Functions

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Reading: relevant chapters of any book on Discrete Maths.
For example, Rosen 7th ed.

Functions

- From calculus, you know the concept of a real-valued function *f*,
  which assigns to each number $x \in \mathbf{R}$
  one particular value $y = f(x)$, where $y \in \mathbf{R}$.
- *Example*: *f* defined by the rule $f(x) = x^2$
- Roughly, functions say “the so-and-so of …”
- Functions are also called operations, mappings, etc.

Functions

- To understand functions more precisely, one needs the mathematical notion of a set
- We assume you are familiar with “naïve set theory” (as opposed to axiomatic set theory).
- In a nutshell:

Reminder of main set concepts

- $\cup$, $\cap$, -, $\in$, $\emptyset$, $\overline{S}$
- =, $\subseteq$, $\subseteq$, $\subset$, $\supset$, $\not\subset$, etc.
- {a,b,...} (def. of a set by enumeration)
  $\{x \mid P(x)\}$ (def. by set builder notation)
- $x \in S$, $S \subseteq T$, $S = T$, $S \subset T$.
- **P**(S) (power set of S),
- A $\times$ B (Cartesian product of A and B)

Reminder of main set concepts

- Important sets of numbers:
  - N are the natural numbers {1, 2, 3, 4, ...}
  - Z are the integers {...-3, -2, -1, 0, 1, 2, 3, ...}
  - Q are the rational numbers $\{x/y: x \in Z, y \in N\}$
  - Irrational numbers cannot be expressed as in Q (factions), e.g. π.
  - R are the real numbers, the rational and irrational
- A relation on A is a subset of A x A. e.g., on, N, the relation < is {(0,1),(0,2), (1,2),…}
- Set equality proof techniques, e.g., to prove A=B, prove each of: $A \subseteq B$ and $B \subseteq A$

Function: formal definition

- A *function (or "mapping") f from A to B, written f* : $A \rightarrow B$, is an assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Generalisations:
  - Functions of *n* arguments:
    f: $(A_1 \times A_2 ... \times A_n) \rightarrow B$.
  - A *partial* (non-*total*) function *f* assigns *zero or one* elements of *B* to each element $x \in A$. In such a case, it is possible that there are elements of A that are not mapped to any element of B. If we say just "function" in the following, we mean a total function.

Functions precisely

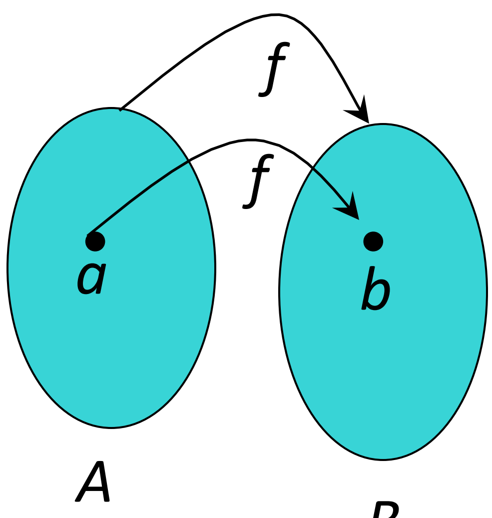
- We can represent a function $f : A \rightarrow B$ as a set of ordered pairs $f = \{(a, f(a)) \mid a \in A\}$. This makes f a relation between A and B: f is a subset of A x B.
- But (total) functions are special:
  - for every $a \in A$, there is at least one pair (a,b).

    Formally: $\forall a \in A\ \exists b \in B\ ((a,b) \in f)$
  - for every $a \in A$, there is at most one pair (a,b).

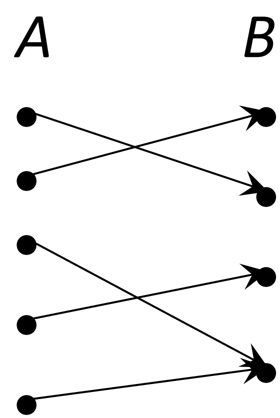
    Formally: $\neg \exists a,b,c\ ((a,b) \in f \wedge (a,c) \in f \wedge b \neq c)$
- A relation over numbers can be represented as a set of points on a plane. (A point is a pair (x,y))
  - A function is then a curve (set of points), with only one y for each x.

Useful diagrams

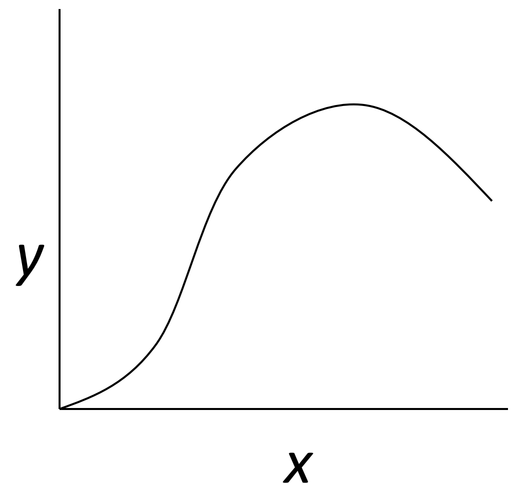
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Familiar functions

- A *set* S over universe U can be viewed as a function from the elements of U to …

Familiar functions

- A *set S* over universe *U* can be viewed as a function from the elements of *U* to …

  … {**T**, **F**}, saying for each element of U whether it is in S or not. This is called the characteristic function of *S*.

  Suppose U={0,1,2,3,4}. Then:

  $S = \{1,3\}$ is

  $S(0) = S(2) = S(4) =$ **F**.

  $S(1) = S(3) =$ **T**.

Familiar functions

- A *set operator,* such as $\cap$ or $\cup$, can be viewed as a function from … to …

Familiar functions

- A *set operator* such as $\cap$ or $\cup$ can be viewed as a function …

… from (ordered) pairs of sets to sets.

Example: $\cap(\{1,3\},\{3,4\}) = \{3\}$

A new notation

- Y^X is the set F of *all* possible functions f: $X \to Y$.
- Thus, $\boldsymbol{f} \in Y^X$, where $\boldsymbol{f}$ (bold-italic) is a particular f *(italic),* is another way of saying $\boldsymbol{f}$: $X \to Y$.
- This notation is especially appropriate, because for finite X, Y, we have $|F| = |Y|^{|X|}$; that is, the number of functions in F is the number of elements in Y to the power of the number of elements in X.

Some function terminology

- If $f : A \rightarrow B$ and $f(a) = b$, where $a \in A$ & $b \in B$, then we say:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
- In general, b may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of f is $R = \{b \mid \exists a\, f(a) = b\}$.

We also say the *signature* of f is $A \rightarrow B$.

Range versus Codomain

- The range of a function may not be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Choosing the right (co)domain

Consider the function f such that f (x) = 100/x

Is f a (total) function from Int to R?

- f is a partial function from Int to R
- f is a (total) function from Int-{0} to R

Consider the function g such that g(x) = √x

Is g a (total) function from R to R?

- g is a total function from R+ to R x R, e.g. g(4)= (2,-2)

Images of sets under functions

- Given f: $A \rightarrow B$, and $S \subseteq A$,
- The *image* of S under f is the set of all images (under f) of the elements of S.

  $f(S) :\equiv \{f(s) \mid s \in S\}$

  $\qquad :\equiv \{b \mid \exists s \in S: f(s) = b\}$.
- The range of f equals the image (under f) of ...

- $:\equiv$ means 'defined as'.

Images of sets under functions

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- The *image* of S under f is the set of all images (under f) of the elements of S.

  $f(S) :\equiv \{f(s) \mid s \in S\}$

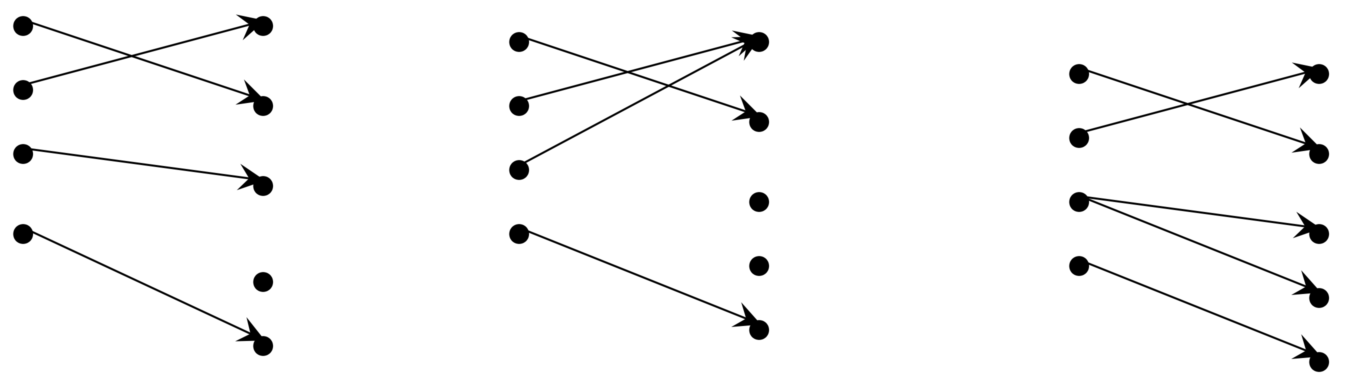
  $:\equiv \{b \mid \exists s \in S\text{: } f(s) = b\}$.
- The range of f equals the image (under f) of the domain of f.

One-to-one functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given f: $A \rightarrow B$, "f is injective" $:\equiv (\neg\exists x,y\colon x \neq y \wedge f(x) = f(y))$.
- In other words: only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - In this case, domain and range have the same cardinality.
- What about codomain? It may be larger.

One-to-one illustration

- Are these relations one-to-one functions?



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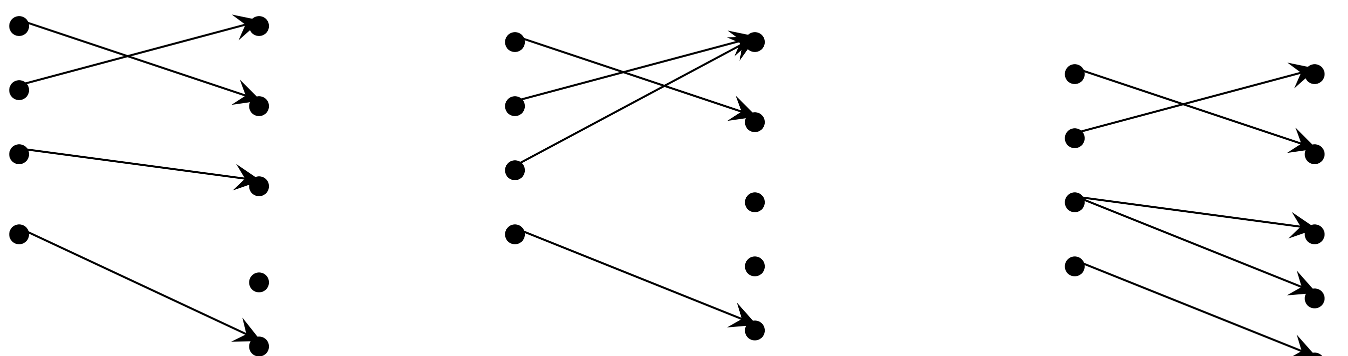

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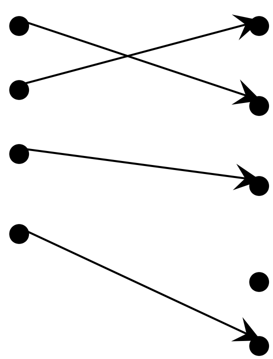
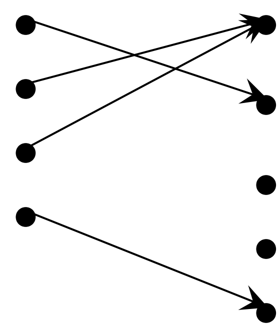
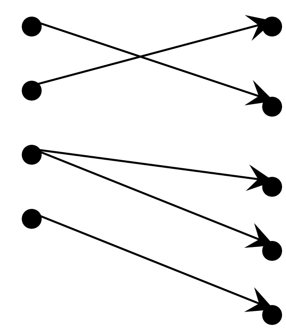


One-to-one



Not one-to-one



Not even a function!

Sufficient conditions for 1-1ness

- For functions f over numbers, we say:
  - f is *strictly increasing* iff $x > y \rightarrow f(x) > f(y)$ for all x,y in domain;
  - f is *strictly decreasing* iff $x > y \rightarrow f(x) < f(y)$ for all x,y in domain;
- If f is either strictly increasing or strictly decreasing, then f must be one-to-one.
  - Does the converse hold?

Onto (surjective) functions

- A function f: $A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain

  $\forall b \in B, \exists a \in A: f(a) = b.$

- Consider *"country of birth of"*: A → B,
  where A=people, B=countries.
  Is this a function?
  Is it an injection?
  Is it a surjection?

Onto (surjective) functions

- A function f: $A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain
- Consider *"country of birth of"*: A → B,
  where A=people, B=countries.
  Is this a function? Yes (always 1 c.o.b.)
  Is it an injection? No (many have same c.o.b.)
  Is it a surjection? Probably yes (every country is the country of birth of someone, but...)

Onto (surjective) functions

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- In predicate logic:

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Onto (surjective) functions

- A function f: $A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b \in B \; \exists a \in A \; f(a) = b$).
- *e.g.*, for domain and codomain Z, the function $f(x) = x+1$ is injective and surjective.

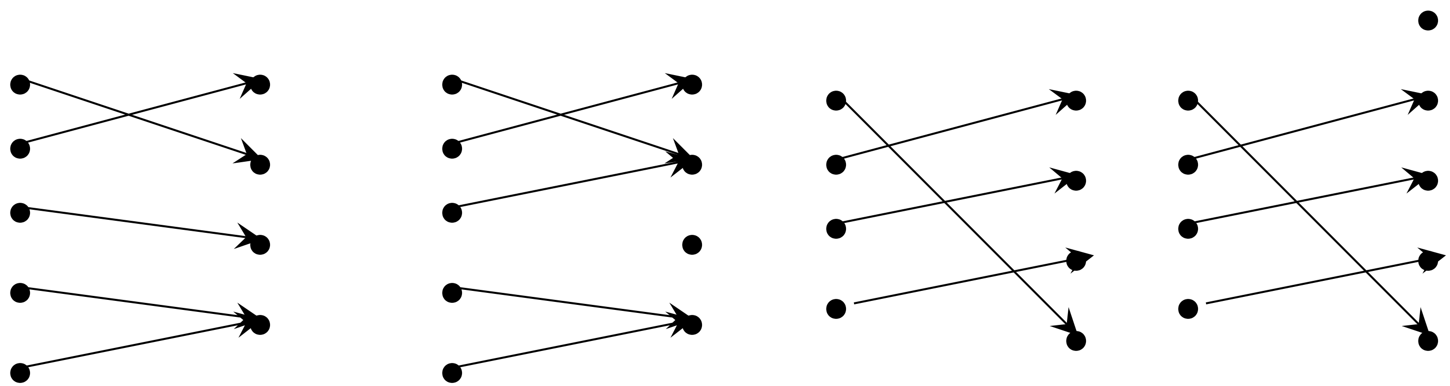
Example

Claim: if f: Z → Z and f(x) = x + 1, then f is 1-to-1 and also onto, where **Z** is the set of **all** integers

- Proof that *f is onto*: Consider any arbitrary element a of Z. We have f(a - 1) = a, where a ∈ Z.
- Proof that *f is 1-to-1*: Suppose f(u) = f(w) = a. In other words, u + 1 = a and w + 1 = a. It follows that u = w.

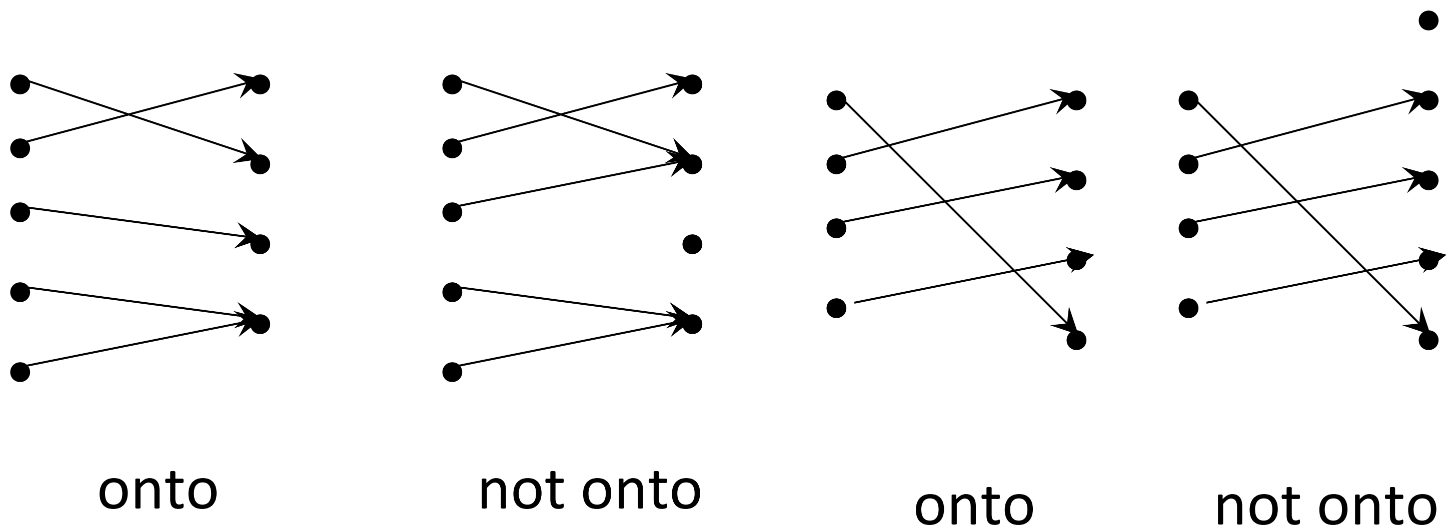
Onto/surjective functions

- Are these functions *onto* their depicted co-domains?



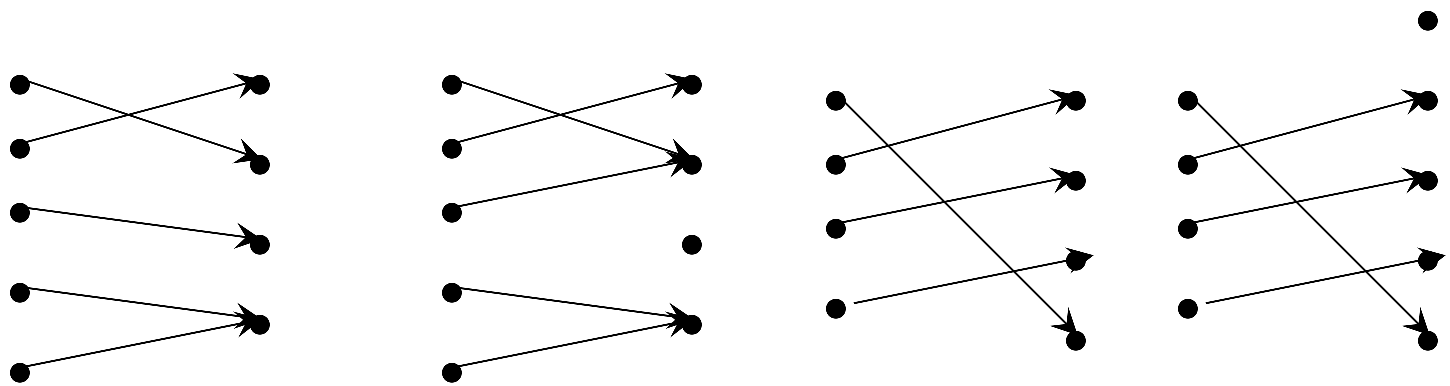
Onto/surjective functions

- Are these functions *onto*?



1-1/injective functions

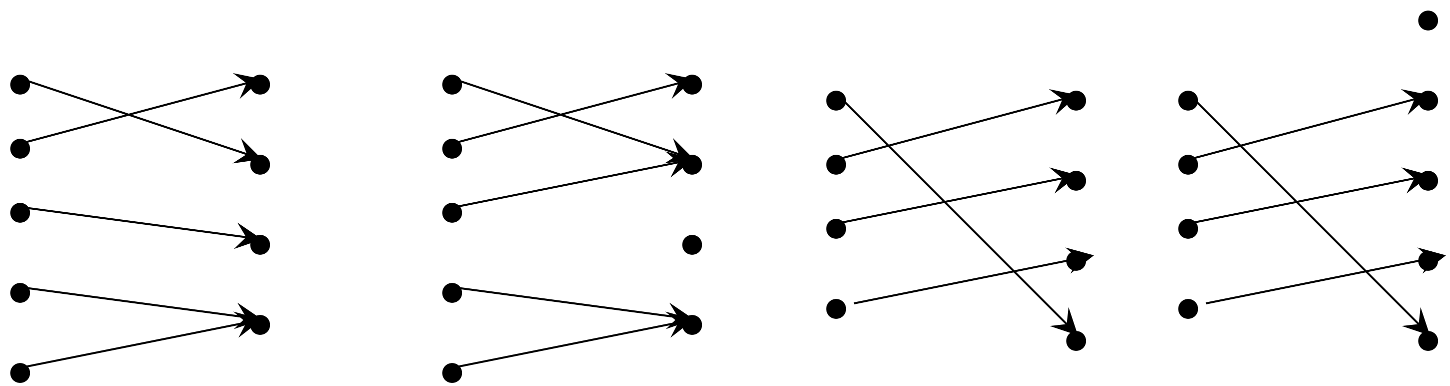
- Are these functions 1-1?



onto   not onto   onto   not onto

1-1/injective functions

- Are these functions 1-1?



not 1-1
onto

not 1-1
not onto

1-1
onto

1-1
not onto

Bijections

- A function is said to be *a one-to-one* ***correspondence***, or *a bijection* iff it is both one-to-one and onto.

Two terminologies for talking about functions

1. injection = one-to-one
2. surjection = onto
3. bijection = one-to-one correspondence

3 = 1 & 2

Bijections

- For bijections $f{:}A \rightarrow B$, there exists a function that is the *inverse* of f, written $f^{-1}: B \rightarrow A$
- Intuitively, this is the function that undoes everything that f does
- Formally, it's the unique function such that

  ...

Bijections

- For bijections $f{:}A \rightarrow B$, there exists an *inverse* of f, written f^{-1}: $B \rightarrow A$
- Intuitively, this is the function that undoes everything that f does
- Formally, it is the unique function such that
  - *f composed with* f^{-1} is the identity function on A, I_A

$$f^{-1} \circ f = I_A$$

  - A function f composed with a function g, f 0 g, is a function where, applied to an argument x, (f o g)(x) = (f(g(x)).
  - The identity function simply returns the input value.

Bijections

- Example 1: Let f: $\mathbf{Z} \rightarrow \mathbf{Z}$ be defined as $f(x) = x + 1$. What is f^{-1} ?
- Example 2: Let g: $\mathbf{Z} \rightarrow \mathbf{N}$ be defined as $g(x) = |x|$. What is g^{-1} ?

Bijections

- Example 1: Let f: **Z** $\rightarrow$ **Z** be defined as f(x) = x + 1. What is f^{-1} ?
- f^{-1} is the function (let's call it h), where h: **Z** $\rightarrow$ **Z** defined as h(x) = x - 1.
- Proof:

$$h \circ f = I$$

h(f(x)) = (x + 1) - 1 = x

Bijections

- Example 2: Let g: $\mathbf{Z} \rightarrow \mathbf{N}$ be defined as $g(x) = |x|$. What is g^{-1} ?
- This was a trick question: there is no such function, since g is not a bijection: There is no function h such that $h(|x|) = x$ and $h(|x|) = -x$
- (NB There is a relation h for which this is true.)

Cardinality (informal)

- The cardinality of a finite set is its number of elements
- E.g., card({a,b,c}) = card({e,f,g}) = 3
- Note: for finite sets X and Y, card(X) = card(Y) if and only if there exists a bijection between X and Y.