Further exercises for the tutorials on logic in CS 3511
(Except where indicated, informal proofs will suffice.)

1. Find a counterexample, if possible, to these statements, where the u.d. consists of all (negative and non-negative) integers:

a. $\forall x \forall y (x^2 = y^2 \rightarrow x = y)$
b. $\forall x \exists y (y^2 = x)$
c. $\forall x \forall y (xy \geq x)$

2. An advert in an Aberdeen supermarket says: "All frozen foods reduced by up to 70%".
a) Is the sentence false if all frozen foods have been reduced by 1%?
b) Propose one or more logical translations of this sentence

3. Use quantifiers to express the associative laws for multiplication of real numbers.

4. Show, using a sequence of equivalences, that the following two statements must have the same truth value:

a) $\neg \exists x \forall y (P(x, y)$
b) $\forall x \exists y \neg P(x, y)$

5. A number is called an upperbound (UB) of a set S of numbers iff it is greater than or equal to every member of S. The number x is called the *least upper bound* (LUB) of a set S of real numbers iff x is an upper bound of S and x is less than or equal to every upper bound of S. (Note that in both cases, x may or may not be a member of S.)

a.) Using quantifiers, say that x is an upper bound of S.
b.) Using quantifiers, say that x is a LUB of S. (This time, feel free to write '$UB(y)$' as short for 'y is an Upper Bound of S'.)
c.) Prove that a set of real numbers can have at most one LUB.
d.) Give an example of a set of real numbers that has a LUB.
e.) Give an example of a set of real numbers that does not have an LUB.
f.) Give an example of a set of rational numbers that has a LUB and contains it (i.e., the LUB is a member of the set)
g.) Give an example of a set of rational numbers that has a LUB but does not contain it.
h.) The notion of an LUB is based on the relation 'greater than'. Can you propose an analogous notion based on the relation 'smaller than'?

For proving that φ and ψ are equivalent, often the best way is to prove separately the implication from left to right ($\varphi \Rightarrow \psi$) and the implication from right to left ($\psi \Rightarrow \varphi$):

6. Show that the following two statements are *not* logically equivalent. Do this by describing (e.g., drawing) a model in which one of the two is true and the other false. (Say also which of the two formulas is true and which is false in the model.)

I.a $\exists x F(x) \land \exists x G(x)$
I.b $\exists x (F(x) \land G(x))$

II.a $\forall x \exists y (R(x, y))$
II.b $\exists y \forall x (R(x, y))$
Hint: when drawing the model, draw an arrow going from a to b to say that $R(a, b)$.

III.a $\forall z F(z)$
III.b $\forall z F(z) \land \exists z F(z)$

IV.a $\exists x \exists y \exists z (x \neq y \land y \neq z)$

IV.b $\exists x \exists y \exists z (x \neq y \wedge y \neq z \wedge x \neq z)$

V.a $\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$
V.b $\exists x \exists y (x \neq y \wedge (\forall z (z = x) \vee \forall z (z = y)))$

VI.a $\forall x \exists y (R(x, y))$
VI.b $\forall x \exists y (R(x, y) \wedge x \neq y)$

VII.a $\forall x \forall y ((P(x) \wedge P(y)) \rightarrow x = y)$
VII.b $\exists xy (P(y) \wedge P(x) \wedge \forall z (z = x \vee z = y))$

VIII.a $\exists x P(x) \wedge \exists x \neg P(x)$
VIII.b $\exists x (P(x) \wedge \neg P(x))$.

7. Show that the following two statements are logically equivalent. Use an informal proof style, based on the meaning of quantified and disjoined formulas.

a) $\forall x P(x) \vee \forall x Q(x)$
b) $\forall x \forall y (P(x) \vee Q(y))$. Why does (b) involve two variables?
c) Construct a model showing that (a) is not logically equivalent to $\forall x (P(x) \vee Q(x))$

8. You will have noticed that it often does not make a difference which variables you choose in your formulas.
a) Give an example of two formulas that differ only in their use of variable, and which are logically equivalent.
b) Give a few examples of two formulas that differ only in their use of variables, and which are not logically equivalent. What different reasons can you see why such pairs of formulas may have different meanings?
c) Formulate a rule that says how any given formula may be transformed into another formula that differs from it in the choice of variables only, such that the result is always logically equivalent to the original.

9. Which of the following arguments are valid? For the ones that are valid, write **formal** proofs (using Natural Deduction- style inference rules):

a.) $p \rightarrow r$ and $r \rightarrow s$, and $\neg s$ therefore $\neg p$
b.) $((p \vee q) \vee r)$ and $\neg q$ and $\neg r$ therefore p
c.) $\forall x \forall y R(x, y)$, therefore $\forall y \forall x R(y, x)$

10. How would you express
10a) in a domain with 5 objects: “Most things have the property P”. (Here it’s ok to use the usual abbreviations for things like “exactly 2”.)
10b) “There are exactly three things that have the property P or the property Q (or both)”. (In this execrcise no abbreviations are allowed.)