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Article *in* SIAM Review · January 1981

DOI: 10.1137/1023006

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SIAM REVIEW
Vol. 23, No. 1, January 1981

0036-1445/81/2301-0006 $01.00/0

A MATHEMATICAL MODEL FOR TRAILER-TRUCK JACKKNIFING*

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Abstract. We present a differential equation which is a model for the position of a trailer relative to the cab which is pulling it. The solution is given for two examples, and the results are generalized in a theorem.

Suppose that a cab is pulling a trailer which is d units long. With suitable scaling, d can be taken to be 1. We can represent the positions of the cab and trailer by two vectors. Let $\mathbf{X}$ be a position vector whose terminal point is at the trailer hitch on the cab, and let $\mathbf{Y}$ be a position vector whose terminal point is at the midpoint between the wheels of the trailer. We represent the truck–trailer combination as shown in Fig. 1. $\mathbf{X}$ will be a given, "smooth" function of time t. We would like to be able to predict $\mathbf{Y}$. That is, for a given path $\mathbf{X}=\mathbf{X}(t)$, we want to know if the truck–trailer will jackknife. Alternatively, we want to determine what conditions we must impose on $\mathbf{X}(t)$ to prevent jackknifing.

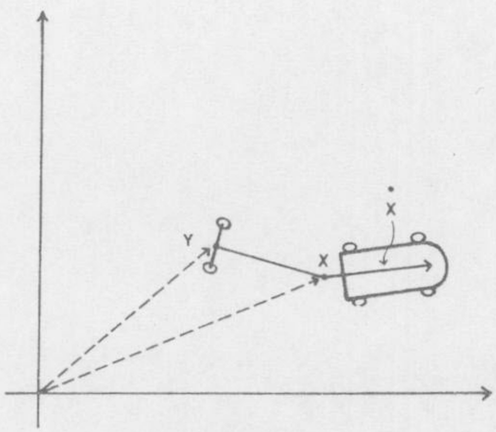

FIG. 1

We model the motion of $\mathbf{Y}$ as follows. First, the trailer length $|\mathbf{X}-\mathbf{Y}|=1$ is constant, which shows that

$$(1) \qquad (\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}-\mathbf{Y})=1.$$

where $\cdot$ is the vector dot product. Also, the wheels of the trailer constrain the vector $\mathbf{Y}$ so that its velocity vector is directed along the trailer's lateral axis $\mathbf{X}-\mathbf{Y}$. That is,

$$(2) \qquad \dot{\mathbf{Y}}=\lambda(\mathbf{X}-\mathbf{Y})$$

for some λ, where $\dot{\mathbf{Y}}=d\mathbf{Y}/dt$.

Differentiating (1) yields $2(\dot{\mathbf{X}}-\dot{\mathbf{Y}})\cdot(\mathbf{X}-\mathbf{Y})=0$ and so

$$(3) \qquad \dot{\mathbf{X}}\cdot(\mathbf{X}-\mathbf{Y})=\dot{\mathbf{Y}}\cdot(\mathbf{X}-\mathbf{Y}).$$

* Received by the editors April 9, 1979, and in revised form August 25, 1979.

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Taking the dot product of (2) on both sides by $\mathbf{X}-\mathbf{Y}$ yields $\dot{\mathbf{Y}}\cdot(\mathbf{X}-\mathbf{Y})=\lambda(\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}-\mathbf{Y})=\lambda$, and so by (3), $\lambda=\dot{\mathbf{X}}\cdot(\mathbf{X}-\mathbf{Y})$. Therefore, (2) becomes

$$\dot{\mathbf{Y}}=[\dot{\mathbf{X}}\cdot(\mathbf{X}-\mathbf{Y})](\mathbf{X}-\mathbf{Y}). \tag{4}$$

We introduce Cartesian coordinates to describe $\mathbf{X}$ and $\mathbf{Y}$. Let $\mathbf{X}=(x_1, x_2)$ and $\mathbf{Y}=(y_1, y_2)$. Equation (4) can be written as the system

$$\begin{aligned} \dot{y}_1&=x_1^2\dot{x}_1-2x_1\dot{x}_1y_1+x_1x_2\dot{x}_2-x_1\dot{x}_2y_2+\dot{x}_1y_1^2-x_2\dot{x}_2y_1+\dot{x}_2y_2y_1,\\ \dot{y}_2&=x_1\dot{x}_1x_2-\dot{x}_1x_2y_1+\dot{x}_2x_2^2-2x_2\dot{x}_2y_2-x_1\dot{x}_1y_2+\dot{x}_1y_1y_2+\dot{x}_2y_2^2. \end{aligned} \tag{5}$$

For convenience, let $\mathbf{Z}=\mathbf{X}-\mathbf{Y}$, so that (4) can be rewritten as

$$\dot{\mathbf{Y}}=(\dot{\mathbf{X}}\cdot\mathbf{Z})\mathbf{Z}. \tag{4'}$$

If the cab is moving forward, we say the cab and trailer are *jackknifed* if $\dot{\mathbf{X}}\cdot\mathbf{Z}<0$; otherwise, we say they are *unjackknifed*. These two configurations are illustrated in Fig. 2. If the cab is backing up, the above situation is reversed. That is, the cab and trailer are *jackknifed* if $\dot{\mathbf{X}}\cdot\mathbf{Z}>0$ and *unjackknifed* otherwise; see Fig. 3.

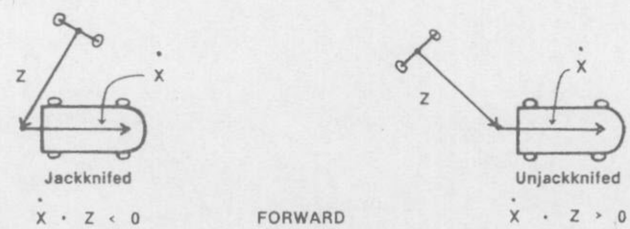

FIG. 2

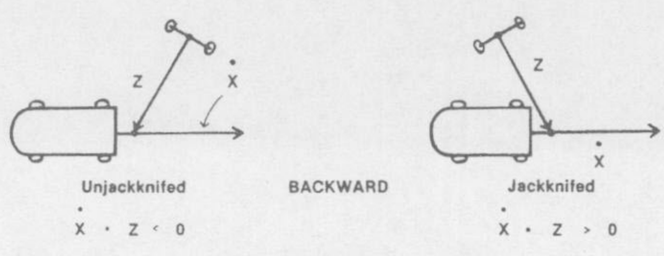

FIG. 3

Example 1. Consider the cab moving forward in a straight line. We may assume the cab travels along the positive x-axis, with $\mathbf{X}(t)=(t, 0)$. Since $|\mathbf{Z}|=1$, the point $\mathbf{Z}(0)=\mathbf{X}(0)-\mathbf{Y}(0)=-\mathbf{Y}(0)$ must lie on the unit circle, so we may assume $Y(0)=(\cos\alpha, \sin\alpha)$ for some real α. Then the system (5) becomes

$$\dot{y}_1=t^2-2ty_1+y_1^2=(t-y_1)^2, \qquad \dot{y}_2=-ty_2+y_1y_2,$$

whose solution is

$$\begin{aligned} y_1(t)&=t+\frac{e^{-2t}+C_1}{e^{-2t}-C_1},\\ y_2(t)&=C_2\exp\left\{\int_0^t (y_1(s)-s)\,ds\right\}. \end{aligned}$$

Applying the initial conditions, $y_1(0)=\cos\alpha$, $y_2(0)=\sin\alpha$, we obtain

$$C_1=\frac{\cos\alpha-1}{\cos\alpha+1},\qquad C_2=\sin\alpha.$$

(Note: If $\alpha=2n\pi$, $C_1=0$. If $\alpha=(2n+1)\pi$, $C_1=\infty$). Thus,

$$y_1(t)=\begin{cases} t+1 & \text{if } \alpha=2n\pi,\\ t-1 & \text{if } \alpha=(2n+1)\pi,\\ t-1+o(1) & \text{if } \alpha\neq n\pi \end{cases}$$

and

$$y_2(t)=\frac{(C_1-1)\sin\alpha}{C_1e^t-e^{-t}}=\begin{cases} 0 & \text{if } \alpha=n\pi,\\ 0+o(1) & \text{if } \alpha\neq n\pi. \end{cases}$$

This solution shows that unless the cab and trailer start in the (rather unrealistic) completely jackknifed position with $\alpha=2n\pi$, the trailer will approach the position following the cab. That is, $\alpha=2n\pi$ is an unstable initial condition with solution $\mathbf{Y}(t)=(t+1,0)$, while $\alpha\neq 2n\pi$ as an initial condition leads to the stable limiting solution $\mathbf{Y}(t)=(t-1,0)$.

Alternatively, we can consider the similar example in which the cab is backing up. In this situation, the solution shows that, *except* for the unstable initial condition $\alpha=2n\pi$, which corresponds to the trailer directly behind the cab, *all* solutions ultimately approach the jackknifed position. Anyone who has attempted to back up a vehicle with a trailer can attest to this fact.

Example 2. Consider the cab traveling along a circle of radius r. For the moment, assume $r>1$. We may assume $\mathbf{X}(t)=(r\cos t, r\sin t)$. Again, since $\mathbf{Z}(0)=\mathbf{X}(0)-\mathbf{Y}(0)$ has length 1, we may assume $\mathbf{Y}(0)=\mathbf{X}(0)+(\cos\alpha,\sin\alpha)$. Intuition, aided by computer graphics, tells us that the trailer should approach an asymptotically stable state. It seems reasonable that the cab–trailer combination should approach the configuration shown in Fig. 4. In fact, direct substitution into (4) or (5) shows that the function

$$\begin{aligned}\mathbf{Y}_1(t)&=c(\cos(t-\theta),\sin(t-\theta))\\ &=(y_{11},y_{21})\end{aligned}\tag{6}$$

is a solution, where

$$c=\sqrt{r^2-1},\quad \sin\theta=1/r,\quad \cos\theta=c/r.$$

In the usual terminology of phase plane analysis, $\mathbf{Y}_1(t)$ represents a periodic solution, and the path of $\mathbf{Y}_1(t)$ represents a stable limit cycle. We verify this in the following way.

The differential equations (5) become

$$\begin{aligned}\dot y_1&=r(y_1\sin t-y_2\cos t)(r\cos t-y_1),\\ \dot y_2&=r(y_1\sin t-y_2\cos t)(r\sin t-y_2).\end{aligned}\tag{7}$$

We introduce new variables by letting

$$u=y_1\sin t-y_2\cos t,\qquad v=y_1\cos t+y_2\sin t.\tag{8}$$

The differential equations for u and v are

$$\dot u=-ru^2+v,\qquad \dot v=(r^2-rv-1)u.\tag{9}$$

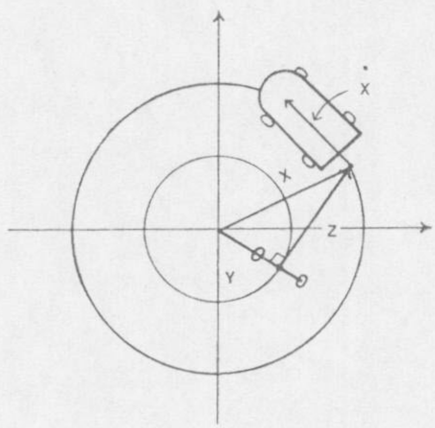

FIG. 4

This system has three critical points; $(0, 0)$, $(c/r, c^2/r)$ and $(-c/r, c^2/r)$, all of which correspond to actual solutions.

The first can be eliminated from consideration, since $(u, v)=(0, 0)$ implies $(y_1, y_2)=(0, 0)$. If this were the case, then $1=|\boldsymbol{X}-\boldsymbol{Y}|=|\boldsymbol{X}|=r>1$, a contradiction. It should be noted that $(0, 0)$ is a saddle point for (9).

The second critical point corresponds to the solution $\mathbf{Y}_1$. The linearized version of (9) about this solution is

$$\dot{w}=z-2cw, \qquad \dot{z}=-cz,$$

where $w=u-c/r$, $z=v-c^2/r$. Both characteristic values ($-c$ and $-2c$) of this system are negative. Therefore, by standard theorems of phase plane analysis (see, for example, [1, Chapt. 11]), the solution $(c/r, c^2/r)$ of (9) is stable, and all other solution paths which start close to it will approach it. The point $(c/r, c^2/r)$ is a stable node for (9).

The third critical point corresponds to another solution

$$\mathbf{Y}_2(t)=c(\cos(t+\theta), \sin(t+\theta))$$

of (7). Analysis similar to that given above shows that this is an unstable (node) solution. In fact, it represents the same orbit as that for $\mathbf{Y}_1$. In this case, the cab–trailer combination is in a jackknifed position initially and will remain in that position. However, any deviation from that initial position will cause the cab–trailer to wander farther away from the initial configuration.

Now, any solution of (4) must satisfy $r-1 \leqq |\mathbf{Y}| \leqq r+1$ since $|\mathbf{X}-\mathbf{Y}|=1$ and $|\mathbf{X}|=r$. Hence, any physically meaningful solution of (9) other than $(-c/r, c^2/r)$ and $(0, 0)$, must approach $(c/r, c^2/r)$. Any bounded solution cannot approach $(-c/r, c^2/r)$ or $(0, 0)$, and there cannot be another periodic solution (limit cycle). If there were, there would have to be one enclosing none of the previously mentioned critical points, and thus there would have to be more critical points. Thus, all bounded solutions of (7) approach the stable periodic solution $\mathbf{Y}_1$, with the exception of $(-c/r, c^2/r)$ and $(0, 0)$. In terms of our original model, the trailer approaches the periodic solution of an unjackknifed trailer shown in Fig. 4. We assume, of course, that the trailer is free to rotate 360° around the hitch without the cab getting in the way. In general, as long as $\mathbf{Y}(0) \neq \mathbf{Y}_2(0)$, then $\mathbf{Y}(t) \to \mathbf{Y}_1(t)$ as $t \to \infty$.

As before, if the cab is backing up, then the jackknifed solution $\mathbf{Y}_2(t)$ is stable, and all others, except $\mathbf{Y}_1(t)$, approach it. The solution $\mathbf{Y}_1(t)$ is the only solution in which the cab–trailer combination remains unjackknifed, yet it is unstable.

In the above example, we assumed that $r>1$. If $r=1$, then $\mathbf{Y}(t)\to(0,0)$. If $r<1$, then the cab–trailer will enter a jackknifed position, even if it started in an unjackknifed position. This will hold whether the cab is going forward or backing up.

We gain some insight into the general situation from the above example. In particular, if r is too small in Example 2, the cab–trailer jackknifes. From this, we postulate the following theorem.

THEOREM. *Assume* $\mathbf{X}$ *is twice continuously differentiable. If the length of the trailer is* 1, *if* $1<r(t)$, *where* $r(t)$ *is the radius of curvature of* $\mathbf{X}$, *and if* $\dot{\mathbf{X}}(0)\cdot\mathbf{Z}(0)>0$, *then* $\dot{\mathbf{X}}(t)\cdot\mathbf{Z}(t)>0$ *for all* $t>0$.

In other words, if the cab is moving forward and the cab–trailer combination is not originally jackknifed, then it will remain unjackknifed. On the other hand, if the cab is moving backward and the cab–trailer is originally jackknifed, then it will remain jackknifed.

Proof. Let $f(t)=\dot{\mathbf{X}}\cdot\mathbf{Z}$. Then $f(0)>0$, and f is continuously differentiable. Suppose the conclusion of the theorem is false. Then there exists $t_1>0$ such that $f(t_1)=0$ and $f'(t_1)\leqq 0$. Assuming that $|\mathbf{Z}|=|\mathbf{X}-\mathbf{Y}|=1$ (trailer has length 1) and $|\dot{\mathbf{X}}|\neq 0$ (cab doesn't stop), we have $\dot{\mathbf{X}}(t_1)\cdot\mathbf{Z}(t_1)=0$ so that $\dot{\mathbf{X}}(t_1)\perp\mathbf{Z}(t_1)$. Also,

$$\begin{aligned}
f'(t)&=\dot{\mathbf{X}}\cdot\dot{\mathbf{Z}}+\ddot{\mathbf{X}}\cdot\mathbf{Z}\\
&=\dot{\mathbf{X}}\cdot(\dot{\mathbf{X}}-\dot{\mathbf{Y}})+\ddot{\mathbf{X}}\cdot\mathbf{Z}\\
&=|\dot{\mathbf{X}}|^2-\dot{\mathbf{X}}\cdot\dot{\mathbf{Y}}+\ddot{\mathbf{X}}\cdot\mathbf{Z}\\
&=|\dot{\mathbf{X}}|^2-(\dot{\mathbf{X}}\cdot\mathbf{Z})\mathbf{Z}\cdot\dot{\mathbf{X}}+\ddot{\mathbf{X}}\cdot\mathbf{Z}\\
&=|\dot{\mathbf{X}}|^2-(\dot{\mathbf{X}}\cdot\mathbf{Z})^2+\ddot{\mathbf{X}}\cdot\mathbf{Z}\\
&=|\dot{\mathbf{X}}|^2-f^2(t)+\ddot{\mathbf{X}}\cdot\mathbf{Z}
\end{aligned}$$

and

$$f'(t_1)=|\dot{\mathbf{X}}(t_1)|^2+\ddot{\mathbf{X}}(t_1)\cdot\mathbf{Z}(t_1).$$

We now use a familiar formula for acceleration (see, for example, [2, p. 423]):

$$\ddot{\mathbf{X}}=\frac{d|\dot{\mathbf{X}}|}{dt}\mathbf{t}+\kappa|\dot{\mathbf{X}}|^2\mathbf{n}\qquad\left(\kappa=\frac{1}{r}\right),$$

where $\mathbf{t}$ and $\mathbf{n}$ are unit tangent and normal vectors respectively. Then $\mathbf{t}(t_1)$ is parallel to $\dot{\mathbf{X}}(t_1)$, and $\mathbf{Z}(t_1)$ is parallel to $\mathbf{n}(t_1)$. Therefore, $\ddot{\mathbf{X}}(t_1)\cdot\mathbf{Z}(t_1)=\pm\kappa(t_1)|\dot{\mathbf{X}}(t_1)|^2$, and

$$\begin{aligned}
f'(t_1)&=|\dot{\mathbf{X}}(t_1)|^2\pm\kappa(t_1)|\dot{\mathbf{X}}(t_1)|^2\\
&=(1\pm\kappa(t_1))|\dot{\mathbf{X}}(t_1)|^2.
\end{aligned}$$

We conclude that $f'(t_1)>0$, since $\kappa(t_1)=1/r(t_1)<1$. This contradicts our assumption that $f'(t_1)\leqq 0$, and the theorem is proved.

Acknowledgment. The authors would like to express their thanks to Professor Otto Ruehr for his valuable assistance.

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