an upper bound of S and x is less than or equal to every upper bound of S.

a.) Using quantifiers, say that x is an upper bound of S.
Answer: $\forall y \in S(x \geq y)$
b.) Using quantifiers, say that x is a LUB of S.
Answer: $\forall y(UB(y) \rightarrow x \leq y)$
c.) Prove that a set of real numbers can have at most one LUB.
Answer: Suppose x_1 **and** x_2 **both are LUBs of** S**. Then one must be bigger than the other, for example** $x_1 > x_2$**. In this case,** x_1 **is not the *lowest* UB of** S
d.) Give an example of a set of real numbers that has a LUB.
Answer: the set $\{1,2,3\}$ **(or any other enumerated set, for that matter)**

e.) Give an example of a set of real numbers that does not have a own LUB.
Answer: The set of all real numbers. Also, the set of all real numbers greater than some value x. It's not the case that all infinite sets qualify, witness for example the set of all (negative or non-negative) numbers smaller than 5.
f.) Give an example of a set of rational numbers that has a LUB and contains it (i.e., the LUB is a member of the set)
Answer: the set $\{1, 2, 3\}$ **again**
g.) Give an example of a set of rational numbers that has a LUB but does not contain it.
Answer: The set of all rational numbers smaller than π**. (**π **itself is the LUB of this set, but it is not an element of the set, because it is not rational.)**
h.) The notion of an LUB is based on the relation 'greater than'. Can you propose an analogous notion based on the relation 'smaller than'?
Answer: One can define the Lower Bound (LB) and Greatest Lower Bound (GLB) analogously, starting with LB(x) is $\forall y(x \leq y)$

6. Next week

7. Next week.

8. Next week.