Assembly of the Global Stiffness Matrix

The assembly methodology for the global stiffness matrix of an exemplar 2-by-2 mesh using two-dimensional bilinear elements is covered below.

Element Global and Local Numbering Convention

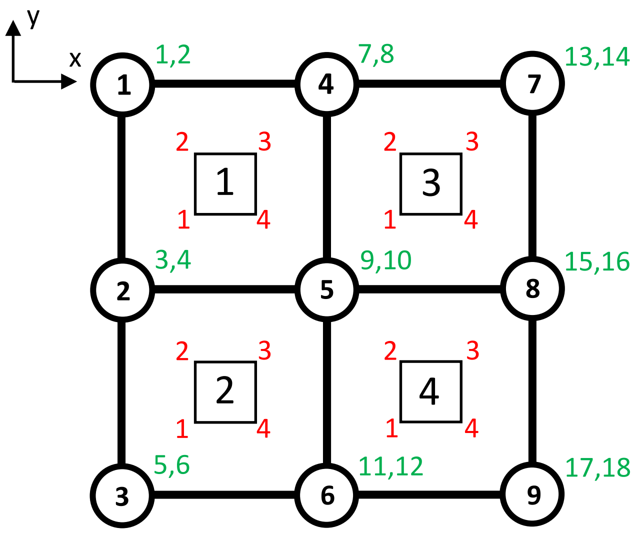
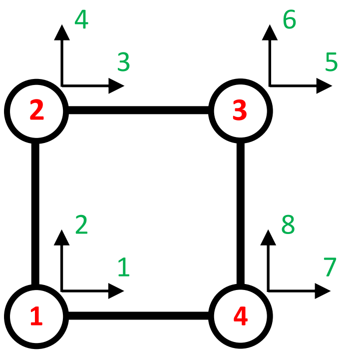


Figure A:
2x2 Global Mesh and Global Numbering Convention



$$[k^e] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,7} & a_{1,8} \\ a_{2,1} & a_{2,2} & & a_{2,7} & a_{2,8} \\ \vdots & & \ddots & & \vdots \\ a_{7,1} & a_{7,2} & & a_{7,7} & a_{7,8} \\ a_{8,1} & a_{8,2} & \cdots & a_{8,7} & a_{8,8} \end{bmatrix}$$

Figure B:
Local Element and Element Stiffness Matrix

A diagram for the 2x2 bilinear elements mesh is shown in Figure A. The global node numbering starts with node 1 at the top left, continuing downwards and moving to the next column once reaching the bottom, with 9 global nodes in total. The corresponding global degrees of freedom (dofs) are denoted in green next to each node, with each node having two dofs in the x-direction and in the y-direction, respectively. The element numbers denoted inside the squares follow a similar convention, with the first element starting at the top left, and the numbering continuing downwards and to the next column.

Figure B shows a single bilinear element, with the local node numbering (denoted in red) convention starting from the bottom left node progressing clockwise, from 1 to 4. Each local node also contains two local dofs in the x-direction and in the y-direction, following the same clockwise convention associated with each local node. Each element will therefore have $4 \times 2 = 8$ local dofs in total, hence the element stiffness matrix will be an 8x8 symmetric matrix, as shown below the diagram in Figure B.

The Assembly

The global mesh is constructed of 9 nodes with 2 dofs each, hence $9 \times 2 = 18$ global dofs in total. Therefore, the global stiffness matrix must be an 18x18 matrix. The global stiffness matrix, $[K]$, will be a matrix such that when multiplied with a global displacement vector, $\{U\}$, in the form of a column vector containing the displacement values for each global dof, will compute a column vector, $[F]$, containing the forces applied in each global dof. This is illustrated in Equation (1) below.

$$\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{17} \\ F_{18} \end{Bmatrix}}_{\substack{18\times1 \\ Global \\ Force \\ Vector,\ \{F\}}} = \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,17} & b_{1,18} \\ b_{2,1} & b_{2,2} & & b_{2,17} & b_{2,18} \\ \vdots & & \ddots & \vdots & \\ b_{17,1} & b_{17,2} & \cdots & b_{17,17} & b_{17,18} \\ b_{18,1} & b_{18,2} & & b_{18,17} & b_{18,18} \end{bmatrix}}_{\substack{18\times18 \\ Global\ Stiffness\ Matrix,\ [K]}} \underbrace{\begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{17} \\ U_{18} \end{Bmatrix}}_{\substack{18\times1 \\ Global \\ Displacement \\ Vector,\ \{U\}}} \tag{1}$$

The assembly process can be pictured as placing the appropriate entries from the local stiffness matrix into the global stiffness matrix through cumulative summation. The global stiffness matrix is firstly initialised as an 18x18 matrix full of 0's. The assembly procedure is broken down into steps performed for each element.

Starting with element 1, it is observed that the local node numbers: 1, 2, 3, and 4 correspond to the global node numbers: 2, 1, 4, and 5, respectively. Hence, the local dofs: 1&2, 3&4, 5&6, and 7&8 correspond to the global dofs: 3&4, 1&2, 7&8, and 9&10, respectively. The deformation characteristics of local node 1 is governed by the information in all 8 local dofs of the element, through shape functions. As the local node 1 (with local dofs 1&2) for element 1 corresponds to the global node 2 (with global dofs 3&4), these entries will be placed into rows 3&4 of the global stiffness matrix, as shown in Equation (2).

$$\begin{matrix} Global \\ node\ 2 \end{matrix} \left\{ \begin{array}{lll} \begin{matrix} Information\ from \\ global\ dofs\ 3\&4 \end{matrix} & \{ & \begin{bmatrix} b_{3,3} & b_{3,4} \\ b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{3,3} & b_{3,4} \\ b_{4,3} & b_{4,4} \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \\ \begin{matrix} Information\ from \\ global\ dofs\ 1\&2 \end{matrix} & \{ & \begin{bmatrix} b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{bmatrix} = \begin{bmatrix} b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{bmatrix} + \begin{bmatrix} a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} \end{bmatrix} \\ \begin{matrix} Information\ from \\ global\ dofs\ 7\&8 \end{matrix} & \{ & \begin{bmatrix} b_{3,7} & b_{3,8} \\ b_{4,7} & b_{4,8} \end{bmatrix} = \begin{bmatrix} b_{3,7} & b_{3,8} \\ b_{4,7} & b_{4,8} \end{bmatrix} + \begin{bmatrix} a_{1,5} & a_{1,6} \\ a_{2,5} & a_{2,6} \end{bmatrix} \\ \begin{matrix} Information\ from \\ global\ dofs\ 9\&10 \end{matrix} & \{ & \begin{bmatrix} b_{3,9} & b_{3,10} \\ b_{4,9} & b_{4,10} \end{bmatrix} = \begin{bmatrix} b_{3,9} & b_{3,10} \\ b_{4,9} & b_{4,10} \end{bmatrix} + \begin{bmatrix} a_{1,7} & a_{1,8} \\ a_{2,7} & a_{2,8} \end{bmatrix} \end{array} \right. \tag{2}$$

This process is repeated for each node in the element, cumulatively summing the entries from the element stiffness matrix into the global stiffness matrix for element 1, as shown in Equation (3).

$$
Element\ 1 \left\{ \begin{array}{ll}
\begin{array}{c} Global \\ node\ 2 \end{array} & \left\{ \begin{array}{l}
\begin{bmatrix} b_{3,3} & b_{3,4} \\ b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{3,3} & b_{3,4} \\ b_{4,3} & b_{4,4} \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \\
\begin{bmatrix} b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{bmatrix} = \begin{bmatrix} b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{bmatrix} + \begin{bmatrix} a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} \end{bmatrix} \\
\begin{bmatrix} b_{3,7} & b_{3,8} \\ b_{4,7} & b_{4,8} \end{bmatrix} = \begin{bmatrix} b_{3,7} & b_{3,8} \\ b_{4,7} & b_{4,8} \end{bmatrix} + \begin{bmatrix} a_{1,5} & a_{1,6} \\ a_{2,5} & a_{2,6} \end{bmatrix} \\
\begin{bmatrix} b_{3,9} & b_{3,10} \\ b_{4,9} & b_{4,10} \end{bmatrix} = \begin{bmatrix} b_{3,9} & b_{3,10} \\ b_{4,9} & b_{4,10} \end{bmatrix} + \begin{bmatrix} a_{1,7} & a_{1,8} \\ a_{2,7} & a_{2,8} \end{bmatrix}
\end{array}\right. \\
\begin{array}{c} Global \\ node\ 1 \end{array} & \left\{ \begin{array}{l}
\begin{bmatrix} b_{1,3} & b_{1,4} \\ b_{2,3} & b_{2,4} \end{bmatrix} = \begin{bmatrix} b_{1,3} & b_{1,4} \\ b_{2,3} & b_{2,4} \end{bmatrix} + \begin{bmatrix} a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{bmatrix} \\
\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} + \begin{bmatrix} a_{3,3} & a_{3,4} \\ a_{4,3} & a_{4,4} \end{bmatrix} \\
\begin{bmatrix} b_{1,7} & b_{1,8} \\ b_{2,7} & b_{2,8} \end{bmatrix} = \begin{bmatrix} b_{1,7} & b_{1,8} \\ b_{2,7} & b_{2,8} \end{bmatrix} + \begin{bmatrix} a_{3,5} & a_{3,6} \\ a_{4,5} & a_{4,6} \end{bmatrix} \\
\begin{bmatrix} b_{1,9} & b_{1,10} \\ b_{2,9} & b_{2,10} \end{bmatrix} = \begin{bmatrix} b_{1,9} & b_{1,10} \\ b_{2,9} & b_{2,10} \end{bmatrix} + \begin{bmatrix} a_{3,7} & a_{3,8} \\ a_{4,7} & a_{4,8} \end{bmatrix}
\end{array}\right. \\
\begin{array}{c} Global \\ node\ 4 \end{array} & \left\{ \begin{array}{l}
\begin{bmatrix} b_{7,3} & b_{7,4} \\ b_{8,3} & b_{8,4} \end{bmatrix} = \begin{bmatrix} b_{7,3} & b_{7,4} \\ b_{8,3} & b_{8,4} \end{bmatrix} + \begin{bmatrix} a_{5,1} & a_{5,2} \\ a_{6,1} & a_{6,2} \end{bmatrix} \\
\begin{bmatrix} b_{7,1} & b_{7,2} \\ b_{8,1} & b_{8,2} \end{bmatrix} = \begin{bmatrix} b_{7,1} & b_{7,2} \\ b_{8,1} & b_{8,2} \end{bmatrix} + \begin{bmatrix} a_{5,3} & a_{5,4} \\ a_{6,3} & a_{6,4} \end{bmatrix} \\
\begin{bmatrix} b_{7,7} & b_{7,8} \\ b_{8,7} & b_{8,8} \end{bmatrix} = \begin{bmatrix} b_{7,7} & b_{7,8} \\ b_{8,7} & b_{8,8} \end{bmatrix} + \begin{bmatrix} a_{5,5} & a_{5,6} \\ a_{6,5} & a_{6,6} \end{bmatrix} \\
\begin{bmatrix} b_{7,9} & b_{7,10} \\ b_{8,9} & b_{8,10} \end{bmatrix} = \begin{bmatrix} b_{7,9} & b_{7,10} \\ b_{8,9} & b_{8,10} \end{bmatrix} + \begin{bmatrix} a_{5,7} & a_{5,8} \\ a_{6,7} & a_{6,8} \end{bmatrix}
\end{array}\right. \\
\begin{array}{c} Global \\ node\ 3 \end{array} & \left\{ \begin{array}{l}
\begin{bmatrix} b_{9,3} & b_{9,4} \\ b_{10,3} & b_{10,4} \end{bmatrix} = \begin{bmatrix} b_{9,3} & b_{9,4} \\ b_{10,3} & b_{10,4} \end{bmatrix} + \begin{bmatrix} a_{7,1} & a_{7,2} \\ a_{8,1} & a_{8,2} \end{bmatrix} \\
\begin{bmatrix} b_{9,1} & b_{9,2} \\ b_{10,1} & b_{10,2} \end{bmatrix} = \begin{bmatrix} b_{9,1} & b_{9,2} \\ b_{10,1} & b_{10,2} \end{bmatrix} + \begin{bmatrix} a_{7,3} & a_{7,4} \\ a_{8,3} & a_{8,4} \end{bmatrix} \\
\begin{bmatrix} b_{9,7} & b_{9,8} \\ b_{10,7} & b_{10,8} \end{bmatrix} = \begin{bmatrix} b_{9,7} & b_{9,8} \\ b_{10,7} & b_{10,8} \end{bmatrix} + \begin{bmatrix} a_{7,5} & a_{7,6} \\ a_{8,5} & a_{8,6} \end{bmatrix} \\
\begin{bmatrix} b_{9,9} & b_{9,10} \\ b_{10,9} & b_{10,10} \end{bmatrix} = \begin{bmatrix} b_{9,9} & b_{9,10} \\ b_{10,9} & b_{10,10} \end{bmatrix} + \begin{bmatrix} a_{7,7} & a_{7,8} \\ a_{8,7} & a_{8,8} \end{bmatrix}
\end{array}\right.
\end{array}\right. \tag{3}
$$

Finally, this procedure is repeated for all 4 elements, completing the assembly of the global stiffness matrix.