MATHEMATICAL MODELING OF THE KINEMATICS OF VEHICLES

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1 Introduction

The availability of various basic technologies drives forward research and development in the scope of autonomous transportation systems. One area of application are transportation operations carried out by vehicles, particularly by single trucks [ZP99] or trucks with trailers [ZB00]. A variety of scientific and technical topics have to be considered to control autonomous movements of such vehicles. One of these topics is the kinematic model of the movements of a truck and of a truck with one axle trailer which will be enrolled here.

Lets develop a scenario of application for autonomously executed transportation operations. At the moment there is no end to see in the augmentation of the volume of transportation. Also the distances that the goods are covering from their starting point to their destination are steadily increasing. Rarely the entire distance of transport is covered in one hop. Instead, the majority of goods is transported in a store-and-forward manner using logistics agencies for intermediate storage. There the incoming goods are newly grouped to form transportation units for the next hop. The administrative part of transport logistics from path planning to tracking the storing position of goods is profoundly supported by information technology. However, the same profoundness of technological support is not given for the automation of the physical transport systems.

A major portion of transportation is carried out by road vehicles, that is to say by trucks without and with trailers. In this case a hop from one logistics agency to the next can typically bee subdivided into the following operations:

1. grouping of goods in the logistics agency

2. driving the vehicle to a certain charging position, in the following called *ramp*

3. charging of the vehicle at the ramp position

4. driving the vehicle to a readiness position waiting for the departure from the logistics agency

5. departure from one logistics agency and driving to the next one

6. arrival of the truck at the logistics agency and placing the vehicle in a readiness position waiting for discharging purposes

7. driving the vehicle to the ramp

8. discharging of the vehicle at the ramp and intermediate storage of the goods in the logistics agency

9. driving the truck in a readiness position waiting for charging purposes

The operations 2, 4, 7, and 9 happen in the well known environment of the logistics center, the haulier's yard. They are the most likely to be entirely automated. This makes sense because of various reasons:

- saving costs of a human driver

Figure 1: Model-truck with one axle trailer

- the permanent availability of the vehicle's facilities

- a higher precision of the driving operation

- a coherent embedding into the administrative part of transport logistics

- the ability to incorporate other tasks such as refueling and maintenance of the vehicle

This may be enough so far to underline the economic interest in the automation of driving operations for vehicles in a known environment. Equally interesting are the technical and scientific aspects of this problem.

So the paper continues with a brief consideration of the technical system and its thriving basic technologies (section 2). However, the emphasis of this paper is on the kinematics of trucks and trucks with one axle trailers.

2   The model of a truck with trailer

The problem to be solved here includes the problem of backing up a truck with trailer to a given position. Various simulations mainly in the scope of neural networks, fuzzy control or genetic algorithms are available (see [NW89], [KK90], [KK92], [SR94], and [CLC95]). The emphasis of these solutions is on the principal solvability of the bakker-upper problem by a certain computational paradigm. In contrast, our major concern is on the construction of a real vehicle (see figure 1) integrating all functionality which is necessary for a deliberate solution in the scope of industrial transportation systems. This includes

- the profound analysis of the vehicle's kinematics

- the integration of various safety aspects

- the predictability of the vehicle's autonomous behavior

- a concept of real-time monitoring for all software components

- a concept for the easy exchange of software components

The real vehicle is a model-truck with a one axle trailer at a scale of $1 : 16$. It incorporates technical solutions to a variety of problems. It has
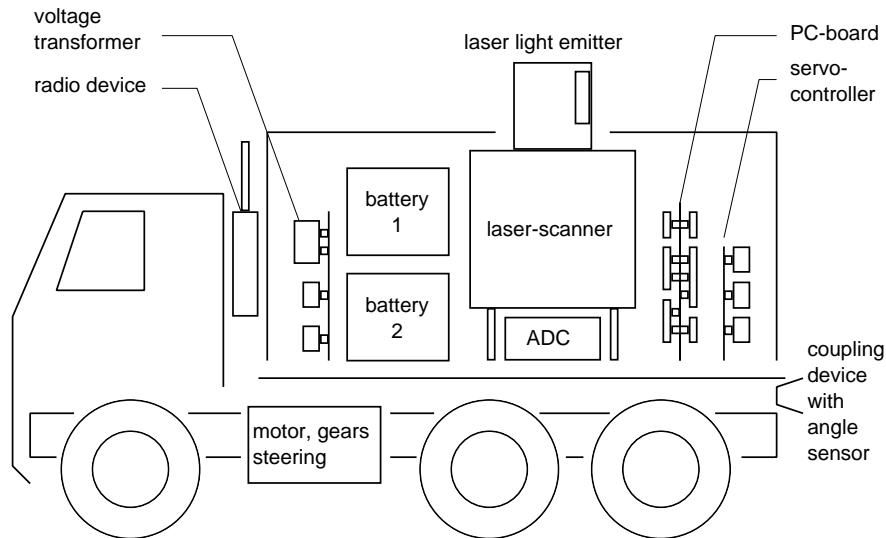
Figure 2: Diagramm of the technical construction of the model-truck

- to control the electric motor, the gears and the steering wheels of the truck via servo interfaces,

- to integrate a position and direction measurement system for the truck with an acceptable precision,

- to sense the angle between truck and trailer,

- to avoid damage by allowing immediate emergency halts triggered by a human supervisor via radio control,

- to supply various levels of battery power ($5V$, $12V$, $24V$).

All these functions (and several others) are integrated into the charging area of the model-truck (as seen in figure 2).

So far only one approach also concerned with a real vehicle is found in literature (see [AS01]). Its objective is to develop a control scheme for tracking a straight line when driving backward. Their approach has a profound background in control theory. In the context of motion planning for robots the definition of non-holonomic vehicles has been derived (see [Lat91]) and a control scheme based on sinusoids has been developed (see [Lau98]). However, this approach is rather complicated and the results are not convincing, yet.

Therefore two other approaches are of interest:

- **anthropomorphic** approach: adopting the rules which a driver learns in a driving-school to maneuver a truck with trailer

- **analytical** approach: deriving the analytical formulas for the motion of the vehicle and using these curves as reference input to the control algorithm.

The results of the anthropomorphic approach will be presented in other papers. They are not considered further in this here. Hence, the focus of this paper is on the analytical approach and will be enrolled in the subsequent sections
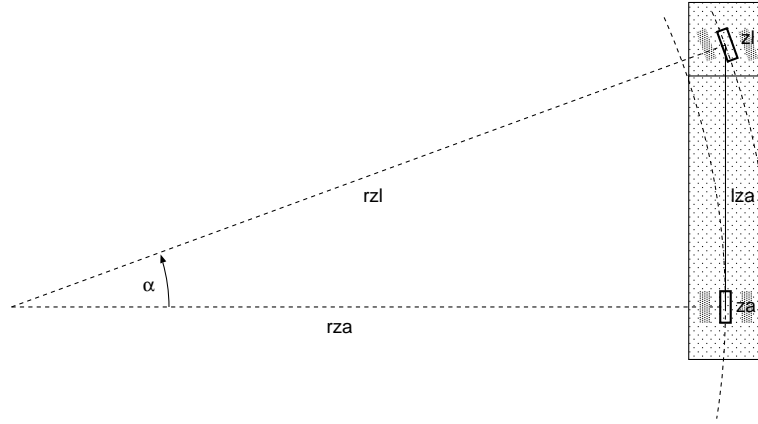
Figure 3: Geometry of the truck

3 The bike model

The movement of a truck depends on its wheels and the way they roll on a given surface. The model which is described in this section is simplified in the following way:

- No kinds of slippage due to high acceleration or deacceleration in forward or lateral directions are considered here. The vehicle is considered as a wheeled object without mass.

- The vehicle in a first abstraction has a pair of steering wheels in front and a pair of fixed wheels behind. All other types of trucks with two pairs of steering wheels or two or three pairs of fixed wheels have to be mapped adequately to the simple model. E.g. for or a truck with a single pair of steering wheels an two pairs of fixed wheels the turning point in curves has to be assumed somewhere between the pair of fixed wheels.

- In a second abstraction it is assumed that a wheels touches the surface in one single point which is the middle of the wheel when seen from above.

- The vehicle in a third abstraction is reduced to the so called *bike model*. That is to say that it consists of one steering wheel and one fixed wheel. When seen from above the position of the wheels in the bike model is exactly in the middle of the corresponding wheels of the former model.

By the bike model the descriptive parameters of the truck reduce to two position parameters:

zl the position of the steering wheel

za the position of the fixed wheel

Position parameters are two-dimensional real values, eg. for zl given by $zl.x$ and $zl.y$. So the length lza of the truck in the bike model is given by the formula:

$$lza \;\;=\;\; \sqrt{(zl.x - za.x)^2 + (zl.y - za.y)^2} \tag{1}$$

Depending on the relative angle α between steering wheel zl and the fixed wheel za the bike either moves straight ahead or on a circle with a certain radius (see figure 3). In this context rzl is the radius of the steering wheel and rza is the radius of the fixed wheel. Steering (forward) to the left is defined to
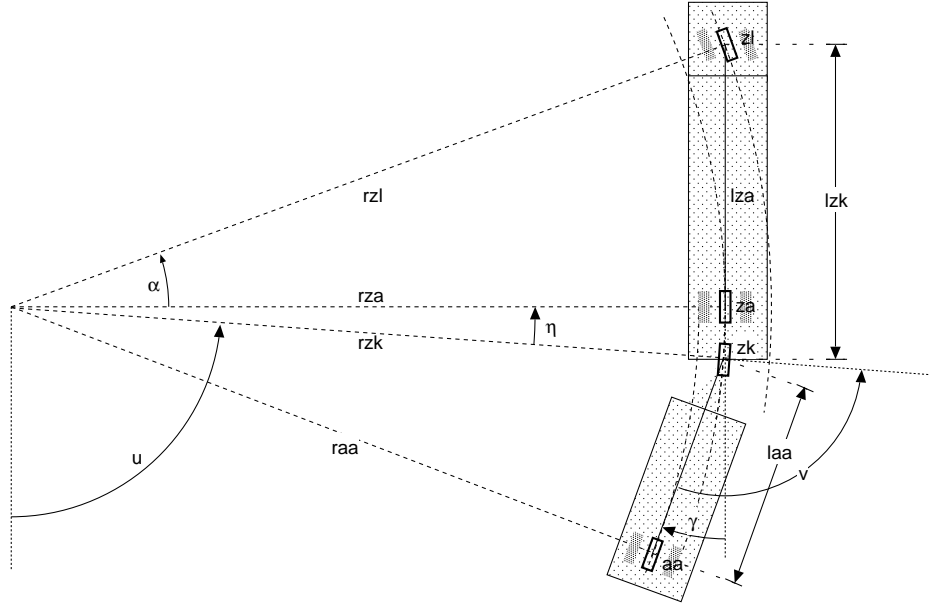
Figure 4: Geometry of the truck with trailer

have a positive value for α and to the right is defined to have a negative value. The relation between the descriptive parameters introduced so far are given by the formulas:

$$lza/rzl = sin(\alpha) \tag{2}$$
$$lza/rza = tan(\alpha) \tag{3}$$

These formulas equally hold for driving forward and backward. For a truck the value of α is somewhere between $-30°$ and $+30°$. In the following it is assumed that α is in the interval $[-\alpha_{max}, \alpha_{max}]$.

The actual direction β of the truck is implicitly determined by za and zl:

$$\frac{zl.x - za.x}{za.y - zl.y} = tan(\beta) \tag{4}$$

The next step to consider is the truck with a trailer. The trailer introduces here is one axle and equally modeled corresponding to bike model of the truck. The trailer has one fixed wheel at position aa and the coupling device between truck and trailer is at position zk. Given an angle $\alpha \neq 0$ of the steering wheel position zk also moves on a circle with the same center as for zl and za. The radius rzk for the coupling device is given by:

$$rzk = rza^2 + (zl.x - za.x)^2 + (zl.y - za.y)^2 \tag{5}$$

Two other length parameters are of interest: the length from the steering wheel to the coupling device lzk and the length from the coupling device to the fixed wheel of the trailer laa.

$$lzk = \sqrt{(zk.x - zl.x)^2 + (zk.y - zl.y)^2} \tag{6}$$
$$laa = \sqrt{(zk.x - aa.x)^2 + (zk.y - aa.y)^2} \tag{7}$$

Depending on the following relations between length parameters this model holds both for a truck with a trailer ($lzk \leq lza$) and for a semi-trailer ($lzk \geq lza$)

A free parameter is the angle γ between truck and trailer. The value of γ is measured from the axis of direction of the truck to the axis of direction of the trailer.

set to be positive for driving a left curve and is negative for driving a right curve.

4 The curves of truck and trailer

One observation when driving forward with some steering angle $\alpha \neq 0$ is that the value of γ converges to some fixed value. Hence, depending on the geometry of truck and trailer there exists a relation between α and γ for keeping both vehicles moving on circles with the same center. Given the lengths lza, lzk, laa and a proposed radius raa for the trailer the relations are as follows:

$$rzk \;=\; \sqrt{laa^2 + raa^2} \tag{8}$$

$$rza \;=\; \sqrt{rzk^2 - (|lzk - lza|)^2} \tag{9}$$

$$\gamma \;=\; 180° - arccos\left(\frac{lzk - lza}{rzk}\right) - arccos\left(\frac{laa}{rzk}\right) \tag{10}$$

$$rzl \;=\; \sqrt{lza^2 + rza^2} \tag{11}$$

$$\alpha \;=\; arcsin(lza/rzl) \tag{12}$$

Sometimes the lengths lza, lzk, laa and the angle α is given. Then raa is computed the following way:

$$rzl \;=\; lza/sin(|\alpha|) \tag{13}$$

$$rza \;=\; lza/tan(|\alpha|) \tag{14}$$

$$rzk \;=\; \sqrt{(lzk - lza)^2 + rza^2}$$

$$\;=\; \sqrt{(lzk - lza)^2 + lza^2/tan^2(|\alpha|)} \tag{15}$$

$$raa \;=\; \sqrt{rzk^2 - laa^2}$$

$$\;=\; \sqrt{(lzk - lza)^2 + lza^2/tan^2(|\alpha|) - laa^2} \tag{16}$$

Example 1 ▷ For the experimental model-truck with trailer the given extensions are: $lzk = 660mm$, $lza = 600mm$ and $laa = 500mm$. The angle α of the steering wheel may be set to $15°$ which is the half of α_{max}. This results in the following values for the radius of the interesting positions on the truck and the trailer: $rza = 2239mm$, $rzk = 2240mm$, $rzl = 2318mm$ and $raa = 2183mm$. The fixed angle γ between truck and trailer is at $14.43°$. ◁

In the following corresponding pairs of α and γ for keeping truck and trialer on circles with a certain radius are denoted by α_{circ} and γ_{circ}.

A special case of driving circles with truck and trailer is driving on a straight line. This is the case for equally $\alpha = 0$ and $\gamma = 0$. However, the most general case is that there is no fixed relation between the inclination of the steering wheel and the angle between truck and trailer. For this cases the trailer follows the truck on so called tractrix curves. Such curves are characterized by a point of traction zk and a point that follows aa. In the situations considered here the point of traction zk moves on a circle or on a straight line. So the question which curve is described by point aa has to be answered. Important for the following is the tractrix for moving the traction point on a circle.

In an xy-coordinate system the truck's traction point is at the angle u in a circle around the origin (see figure 4). The actual direction of the trailer is at an angle v from angle u. The position aa of the fixed wheel of the trailer relative to the origin is at:

$$aa.x(u,v) \;=\; rzk \, cos(u) + laa \, cos(u - v) \tag{17}$$

$$aa.y(u,v) \;=\; rzk \, sin(u) + laa \, sin(u - v) \tag{18}$$

Dragging from position zk lead to changes of position aa:

$$\frac{\delta(aa.y)}{\delta(aa.x)} \;=\; tan(u - v) \tag{19}$$

The formulas (17), (18) and (19) determine the tractrix curve.

The left hand sides of (19) can also be computed by partial derivation of formulas (17) and (18) (see [BS91], page 284):

$$\frac{\delta(aa.y)}{\delta(aa.x)} \quad = \quad \frac{\frac{\delta(aa.y)}{\delta v} + \frac{\delta(aa.y)}{\delta u}\frac{\delta u}{\delta v}}{\frac{\delta(aa.x)}{\delta v} + \frac{\delta(aa.x)}{\delta u}\frac{\delta u}{\delta v}} \tag{20}$$

$$= \quad \frac{-laa\,cos(u-v) + [rzk\,cos(u) + laa\,cos(u-v)]\frac{\delta u}{\delta v}}{laa\,sin(u-v) + [-rzk\,sin(u) - laa\,sin(u-v)]\frac{\delta u}{\delta v}} \tag{21}$$

With formula (19) it follows:

$$\tan(u-v) \quad = \quad \frac{sin(u-v)}{cos(u-v)} \tag{22}$$

$$= \quad \frac{-laa\,cos(u-v) + [rzk\,cos(u) + laa\,cos(u-v)]\frac{\delta u}{\delta v}}{laa\,sin(u-v) + [-rzk\,sin(u) - laa\,sin(u-v)]\frac{\delta u}{\delta v}} \tag{23}$$

By multiplication and simplification with $(sin(x))^2 + cos(x)^2 = 1$ the formula reduces to:

$$laa \quad = \quad (laa + rzk(cos(u)cos(u-v) + sin(u)sin(u-v)))\frac{\delta u}{\delta v} \tag{24}$$

By the trigonometric equation $cos(u - (u-v)) = cos(u)cos(u-v) + sin(u)sin(u-v)$ the formula is simplified:

$$laa \quad = \quad [laa + rzk\,cos(v)]\frac{\delta u}{\delta v} \tag{25}$$

This finally leads to the simple differential equation:

$$\frac{\delta u}{\delta v} \quad = \quad \frac{laa}{laa + rzk\,cos(v)} \tag{26}$$

Typically for truck and trailer is the relation $rzk > laa$. For this case the formula above has the following solution ([BS91], page 55):

$$u(v) \quad = \quad \frac{laa}{\sqrt{rzk^2 - laa^2}}\,ln\left|\left(\frac{(rzk - laa)\,tan(v/2) + \sqrt{rzk^2 - laa^2}}{(rzk - laa)\,tan(v/2) - \sqrt{rzk^2 - laa^2}}\right)\right| \tag{27}$$

This formula reads simpler with $raa = \sqrt{rzk^2 - laa^2}$:

$$u(v) \quad = \quad \frac{laa}{raa}\,ln\left|\left(\frac{(rzk - laa)\,tan(v/2) + raa}{(rzk - laa)\,tan(v/2) - raa}\right)\right| \tag{28}$$

The ln-function does not allow for the value $2\,arctan(raa/(rzk/laa))$ for angle v. As the trailer depends on the movement of the truck it is better to have v depend on u. Depending on the value of v there are two different results (see also figure 5).

First result for $|v| < 2\,arctan(raa/(rzk - laa))$:

$$v(u) \quad = \quad 2\,arctan\left(\frac{raa}{rzk - laa}\frac{e^{\frac{raa}{laa}u} - 1}{e^{\frac{raa}{laa}u} + 1}\right) \tag{29}$$
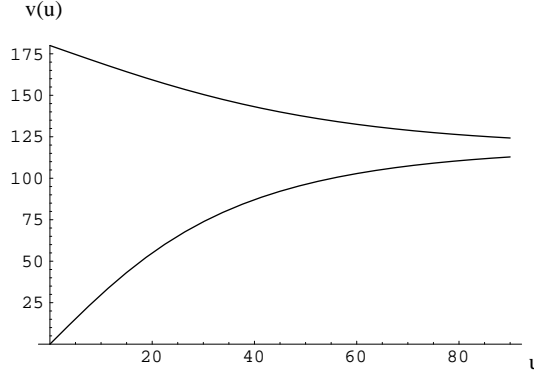
v(u)

Figure 5: There are two formulas for $v(u)$, the lower one corresponding to formula 29 and the upper to formula 30
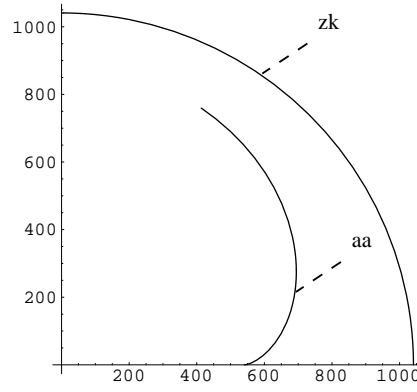

Figure 6: The inner traction of position aa by position zk for the angles $0 \leq u \leq \pi/2$.

Second for result for $|v| > 2\,arctan(raa/(rzk - laa))$:

$$v(u) \quad = \quad 2\,arctan\left(\frac{raa}{rzk - laa}\frac{e^{\frac{raa}{laa}u} + 1}{e^{\frac{raa}{laa}u} - 1}\right) \tag{30}$$

The two different results correspond to two different kinds of initial positions of truck and trailer. For increasing values for u formula 30 represents the traction where the trailer is already in the circle which is described by the motion of the truck (see figure 6).

Instead, formula 29 represents the traction where the trailer is outside the circle which is described by the motion of the truck (see figure 7). This is the typical situation when a truck is backing up. So, in the following $v(u)$ represents formula 29.

For the following the relation between the two angles γ and v has to be discussed. For a give α of the steering wheels the truck moves on a circle. From the center of this circle there is an angle η between the fixed wheel za and the coupling device zk. This angle is given by formula:

$$\eta = arcsin((lzk - lza)/rzk) \tag{31}$$

For $u = 0$ angle η denotes the deviation of the truck from the vertical. This holds both for truck with trailer ($\eta \geq 0$) and for a semi-trailer ($\eta \leq 0$). Angle η is also necessary for for the relation between γ
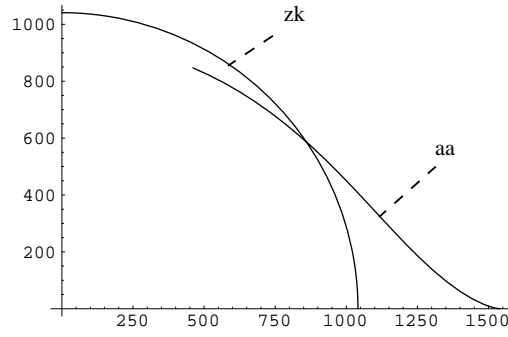
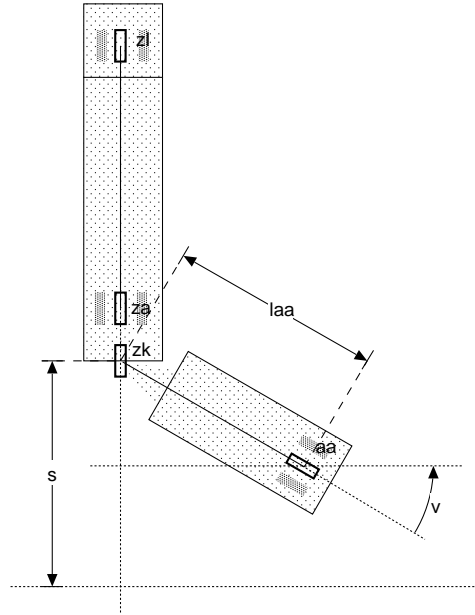Figure 7: The outer traction of position aa by position zk for the angles $0 \leq u \leq \pi/2$.

Figure 8: Geometry of the truck

and v which is:

$$\gamma = 90° - v - \eta \tag{32}$$

5 Comparison to the classical tractrix problem

The classical tractrix problem dates back to Gottfried Wilhelm Leibniz (1646-1716) and assumes that the traction point moves on a straight line. Instead of an angle u the position zk of the coupling device as traction point is determined by the distance s. Depending on s the trailer's axis aa and its direction v changes as seen in figure 8. The relations between s, v and aa are expressed in the following formulas:

$$aa.x(s,v) = s - laa\,sin(v) \tag{33}$$
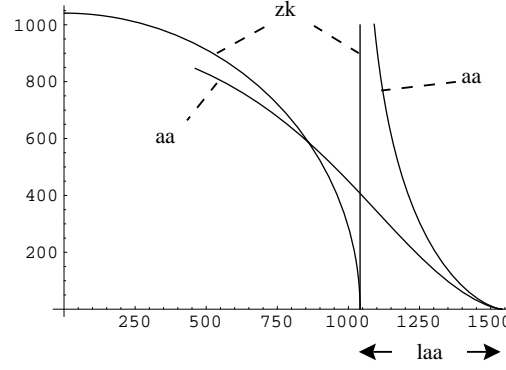$$aa.y(s,v) = laa\,cos(v) \tag{34}$$

Figure 9: Comparison of the movement of the axis aa of the trailer depending on the traction point zk moving either on a circle or on a straight line.

The differential quotient $\delta v / \delta s$ is equal to $cos(v)/laa$. So we have:

$$s(v) \;=\; laa \int \frac{\delta v}{cos(v)} \tag{35}$$

$$=\; laa\, ln \left| tan \left(\frac{v}{2} + \frac{\pi}{4} \right) \right| \tag{36}$$

The corresponding inverse function is for $\pi/2 > v \geq -\pi/2$:

$$v(s) \;=\; 2\, arctan(e^{\frac{s}{laa}}) - \frac{\pi}{2} \tag{37}$$

Of practical importance are only not negative angles v which correspond to not negative distances s as depicted in figure 9.

6 Maneuvers for truck and trailer

As pointed out in the introduction (section 1) the principal concern is autonomous transportation control for a single truck or for a truck with one axle trailer moving these vehicles from some starting position to some final position, given a certain yard. As already derived before (section 2 and 3) the geometry of their movements is different. Accordingly the routes from a starting position to a final position allow for different strategies to be composed of elementary curves:

single truck: The entire route can easily be composed of straight lines and circles

truck with one axle trailer: Equally straight lines and circles are possible for truck and trailer. However, in between some maneuvering has to be done to change the angle γ (see figure 4) from a value 0 for straight movements to γ_{circ} for cyclic movements and vice versa.

With respect to both forms of vehicles the most challenging problems have to be solved for the latter one. An adequate strategy to rule the problem is to propose defined maneuvers composed of several curves which have to be followed in corresponding maneuver phases.

The absolute angle of the trailer is denoted by ϕ[1]. In the beginning of a maneuver (state 1) the trailer is straight behind the truck ($\gamma = 0$) with some direction ϕ_1 and at the end of a maneuver (state 4) the trailer is straight behind the truck again with some other direction ϕ_4. In between are three maneuver phases, in the following explained in detail for the case of a right curve ($\phi_1 < \phi_4$):

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[1] equal to value $u - v$

1-2 : the truck drives backward on a left circle with the steering wheels at the angle $\alpha = \alpha_{max}$. This phase of the maneuver starts with $\gamma = 0$ and lasts until $\gamma = \gamma_{circ}$ which allows to proceed on a circle with a given radius raa as described in formulas (8 - 16). Meanwhile the trailer changed its direction by an angle $\Delta\phi_{1-2}$.

2-3 : Instantaneously the steering wheels are turned to the angle $\alpha = \alpha_{circ}$ for driving right on a sector of a circle with radius raa. This phase lasts until the trailer has changed its direction by the angle $\Delta\phi_{2-3}$ which is

$$\Delta\phi_{2-3} = \phi_4 - \phi_1 - \Delta\phi_{1-2} - \Delta\phi_{3-4} \tag{38}$$

3-4 : The angle $\Delta\phi_{3-4}$ missing in the formula above is gained by driving backward in the last phase with the steering wheels at an angle $\alpha = -\alpha_{max}$. This phase starts with $\gamma = \gamma_{circ}$ and ends when the angle $\gamma = 0$ is reached, that is to say when the trailer is straight behind the truck again.

Example 2 ▷ The starting direction of truck and trailer may be at $\phi_1 = 90°$ and the final direction at $\phi_4 = 135°$. The difference has to be overcome by a maneuver where several free parameters have to be fixed. Referring to example 1 the projected radius for phase 2-3 is $raa = 2183mm$. As a consequence phase 1-2 ends when angle $\gamma = \gamma_{circ} = 14.43°$ is reached. Then the steering wheels instantly turn to angle $\alpha = alpha_{circ} = -15°$ for phase 2-3. Finally for phase 3-4 the steering wheels are turned to angle $\alpha = -\alpha_{max}$. Given these parameters the angles $\Delta\phi_{1-2}$ and $\Delta\phi_{3-4}$ are exactly specified, whereas angle $\Delta\phi_{2-3}$ has to be filled in as described by formula (38). ◁

If the parameters for a maneuver are fixed as in example 2 also the geometry of phase 1-2 and phase 3-4 is determined and as a consequence of formula (38) this also determines the duration of phase 2-3. However, the relevant parts of the tractrix curve and the angles $\Delta\phi_{1-2}$ and $\Delta\phi_{3-4}$ have still to be computed.

Whereas angle α_{max} depends on the physics of the model-truck, the angle α_{circ} has to be chosen carefully within the bounds $0°$ and α_{max}. If chosen near to $0°$ the circle during phase 2-3 becomes huge and hinders to find a path from the truck's starting point to its final point. Otherwise, choosing α_{circ} near to α_{max} implies narrow circles for phase 2-3, but the maneuver phase 3-4 to get out of the circle and to put the trailer straight behind the truck becomes rather long which also hinders to find an adequate path. So α_{circ} has to be chosen somewhere far from these bounds, for example in the middle: $\alpha_{circ} = \alpha_{max}/2$. The decision for a certain steering angle for maneuver phase 2-3 determines an angle γ_{circ} between truck and trailer which is given by formula (10).

It has to be noticed here that the turning point of the tractrix curve does not necessarily coincide with the state that the trailer is straight behind the truck. This is only the case when the fixed wheel za coincides with the coupling device zk. In general the turning point is characterized in that the direction of the traction by zk which is $u + 90°$ equals the direction of the trailer which is $(u - v) + 180°$. Hence, the turning point is found for $v = 90°$ and depending on traction point zk at angle $u(90°)$.

However, the maneuver phases are not determined by the turning points of the tractrix curve, but by certain angles $\gamma = 0°$ and $\gamma = \gamma_{circ}$. And as seen in figure 4 angle γ depends on angle u in the following way:

$$\gamma(u) \quad = \quad 90° - v(u) - \eta \qquad u \geq 0 \tag{39}$$

This is the formula for a left curve and the formula for a right curve is:

$$\gamma(u) \quad = \quad -90° + v(-u) + \eta \qquad u \leq 0 \tag{40}$$

For a left curve the maneuver phase 1-2 starts with $v_1 = 90° - \eta$ and lasts until $v_2 = 90° - \eta - \gamma_{circ}$ thereby reaching the desired angle γ_{circ} between truck and trailer. The reverse maneuver phase 3-4 starts with $v_3 = -90° + \eta - \gamma_{circ}$ and lasts until $v_4 = -90° + \eta$. Here it has to be noticed that phase
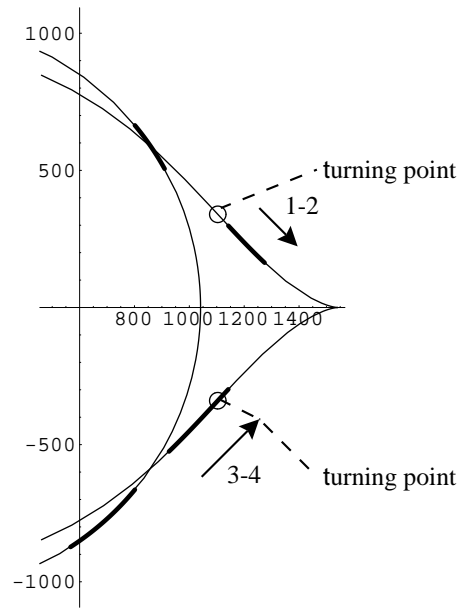
Figure 10: Both tractrix curves for maneuver phase 1-2 (left curve) and for maneuver phase 3-4 and the sense of movement for traction point zk.

1-2 belongs to a left curve, because of $\alpha = \alpha_{max}$, and phase 3-4 belongs to a right curve, because of $\alpha = -\alpha_{max}$.

Seen from traction point zk the maneuver phase 1-2 covers the interval $[u(v_1), u(v_2)]$ and maneuver phase 3-4 the interval $[u(v_3), u(v_4)]$. By these values the angles $\Delta\phi_{1-2}$ and $\Delta\phi_{3-4}$ compute as follows:

$$\Delta\phi_{1-2} = (u(v_2) - v_2) - (u(v_1) - v_1) \tag{41}$$

$$\Delta\phi_{3-4} = (u(v_4) - v_4) - (u(v_3) - v_3) \tag{42}$$

Example 3 ▷ Based on the parameters given by the examples above the angles for maneuver phase 1-2 run from $v_1 = 86.70°$ to $v_2 = 72.26°$. Thereby the coupling device as traction point covers the interval $[39.65°, 29.06°]$. This corresponds to an angle $\Delta\phi_{1-2} = 3.84°$. For maneuver phase 3-4 the angles run from $v_3 = -101.13°$ to $v_4 = -86.70°$. Thereby zk covers the interval $[-57.01°, -39.65°]$. This corresponds to an angle $\Delta\phi_{3-4} = 2.93°$. As a consequence the circle in maneuver phase 2-3 covers the angle $\Delta\phi_{2-3} = 38.23°$. See also the bold curves for zk in figure 10. ◁

It should be noticed that the curve for maneuver phase 3-4 contains the turning point. This means that the axis of the trailer aa crosses over the line with the final direction ϕ_4. This is always the case for trucks with $lza < lzk$. Otherwise – this applies to semi-trailers – the cross over happens in maneuver phase 1-2.

The truck is the control device. Therefore switching to a certain maneuver phase should depend on the movement of the truck. During any maneuver phases the truck and particularly the coupling device zk moves on circles with an actual radius corresponding to steering angles α_{max} or α_{circ}, depending on the maneuver phase. The distance d of any phase is:

$$d_{1-2} = |u(v_2) - u(v_1)|\, rzk_{1-2} \tag{43}$$

$$d_{2-3} = \Delta\phi_{2-3}\, rzk_{2-3} \tag{44}$$

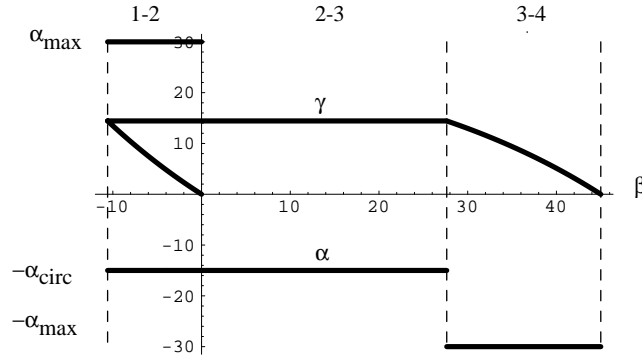$$d_{3-4} = |u(v_3) - u(v_4)|\, rzk_{3-4} \tag{45}$$

Figure 11: Diagram of the the angle γ between truck and trailer depending on different angles α of the steering wheel during the three maneuver phases and the actual direction β of the truck.
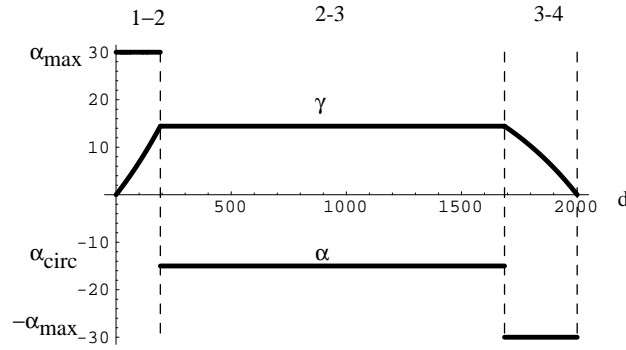

Figure 12: Diagram of the the angle γ between truck and trailer depending on different angles α of the steering wheel during the three maneuver phases and the actual distance d covered by the coupling device

Let β be the direction of the truck. In any maneuver phase the distance is proportional to the change of direction of the truck. For maneuver phase 1-2 the truck turns backwards to the left, which is a change of $u(v_2) - u(v_1)$ which is a decrease in the value of direction β. The next two maneuver phases increase the direction by $\Delta\phi_{2-3} + (u(v_4) - u(v_3))$. All changes of direction sum up to $\phi_4 - \phi_1$ (for $\phi_1 < \phi_4$).

Example 4 ▷ For driving backwards as proposed by example 2 the change of direction $\phi_4 - \phi_1$ equals $45°$ which corresponds to a right curve for truck and trailer. The necessary maneuver phases have to controlled by the truck moving backwards at different steering angles α. For maneuver phase 1-2 the direction of the truck decreases by $u(v_2) - u(v_1) = -10.59°$. Then in maneuver phases 2-3 and 3-4 the direction of the truck increases again by $\Delta\phi_{2-3} = 38.22°$ and $u(v_4) - u(v_3) = 17.36°$ respectively. See figure 11 where angle γ depends on the actual angle of the steering wheel α and the actual direction β of the truck.

The radius for maneuver phases 1-2 and 3-4 is $rzk_{1-2} = rzk_{3-4} = 1040mm$ whereas for maneuver phase 2-3 it is $rzk_{2-3} = 2240mm$ (see example 1). The distance d which is covered in the different phases is:

$$d_{1-2} = 192mm \tag{46}$$
$$d_{2-3} = 1494mm \tag{47}$$
$$d_{3-4} = 315mm \tag{48}$$

Figure 12 shows angle γ depending on the actual angle of the steering wheel α and the distance d of the coupling device. ◁

Example 5 ▷ So far the steering angle α_{circ} for driving the on a circle has been chosen arbitrarily at a value $\pm\alpha_{max}/2$. In order to reduce the distance of driving it should be considered to drive on a narrower circle during
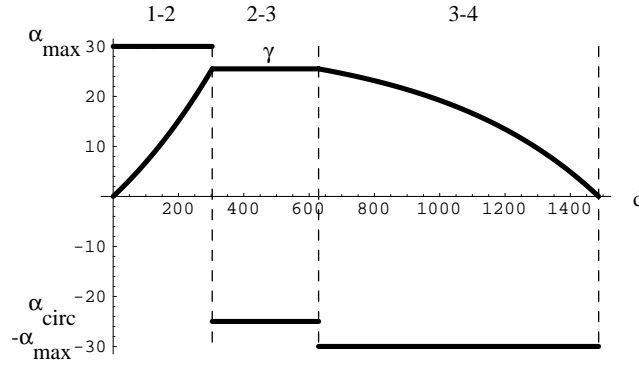
Figure 13: Diagram of the the angle γ between truck and trailer depending on different angles α of the steering wheel during the three maneuver phases and the actual distance d covered by the coupling device

phase 2-3. Setting

$$\alpha_{circ} \quad = \quad \pm\frac{5}{6}\alpha_{max} \tag{49}$$

results in reducing the distance covered by the maneuver from $2002mm$ to $1487mm$. Also the difference between phase 1-2 and phase 3-4 becomes obvious (see figure 13). The angle γ increases rapidly during phase 1-2 ($d_{1-2} = 303mm$) whereas phase 3-4 maneuvering truck and trailer into a straight line uses a far longer distance $d_{3-4} = 858mm$. Extrapolating the the strategy of increasing the steering angle α_{circ} introduces two problems:

- Latencies in changing from phase 1-2 to phase 3-4 result in enormous differences of the angle γ and its projected value γ_{circ}.

- Control deviations during phase 3-4 result in far longer distances d_{3-4} until truck and trailer are in a straight line again.

These observations underline the chaotic character of driving backward with truck and trailer. ◁

7 The bending of the tractrix-curve

An important characterization for the vehicle moving backward in different maneuver phases is the bending of the axis of the trailer, that is to say point aa. Obviously the bending is 0 when moving straight backward. Equally constant is the bending in maneuver phase 2-3 with a value $1/raa$. However, for the maneuver phases 1-2 and 3-4 the bending is steadily changing and has to be computed applying its mathematical definition. For parametric functions in the x- and y-direction it is given by:

$$\frac{x'y'' - x''y'}{\left(x'^2 + y'^2\right)^{3/2}} \tag{50}$$

The particular functions are $aa.x$ and $aa.y$ parameterized in u:

$$aa.x(u) \quad = \quad rzk\,cos(u) + laa\,cos(u - v(u)) \tag{51}$$
$$aa.y(u) \quad = \quad rzk\,sin(u) + laa\,sin(u - v(u)) \tag{52}$$

In the case of $aa.x(u)$ the derivatives for solving the bending function are:

$$aa.x'(u) \quad = \quad -rzk\,sin(u) - laa\,sin(u - v(u))\,(1 - v'(u)) \tag{53}$$
$$aa.x''(u) \quad = \quad -rzk\,cos(u) - laa\,cos(u - v(u))\,(1 - v'(u))^2 \tag{54}$$
$$+\,laa\,sin(u - v(u))\,v''(u)$$

The corresponding derivatives for $aa.y$ are:

$$aa.y'(u) = rzk\,cos(u) + laa\,cos(u - v(u))\,(1 - v'(u)) \tag{55}$$

$$aa.y''(u) = -rzk\,sin(u) - laa\,sin(u - v(u))\,(1 - v'(u))^2 \tag{56}$$
$$- laa\,cos(u - v(u))\,v''(u)$$

For the evaluation of these formulas the first and second derivative of $v(u)$ is needed. The following substitutions are introduced:

$$f_1(u) = e^{\frac{raa}{laa}u} - 1 \tag{57}$$

$$f_2(u) = e^{\frac{raa}{laa}u} + 1 \tag{58}$$

Let $f'(u)$ denote the common derivative of $f_1'(u)$ and $f_2'(u)$:

$$f_1'(u) = f_2'(u) = f'(u) = \frac{raa}{laa}\,e^{\frac{raa}{laa}u} \tag{59}$$

Additionally $h(u)$ substitutes the argument of the $arctan$-Function in formula 29:

$$h(u) = \frac{raa}{rzk - laa}\frac{f_1(u)}{f_2(u)} \tag{60}$$

In advance the first and second derivatives of $h(u)$ are computed:

$$h'(u) = 2\frac{raa}{rzk - laa}\frac{f'(u)}{(f_2(u))^2} \tag{61}$$

$$h''(u) = 2\frac{raa}{rzk - laa}\frac{f''(u)f_2(u) - 2(f'(u))^2}{(f_2(u))^3} \tag{62}$$

Finally the first and second derivatives of $v(u)$ can be computed:

$$v'(u) = 2\frac{h'(u)}{1 + (h(u))^2} \tag{63}$$

$$v''(u) = 2\frac{h''(u)(1 + (h(u))^2) - 2h(u)(h'(u))^2}{(1 + (h(u))^2)^2} \tag{64}$$

Given function $v(u)$ it is not necessary to apply the definition of bending for parametric functions. The general definition is based on the the following quotient for the points A and B on the curve:

$$b(A) = \lim_{B \to A}\frac{\text{angle between tangent in } A \text{ and tangent in } B}{\text{length of the curve between } A \text{ and } B} \tag{65}$$

Here the numerator is immediately given as the change of the direction of the trailer: $(u - v(u))$. The denominator is is determined by the change in x and y direction. So the formula is given by:

$$b(u) = \frac{1 - v'(u)}{\sqrt{(aa.x'(u))^2 + (aa.y'(u))^2}} \tag{66}$$

Hence, bending can be computed based on formula (50) or formula (66). The radius r of the bending circle is often more convenient and can be computed by: $r(u) = 1/b(u)$.

Example 6 ▷

Bending is a steady function (see figure 14) except for $u = 0$. For a left curve bending is set to a positive value and for a right curve to a negative value. For $u = 0$ there is a left value $-\infty$ and a right value $+\infty$. Bending is 0 for $v = \pm 90°$ and $u(v) = \pm 42.76°$ This corresponds to the turning point in figure 10.

The bending of the coupling device is an unsteady movement with respect to the different maneuver phases (see figure 15. There is an abrupt change in bending with any change of the maneuver phase that is to say from state 1 to state 4. The following table shows the change of radius of the traction point zk and of the axis of the trailer aa in states 1 to 4 of the maneuver:
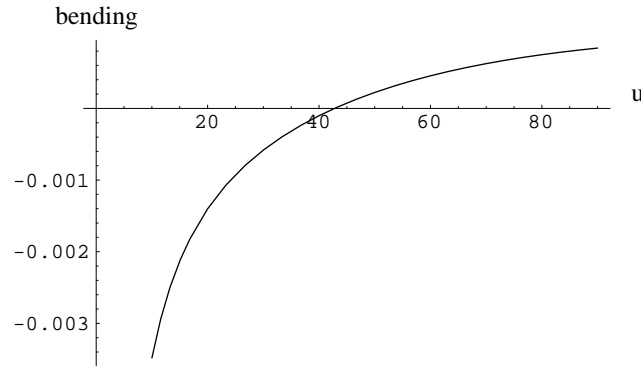
Figure 14: The bending of the tractrix curve for u running from $10°$ to $90°$. Notice that for instance the bending value 0.001 corresponds to a bending radius of $1000mm$.
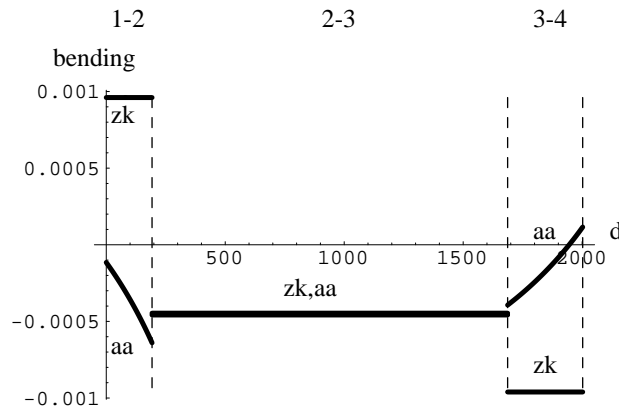

Figure 15: The bending of the tractrix curve depending on the distance d covered by the traction point zk and the axis of the trailer aa.

state	$zk[mm]$	$aa[mm]$
1	from ∞ to -8660	from ∞ to 913
2	from -1556 to -2240	from 913 to -2183
3	from -2240 to -2551	from -2183 to -913
4	from 8660 to ∞	from -913 to ∞

$\triangleleft$

8 Embedding of maneuvers

A convenient method for finding a path from the vehicle's actual position to the ramp is based on the construction of an adequate polygon of straight lines. This implies that maneuvers have to be executed whenever truck and trailer change from one straight line to the next in the sequence of polygons. A trailer which is straight behind the truck at a certain position $(aa.x, aa.y)$ in direction ϕ_1 drives straight forward or straight backward to the starting position $(aa.x_1, aa.y_1)$ of the three maneuver phases. After the maneuver the trailer is again straight behind the truck at position $(aa.x_4, aa.y_4)$ in direction ϕ_4 and a subsequent driving operation based on a straight line and a maneuver can start. It should be noticed
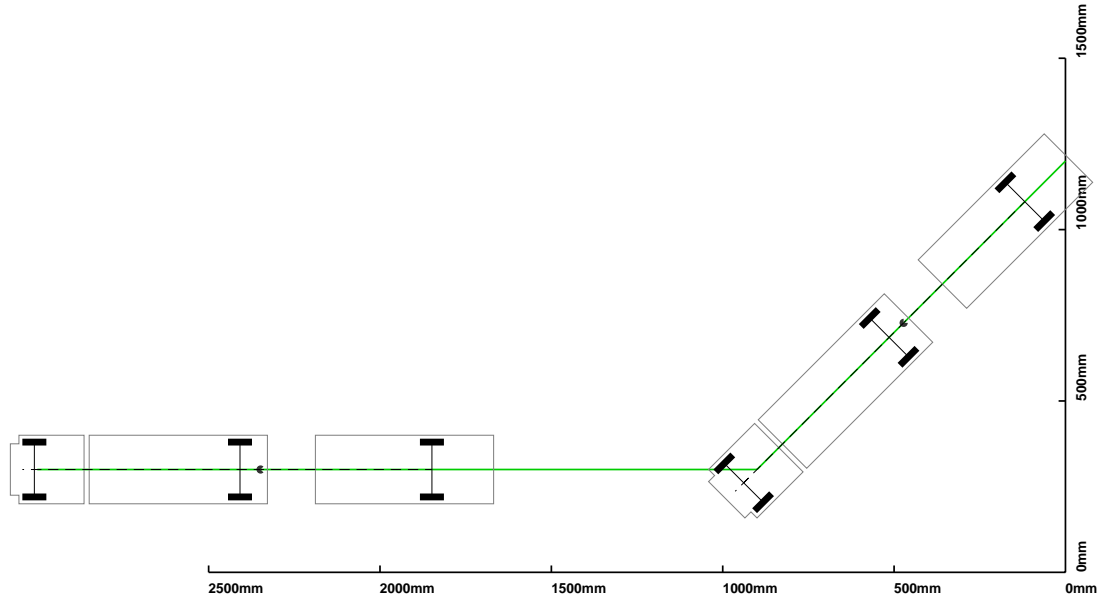
Figure 16: The proposed change from direction $\phi_1 = 90°$ to $\phi_4 = 135°$ for both truck and trailer.

here that the maneuver can also be applied in a forward direction of the vehicle starting at position $(aa.x_4, aa.y_4)$ in direction ϕ_4 and ending at position $(aa.x, aa.y)$ in direction ϕ_1.

Example 7 ▷ It is assumes that the truck and trailer have direction $\phi_1 = 90°$ and should – by means of a maneuver – drive backward to reach direction $\phi_4 = 135°$ (see figure 16). ◁

Before discussing strategies for finding an adequate path for the vehicle it is necessary to develop some basic operations. In this context it is interesting to compute for a pair (l_a, l_b) of subsequent straight lines the positions where a maneuver has to start on the first line such that it ends on the second one. In general for a pair of straight lines there are at least four different maneuvers which can be categorized by the following criteria:

- two are driven in forward direction and two in backward direction

- two of them cover an angle

$$\Delta\phi_{1-2} + \Delta\phi_{2-3} + \Delta\phi_{3-4} \leq 180° \tag{67}$$

and two an angle of

$$\Delta\phi_{1-2} + \Delta\phi_{2-3} + \Delta\phi_{3-4} \geq 180° \tag{68}$$

With respect to the last item it seems to be appropriate not to consider angles $> 180°$. To exclude any indeterminism with respect to the first item the tuple of straight lines is augmented by the direction of driving, for forward and for backward.

Without loss of generality we assume that a straight line l is given by the triple (x, y, ϕ) where position (x, y) is some point on the line l and ϕ the angle in forward direction of the truck. With formula (38) the angle between two straight lines should at least have a value of $\Delta\phi_{1-2} + \Delta\phi_{3-4}$.[2]

---

[2]In the following the discriminator sign . is used for projecting a vector to a scalar (as already done):

$$l.x \;=\; x \qquad \text{for} \qquad l = (x, y, \phi)$$

Based on this notation a tuple (l_a, l_b, bw) results in the same maneuver as tuple (l_b, l_a, fw). Hence, it is sufficient to consider the tuple (l_a, l_b, bw) for computing the corresponding starting point and the end point of the maneuver. This is realized step by step based on a sequence of basic operations But beforehand the definition of a curve.

The maneuver will be a curve which is composed by curves for the phases 1-2, 2-3 and 3-4. Decisive for the composition ist that they are tangentially joined respecting the direction of driving. Lines – as noted above – contain this direction.

A curve k which is part of the vehicle trajectory has a starting point with a given direction and an end point with a given direction. This fits conveniently into the data structure give by a couple of lines (l_a, l_b). Therefore a curve k is denoted by two lines (l_a, l_b).

Tangentially joining a pair of curves is based on a few basic functions:

- turning a line l around a point (x, y) for a certain angle ϕ:

$$dl(l, x, y, \phi) \quad \mapsto \quad m \tag{69}$$

For $l = (x_l, y_l, \phi_l)$ the resulting line $m = (x_m, y_m, \phi_m)$ is computed in the following way:

$$
\begin{aligned}
x_m &= (x_l - x)\cos(\phi) + (y_l - y)\sin(\phi) + x & (70) \\
y_m &= -(x_l - x)\sin(\phi) + (y_l - y)\sin(\phi) + y & (71) \\
\phi_m &= \phi_l + \phi & (72)
\end{aligned}
$$

- adding two curves $k_1 = (l_a, l_b)$ and $k_2 = (l_c, l_d)$ tangentially computing the new final straight line l of the composite curve k:

$$ak(k_1, k_2) \quad \mapsto \quad l \tag{73}$$

The new curve k is equal to $(k_1.l_a, l)$ where l is computed – by using the auxiliary variable m – in the following way:

$$
\begin{aligned}
m &= (l_d.(x, y) + l_b.(x, y) - l_c.(x, y), l_d.\phi) & (74) \\
l &= dl(m, l_b.(x, y), l_b.\phi - l_c.\phi) & (75)
\end{aligned}
$$

- computing the intersection (x, y) of two straight lines (l_a, l_b):

$$sl(l_a, l_b) \quad \mapsto \quad (x, y) \tag{76}$$

The coordinates x and y are computed – by using the auxiliary variable r – in the following way:

$$
\begin{aligned}
r &= \frac{\sin(l_b.\phi)\,(l_b.x - l_a.x) + \cos(l_b.\phi)\,(l_a.y - l_b.y)}{\cos(l_a.\phi)\sin(l_b.\phi) - \sin(l_a.\phi)\cos(l_b.\phi)} & (77) \\
x &= l_a.x + r\cos(l_a.\phi) & (78) \\
y &= l_a.y + r\sin(l_a.\phi) & (79)
\end{aligned}
$$

The objective is to fit a maneuver into a pair of straight lines. The maneuver covers the angle $\Delta\phi_{1-4} = \phi_4 - \phi_1$ and is composed of three curves:

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or for projecting a vector to a vector with a lower dimension:

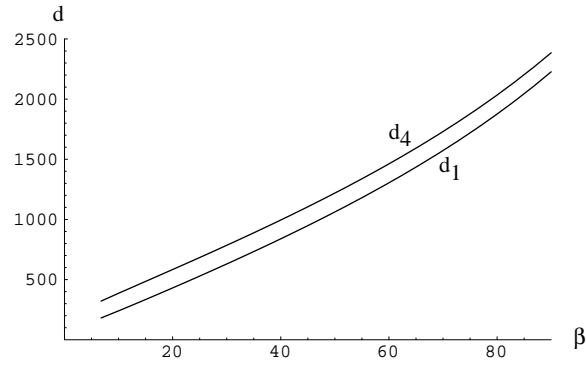$$l.(x, y) \quad = \quad (x, y) \qquad \text{for} \qquad l = (x, y, \phi)$$

Figure 17: The distance of the starting point and the end point of the maneuver from the intersection of two straight lines with angle β between $\Delta\phi_{1-2} + \Delta\phi_{3-4}$ and $90°$.

$k_{1-2} = (l(u_1), l(u_2))$ where $l(u_1) = (aa.x(u_1), aa.y(u_1), \pi + (u_1 - v(u_1))$ and $l(u_2)$ analogous

$k_{2-3}(\psi) = (0, raa, 0, dl(0, 0, \psi, 0, raa, 0).(x, y), \psi)$

$k_{3-4} = (l(u_3), l(u_4))$ where $l(u_3) = (aa.x(u_3), aa.y(u_3), \pi + (u_3 - v(u_3))$ and $l(u_4)$ analogous

The complete maneuver including all three phases ends in line $l_{1-4}(\psi)$ and consisting of curves k_{1-2}, $k_{2-3}(\psi)$ and k_{3-4} composed by function ak:

$$k_{1-4}(\psi) = ak(k_{1-2}, 0, raa, 0, ak(k_{2-3}(\psi), k_{3-4})) \tag{80}$$

The whole curve $k_{1-4}(\psi)$ is given by the tuple $l(u_1), l_{1-4}(\psi)$.

Attaching curve $k_{1-4}(\psi)$ to a given line l is done by function $an(l, \zeta)$:

$$an(l, \zeta) = ak(l, l, l(u_1), k_{1-4}(\zeta - l.(\phi) - \Delta\phi_{1-2} - \Delta\phi_{3-4})) \tag{81}$$

The main objective is to compute the starting point p_1 and the end point p_4 for a given maneuver given to lines l_a and l_b:

$$p_1(l_a, l_b) = l_a.(x, y) + sl(l_a, l_b) - sl(l_a, an(l_a, l_b.\phi)) \tag{82}$$
$$p_4(l_a, l_b) = an(l_a, l_b.\phi) + sl(l_a, l_b) - sl(l_a, an(l_a, l_b.\phi)) \tag{83}$$

The two functions $d_1(\beta)$ and $d_4(\beta)$ represent the distance of the starting point and the end point of the maneuver from the intersection of the two straight lines l_a and l_b. They are defined as:

$$d_1(\beta) = |sl(l_a, l_b) - p1(l_a, l_b)| \quad \text{for} \quad \beta = |l_a.\phi - l_b.\phi| \tag{84}$$
$$d_4(\beta) = |sl(l_a, l_b) - p4(l_a, l_b)| \quad \text{for} \quad \beta = |l_a.\phi - l_b.\phi| \tag{85}$$

Example 8 ▷ For a difference $\beta = 45°$ between a pair l_a and l_b of straight lines the functions d_1 and d_4 give the absolute distance of the starting point of maneuver phase 1-2 and the end point of maneuver phase 4 from the point of intersection between l_a and l_b. For the model-truck presented in example 1 executing the maneuver as described in example 4 the distances are:

$$d_1(45°) = 948mm \tag{86}$$
$$d_4(45°) = 1105mm \tag{87}$$

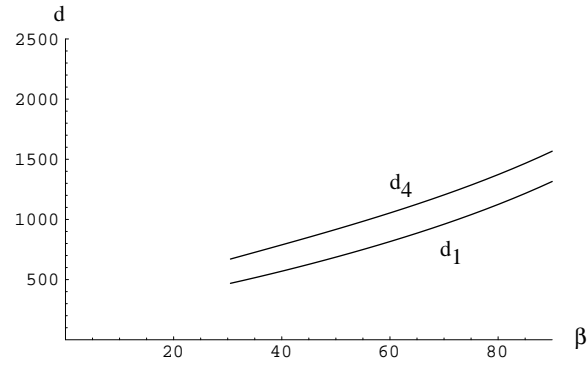The dependency of d_1 and d_4 on β can be seen in figure 17. ◁

Figure 18: The distance of the starting point and the end point of the maneuver from the intersection of two straight lines with angle β between $\Delta\phi_{1-2} + \Delta\phi_{3-4}$ and $90°$.

Example 9 ▷ In example 5 the angle α_{circ} is set to $25°$. This results in shorter distances for an angle $\beta = 45°$ between a pair l_a and l_b of straight lines:

$$d_1 \quad = \quad 27mm \tag{88}$$
$$d_4 \quad = \quad 852mm \tag{89}$$

The dependency of d_1 and d_4 on β can be seen in figure 18. ◁

9 Reflection and outlook

The major results of the analytic approach so far consider the curves for a truck with a one axle trailer:

- the derivation of a closed formula for the curves of truck with one axle trailer

- the selection of the decisive parts of this curve for the formation of maneuver phases

- the dependencies of the angle γ between truck and trailer of the distance d and the angle β of the truck (see figures 12 and 11) which are important as reference input for control algorithms

- the property of bending to be unsteady from one maneuver phase to the next

- the computation of starting points and end points for maneuvering a vehicle from one straight line to the next

So far a big deal of necessities for the specification and execution of defined movements of vehicles is achieved. These results constitute primitives in the context of more general operations which are necessary to control the vehicles on a yard of a logistics center:

- the computation of a polygon of straight lines from a vehicles actual position to a final position, a path finding algorithm

- a function for the assessment of paths, e.g. preferring lines which are tracked in forward direction to straight lines which are tracked in backward direction

- the computation of sleeves – a metaphor for the space covered by the movements of truck and trailer – at two levels of strictness

– necessary sleeves: a geometric description of the exact space covered by the most extreme position of a vehicle

– sufficient sleeve: a geometric description of a left and right hand function which is never crossed over by any part of the real vehicle which does not behave ideally

• the dynamic schedule for the collision-free movement of different vehicles present on the yard of a logistics center

The analytical approach has been restricted to trucks with one axle trailer. Consequently, as a next step the two axle trailer has to be considered. This will result in far more complex formulas for the description of curves as well as more sophisticated maneuver phases.

10 Appendix

10.1 Important definitions for the kinematic model

10.2 Figures of the maneuver phases

The figures refer to the the truck as described in example 1. For driving backward as proposed by example 2 the change of direction $\phi_4 - \phi_1$ equals $45°$ which corresponds to a right curve for truck and trailer.

Figure 19 shows the curve for the three maneuver phases for the trailers axis for the maneuver as described by example 8 with the starting point and end point as depicted by figure 16.
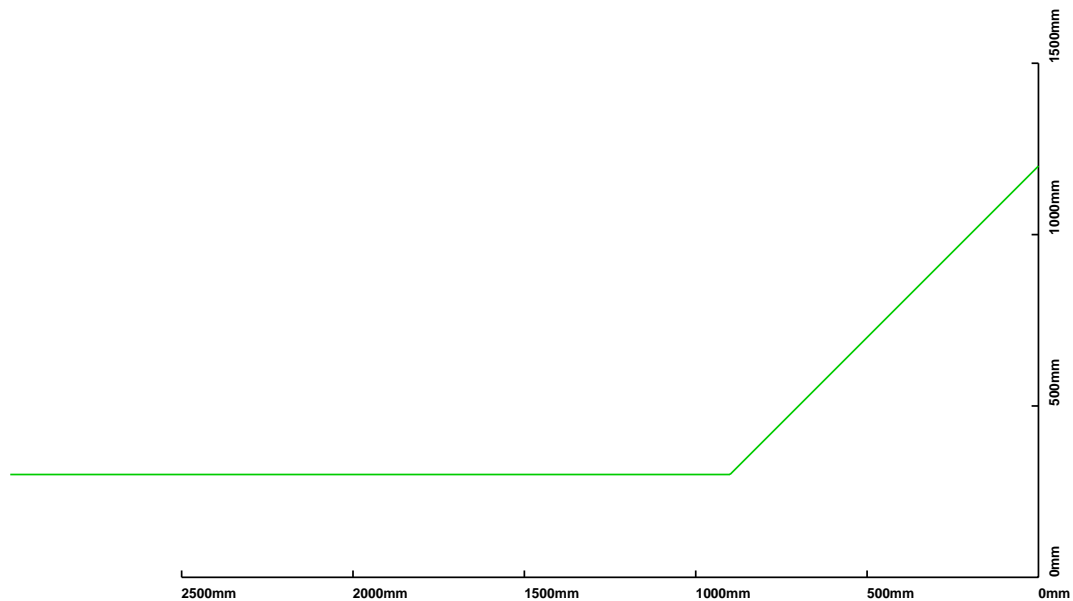
Figure 19: The curve describes the position of the axis of the trailer during the three maneuver phases.

Figure 20 shows the truck in a sequence of positions corresponding to the beginning of a maneuver phase. the angle of the steering wheel which can be seen in the figure is kept constant during a maneuver phase.

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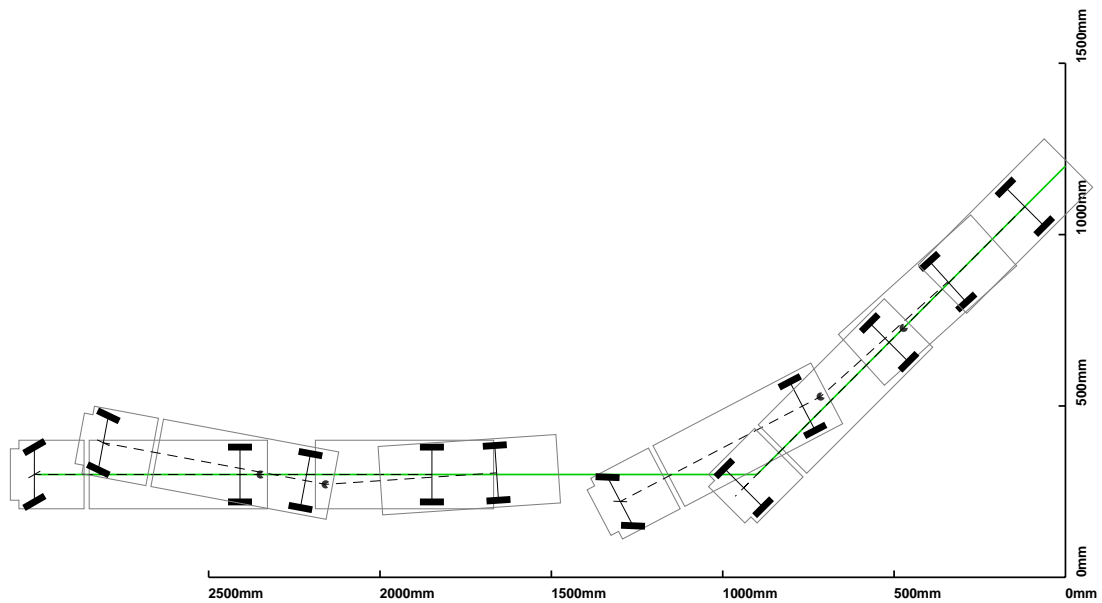
Figure 20: Positions of truck and trailer during at the beginning of maneuver phases.

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