Logique intuitionniste (LJ)

$$\frac{}{\Gamma, A \vdash A}\,\text{ax}$$

$$\frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,\text{cut}$$

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\text{cont}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}\,\Rightarrow_{\text{left}}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,\Rightarrow_{\text{right}}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \Leftrightarrow B \vdash C}\,\Leftrightarrow_{\text{left1}}$$

$$\frac{\Gamma \vdash B \quad \Gamma, A \vdash C}{\Gamma, A \Leftrightarrow B \vdash C}\,\Leftrightarrow_{\text{left2}}$$

$$\frac{\Gamma, A \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\text{right}}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \land B \vdash C}\,\land_{\text{left}}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B}\,\land_{\text{right}}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \lor B \vdash C}\,\lor_{\text{left}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \lor B}\,\lor_{\text{right1}}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \lor B}\,\lor_{\text{right2}}$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\,\neg_{\text{left}}$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\,\neg_{\text{right}}$$

$$\frac{}{\Gamma, \bot \vdash A}\,\bot_{\text{left}}$$

$$\frac{}{\Gamma \vdash \top}\,\top_{\text{right}}$$

$$\frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x.A(x) \vdash B}\,\forall_{\text{left}}$$

$$\frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x.A(x)}\,\forall_{\text{right}}, x \notin \Gamma$$

$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A(x) \vdash B}\,\exists_{\text{left}}, x \notin \Gamma, B$$

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A(x)}\,\exists_{\text{right}}$$

Avec tiers-exclu

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\,\text{em}$$

Logique classique (LK)

$$\frac{}{\Gamma, A \vdash \Delta, A}\,\text{ax}$$

$$\frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,\text{cut}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\text{cont}_{\text{left}}$$

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,\text{cont}_{\text{right}}$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\,\Rightarrow_{\text{left}}$$

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B}\,\Rightarrow_{\text{right}}$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma, A \Leftrightarrow B \vdash \Delta}\,\Leftrightarrow_{\text{left1}}$$

$$\frac{\Gamma \vdash \Delta, B \quad \Gamma, A \vdash \Delta}{\Gamma, A \Leftrightarrow B \vdash \Delta}\,\Leftrightarrow_{\text{left2}}$$

$$\frac{\Gamma, A \vdash \Delta, B \quad \Gamma, B \vdash \Delta, A}{\Gamma \vdash \Delta, A \Leftrightarrow B}\,\Leftrightarrow_{\text{right}}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\,\land_{\text{left}}$$

$$\frac{\Gamma \vdash \Gamma, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \land B}\,\land_{\text{right}}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \lor B \vdash \Delta}\,\lor_{\text{left}}$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \lor B}\,\lor_{\text{right}}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\,\neg_{\text{left}}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,\neg_{\text{right}}$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\,\bot_{\text{left}}$$

$$\frac{}{\Gamma \vdash \Delta, \top}\,\top_{\text{right}}$$

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x.A(x) \vdash \Delta}\,\forall_{\text{left}}$$

$$\frac{\Gamma \vdash \Delta, A(x)}{\Gamma \vdash \Delta, \forall x.A(x)}\,\forall_{\text{right}}, x \notin \Gamma, \Delta$$

$$\frac{\Gamma, A(x) \vdash \Delta}{\Gamma, \exists x.A(x) \vdash \Delta}\,\exists_{\text{left}}, x \notin \Gamma, \Delta$$

$$\frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x.A(x)}\,\exists_{\text{right}}$$

Priorités

- $\neg \succ \land \succ \lor \succ \Rightarrow \succ \Leftrightarrow$
- $\land, \lor, \Leftrightarrow$: associatif à gauche.
- $\Rightarrow$: associatif à droite.
- $\forall, \exists$: jusqu'au bout du groupe courant.

Logique implicative minimale

$$\frac{}{\Gamma, A \vdash A}\,\text{ax}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,\Rightarrow_I$$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\,\Rightarrow_E$$

Règles de typage du λ-calcul

$$\frac{(x, \tau) \in \Gamma}{\Gamma \vdash x : \tau}\,\text{Var}$$

$$\frac{\Gamma, (x, \tau_1) \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1.t : \tau_1 \to \tau_2}\,\text{Fun}$$

$$\frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1 t_2 : \tau_2}\,\text{App}$$

Logique de Hoare

$$\frac{}{\{P\}\ \text{skip}\ \{P\}}\,\text{skip}$$

$$\frac{}{\{P(e)\}\ x := e\ \{P(x)\}}\,:=$$

$$\frac{\{P\}\ i_1\ \{Q\} \quad \{Q\}\ i_2\ \{R\}}{\{P\}\ i_1; i_2\ \{R\}}\,;$$

$$\frac{\{P \land e\}\ i_1\ \{Q\} \quad \{P \land \neg e\}\ i_2\ \{Q\}}{\{P\}\ \text{if } e \text{ then } i_1 \text{ else } i_2\ \{Q\}}\,\text{if}$$

$$\frac{\{I \land e\}\ i\ \{I\}}{\{I\}\ \text{while } e \text{ do } i\ \{I \land \neg e\}}\,\text{while}$$

$$\frac{\{P'\}\ i\ \{Q'\} \quad P \Rightarrow P' \quad Q' \Rightarrow Q}{\{P\}\ i\ \{Q\}}\,\text{Aff}$$

Exemples de λ-termes

- $A \Rightarrow (A \Rightarrow B) \Rightarrow B$:
  $\lambda x : \iota_A . \lambda y : \iota_A \to \iota_B . yx : \iota_A \to (\iota_A \to \iota_B) \to \iota_B$
- $(A \Rightarrow B \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$:
  $\lambda x : \iota_A \to \iota_B \to \iota_C . x : (\iota_A \to \iota_B \to \iota_C) \to \iota_A \to \iota_B \to \iota_C$

Exemple de schéma d'induction fonctionnelle

Pour la fonction $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ définie récursivement par

$$\begin{aligned} f(x, 0) &= 1 \\ f(x, n+1) &= x \times f(x, n), \end{aligned}$$

le schéma d'induction fonctionnelle est

$$\begin{aligned} &\forall P \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \text{Prop.} \\ &(\forall x \in \mathbb{N}. P(x, 0, 1)) \\ \Rightarrow &(\forall x, n \in \mathbb{N}. \\ &\quad P(x, n, f(x, n)) \Rightarrow P(x, n+1, x \times f(x, n))) \\ \Rightarrow &\forall x, n \in \mathbb{N}. P(x, n, f(x, n)) \end{aligned}$$

Exemple de preuve dans la logique de Hoare

$$\dfrac{\dfrac{\dfrac{}{\{0 = 0\}\ i := 0\ \{i = 0\}}\ {:=} \qquad \top \Rightarrow 0 = 0}{\{\}\ i := 0\ \{i = 0\}}\ \text{Aff}}{}$$

$$\dfrac{\dfrac{}{\{i = 0 \wedge 0 = 0\}\ r := 0\ \{i = 0 \wedge r = 0\}}\ {:=} \qquad i = 0 \Rightarrow i = 0 \wedge 0 = 0}{\{i = 0\}\ r := 0\ \{i = 0 \wedge r = 0\}}\ \text{Aff}$$

$$\dfrac{\dfrac{}{\left\{r + i + 1 = \frac{(i+1)(i+2)}{2}\right\}\ i := i + 1;\ \left\{r + i = \frac{i(i+1)}{2}\right\}}\ {:=} \qquad \dfrac{}{\left\{r + i = \frac{i(i+1)}{2}\right\}\ r := r + i;\ \left\{r = \frac{i(i+1)}{2}\right\}}\ {:=}}{\left\{r + i + 1 = \frac{(i+1)(i+2)}{2}\right\}\ i := i + 1; r := r + i;\ \left\{r = \frac{i(i+1)}{2}\right\}}\ ;$$

$$\dfrac{\left\{r + i + 1 = \frac{(i+1)(i+2)}{2}\right\}\ i := i + 1; r := r + i;\ \left\{r = \frac{i(i+1)}{2}\right\} \qquad r = \frac{i(i+1)}{2} \Rightarrow r + i + 1 = \frac{(i+1)(i+2)}{2}}{\left\{r = \frac{i(i+1)}{2} \wedge i \neq n\right\}\ i := i + 1; r := r + i;\ \left\{r = \frac{i(i+1)}{2}\right\}}\ \text{Aff}$$

$$\dfrac{\left\{r = \frac{i(i+1)}{2} \wedge i \neq n\right\}\ i := i + 1; r := r + i;\ \left\{r = \frac{i(i+1)}{2}\right\}}{\left\{r = \frac{i(i+1)}{2}\right\}\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{i(i+1)}{2} \wedge \neg(i \neq n)\right\}}\ \text{while}$$

$$\dfrac{\left\{r = \frac{i(i+1)}{2}\right\}\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{i(i+1)}{2} \wedge \neg(i \neq n)\right\} \qquad i = 0 \wedge r = 0 \Rightarrow r = \frac{i(i+1)}{2} \qquad r = \frac{i(i+1)}{2} \wedge \neg(i \neq n) \Rightarrow r = \frac{n(n+1)}{2}}{\{i = 0 \wedge r = 0\}\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{n(n+1)}{2}\right\}}\ \text{Aff}$$

$$\dfrac{\{i = 0\}\ r := 0\ \{i = 0 \wedge r = 0\} \qquad \{i = 0 \wedge r = 0\}\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{n(n+1)}{2}\right\}}{\{i = 0\}\ r := 0;\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{n(n+1)}{2}\right\}}\ ;$$

$$\dfrac{\{\}\ i := 0\ \{i = 0\} \qquad \{i = 0\}\ r := 0;\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{n(n+1)}{2}\right\}}{\{\}\ i := 0; r := 0;\ \text{while } i \mathrel{!}= n \text{ do } i := i + 1; r := r + i;\ \left\{r = \frac{n(n+1)}{2}\right\}}\ ;$$