In[1]:= `ClearAll["Global`*"]`

This workbook contains the details of the calculations presented in Section J of the Supplementary material, as well as code that can be used to prove stability of a 3-dimensional replicator equation.

This function takes a replicator equation with matrix A and equilibrium p in the interior of the simplex, and transforms it into the equation with matrix B with form $B = \begin{pmatrix} 0 & a & -a+k \\ -b+k & 0 & b \\ c & -c+k & 0 \end{pmatrix}$

It returns the new matrix as well as the values of a, b, c, and k.

In[2]:=
```
abckForm[A_, p_] := Module[{A1, A2, A3, k1, a1, b1, c1, k2},
  A1 = A.DiagonalMatrix[p]; A2 = A1 - Outer[Times, {1, 1, 1}, Diagonal[A1]];
  k1 = Total[A2[[1]]]; A3 = If[k1 == 0, A2, A2 / Abs[k1]]; k2 = If[k1 == 0, 0, k1 / Abs[k1]];
  a1 = A3[[1, 2]];
  b1 = A3[[2, 3]];
  c1 = A3[[3, 1]];
  {B → A3, {a → a1, b → b1, c → c1, k → k2}}]
```

For example, this is the transformed matrix for the example by Taylor and Jonker (1978)

In[3]:= `TJ = abckForm[{{2, 1, 5}, {5, 1, 0}, {1, 4, 3}}, {15, 11, 9} / 35]`

Out[3]= $\left\{B \to \left\{\{0, 0, 1\}, \left\{\frac{5}{2}, 0, -\frac{3}{2}\right\}, \left\{-\frac{5}{6}, \frac{11}{6}, 0\right\}\right\}, \left\{a \to 0, b \to -\frac{3}{2}, c \to -\frac{5}{6}, k \to 1\right\}\right\}$

These are the matrices corresponding to the other examples:
Weibull 1997

In[4]:= `W = abckForm[{{1, 5, 0}, {0, 1, 5}, {5, 0, 4}}, {3, 8, 7} / 18]`

Out[4]= $\left\{B \to \left\{\{0, 8, -7\}, \left\{-\frac{3}{4}, 0, \frac{7}{4}\right\}, \{3, -2, 0\}\right\}, \left\{a \to 8, b \to \frac{7}{4}, c \to 3, k \to 1\right\}\right\}$

Cressman and Tao 2014

In[5]:= `CT = abckForm[{{0, 6, -4}, {-4, 0, 4}, {2, -2, 0}}, {10, 8, 11} / 29]`

Out[5]= $\{B \to \{\{0, 12, -11\}, \{-10, 0, 11\}, \{5, -4, 0\}\}, \{a \to 12, b \to 11, c \to 5, k \to 1\}\}$

Zeeman 1980

In[6]:= `Z = abckForm[{{0, 1, 1}, {-1, 0, 3}, {1, 1, 0}}, {1, 1, 1} / 3]`

Out[6]= $\left\{B \to \left\{\left\{0, \frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, 0, \frac{3}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, 0\right\}\right\}, \left\{a \to \frac{1}{2}, b \to \frac{3}{2}, c \to \frac{1}{2}, k \to 1\right\}\right\}$

Hofbauer and Sigmund 1998

In[7]:= `HS = abckForm[{{0, 6, -4}, {-3, 0, 5}, {-1, 3, 0}}, {1, 1, 1} / 3]`

Out[7]= $\left\{B \to \left\{\{0, 3, -2\}, \left\{-\frac{3}{2}, 0, \frac{5}{2}\right\}, \left\{-\frac{1}{2}, \frac{3}{2}, 0\right\}\right\}, \left\{a \to 3, b \to \frac{5}{2}, c \to -\frac{1}{2}, k \to 1\right\}\right\}$

Having transformed the matrix in this way, we search for weights w, such that H(Ã) is copositive; the matrix H(Ã) is the symmetric part of $\tilde{A} = (1w^T - I)AD(w)^{-1}$, where w is a positive vector with components summing to 1

In[8]:=
```
B := {{0, a, k - a}, {k - b, 0, b}, {c, k - c, 0}}
```

In[9]:=
```
Ã := (Outer[Times, {1, 1, 1}, {w1, w2, 1 - w1 - w2}] - IdentityMatrix[3]).
   B.FullSimplify[Inverse[DiagonalMatrix[{w1, w2, 1 - w1 - w2}]]]
```

In[10]:=
```
HÃ := FullSimplify[(Ã + Transpose[Ã]) / 2]
```

These are the conditions for copositivity of HÃ, assuming that its determinant is zero (which is always true), k = 1 (a necessary condition for stability), and that w1>0, w2>0, 1-w1-w2>0

In[11]:=
```
HÃ1 := FullSimplify[HÃ /. k → 1]
```

These are all the conditions for copositivity of H(Ã)

In[12]:=
```
copositivity := FullSimplify[Reduce[{HÃ1[[1, 1]] ≥ 0, HÃ1[[2, 2]] ≥ 0, HÃ1[[3, 3]] ≥ 0,
      HÃ1[[1, 2]] + Sqrt[HÃ1[[1, 1]] × HÃ1[[2, 2]]] ≥ 0, HÃ1[[1, 3]] + Sqrt[HÃ1[[1, 1]] × HÃ1[[3, 3]]] ≥ 0,
      HÃ1[[2, 3]] + Sqrt[HÃ1[[2, 2]] × HÃ1[[3, 3]]] ≥ 0, w1 > 0, w2 > 0, 1 - w1 - w2 > 0}],
    Assumptions → {a ∈ ℝ, b ∈ ℝ, c ∈ ℝ, w1 ∈ ℝ, w2 ∈ ℝ}] // LogicalExpand
```

There are three sets of conditions: two represent specific choices of w1 and w2, while the third is a set of nonlinear inequalities

In[13]:=
```
Length[copositivity]
```

Out[13]=

3

First set of conditions

In[14]:=
```
c1 = copositivity[[1]]
```

Out[14]=

$$b + \frac{(-1 + c)\,(-1 + w1 + w2)}{w2} == 0 \;\&\&\; 1 + a\,(-1 + w1) + \frac{1}{(-1 + w1)\,w1}\Bigg((-1 + b)\,(-1 + w2)\,w2 + (-1 + b)\,w1\,w2\,(1 + w2) + 2\,w1\,\sqrt{\left(\frac{1}{w1}\,w2\,(-1 + w1 + w2)\,\big((-1 + c)\,(-1 + w1) + (-1 + b + c)\,w2\big)\,\big(c\,(-1 + w1) + (-1 + b + c)\,w2\big)\right)} + c\,(-1 + w1 + w2)\,\big(w2 + w1\,(-1 + w1 + w2)\big)\Bigg) == w1 + w2 \;\&\&\; w1 > 0 \;\&\&\; w2 > 0 \;\&\&\; w1 + w2 < 1$$

Second set of conditions

In[15]:= `c2 = copositivity[[2]]`

Out[15]=

$$b + \frac{c\,(-1 + w1 + w2)}{w2} == 1 \,\&\&\, 1 + a\,(-1 + w1) + \frac{1}{(-1 + w1)\,w1}\Bigg((-1 + b)\,(-1 + w2)\,w2 + (-1 + b)\,w1\,w2\,(1 + w2) + 2\,w1\,\sqrt{\left(\frac{1}{w1}\,w2\,(-1 + w1 + w2)\,\big((-1 + c)\,(-1 + w1) + (-1 + b + c)\,w2\big)\,\big(c\,(-1 + w1) + (-1 + b + c)\,w2\big)\right)} + c\,(-1 + w1 + w2)\,\big(w2 + w1\,(-1 + w1 + w2)\big)\Bigg) == w1 + w2 \,\&\&\, w1 > 0 \,\&\&\, w2 > 0 \,\&\&\, w1 + w2 < 1$$

We can solve the first two equations of c1 for w1 and w2; these will be valid choices as long as the inequalities w1>0, w2>0, 1-w1-w2>0 are satisfied

In[16]:= `p1 = FullSimplify[Solve[c1[[{1, 2}]], {w1, w2}]][[1]]`

Solve: There may be values of the parameters for which some or all solutions are not valid.

Out[16]=

$$\left\{w1 \to \frac{b\,(-1 + c)}{(-1 + b)\,(-1 + c) + a\,(-1 + b + c)},\ w2 \to \frac{(-1 + c)\,\big(1 - c + a\,(-1 + b + c)\big)}{(-1 + b + c)\,\big((-1 + b)\,(-1 + c) + a\,(-1 + b + c)\big)}\right\}$$

In[17]:= `p1inequalities = Reduce[{w1 > 0, w2 > 0, 1 - w1 - w2 > 0} /. p1]`

Out[17]=

$$\left(c < 1 \,\&\&\, b < 0 \,\&\&\, a < \frac{-1 + c}{-1 + b + c}\right) \,||\, \left(c > 1 \,\&\&\, b > 0 \,\&\&\, a > \frac{-1 + c}{-1 + b + c}\right)$$

The same can be done for the second set of conditions

In[18]:= `p2 = FullSimplify[Solve[c2[[{1, 2}]], {w1, w2}]][[1]]`

Solve: There may be values of the parameters for which some or all solutions are not valid.

Out[18]=

$$\left\{w1 \to \frac{(-1 + b)\,c}{(-1 + b)\,(-1 + c) + a\,(-1 + b + c)},\ w2 \to \frac{c\,\big(1 - b + a\,(-1 + b + c)\big)}{(-1 + b + c)\,\big((-1 + b)\,(-1 + c) + a\,(-1 + b + c)\big)}\right\}$$

In[19]:= `p2inequalities = Reduce[{w1 > 0, w2 > 0, 1 - w1 - w2 > 0} /. p2]`

Out[19]=

$$\left(c < 0 \,\&\&\, b < 1 \,\&\&\, a < \frac{-1 + b}{-1 + b + c}\right) \,||\, \left(c > 0 \,\&\&\, b > 1 \,\&\&\, a > \frac{-1 + b}{-1 + b + c}\right)$$

Finally, the third set of conditions is a complicated system of nonlinear inequalities

In[20]:=
```
c3 = copositivity[[3]]
```

Out[20]=

$$w1 > 0 \,\&\&\, w2 > 0 \,\&\&\, b + \frac{(-1 + c)\,(-1 + w1 + w2)}{w2} > 0 \,\&\&\, w1 + w2 < 1 \,\&\&\, b + \frac{c\,(-1 + w1 + w2)}{w2} < 1 \,\&\&\, a \le \frac{1}{(-1 + w1)^2\, w1} \Bigg(-\left((-1 + c)\, w1^3\right) - (-1 + b + c)\,(-1 + w2)\, w2 + w1^2 \left(-2 - 2\,c\,(-1 + w2) + w2\right) + w1 \Bigg(1 - c + (-b + c)\, w2 - (-1 + b + c)\, w2^2 + 2\sqrt{\left(\frac{1}{w1}\, w2\,(-1 + w1 + w2)\left((-1 + c)\,(-1 + w1) + (-1 + b + c)\, w2\right)\left((-1 + b)\, w2 + c\,(-1 + w1 + w2)\right)\right)}\Bigg)\Bigg) \,\&\&\, -\frac{1}{(-1 + w1)^2\, w1} \Bigg((-1 + c)\, w1^3 + w1^2 \left(2 + 2\,c\,(-1 + w2) - w2\right) + (-1 + b + c)\,(-1 + w2)\, w2 + w1 \Bigg(-1 + c + c\,(-1 + w2)\, w2 + w2\left(b + (-1 + b)\, w2\right) + 2\sqrt{\left(\frac{1}{w1}\, w2\,(-1 + w1 + w2)\left((-1 + c)\,(-1 + w1) + (-1 + b + c)\, w2\right)\left(c\,(-1 + w1) + (-1 + b + c)\, w2\right)\right)}\Bigg)\Bigg) \le a$$

This function plots possible solutions for w1 and w2 given the values of a b c and for k = 1, as well as the two points given by the first two conditions (if they represent valid solutions)

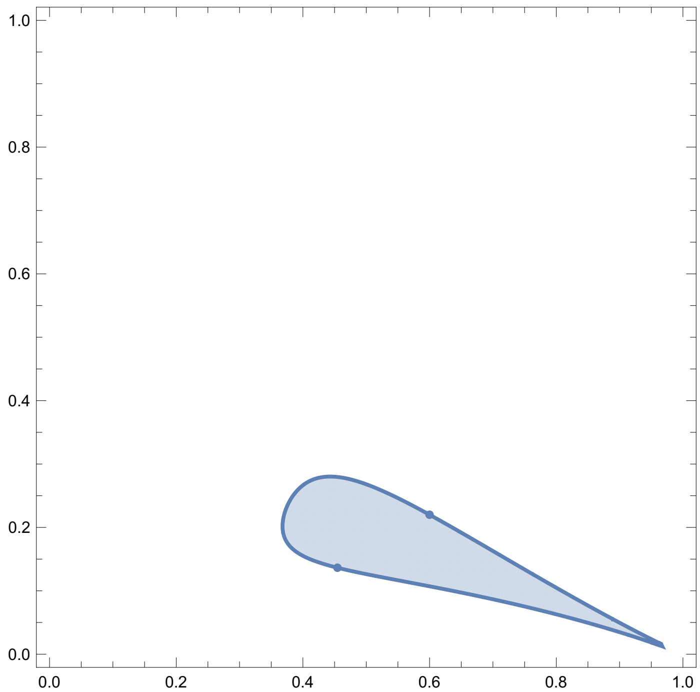
In[21]:=
```
plotSolutions[rules_] :=
 Module[{pl, lp}, pl = RegionPlot[c3 /. rules, {w1, 0, 1}, {w2, 0, 1}, PlotPoints → 100];
  lp = ListPlot[{{w1, w2} /. p1 /. rules, {w1, w2} /. p2 /. rules}];
  Show[{pl, lp}]]
```

In[22]:=
```
getSolution[rules_] := FindInstance[copositivity /. rules, {w1, w2}][[1]]
```

This is the valid region of weights for the example of Taylor and Jonker

In[23]:= `plotSolutions[TJ[[2]]]`

Out[23]=



This extracts a valid solution from the set of valid solutions

In[24]:= `getSolution[TJ[[2]]]`

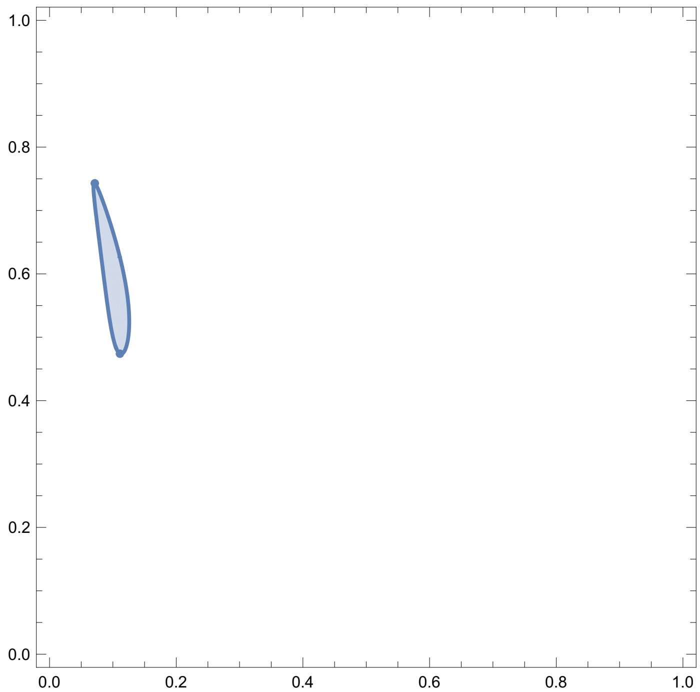
Out[24]=

$$\left\{w1 \to \frac{3}{5},\ w2 \to \frac{11}{50}\right\}$$

The example by Weibull 1997

In[25]:= `plotSolutions[W[[2]]]`

Out[25]=



In[26]:= `getSolution[W[[2]]]`

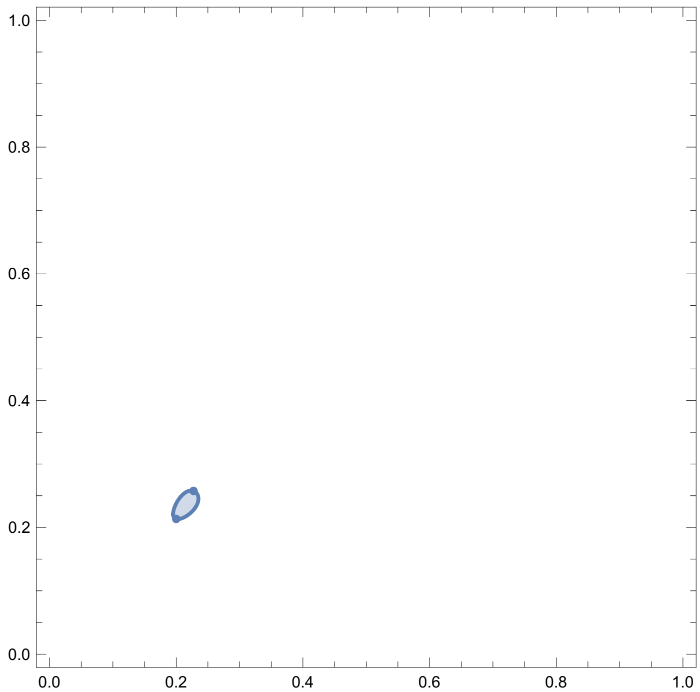
Out[26]=

$$\left\{w1 \to \frac{1}{9},\ w2 \to \frac{64}{135}\right\}$$

Cressman and Tao 2014

In[27]:=
```
plotSolutions[CT[[2]]]
```

Out[27]=



In[28]:=
```
getSolution[CT[[2]]]
```

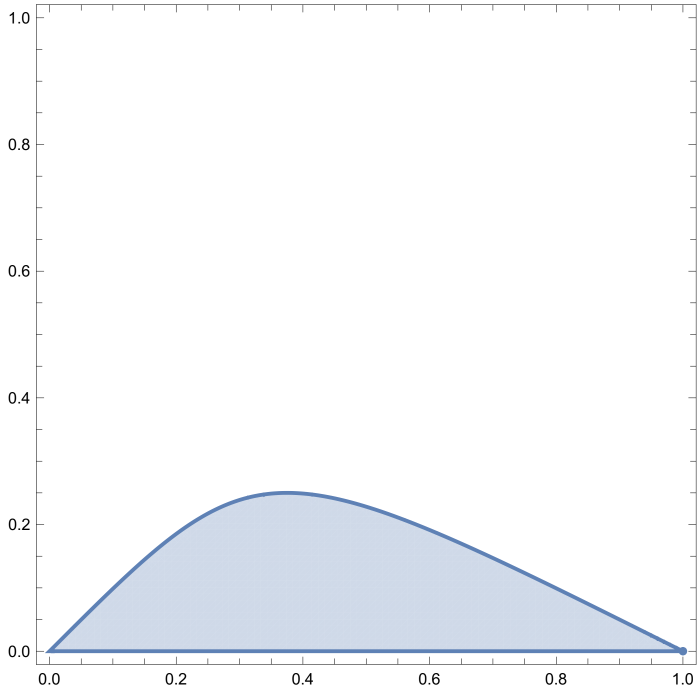
Out[28]=

$$\left\{w1 \to \frac{1}{5},\ w2 \to \frac{16}{75}\right\}$$

Zeeman 1980

In[29]:=
```
plotSolutions[Z[[2]]]
```

Out[29]=



In[30]:=
```
getSolution[Z[[2]]]
```

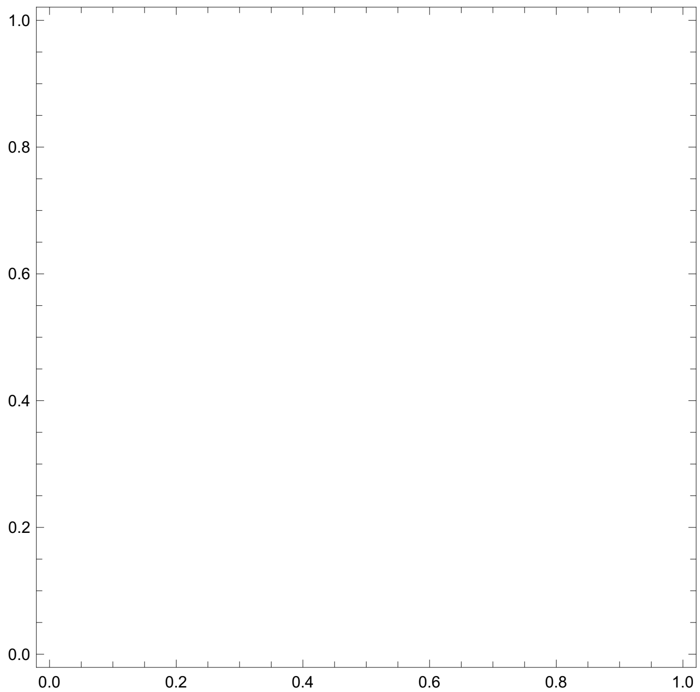
Out[30]=

$$\left\{w1 \to \frac{7}{16},\ w2 \to \frac{1}{8}\right\}$$

For the example of Hofbauer and Sigmund 1998, the equilibrium is not globally stable; accordingly, there are no valid choices of w1 and w2 that make it an ESS

In[31]:=
```
plotSolutions[HS[[2]]]
```

Out[31]=



In fact, the conditions are never satisfied

In[32]:=
```
Reduce[copositivity /. HS[[2]]]
```

Out[32]=
```
False
```