CS2013 Tutorial questions on mathematical induction

Aim: to let you get some practice applying mathematical induction to different kinds of problems. These exercises (which will probably take you a few hours) assume you do know what mathematical induction is and how it is used to prove things.

For background on mathematical induction, consult any book on discrete mathematics, or the lecture on induction that we have recently put on the web site of the course. A rich source is Rosen, Discrete Mathematics And Its Applications (any edition), from which some of the exercises below were taken.

Advice: proofs by induction can be difficult to read. The best way of understanding them is often to try to prove the theorem yourself first.

Questions

1. *(Number theory.)* Prove: The n-th positive even number equals 2n.

2. *(Number theory.)* Prove: 3^n < n! for n>=7.

3. *(Set theory)*

Prove that if A1,A2,..,An and B1,B2,..,Bn are sets such that A_j is a subset of B_j
for each j=1,2,..,n, then
(A1 intersection ... intersection An) is s subset of (B1 intersection ... intersection Bn).

4. (*Propositional logic*, using - for negation and ^ for conjunction)

Prove that -(p1 v p2 v ... v p_n) <==> -p1 ^ -p2 ^ ... ^ -p_n.

5. (*Number theory*) What is wrong with this “proof”?

“<u>Theorem</u>”: For every positive integer n, if x and y are positive integers with max(x,y)=n, then x=y.
<u>Base step:</u> Let n=1. If max(x,y)=1 then x=y=1.
<u>Induction step:</u> Let k be a positive integer. Suppose that whenever x and y are positive integers with max(x,y)=k, then x=y. Now let max(x,y)=k+1, where x and y are positive integers. Then max (x-1,y-1)=k, hence (by the induction hypothesis) x-1=y-1, from which it follows that x=y.

6. (*Formula induction*)

Prove that all statements of propositional logic have an even number of brackets.
(Formulated and proven in class.)

Sample answers on the next page.

Sample answers

1. Prove: The n-th positive even number equals $2n$.

Proof using the first principle of mathematical induction:
Let's write $E(n)$ as short for "the n-th positive even number".
The theorem can then be stated briefly as $E(n)=2n$.

- Basis step: We need to prove that $E(1) = 2.1 = 2$, which is true, because $E(1)=2$.

- Induction step: Suppose (this is the Induction Hypothesis, IH) that for certain n, it is true that $E(n)=2n$. We now need to prove that it follows that $E(n+1)=2(n+1)$. So let's focus on $E(n+1)$.

  Evidently, $E(n+1)=E(n)+2$. By writing $E(n+1)$ in this form (in which $E(n)$ occurs!), we are now able to apply IH.
  For, by IH, it follows from $E(n+1)=E(n)+2$ that $E(n+1)=2n+2$.
  But from this it follows directly that $E(n+1)=2(n+1)$, which is what we wanted to prove. QED

NB Although this written version of the proof is slightly over-elaborate, it will help you (as well as whoever wants to read and understand your proofs!) enormously if you do write explicitly
- what the Basis Step is
- what the Induction Hypothesis (IH) is
- which step(s) in the proof of the induction step hinge on the IH.

2. Prove: $3^n < n!$ for $n>=7$.

. Base step: Prove that $3^7 < 7!$ Proof is trivial since $3^7=2187$ and $7!=5040$.

. Inductive step: From the Induction Hypothesis $3^n < n!$, we have to prove that $3^{(n+1)} < (n+1)!$, given that $n>=7$.

  Proof of Inductive Step: Comparing the two terms in the inequality, we see that

$$3^{(n+1)} = 3^n * a \quad \text{where } a=3, \text{ and}$$
$$(n+1)! = n! * b \quad \text{where } b=n+1.$$

  But the Induction Hypothesis tells us that $3^n < n!$
  And $n>= 7$, hence $a<b$. It follows that $3^n * a < n! * b$, proving the Base Step.

QED

3. Prove that if A1,A2,..,An and B1,B2,..,Bn are sets such that A_j is a subset of B_j for each j=1,2,..,n, then
(A1 intersection ... intersection An) is s subset of (B1 intersection ... intersection Bn).

-- The Basis Step for this proof by induction says that A1 is a subset of B1. This follows directly from the assumptions made in the theorem.

-- Induction Step: From the Induction Hypothesis (IH) that
(A1 intersection ... intersection An) is s subset of (B1 intersection ... intersection Bn),
we prove that
(A1 intersection ... intersection An intersection An+1) is s subset of
(B1 intersection ... intersection Bn intersection Bn+1).

The first of these two formulas can be written as
(A1 intersection ... intersection An) intersection An+1)),
which is the intersection of two sets. By IH, its first part,
(A1 intersection ... intersection An) is a subset of (B1 intersection ... intersection Bn).

Likewise, its second part, An+1, is a subset of Bn+1. It follows from simple set theory that
(A1 intersection ... intersection An) intersection An+1) is s subset of
(B1 intersection ... intersection Bn) intersection Bn+1).

The structure of the last part of the proof is as follows:
We prove that (X intersection An+1) is a subset of (Y intersection Bn+1) by arguing that
X is a subset of Y, and An+1 is a subset of Bn+1.

4. Prove: -(p_1 v .. v p_n) = -p_1 & .. & -p_n.

. Base step: n=1. Proof is trivial, since this amounts to -p_1= -p_1

. Inductive step: suppose (Induction Hypothesis) -(p_1 v .. v p_n) = -p_1 & .. & -p_n.
Prove from this that -(p_1 v .. v p_n v p_n+1) = -p_1 & .. & -p_n & -(p_n+1).

Proof of inductive step:
-(p_1 v .. v p_n v p_n+1) =
-((p_1 v .. v p_n) v p_n+1) =
-(p_1 v .. v p_n) & -(p_n+1) (De Morgan) =
-p_1 & .. & -p_n & -(p_n+1) (Induction hypothesis)

QED

5. (the purported proof of the “theorem” that, for every positive integer n, if x and y are positive integers with max(x,y)=n, then x=y.)

Evidently the theorem cannot be true. (It’s always a good idea to use your common sense as well as your technical abilities!) As for the purported proof: It rests on the assumption that x-1 and y-1 are both positive integers (since these are what the Induction Hypothesis is about). But if x and y are positive integers, x-1 and/or y-1 may not be positive. Concretely, things go wrong when max(x,y)=n=2, for example x=2, y=1. In this case, (x-1,y-1)=(1,0), so y-1 is not positive. In other words, the Induction Step does not hold.

6. See CS2013 lecture slides