CS2013

Introduction to probability

UNIVERSITY OF ABERDEEN

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The story so far ...

Some basic concepts of **statistics** : tools for *summarising data*

- kinds of data/variables
- difference between sample and sample space
- various kinds of graphs
- the sample mean, median and mode
- skew
- measures of spread: variance and standard deviation
- upper and lower quartiles

Now a related but more theoretical topic: **probability**

Initial thoughts

A person tosses a coin five times. Each time it comes down heads. What is the probability that it will come down heads on the sixth toss?

½ *or* '**1 chance in 2**' *or* **50%**

- assumed that the coin is 'fair'
- ignored empirical evidence of the first five tosses

less than ½ *or* **...**

- assumed that the coin is 'fair'
- thought about 'law of averages' – in the long run, half heads, half tails
- but dice have no memory!

more than ½ *or* ...

- cynical – assumed biased coin

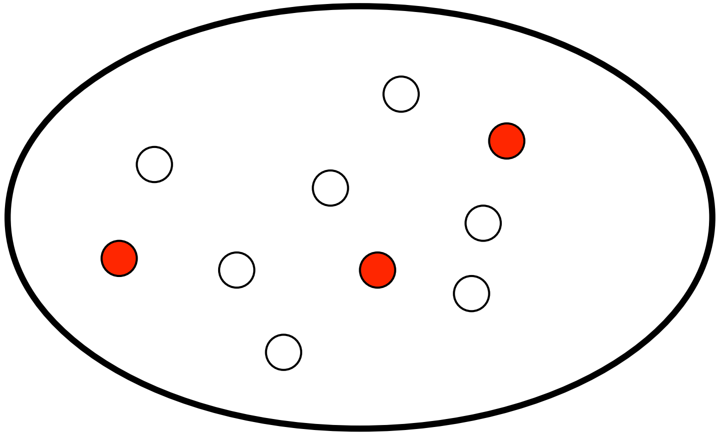
Probability can be defined in different ways, some of which are more suitable in particular situations.

First, let's look at a relatively simple situation, based on complete information.

One definition of probability

Probability: the extent to which an event is likely to occur, **measured by the ratio** of the **number of favourable outcomes** to the **total number of possible outcomes**.

10 balls in a bag
7 white; 3 red



- close eyes
- shake bag
- put in hand and pick a ball
- 10 possible balls (outcomes)

Might there be a reason why any one ball is picked rather than any other?

If not, then all outcomes are **equally likely**.

Probability of picking a red ball

- favourable outcome = red ball
- number of favourable outcomes = 3
- number of outcomes = 10
- probability = 3/10 (i.e. 0.3)

Note:

- probability of picking a white ball is 7/10 (0.7)
- probabilities lie between 0.0 and 1.0
- we have two mutually exclusive outcomes
  (can’t be both red and white)
- outcomes are exhaustive (no other colour there)
- in this case the probabilities add to 1.0 (0.3 + 0.7)
- a probability is a **prediction**

Useful concepts

Trial

- action which results in one of several possible outcomes;
- e.g. picking a ball from the bag

Experiment

- a series of trials (or perhaps just one)
- e.g. picking a ball from the bag twenty times

Event

- a set of outcomes with something in common
- e.g. a **red** ball

Probability derived *a priori*

Suppose each trial in an experiment can result in one (and only one) of n equally likely (as judged by thinking about it) outcomes, r of which correspond to an event E.

The probability of event E is:

$$P(E) = \frac{r}{n}$$

$a\ priori$ *means "without investigation or sensory experience"*

More complex *a priori* probabilities

What's the probability of throwing heads with a fair coin? 1/2

More complex *a priori* probabilities

What's the probability of throwing heads with a fair coin? ½

What's the probability of throwing exactly one head with two fair coins?

More complex *a priori* probabilities

What's the probability of throwing a head with a fair coin? ½

What's the probability of throwing exactly one head with two fair coins? 1/2

More complex *a priori* probabilities

What's the probability of throwing heads with a fair coin? ½

What's the probability of throwing exactly one head with two fair coins? ½

Proof:

- set of outcomes = {hh,tt,ht,th}, so n=4
- favourable outcomes = {ht,th}, so r=2
- therefore, P(E)=2/4=1/2

More complex *a priori* probabilities

What's the probability of throwing heads with a fair coin? ½

What's the probability of throwing at least one head with two fair coins? 3/4

Proof:

- set of outcomes = {hh,tt,ht,th}, so n=4
- favourable outcomes = {ht,th,hh}, so r=3
- therefore, P(E)=3/4

More complex *a priori* probabilities

What's the probability of throwing 6 with one fair dice? 1/6

What's the probability of throwing 8 with two fair dice?

More complex *a priori* probabilities

Probability of throwing 8 with two dice:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	**8**
3	4	5	6	7	**8**	9
4	5	6	7	**8**	9	10
5	6	7	**8**	9	10	11
6	7	**8**	9	10	11	12

- 5 outcomes correspond to event “throwing 8 with two dice”
- 36 possible outcomes
- probability = 5/36

Probability derived from experiment

Toss a drawing pin in the air - two possible outcomes:
point **up,** point **down**

Can we say a priori what the relative probabilities are? (c.f. ‘fair’ coin)

If not, then experimentation is a possible <u>source</u> for determining probabilities. (This gets us back to statistics!)

Probability based on relative frequencies (from experimental data)

If, in a large number of trials (n), r of these result in an event E, the probability of event E is:

$$P(E) = \frac{r}{n}$$

Probability derived from experiment

- number of trials must be 'large' (how large?)
- trials must be 'independent' - the outcome of any one toss must **not** depend on any previous toss
- no guarantee that the value of r/n will settle down
- compare with the relative frequencies for existing data
- if we have enough experiments then we believe that the relative frequency contains a general 'truth' about the system we are observing.

a priori and Experimental Probabilities Ranges and Odds

- In both cases the **minimum** probability is **0**
  - a priori – event cannot occur
    - a blue ball is drawn
    - throwing 7 on an ordinary dice
  - experimental – event has not occurred

- In both cases the **maximum** probability is **1**
  - a priori – event must occur
    - sun will rise tomorrow (knowledge of physics)
  - experimental – the event always has occurred
    - the sun has risen every day in my experience

- For two events, express as **odds**:
  - odds on a red ball are 3/10 to 7/10 i.e. 3 to 7

Question

I think there's a better than evens chance that *English Jim* will win the 2:50 at Exeter today.

How likely is it to snow tomorrow?

What kind of probability are we talking about here?

- a priori?
- experimental?

Neither

- future event so no experimental evidence (and only happens once)
- reasoning from first principles – what principles?

Probability of two independent events

Experiment

- trial 1: pick a ball – event E_1 = the ball is red
- replace the ball in the bag
- trial 2: pick a ball – event E_2 = the ball is white

What is the probability of the event E:

E = first picking a red ball, then a white ball (with replacement).

$$P(E) = P(E_1 \text{ and } E_2) \quad \text{or} \quad P(E_1 \cap E_2)$$

Probability of two independent events

Experiment

- trial 1: pick a ball – event E_1 = the ball is red
- replace the ball in the bag
- trial 2: pick a ball – event E_2 = the ball is white

What is the probability of the event E:

E = first picking a red ball, then a white ball (with replacement).

$$P(E) = P(E_1 \text{ and } E_2) \quad \text{or} \quad P(E_1 \cap E_2)$$

If the two trials are **independent** (the outcome of the first trial does not affect the outcome of the second) then:

$$P(E_1 \text{ and } E_2) = P(E_1) * P(E_2)$$ - the **multiplication** law

Probability of two red balls

	R	R	R	W	W	W	W	W	W	W
R	Y	Y	Y	N	N	N	N	N	N	N
R	Y	Y	Y	N	N	N	N	N	N	N
R	Y	Y	Y	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N	N

- 9 favourable outcomes out of 100 possibilities
- all equally likely
- probability of two reds = 9/100 (= 3/10 * 3/10)

Probability of two non-independent events

Experiment

- trial 1: pick a ball – event E_1 = the ball is red
- **do not replace** the ball in the bag
- trial 2: pick a ball – event E_2 = the ball is red

What is the probability of the event E, where
E = picking two red balls on successive trials (**without replacement**)?

In this case, the probability of E_2 is affected by the outcome of trial 1.

See the following table, which visualises the situation where trial 1 results in E_1 being true.

Probability of two red balls

second draw – assuming red chosen first time

first draw

	R	R	W	W	W	W	W	W	W
R	Y	Y	N	N	N	N	N	N	N
R	Y	Y	N	N	N	N	N	N	N
R	Y	Y	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N

- 6 favourable outcomes out of 90 possibilities
- all equally likely
- probability of two reds = 6/90

Conditional probabilities

$P(E_2 | E_1)$ =

- the probability that E_2 will happen **given that** E_1 has already happened.

If E_2 and E_1 are (conditionally, statistically) independent then

$$P(E_2 | E_1) = P(E_2)$$

Be careful about assuming independence.

Probability of two non-independent events

Instead of drawing a table, you can also use a more general formula:

$$P(E_1 \, and \, E_2) = P(E_1) * P(E_2 | E_1)$$

NB: still the general form of the **multiplication** law

See the multiplication in red in the following slide:

Probability of two red balls

second draw – assuming red chosen first time

first draw

	R	R	W	W	W	W	W	W	W
R	Y	Y	N	N	N	N	N	N	N
R	Y	Y	N	N	N	N	N	N	N
R	Y	Y	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N
W	N	N	N	N	N	N	N	N	N

- 6 favourable outcomes out of 90 possibilities
- all equally likely
- probability of two reds = 6/90 (= 3/10 * 2/9)

Probability of one event or another

Experiment

- bag contains 3 red balls, 7 white balls and 5 blue balls
- trial: pick a ball
- event E_1 = the ball is red
- event E_2 = the ball is blue

\- What is the probability of the event E = picking **either** a red ball **or** a blue ball?

In this case, it's just $P(E_1)+P(E_2)=3/15+5/15=8/15$

\- What's the probability of throwing either 8 or 9 with two dice?

Additive probabilities

Probability of throwing 8 (E_1) **or** 9 (E_2) with two dice:

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- 9 outcomes correspond to event $E = E_1 \ or \ E_2$
- 36 possible outcomes
- probability = 9/36 (= ¼) = 5/36 + 4/36 = $P(8) + P(9)$

Probability of two mutually exclusive events

If the two events are mutually **exclusive** (they can't both happen in the same trial) then:

$$P(E_1 \, or \, E_2) = P(E_1) + P(E_2)$$

If this cannot be assumed, you need a slightly more complex formula:

$$P(E_1 \, or \, E_2) = P(E_1) + P(E_2) - P(E_1 \, and \, E_2)$$

Of course if $P(E_1 \, and \, E_2) = 0$ then the two formulas are the same

Probability of two non-mutually exclusive events

E = Probability of throwing at least one 5 with two dice
E1 = 5 on first dice; *E2* = 5 on second dice

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	**1,5**	1,6
2	2,1	2,2	2,3	2,4	**2,5**	2,6
3	3,1	3,2	3,3	3,4	**3,5**	3,6
4	4,1	4,2	4,3	4,4	**4,5**	4,6
5	**5,1**	**5,2**	**5,3**	**5,4**	**5,5**	**5,6**
6	6,1	6,2	6,3	6,4	**6,5**	6,6

- 11 outcomes correspond to event E

  $E = E_1$ **or** E_2 **or** *[implicitly]* $(E_1$ *and* $E_2)$
- 36 possible outcomes
- probability = 11/36 = 6/36 + 6/36 – 1/36

Counting/Combinatorics

Drawing tables is a good way of visualising a set of outcomes.

If the number of outcomes is very large, this won't work very well.

Luckily, there is an area of mathematics that focusses on questions of counting.

For example, *"How many ways are there to pick n elements from a set of m elements (where n<m)"?*

This is sometimes called *Combinatorics* (also "Permutations and Combinations"). We're not going into this here.

Probability so far: summing up

- Probability as a fraction *r / n* (i.e., *favourable/total*)
- Visualising all total and all favourable outcomes in a table
- Relation between an event and its components. E.g.,
  - E = E1 and E2
  - E = E1 or E2
- Questions to consider:
  - Are E1 and E2 independent of each other?
  - Are E1 and E2 mutually exclusive?
- *Conditional* probability: the probability of E1|E2