

Exercises for CS2013

Nov 5, 2013

Introduction

There will be three practicals (unless modified) for the logic section of the course. The first covers Propositional Logic and part of Predicate Logic; the second covers more about Predicate Logic; the third covers Proof.

There is more on this practical than can be done in one session. It is **best** if you work through all this material **before** the practical, so that you can use the practical to ask questions or identify problems. In the practical, we will not work out all the answers to all the problems below, but only on those **selected with *****.

If you work best on your own, fine. But, if you work better with others, form peer groups. See to it that you individually know what to do and how, for at the "end of the day" the exam is you and the exam.

Propositional Logic

1. Translate into propositional logic (abbreviating 'it has rained' as r , 'it's been cold' as c and 'the plant is dead' as d). Use truth tables to determine whether one of these is logically equivalent to the other.
 - a. If it has rained and it's been cold then the plant is dead.
 - b. ***If it has rained then either it hasn't been cold or the plant is dead.
2. *Determine the status of each of these formulas (contingent, tautologous, or contradictory). Prove your case by using truth tables. (\oplus is the exclusive disjunction.)*
 - a. $p \oplus \neg p$
 - b. *** $p \oplus p$
 - c. *** $p \oplus (p \vee \neg p)$
 - d. $p \oplus (p \wedge \neg p)$
3. *Express each formula using only (at most) the connectives listed. In each case use a truth table to prove the equivalence. (Note: \oplus is exclusive 'or')*
 - a. ***Formula: $p \rightarrow q$. Connectives: $\{\neg, \vee\}$.
 - b. ***Formula: $p \oplus q$. Connectives: $\{\neg, \vee, \wedge\}$.
 - c. Formula: $p \leftrightarrow q$. Connectives: $\{\rightarrow, \wedge\}$.
 - d. Formula: $(p \rightarrow q) \wedge ((\neg p) \rightarrow q)$. Conn: $\{\neg, \vee\}$.
 - a. ***Formula: $\neg p$. Conn: $\{ | \}$ (the Sheffer stroke). The truth table for $|$ is:

| p | | q |
|----------|----------|----------|
| T | F | T |
| T | T | F |
| F | T | T |
| F | T | F |

4. Which of these are tautologies? Please prove your claims, using truth tables.
 - a. $p \rightarrow (q \rightarrow p)$
 - b. *** $p \rightarrow (\neg p \rightarrow p)$
 - c. $(q \rightarrow p) \rightarrow (p \rightarrow q)$
 - d. *** $(q \rightarrow p) \vee (p \rightarrow q)$
 - e. *** $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$
5. ***Reading formulas off truth tables. In class, a proof was sketched for the claim that every propositional logic formula can be expressed using the connectives $\{\wedge, \neg\}$. The proof proceeded essentially by “reading” the correct formula off the truth table. Use this meticulous method to construct a formula equivalent to $p \oplus q$.
6. Reading formulas off truth tables. As Exercise 3, but construct a formula equivalent to p/q . Does the method always produce the shortest answer (i.e. the shortest formula that is logically equivalent to the original while still only using negation and conjunction)?

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Predicate Logic

1. ***Suppose $\mathbf{M1} = (\mathbf{D}, \mathbf{I})$ where $\mathbf{D} = \{\text{bill, jill, mary}\}$ and $\mathbf{I}(\text{jill}') = \text{jill}$, $\mathbf{I}(\text{mary}') = \text{mary}$, $\mathbf{I}(\text{bill}') = \text{bill}$,
 $\mathbf{I}(\mathbf{B}) = \{\text{bill, jill}\}$
 $\mathbf{I}(\mathbf{G}) = \{\text{mary}\}$
 $\mathbf{I}(\mathbf{A}) = \{(\text{bill, jill}), (\text{mary, bill})\}$

Formally evaluate the following:

- Is $\exists y \mathbf{B}(y)$ true in $\mathbf{M1}$?
- Is $\exists y [\mathbf{B}(y) \wedge \mathbf{G}(y)]$ true in $\mathbf{M1}$?
- Is $\exists y [\mathbf{B}(y) \wedge \mathbf{A}(y, \text{bill}')]]$ true in $\mathbf{M1}$?

2. ***Suppose $\mathbf{M2} = (\mathbf{D}, \mathbf{I})$ where $\mathbf{D} = \{\text{bill, jill, mary}\}$ and $\mathbf{I}(\text{jill}') = \text{jill}$, $\mathbf{I}(\text{mary}') = \text{mary}$, $\mathbf{I}(\text{bill}') = \text{bill}$,
 $\mathbf{I}(\mathbf{B}) = \{\text{bill, jill}\}$
 $\mathbf{I}(\mathbf{G}) = \{\text{mary}\}$
 $\mathbf{I}(\mathbf{A}) = \{(\text{bill, jill}), (\text{mary, bill}), (\text{jill, jill})\}$

Formally evaluate the following:

- Is $\forall y \mathbf{B}(y)$ true in $\mathbf{M2}$?
- Is $\forall y [\mathbf{B}(y) \vee \mathbf{G}(y)]$ true in $\mathbf{M2}$?
- Is $\forall y [\mathbf{B}(y) \wedge \mathbf{A}(y, \text{bill}')]]$ true in $\mathbf{M2}$?
- Is $\forall x [\mathbf{B}(x) \rightarrow \mathbf{A}(x, \text{jill})]$ true in $\mathbf{M2}$?

3. Suppose $\mathbf{M3} = (\mathbf{D}, \mathbf{I})$, where $\mathbf{D} = \{\text{jill, bill, phil, will, mary}\}$ and
 $\mathbf{I}(\text{is_happy}') = \{\text{jill, bill, phil}\}$
 $\mathbf{I}(\text{is_hungry}') = \{\text{jill, bill, phil, will, mary}\}$
 $\mathbf{I}(\text{jill}') = \text{jill}$ (and so on).

Formally show with the model and analysis the following:

- $\forall x \text{is_hungry}'(x)$ and $\forall z \text{is_hungry}'(z)$ are both true in $\mathbf{M3}$.
- $\exists x \text{is_happy}'(x)$ and $\exists z \text{is_happy}'(z)$ are both true in $\mathbf{M3}$.
- $\forall x \text{is_hungry}'(x)$ entails $\exists z \text{is_hungry}'(z)$ in $\mathbf{M3}$.
- $\text{is_happy}'(x)$ and $\text{is_happy}'(z)$ are both true in $\mathbf{M3}$.

What do your observations about these examples lead you to understand about quantifiers and variables?

4. Determine the free and bound variables.

- *** $\exists x \neg P(x)$
- $\neg \forall x \neg P(x)$

- c. ******* $\exists y Q(x)$
- d. ******* $\forall x P(b)$
- e. $\forall x (\exists y R(x,y))$
- f. ******* $\forall x (\exists y R(x,z))$
- g. $\forall x \exists x \neg P(x)$
- h. ******* $\forall x (P(x) \wedge Q(x))$
- i. $\exists y Q(y) \wedge \forall x Q(x)$

5. *******Formally calculate the Truth values of the following expressions in the model:

M4: a model where $D = \{a, b\}$, $I(Q) = \{a\}$, $I(P) = \{a, b\}$.

- a. $\exists x (Q(x) \wedge P(x))$
- b. $\forall x (Q(x) \wedge P(x))$
- c. $\forall x (Q(x) \rightarrow P(x))$
- d. $\exists x (Q(x) \rightarrow P(x))$

Predicate Logic Formal Semantics

A model **M** is an ordered pair $\langle \mathbf{D}, \mathbf{I} \rangle$, where **D** is a set of entities, and **I** is an *interpretation function*. **M** defines gives a specific 'meaning' to the non-logical symbols (logical symbols are $\neg \vee \wedge \rightarrow \forall \exists$).

If *a* is an individual constant then $\mathbf{I}(a) \in \mathbf{D}$. Assume for every entity in **D** there is a constant.

If *P* is a 1-place predicate then $\mathbf{I}(P) \subseteq \mathbf{D}$

If *R* is a 2-place predicate, then $\mathbf{I}(R) \subseteq \{(\alpha, \beta): \alpha \in \mathbf{D} \text{ and } \beta \in \mathbf{D}\}$

So on for 3-place predicates etc.

A formula of the form *P(a)* is true with respect to **M** iff $\mathbf{I}(a) \in \mathbf{I}(P)$

A formula of the form *R(a,b)* is true with respect to **M** iff $(\mathbf{I}(a), \mathbf{I}(b)) \in \mathbf{I}(R)$.

A formula of the form $\neg \varphi$ is true wrt **M** iff φ is false wrt **M**.

A formula of the form $\varphi \vee \psi$ is true wrt **M** iff φ is true wrt **M** or ψ is true wrt **M** or both.

A formula of the form $\varphi \wedge \psi$ is true wrt **M** iff φ is true wrt **M** and ψ is true wrt **M**.

A formula of the form $\varphi \rightarrow \psi$ is true wrt **M** iff φ is false wrt **M** or ψ is true wrt **M**.

A formula of the form $\exists x \varphi$ is true wrt **M** iff there is an $\mathbf{I}(a) \in \mathbf{D}$ such that $\varphi(x:=a)$ is true wrt **M**.

A formula of the form $\forall x \varphi$ is true wrt **M** iff for every $\mathbf{I}(a) \in \mathbf{D}$, $\varphi(x:=a)$ is true wrt **M**.