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A SIMPLE MODEL FOR OPTION PRICING WITH JUMPING STOCHASTIC VOLATILITY

STEFANO HERZEL
Ist. Mat. Gen. Fin., Università di Perugia,
Facoltà di Economia, Via A. Pascoli 1, 06100 Perugia, Italy

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This paper proposes a simple modification of the Black–Scholes model by assuming that the volatility of the stock may jump at a random time τ from a value σ_a to a value σ_b. It shows that, if the market price of volatility risk is unknown, but constant, all contingent claims can be valued from the actual price C_0, of some arbitrarily chosen "basis" option. Closed form solutions for the prices of European options as well as explicit formulas for *vega* and *delta* hedging are given. All such solutions only depend on σ_a, σ_b and C_0. The prices generated by the model produce a "smile"-shaped curve of the implied volatility.

1. Introduction

We study a model for option pricing in which the process of the underlying security price is a diffusion with stochastic volatility. It is an apparently small modification of the celebrated Black–Scholes model (from now on: BS) which considers the possibility of a single jump for the volatility parameter σ.

The BS model is widely used for trading and hedging options, mostly because of the simplicity of its assumptions and, consequently, of its derivations. However, it is often the case that, because of the excessive simplicity of the assumptions, the model fails to explain reality. In particular, the constant volatility assumption for the underlying security seems to be contradicted by statistical evidence.

Several studies on implied volatility have shown evidence of the "smile" effect which is not consistent with constant volatility. Moreover, option traders do not seem to agree on such an assumption either. In fact, they often use a trading strategy called *vega-hedging* whose goal is to hedge the risk associated with the change in volatility of the underlying asset. Usually this is accomplished by taking a long position in one option and a short position in another option, written on the same underlying asset, but with different contract specifications.

Many models have been proposed in which the volatility is not a constant but a deterministic function of time (as in [17]) or of the stock price (e.g. [10]). All such cases lead to a market model with a single risk factor, where the riskiness of the option can be hedged away with a dynamic, self-financing, trading strategy that involves only the underlying security and the money market account. Therefore, they still do not justify the use of vega-hedging. A model which does justify vega-hedging must consider a stochastic process for volatility which is not measurable with respect to the Brownian motion driving the stochastic dynamics of the stock price.

Hull and White [15] modeled the squared volatility as a log-normal diffusion. Under the hypothesis that volatility is independent of the stock price, and assuming the premium for volatility risk is zero, they determined the arbitrage-free price of a call option.

A more general diffusion process for volatility was studied by Wiggins [22]; he derived a partial differential equation for the option price, for which he proposed a numerical integration technique that seems to be fairly computationally intensive.

Stein and Stein [21] suggested a mean reverting diffusion process and used a Fourier inverse transformation to integrate the resulting partial differential equation under the hypothesis that the correlation with the stock price process was zero. The assumption of no correlation was relaxed by Heston [14], who was also able to derive closed form solutions for the option price.

Heston [14] and Ball and Roma [3] proposed a square root diffusion for stochastic volatility whose form is similar to that used in a different setting to model the term structure of interest rate. All these papers assume that the market price of volatility risk is constant and exogenous.

For an extensive review of the most important contributions to stochastic volatility models see Ball [1].

In our setting the volatility of the stock jumps at a random time τ from a value σ_a to a value σ_b.

The basic difference between our model and all those examined so far is that we do not model stochastic volatility as a diffusion, but as a jump process. Empirical evidence has documented jump type behavior of volatility, particularly in foreign exchanges rates; see [4, 16, 5].

In order to retain one of the prominent attractions of the Black–Scholes model, simplicity, we have chosen the simplest of the jump processes where only the time of the jump is random. Such a hypothesis lets us derive very tractable closed form solutions for options pricing.

Boyle and Lee [8] presented a model of deposit insurance for banks where the volatility could take one of a given set of values at a time t, when the bank's asset price hit a previously fixed barrier. However, theirs is not a stochastic volatility model.

Naik [18] developed a model in which the volatility of the risky asset is subject to random jumps. His model is more general than ours because it lets volatility switch back and forward from one state to the other and also considers jumps for the stock price. Of course, less tractable formulas are the price to be paid for a greater generality.

A class of models for stochastic volatility which contains all of the models cited so far has been studied in Herzel [13].

A nice feature of our model is that, unlike any other model we are aware of, the market price for volatility risk does not have to be exogenous. In fact, in our case it is enough to assume that it is constant through time. Not only should this assumption permit an easier understanding of the model economy, but also it avoids the difficult task of evaluating the market price of risk.

Under some hypotheses on the probability distribution of the jump time τ, we show that the equivalent martingale measure depends on a real parameter λ_Q, which can be determined from the actual price of a derivative security. From an operational point of view it should be remarked that, in principle, there are no constraints on the choice of such a security (as long as it is volatility-dependent). In particular, we will work out the computations for the case of a European call option.

The paper is structured as follows: in Sec. 2 we specify the market model. In Sec. 3 we investigate the properties of the equivalent martingale measure and of the market prices of risks. In Sec. 4 we determine closed form solutions for pricing and hedging options when the market price for volatility risk is constant. Section 5 contains conclusions and some possible extensions of our analysis.

2. The Model

We consider an arbitrage-free, perfect market with continuous-time trading. In this section we will specify the stochastic dynamics of two traded securities: the stock and the money market account, whose prices at time t will be indicated by S_t and B_t respectively. Later, to get completeness of the market, we will assume the existence of a third traded asset X_t, whose dynamics, however, will not be assumed as an input data of the model.

First we need to construct a probability space which supports a Brownian motion and a jump process who are independent; therefore, we proceed with some definitions. Let W_t^* be a one dimensional Brownian motion on a filtered probability space $(\Omega^W, \mathcal{F}^W, \{\mathcal{F}_t^W\}_{t=0}^\infty, P^W)$, where $\{\mathcal{F}_t^W\}_{t=0}^\infty$ is the filtration generated by W_t^*, completed with the set of measure zero. Moreover, let τ^* be a random variable defined on a second probability space $(\Omega^\tau, \mathcal{F}^\tau, P^\tau)$. We assume that τ^* is exponentially distributed with parameter $\lambda > 0$, that is, for any $t \geq 0$,

$$P^\tau(\tau^* \leq t) = 1 - e^{-\lambda t}\,.$$

We indicate with $\{\mathcal{F}_t^\tau\}_{t=0}^\infty$ the (completed) filtration generated by the process $H_t := \mathbf{1}_{\{\tau^* \leq t\}}$, where, here and in the sequel, $\mathbf{1}_A$ is the indicator function of the set A. The probability space underlying the model is

$$(\Omega, \mathcal{F}, P) := (\Omega^W \times \Omega^\tau, \mathcal{F}^W \bigotimes \mathcal{F}^\tau, P^W \bigotimes P^\tau)\,,$$

endowed with the (completed) filtration

$$\{\mathcal{F}_t\}_{0\le t\le\infty} := \{\mathcal{F}_t^\tau \bigotimes \mathcal{F}_t^W\}_{0\le t\le\infty} .$$

We remark that the filtration $\mathcal{F}_t$ is generated by a two-dimensional Lèvy-process and it is completed with the set of measure zero; therefore it is right continuous [19, Theorem 1.4.31]. It is easy to show that $W_t(\omega_1,\omega_2) := W_t^*(\omega_1)$ is a one dimensional Brownian motion on the product space and that $\tau(\omega_1,\omega_2) := \tau^*(\omega_1)$ is exponentially distributed with parameter λ.

Now we can define the dynamics of the assets in the market model. Let S_t be the price at time t of a stock, called S. We assume that it satisfies

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t \,, \tag{2.1}$$

with the actual price (i.e. at time $t = 0$) being equal to S^0. Here μ is a constant and σ_t, the volatility of the stock price process is given by

$$\sigma_t := \sigma_a \mathbf{1}_{t\le\tau} + \sigma_b \mathbf{1}_{t>\tau} \,, \tag{2.2}$$

where σ_a and σ_b are strictly positive constants.

In words, σ_t is a stochastic process, independent of W_t, whose sample paths are equal to σ_a up to time $\tau(\omega)$ and to σ_b afterwards. The parameter λ regulates the expected value of the time of the jump of the volatility; in fact

$$E^P\tau = \frac{1}{\lambda} ,$$

that is, the smaller the value of λ, the later the jump is expected. While the values of σ_a and σ_b are assumed as given (i.e. they are input data), we want to stress that the value of λ (as well as that of μ), does not need to be given as an input.

We remark that, unlike σ_t, the price process S_t has continuous sample paths.

A second asset in the market is the "money market account" B, whose price process B_t satisfies

$$dB_t = rB_t dt \,, \tag{2.3}$$

where $B_0 = 1$ and r, the *risk-free interest rate*, is a positive constant, given as an input data. Of course the assumption of a constant and non-random interest rate could be removed in the same way as in some modifications of the classical BS model, with some detriment to the simplicity of the pricing formulas.

Of course, the two assets S and B alone are not enough to complete the market model. To recover market completeness (and hence a unique no-arbitrage pricing function for derivatives securities), we assume the existence a third traded asset X, whose price process X_t is the unique strong solution of

$$\frac{dX_t}{X_{t-}} = \bar{\alpha}_t dt + \bar{\beta}_t dW_t + \gamma_t dG_t \,, \tag{2.4}$$

with $\bar{\alpha}_t$, $\bar{\beta}_t$ and γ_t predictable, bounded, and regular enough for X_t to be well defined.

Here G_t is a P-martingale with cadlag (or Right Continuous with Left Limits) sample paths, associated with the exponential time τ,

$$\begin{aligned} G_t &:= E^P(\tau|\mathcal{F}_t) - \frac{1}{\lambda} \\ &= t\mathbf{1}_{t<\tau} + \left(\tau - \frac{1}{\lambda}\right)\mathbf{1}_{t\geq\tau} . \end{aligned}$$

We note that G is a martingale independent of W and that its sample paths jump at time τ. The amplitude of the jump is

$$\Delta G_\tau := G(\tau) - G(\tau-) = -\frac{1}{\lambda} .$$

Remark 1. The martingale G has been extensively studied by Chou and Meyer [9][a] They show (Proposition 1) that every (local) G-measurable martingale can be represented as a stochastic integral with respect to G. Therefore (with a minor abuse of notation), we can say that G has the "representation property" (for a definition, see [19, Sec. IV.3]), in the factor probability space $(\Omega^\tau, \mathcal{F}^\tau, \{\mathcal{F}_t^\tau\}_{t=0}^\infty, P^\tau)$.

We emphasize that we do not require an explicit formulation of $\bar{\alpha}_t$, $\bar{\beta}_t$ and γ_t. In fact the modeling of the price process for this third security X could be a rather hard task, expecially for a trader who is more used to the BS model and prefers to think in terms of changes in volatility of the underlying asset S. With a self-financing trading strategy in X_t and S_t one can get a portfolio whose price process dynamics Y_t are given by

$$\frac{dY_t}{Y_{t-}} = \alpha_t dt + \gamma_t dG_t . \tag{2.5}$$

Of course, we have to assume that α_t and γ_t are regular enough for the existence of a strong solution Y of Eq. (2.5) and that they are (almost surely) bounded. Note that the solution to (2.5) with initial condition Y_0 is given by the stochastic exponential (see [19, Theorem 2.8.36]),

$$Y_t = Y_0 e^{\int_0^t \alpha_s ds}\mathbf{1}_{\{t<\tau\}} e^{\int_0^t \gamma_s ds} + \mathbf{1}_{\{t\geq\tau\}}\left(1 - \frac{\gamma_\tau}{\lambda}\right) e^{\int_0^\tau \gamma_s ds} .$$

The setting is now complete. We have defined a set of "input data", e.g. S_0, r, σ_a, σ_b and have just made some assumptions on other parameters without completely specify them (i.e. we have stated that τ is exponentially distributed, but we have not assumed as an input data the parameter λ, we have left unspecified the coefficients driving the dynamics of the asset Y, etc.) In the next section we will formulate a minimal set of assumptions to get a unique pricing formula for derivative securities written on S.

[a] Thanks to Monique Jeanblanc-Picqué for having mentioned this paper to me.

3. The Market Prices of Risks

We recall that Q is an "equivalent martingale measure" if it is equivalent to P and if the discounted price process of any traded security is a Q-(local) martingale. Such a measure is fundamental for no-arbitrage pricing (see [12]). We want to determine a minimal set of conditions on the "coefficients" of the market model to get the existence of a unique martingale measure Q.

It follows from the equivalence of P and Q that there exists a Radon–Nykodin derivative, that is a strictly positive, integrable random variable Z, such that $\frac{dQ}{dP} = Z$ and $E^P Z = 1$. Let

$$Z_t := E^P(Z|\mathcal{F}_t)$$

be the right continuous version. Then Z_t is a uniformly integrable P-martingale which is sometimes called "the likelihood process".

Remark 2. We have already noted (in Remark 1), that G has the representation property, in the first factor space $(\Omega^\tau, \mathcal{F}^\tau, \{\mathcal{F}_t^\tau\}_{t=0}^\infty, P^\tau)$. [19, Theorem IV.3.42], proves that the Brownian Motion W^* has the analogous property in the second factor space $(\Omega^W, \mathcal{F}^W, \{\mathcal{F}_t^W\}_{t=0}^\infty, P^W)$. Then, by using the same argument as in [7, Remark 3.2], it can be proved that every local martingale on the product space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^\infty, P)$ may be represented as a stochastic integral with respect to the pair (W, G).

From the previous remark follows that the likelihood process Z, being a strictly positive martingale, satisfies

$$\begin{aligned} dZ_t &= Z_t \Phi_t dW_t + Z_{t-} \Psi_t dG_t \,, \\ Z_0 &= 1 \,, \end{aligned} \tag{3.6}$$

where Φ_t and Ψ_t are predictable processes. The explicit solution to (3.6) is

$$Z_t = Z_t^\Phi Z_t^\Psi \,, \tag{3.7}$$

where

$$Z_t^\Phi := \exp\left(\int_0^t \Phi_s dW_s - \frac{1}{2}\int_0^t \Phi_s^2 ds\right) \tag{3.8}$$

$$Z_t^\Psi := \mathbf{1}_{\{t<\tau\}} e^{\int_0^t \Psi_s ds} + \mathbf{1}_{\{t\geq\tau\}} \left(1 - \frac{\Psi_\tau}{\lambda}\right) e^{\int_0^\tau \Psi_s ds} \,. \tag{3.9}$$

Since the paths of Z_t are strictly positive, the following condition must be satisfied by Ψ_t:

$$\Psi_t < \lambda \,, \quad 0 \leq t \leq \tau \,. \tag{3.10}$$

We can now state the first result relating the dynamics of the likelihood process Z_t to the parameters of the assets price processes.

Theorem 1. *Q is an equivalent martingale measure if and only if the following equations are satisfied a.s. for each t:*

$$\sigma_t \Phi_t = r - \mu\,, \tag{3.11}$$

$$\mathbf{1}_{\{t \le \tau\}} \gamma_t \Psi_t = \lambda(r - \alpha_t)\,. \tag{3.12}$$

Proof. A process χ_t is a Q-martingale if and only if $Z_t \chi_t$ is a P-martingale. From Corollary 2.6.2 of [19], it follows that

$$d(Z\chi)_t = Z_{t-} d\chi_t + \chi_{t-} dZ_t + d[\chi, Z]_t\,,$$

where $[\chi, Z]_t$ is the quadratic covariation of χ_t, Z_t.

Let us define

$$\chi_t := (S/B)_t\,,$$

from which, by Ito's Lemma we get

$$d(ZS/B)_t = \frac{S_t Z_{t-}}{B_t}(\mu - r + \sigma_t \Phi_t)dt + \frac{S_t Z_{t-} \sigma_t}{B_t} dW_t + \frac{S_t}{B_t} dZ_t\,.$$

A necessary and sufficient condition for χ_t to be a Q-martingale is that

$$\mathcal{M} := \int_0 \frac{S_t Z_{t-}}{B_t}(\mu - r + \sigma_t \Phi_t)dt$$

is also a Q-martingale. But $\mathcal{M}$ has continuous paths with finite variation on compact sets. Therefore [19, Theorem II.6.27] it is a.s. zero. This proves (3.11).

Now we set

$$\chi_t := (Y/B)_t\,,$$

from which we get

$$\begin{aligned} d(ZY/B)_t &= \frac{Z_{t-} Y_{t-}}{B_t}([\alpha_t - r]dt + \gamma_t dG_t) + Y_{t-} dZ_t + \frac{Z_{t-} Y_{t-} \gamma_t \Psi_t}{B_t} d[G, G]_t \\ &= \frac{Z_{t-} Y_{t-}}{B_t}[(\alpha_t - r) + \mathbf{1}_{t \le \tau} \gamma_t \Psi_t / \lambda]dt + dM_t\,, \end{aligned}$$

where M is a P-martingale. Note that in the last equality we have used the fact that

$$d[G, G]_t = \frac{1}{\lambda}(\mathbf{1}_{t \le \tau} dt - dG_t)\,.$$

Hence, from the same argument as above, $Z\hat{\chi}$ is a P-martingale if and only if (3.12) holds. $\square$

Note that (3.11) and the fact that σ_t is strictly positive imply the unicity of the process Φ_t.

The next corollary gives necessary and sufficient conditions on the coefficients of the process Y for the existence of a unique martingale measure, that is for the model to be complete and arbitrage-free.

Corollary 1. *There exists a unique equivalent martingale measure if and only if the following conditions hold* (*a.s.*)

$$\gamma_t \neq 0\,, \quad t \leq \tau \tag{3.13}$$

$$\frac{r-\alpha_t}{\gamma_t} < 1\,, \quad t \leq \tau \tag{3.14}$$

$$\alpha_t = r\,, \quad t > \tau \tag{3.15}$$

Proof. From Theorem 1 follows that an equivalent martingale measure exists and is unique if and only if (3.11) and (3.12) uniquely define, through (3.7), a likelihood process Z_t. While (3.11) uniquely defines a process Φ_t, relations (3.13) and (3.15) are necessary and sufficient for the existence of a solution to (3.12) which has the form

$$\Psi_t = \frac{r-\alpha_t}{\gamma_t}\lambda \mathbf{1}_{\{t\leq\tau\}} + \mathcal{H}_t \mathbf{1}_{\{t>\tau\}}\,,$$

where $\mathcal{H}_t$ is a predictable process. From (3.7) follows that all such Ψ_t lead, for any choice of $\mathcal{H}_t$, to a unique likelihood process Z_t. Finally, relation (3.14) is equivalent to condition (3.10). □

Formula (3.15) has an immediate economic interpretation: it states that after the jump, an asset that was only subject to that risk, must have, to prevent arbitrage, the same rate of return as the risk-free asset.

Formula (3.13) states that, to complete the market, the third security must be sensible to changes in the source of risk G_t, i.e. must always react to changes in the volatility of S. Later we will show that any European option written on S has such a feature.

Let P_t be the price at time t of some security whose stochastic dynamics are given by

$$\frac{dP_t}{P_{t-}} = a_t dt + b_t dW_t + c_t dG_t\,.$$

Then the discounted price process $\bar{P}_t := P_t/B_t$ is the solution of

$$\frac{d\bar{P}_t}{\bar{P}_{t-}} = (a_t - r)dt + b_t dW_t + c_t dG_t\,.$$

From the same argument as in the proof of Theorem 1 we get that $\bar{P}_t$ is a Q-martingale if and only if, for all t and a.s.,

$$a_t = r + b_t(-\Phi_t) + c_t \mathbf{1}_{t\leq\tau}(-\Psi_t)\,. \tag{3.16}$$

Hence, by setting $c_t = 0$ and $b_t \neq 0$, it follows that $-\Phi_t$ is the market price for the *diffusion* risk; on the other hand, setting $b_t = 0$ and $c_t \neq 0$, we get that $-\Psi_t$ is the market price for the *volatility* risk.

In our setting we have made some assumptions on the probability distribution of the diffusion process S_t and of the jumping time τ under the probability measure P. The next theorem examines what happens under an equivalent martingale measure Q.

Theorem 2. *Let Q be an equivalent martingale measure. Then*

$$\tilde{W}_t = W_t - \int_0^t \Phi_s ds \tag{3.17}$$

is a Q-Brownian motion. Moreover, if

$$\mathcal{K}_t := \frac{r - \alpha_t}{\gamma_t}$$

is a deterministic function of t, then τ is Q-independent of $\tilde{W}_t$ and

$$Q(\tau > t) = \exp\left[-\int_0^t \lambda(1 - \mathcal{K}_u)du\right] . \tag{3.18}$$

Proof. From Girsanov's Theorem [19, Theorem III.6.20], it follows that

$$M_t := W_t - \int_0^t \frac{1}{Z_s} d[Z, W]_s$$

is a Q-local martingale. Since

$$d[Z, W]_t = Z_t \Phi_t dt ,$$

$\tilde{W}_t$ is a local martingale. Moreover, since $\int_0^t \Phi_s ds$ is a finite variation process

$$d[\tilde{W}, \tilde{W}]_t = d[W, W]_t = dt .$$

Hence, from Levy's Theorem [19, Theorem II.8.38], $\tilde{W}_t$ is a Q-Brownian motion.

The likelihood process Z_t, being the solution of (3.7), is the stochastic exponential

$$Z_t = Z_t^{\Phi} Z_t^{\Psi} ,$$

where Z_t^{Φ} and Z_t^{Ψ} are given by (3.8) and by (3.9).

It follows from Theorem 1 that

$$\Psi_t = \lambda \mathcal{K}_t \mathbf{1}_{t \le \tau} .$$

Hence

$$\begin{aligned} Q(t < \tau) &= E^P(Z_t \mathbf{1}_{t<\tau}) \\ &= E^P\left(\exp\left(\Phi^a W_t - \frac{t {\Phi^a}^2}{2}\right) \exp\left(\int_0^t \lambda \mathcal{K}_s ds\right) \mathbf{1}_{t<\tau}\right) , \end{aligned}$$

where we have defined

$$\Phi^a := \frac{r - \mu}{\sigma_a} .$$

From the P-independence of τ and W and the assumption of a deterministic $\mathcal{K}_t$:

$$Q(t < \tau) = \exp\left(\int_0^t \lambda \mathcal{K}_s ds\right) E^P \exp\left(\Phi^a W_t - \frac{t\Phi^{a2}}{2}\right) E^P(\mathbf{1}_{t<\tau}).$$

Now observe that

$$X_t := \exp\left(\Phi^a W_t - \frac{t\Phi^{a2}}{2}\right)$$

is the unique strong solution of

$$dX_t = X_t \Phi^a dW_t, \tag{3.19}$$

$$X_0 = 1. \tag{3.20}$$

Hence it is a P-martingale and its expected value is always equal to one; therefore,

$$\begin{aligned} Q(t < \tau) &= \exp\left(\int_0^t \lambda \mathcal{K}_s ds\right) P(t < \tau) \\ &= \exp\left(\int_0^t \lambda \mathcal{K}_s ds - \lambda t\right) \end{aligned}$$

To prove independence it suffices to prove that, for any $s, t \in \Re$ and B a Borel set of $\Re$,

$$Q(\tau < s, \tilde{W}_t \in B) = Q(\tau < s) Q(\tilde{W}_t \in B).$$

Let's first consider the case when $s \leq t$; from the definition of the product space:

$$(\Omega, \mathcal{F}, P) := (\Omega^W \times \Omega^\tau, \mathcal{F}^W \bigotimes \mathcal{F}^\tau, P^W \bigotimes P^\tau),$$

we get

$$\begin{aligned} Q(\tau < s, \tilde{W}_t \in B) &= E^P(Z_t \mathbf{1}_{\tau<s} \mathbf{1}_{\tilde{W}_t \in B}) \\ &= \int_{\Omega^\tau} dP^\tau(\omega^\tau) Z_t^\Psi(\omega^\tau) \mathbf{1}_{\tau<s}(\omega^\tau) \\ &\quad \times \int_{\Omega^W} dP^W(\omega^W) Z_t^\Phi(\omega^W, \omega^\tau) \mathbf{1}_{\tilde{W}_t \in B}(\omega^W, \omega^\tau). \end{aligned} \tag{3.21}$$

From Girsanov's Theorem it follows that for each $\omega^\tau \in \Omega^\tau$ there is a probability measure $Q(\omega^\tau)$ on Ω^W whose likelihood process (with respect to P^W) satisfies

$$\frac{dZ_t^W}{Z_t^W} = \Phi_t(\omega^\tau) dW_t^*$$

and such that

$$\tilde{W}_t^* := W *_t - \int_0^t \Phi_s(\omega^\tau) ds$$

is a $Q(\omega^\tau)$ Brownian motion.

Therefore, for each $\omega^\tau \in \Omega^\tau$,

$$\int_{\Omega^W} dP^W(\omega^W) Z_t^\Phi(\omega^W, \omega^\tau) \mathbf{1}_{\tilde{W}_t \in B}(\omega^W, \omega^\tau) = P^W(W_t^* \in B)$$
$$= Q(\tilde{W}_t \in B)\,,$$

where the last equality follows from the fact that $\tilde{W}_t$ is a Q Brownian motion.

Hence, substituting into (3.21), we get

$$\begin{aligned} Q(\tau < s, \tilde{W}_t \in B) &= Q(\tilde{W}_t \in B) \int_{\Omega^\tau} dP^\tau(\omega^\tau) Z_t^\Psi(\omega^\tau) \mathbf{1}_{\tau < s}(\omega^\tau) \\ &= Q(\tilde{W}_t \in B) \exp\left(\int_0^s \lambda \mathcal{K}_t dt \right) P(\tau < s) \\ &= Q(\tilde{W}_t \in B) Q(\tau < s) \end{aligned}$$

When $s > t$ we get

$$\begin{aligned} &Q(\tau < s, \tilde{W}_t \in B) = E^P(Z_s \mathbf{1}_{\tau < s} \mathbf{1}_{\tilde{W}_t \in B}) \\ &\quad = \int_{\Omega^\tau} dP^\tau(\omega^\tau) Z_s^\Psi(\omega^\tau) \mathbf{1}_{\tau < s}(\omega^\tau) \int_{\Omega^W} dP^W(\omega^W) Z_s^\Phi(\omega^W, \omega^\tau) \mathbf{1}_{\tilde{W}_t \in B}(\omega^W, \omega^\tau)\,. \end{aligned}$$

Now observe that, for each ω^τ in Ω^τ,

$$Z_s^\Phi(\cdot, \omega^\tau) = Z_t^\Phi(\cdot, \omega^\tau) \zeta_{t,s}^\Phi(\cdot, \omega^\tau)\,, \tag{3.22}$$

where

$$\zeta_{t,s}^\Phi(\cdot, \omega^\tau) = \exp\left[\left(\int_t^s \Phi_u(\omega^\tau) dW_u^* \right)(\cdot) - \frac{1}{2} \int_t^s \Phi_u^2(\omega^\tau) du \right].$$

Note that, for each ω^τ, $\zeta_{t,s}^\Phi(\cdot, \omega^\tau)$ is P^W-independent of $Z_t^\Phi(\cdot, \omega^\tau) \mathbf{1}_{\tilde{W}_t \in B}(\cdot, \omega^\tau)$ (since Brownian motion has independent increments).

Hence, for each ω^τ,

$$\begin{aligned} &\int_{\Omega^W} dP^W(\omega^W) Z_s^\Phi(\omega^W, \omega^\tau) \mathbf{1}_{\tilde{W}_t \in B}(\omega^W, \omega^\tau) \\ &\quad = \int_{\Omega^W} dP^W(\omega^W) \zeta_{t,s}^\Phi(\omega^W, \omega^\tau) \int_{\Omega^W} dP^W(\omega^W) Z_t^\Phi(\omega^W, \omega^\tau) \mathbf{1}_{\tilde{W}_t \in B}(\omega^W, \omega^\tau) \\ &\quad = Q(\tilde{W}_t \in B)\,, \end{aligned}$$

where the last equality follows from the fact that, for each ω^τ,

$$\int_{\Omega^W} dP^W(\omega^W) \zeta_{t,s}^\Phi(\omega^W, \omega^\tau) = 1$$

which can easily be shown by using (3.22) and the P^W-independence of $\zeta_{t,s}^\Phi(\cdot, \omega^\tau)$ and $Z_t^\Phi(\cdot, \omega^\tau)$.

Therefore, for any s and t,

$$Q(\tau < s, \tilde{W}_t \in B) = Q(\tau < s) Q(\tilde{W}_t \in B)$$

and the proof is complete. □

The last result states that both the independence of τ from the Brownian motion and the exponential distribution of τ are preserved under the equivalent martingale measure Q if the market price of volatility risk is independent of the diffusion risk. Hence such properties depend on investor's preferences. This should not come as a surprise: in fact, if the market price of volatility risk depends on the diffusion risk, then the prices of the Arrow–Debreu securities for each of these risk factors cannot be independent. It should also be seen as a warning to those studies who make assumptions on such things like distributions and independence directly under the equivalent martingale measure without investigating the implications of such hypotheses on the preferences of investors.

In the next section we will work out computations explicitly for a particular form of the market price of volatility risk.

4. Hedging and Pricing Options

In this section we will be using the equivalent martingale measure theory as a tool to assign a no-arbitrage price to some contingent claims.

We assume that the market price of volatility risk is given by

$$\Psi_t := \beta\lambda\mathbf{1}_{t\leq\tau}\,, \tag{4.23}$$

where β is some unknown constant. From (3.14) it follows that β must be strictly smaller than 1.

A possible (although only psychological) justification of such an assumption is that, if all investors agree on the probability model depicted so far, they should not ask for a risk premium for volatility risk that varies with time or is connected to the price of the stock. From the point of view of a user of our model, it should be much easier making (and understanding) such hypothesis than providing the stochastic process for the third security X_t. In fact we will see that this is all we need to price and hedge derivative securities.

From assumption (4.23) and the results of the previous section it follows that

(i) the stochastic dynamics of S_t under Q are given by

$$\frac{dS_t}{S_t} = rdt + \sigma_t d\tilde{W}_t\,, \tag{4.24}$$

with $\tilde{W}_t$ being a Brownian motion with respect to Q.

(ii) τ is independent of $\tilde{W}_t$ and its distribution with respect to Q is exponential with parameter

$$\lambda_Q = \lambda(1-\beta)\,.$$

We remark that, since λ and β are not input data, the constant λ_Q is unknown; however, we will show that it can be determined from the market price of some derivative security.

The process S_t satisfying (4.24) is given by

$$
\begin{aligned}
S_t &= S_0 \mathbf{1}_{t<\tau} \exp\left(\left(r - \frac{\sigma_a^2}{2}\right) t + \sigma_a \tilde{W}_t\right) \\
&\quad + S_0 \mathbf{1}_{t\geq\tau} \exp\left(\left(r - \frac{\sigma_a^2}{2}\right)\tau + \left(r - \frac{\sigma_b^2}{2}\right)(t-\tau) + \sigma_a \tilde{W}_\tau + \sigma_b(\tilde{W}_t - \tilde{W}_\tau)\right) \\
&= S_0 \mathbf{1}_{t<\tau} e^{(r-\frac{\sigma_a^2}{2})t+\sigma_a \tilde{W}_t} + S_\tau \mathbf{1}_{t\geq\tau} e^{(r-\frac{\sigma_b^2}{2})(t-\tau)+\sigma_b(\tilde{W}_t-\tilde{W}_\tau)} .
\end{aligned}
$$

We observe that, for $\tau = t < T$, S_T has the same distribution as

$$
S_0 \exp\left(\left(r - \frac{\bar{\sigma}(t,T)^2}{2}\right) T + \bar{\sigma}(t,T)\tilde{W}_T\right) ,
$$

where

$$
\bar{\sigma}(t,T) = \left(\frac{\sigma_a^2 t + \sigma_b^2 (T-t)}{T}\right)^{1/2} . \tag{4.25}
$$

Hence, when the time of volatility's jump is not stochastic, the model reduces to a BS-like setting with volatility given by the average $\bar{\sigma}(t,T)$.

The arbitrage-free price at time t of a European call option, expiring at $T > t$, with strike price equal to K, is given by

$$
C_t = e^{-r(T-t)} E_t^Q (S_T - K)^+
$$

where E_t^Q is the Q-expectation conditioned to $\mathcal{F}_t$. Hence the actual (i.e. $t = 0$) arbitrage-free price is given by

$$
\begin{aligned}
C_0 &= E_0^Q[(S_T-K)^+ | \tau \leq T](1 - e^{-\lambda_Q T}) \\
&\quad + E_0^Q[(S_T-K)^+ | \tau > T] e^{-\lambda_Q T} .
\end{aligned}
$$

Let $J_s(\sigma)$ be the BS price of the same option at a time $s < T$, when the volatility of the underlying process is a given σ, i.e.

$$
J_s(\sigma) := S_s \mathcal{N}(d_s^1(\sigma)) - K e^{-r(T-s)} \mathcal{N}(d_s^2(\sigma)) .
$$

Here $\mathcal{N}(\cdot)$ is the cumulative, standard, normal distribution function,

$$
d_s^1(\sigma) := \frac{[\log(S_s/K) + (r + \sigma^2/2)(T-s)]}{\sigma\sqrt{T-s}} ,
$$

and

$$
d_s^2(\sigma) := d_s^1(\sigma) - \sigma\sqrt{T-s} .
$$

From the independence of τ and the Brownian motion, we get

$$
C_0 = \lambda_Q \int_0^T e^{-\lambda_Q t} J_0(\bar{\sigma}(t,T)) dt + e^{-\lambda_Q T} J_0(\sigma_a) , \tag{4.26}
$$

where $\bar{\sigma}(t,T)$ is the average volatility defined in (4.25). After integrating by parts, we get a better expression for the price of the call

$$C_0 = J_0(\sigma_b) + \int_0^T e^{-\lambda_Q t} D_0(\bar{\sigma}(t,T))dt\,, \tag{4.27}$$

where

$$\begin{aligned} D_0(\bar{\sigma}(t,T)) &:= \frac{d}{dt} J_0(\bar{\sigma}(t,T)) \\ &= \frac{S_0(\sigma_a^2 - \sigma_b^2)}{\bar{\sigma}(t,T)\sqrt{8\pi T}} \exp(-d_0^1(\bar{\sigma}(t,T))^2/2)\,. \end{aligned}$$

The first derivative of the option value with respect to the volatility of the underlying asset is called the *vega* of the option and it is usually indicated with the symbol Λ, that is

$$\Lambda(\bar{\sigma}(t,T)) = \sqrt{\frac{T}{2\pi}} S_0 \exp(-d_0^1(\bar{\sigma}(t,T))^2/2)\,.$$

Substituting into the formula for C_0 we get:

$$C_0 = J_0(\sigma_b) + \frac{\sigma_a^2 - \sigma_b^2}{T} \int_0^T \frac{e^{-\lambda_Q t}\Lambda(\bar{\sigma}(t,T))}{2\bar{\sigma}(t,T)} dt\,.$$

We note that $D_0(\bar{\sigma}(t,T))$ is positive if and only if $\sigma_a > \sigma_b$. Therefore $C_0 > J_0(\sigma_b)$ if and only if $\sigma_a > \sigma_b$, that is if it is believed that the volatility may decrease sometime in the future. Analogously, from (4.26) we can also get

$$C_0 = J_0(\sigma_a) + \int_0^T \lambda_Q e^{-\lambda_Q t}[J_0(\bar{\sigma}(t,T)) - J_0(\sigma_a)]dt\,.$$

From this last relation it follows that $C_0 < J_0(\sigma_a)$ if and only if $\sigma_a > \sigma_b$. This is in accordance with what is proved by El Karoui *et al.* [11, Theorem 6.2], on the bounds of the option prices with a stochastic volatility.

Figure 1 was obtained by considering a sequence of prices generated by our model for European call options with different strike prices but same maturity and the corresponding implied volatilities. It shows that our model produces the often observed effect of the "volatility smile". This is in accordance with [20, Theorem 4.2], who showed that the smile effect is a consequence of the stochastic volatility feature when the processes for S and σ are Q-independent.

Formula (4.27) is easily generalized to get the stochastic process for the option price: for $0 \le s \le T$,

$$C_s = J_s(\sigma_b) + \mathbf{1}_{s<\tau} \int_s^T e^{-\lambda_Q t} D_s(\bar{\sigma}(t, T-s))dt\,. \tag{4.28}$$

Hence, if the volatility jumps at $t < T$, the option price also jumps by an amount

$$\begin{aligned} \Delta C_t &:= C_t - C_{t-} \\ &= -\int_t^T e^{-\lambda_Q s} D_t(\bar{\sigma}(s, T-t))ds\,. \end{aligned} \tag{4.29}$$

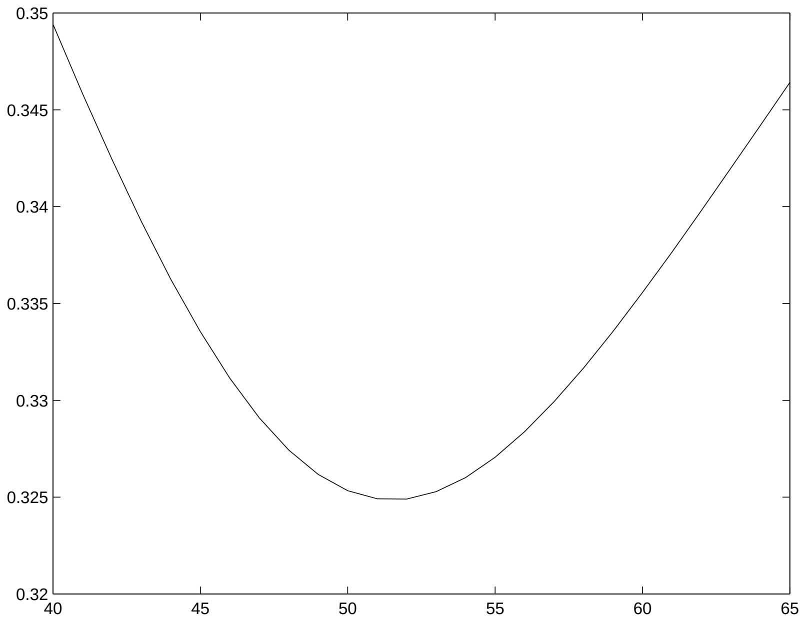

Fig. 1. Black–Scholes implied volatility for a European call option as a function of the strike price when $S_0 = 50$, $T = 0.6$, $r = 0.05$, $\sigma_a = 0.5$, $\sigma_b = 0.1$, $\lambda = 3$.

This observation and Corollary 1 shows that any European option completes the market. In fact, using the notation of Corollary 1, we have

$$\gamma_t = -\lambda \Delta C_t \,,$$

and condition (3.13) is satisfied. Condition (3.15) is also satisfied because otherwise there would be an arbitrage opportunity, while condition (3.14) is fulfilled by the construction of Q (see (4.23)). This result is in the line of what proved by Bajeux–Besnainou and Rochet [2, Proposition 5.2], who showed that European Options are an appropriate instrument to get market completeness in a stochastic volatility model where the volatility is a diffusion process.

Since the parameter λ_Q drives the speed of the change in σ and $J(\sigma)$ is a monotone function, C_0 should be expected to be a monotone function of λ_Q: that is what next proposition proves.

Proposition 1. *Let C_0 be given by expression* (4.27). *Then if $\sigma_a < \sigma_b$, it is a monotone increasing function of λ_Q, otherwise it is monotone decreasing. Moreover*,

$$\lim_{\lambda_Q \to 0} C_0 = J_0(\sigma_a)\,, \tag{4.30}$$

$$\lim_{\lambda_Q \to \infty} C_0 = J_0(\sigma_b)\,. \tag{4.31}$$

Proof. By taking the derivative of C_0 with respect to λ_Q we get

$$\frac{\partial C_0}{\partial \lambda_Q} = \frac{\sigma_b^2 - \sigma_a^2}{T} \int_0^T \frac{te^{-\lambda_Q t}\Lambda(\bar{\sigma}(t,T))}{2\bar{\sigma}(t,T)} dt \,,$$

which is a function of λ_Q with the same sign as $\sigma_b^2 - \sigma_a^2$.

From a straightforward application of the Dominated Convergence Theorem we get (4.30) and (4.31). □

Let C_0 be the observed price of our favorite call option. We call it the *basis* option. In practical applications it should be chosen among the most representative options actually traded on the market. From Proposition 1 it follows that, if C_0 is between $J_0(\sigma_a)$ and $J_0(\sigma_b)$, the value of the unknown parameter λ_Q is uniquely determined from the price of the basis option, by inverting (4.27) with respect to λ_Q.

Once λ_Q has also been determined, we have all the parameters needed for no-arbitrage pricing and hedging of contingent claims. In fact, we can construct a self-financing trading strategy to hedge all the risk factors considered by our model. We will show how to do it with an example.

Let E_t and F_t be the prices at time t of two derivative securities, called E and F, both written on S. We assume that $E_t = f^E(t, S_t, G_t)$ and $F_t = f^F(t, S_t, G_t)$ are caglad processes, where f^E and f^F are regular enough to apply Ito's Lemma [19, Theorem II.33].

Let $(\Theta_t^S, \Theta_t^E, \Theta_t^F)$ be a self-financing trading strategy (a caglad process), whose components represent, respectively, the shares of S, E and F held in a portfolio P at time t. The value at time t of the portfolio P is given by

$$P_t = \Theta_t^S S_t + \Theta_t^E E_t + \Theta_t^F F_t \,,$$

while the self-financing condition gives

$$dP_t = \Theta_t^S dS_t + \Theta_t^E dE_t + \Theta_t^F dF_t \,.$$

Using Ito's Lemma we can get an expression of dP_t in terms of the processes W and G:

$$dP_t = a_t dt + b_t S_t \sigma_t dW_t - \lambda c_t dG_t \,,$$

where

$$\begin{aligned} b_t &= \Theta_t^S + \Theta_t^E \frac{\partial f^E}{\partial S}(t, S_t, G_t) + \Theta_t^F \frac{\partial f^F}{\partial S}(t, S_t, G_t) \\ c_t &= \Theta_t^E \Delta E_t + \Theta_t^F \Delta F_t \,, \end{aligned}$$

with

$$\begin{aligned} \Delta E_t &:= E_t - E_{t-} \\ \Delta F_t &:= F_t - F_{t-} \,, \end{aligned}$$

(for a_t because it is not necessary for hedging purposes). Hence for a perfect hedging we must choose a trading strategy $(\Theta_t^S, \Theta_t^E, \Theta_t^F)$ that satisfies

$$\Theta_t^S + \Theta_t^E \frac{\partial f^E}{\partial S}(t, S_t, G_t) + \Theta_t^F \frac{\partial f^F}{\partial S}(t, S_t, G_t) = 0\,, \tag{4.32}$$

$$\Theta_t^E \Delta E_t + \Theta_t^F \Delta F_t = 0\,. \tag{4.33}$$

Therefore, to construct a perfectly hedged portfolio, we must be able to compute the partial derivatives of E and F with respect to the underlying security (i.e. their "Delta's") as well as their "jump terms", ΔE_t and ΔF_t.

As an example of application of the hedging formulas we can imagine the situation of an investor who is short one call option C^K with strike price K and maturity T. Suppose that she wants to construct an hedged portfolio by using a second call option C^H on the same underlying asset and with the same maturity, but with a different strike price, H. From the option pricing formula (4.28) we get

$$\frac{\partial C^H}{\partial S}(t, S_t, G_t) = \frac{\partial J_t^H(\sigma_b)}{\partial S} + \mathbf{1}_{t<\tau} \int_t^T e^{-\lambda_Q u} \frac{\partial D_t^H(\bar\sigma(u, T-t))}{\partial S} du\,,$$

$$\Delta C_t^H = -\int_t^T e^{-\lambda_Q s} D_t^H(\bar\sigma(s, T-t)) ds\,.$$

From (4.32) and (4.33) it follows that she should hold

$$\bar\Theta_t^H = \mathbf{1}_{t\le\tau} \frac{\int_t^T e^{-\lambda_Q u}\Lambda(\bar\sigma(u, T-u), K) du}{\int_t^T e^{-\lambda_Q u}\Lambda(\bar\sigma(u, T-u), H) du}$$

shares of C^H and

$$\begin{aligned}\bar\Theta_t^S := -\bar\Theta_t^H &\left(\Delta(\sigma_b, H) - (\sigma_a^2 - \sigma_b^2)\int_t^T e^{-\lambda_Q u}\frac{\Lambda'(\bar\sigma(u, T-u), H)}{2(T-u)\bar\sigma(u, T-u)} du\right) \\ &- \left(\Delta(\sigma_b, K) - \mathbf{1}_{t\ge\tau}(\sigma_a^2 - \sigma_b^2)\int_t^T e^{-\lambda_Q u}\frac{\Lambda'(\bar\sigma(u, T-u), K)}{2(T-u)\bar\sigma(u, T-u)} du\right)\end{aligned}$$

shares of S.

In the last formula we have used the classical (for traders) "greek" notation: $\Lambda(\sigma, H)$ and $\Delta(\sigma, H)$ are the partial derivatives of the BS option price taken with respect to, respectively, the volatility and the underlying asset (i.e. *vega* and *delta*). For the sake of an easier notation, the only variables we have explicitly written here are the volatility σ and the strike price H (or K). We indicated with $\Lambda'(\sigma, H)$ is the partial derivative of $\Lambda(\sigma, H)$ taken with respect to the underlying asset. Of course, even if we have not written it explicitly, all such values are functions of S_t, the actual stock price.

Note that after the jump, i.e. for $t > \tau$, vega-hedging is not needed any more and we recover the usual BS delta-hedging formula.

5. Conclusions

We derived a formal model for vega-hedging in a very simple setting. We examined a market model with two risk factors, the "diffusion" and the "volatility" risks. We assumed as given the stochastic dynamics of one asset S_t and of the money market account B_t as in the BS setting. Then we determined some necessary and sufficient conditions on the coefficients of the model to get a complete and arbitrage-free market. Afterwards we required the market price of volatility risk to be some positive, albeit unknown, constant. Under such a hypothesis (much weaker than assuming as given the stochastic dynamics of the price of a third security), we derived a characterization of the equivalent martingale measure Q in terms of the parameter λ_Q of the exponential distribution of the time of the volatility jump, τ. We proved that such a parameter is uniquely determined from the actual price of a "basis" option on S_t. With this we determined closed formulas for option pricing and hedging. Moreover, we observed that the model prices for a sequence of call options with different strike prices are consistent with the "volatility smile" effect.

We stress that, after accepting the assumption of a constant market price for the volatility risk, the only data needed as an input for our model are the two values σ_a and σ_b for the volatility, the market price S_0 of the stock, the risk-free interest rate r and the market price C_0 of the basis option.

At a first stage of approximation the values for σ_a and σ_b may be derived as the implicit volatilities of options with different maturities. However, it would be interesting to implement a statistical method for estimating those values of σ_a, σ_b and λ_Q which best fit the observed prices. Also, as suggested by an anonymous referee, it would be of some interest to analyze the connections between the parameters in the option price formula and the shape of the volatility smile.

For some extensions of the model, like considering more than one jump, a jumping asset price process S_t, or a more general structure for the market price of volatility risk, see [13].

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