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A numerical method for the stability analysis of quasi-polynomial vector fields

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Abstract

This paper shows the sufficient conditions for the existence of a Lyapunov function in the class of quasi-polynomial dynamical systems. We focus on the cases where the system's parameters are numerically specified. A numerical algorithm to analyze this problem is presented, which involves the resolution of a linear matrix inequality (LMI). This LMI is collapsed to a *linear programming problem*. From the numerical viewpoint, this computational method is very useful to search for sufficient conditions for the stability of non-linear systems of ODEs. The results of this paper greatly enlarge the scope of applications of a method previously presented by the authors.

Keywords: Numerical methods; Linear matrix inequalities; Lyapunov functions

1. Introduction

Determining the conditions for global stability of fixed points is a major problem in the qualitative theory of dynamical systems. Lyapunov's method ensures stability of a fixed point and determines its *basin of attraction*, provided that a *Lyapunov function* exists. Moreover, application of LaSalle's invariant principle implies restrictions on the asymptotic behavior of the system. However, lack of a general recipe to construct such a function is a major drawback of such an approach [4]. Hamiltonian flows are exceptions. Here, total energy of the system can be a natural candidate to replace a Lyapunov function.

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Elsewhere, we present a method to obtain sufficient conditions for stability of interior fixed points to a class of generalized dynamical systems known as *quasi-polynomial* (QP) systems [5,7]. However, such a method is not useful as far the free parameters of the system are numerically specified and, accordingly, a more general approach is needed. This paper advances such a method. Together with previous works on the subject [5,7], a novel methodology for evaluating stability of quasi-polynomial systems—which works with numerical and algebraic situations alike—is put forward. Since QP systems in non-linear models of physics and chemistry are widespread, the applicability of our methodology is broad.

The paper is organized as follows. Section 2 briefly reviews our previous results [5,7]. Section 3 presents the novel methodology. Section 4 applies the methodology to a system describing the non-linear interaction of three waves. Section 5 concludes.

2. Stability problem in quasi-polynomial systems

Elsewhere [5,7], we have derived the sufficient conditions for the existence of a Lyapunov function in a general class of non-linear systems known as (QP) systems, defined by the following set of equations:

$$\dot{x}_i = l_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \quad i = 1,\dots,n, \;\; x \in \mathfrak{R}^n, \tag{1}$$

where $B=[B_{ij}]$ and $A=[A_{ij}]$ are constant real matrices. The number m is related to the number of quasi-monomials in the vector field of Eq. (1). The label *quasi-monomial* is used because the entries of matrix B can assume real values. This is also the reason for the label *quasi-polynomial* systems. A QP-system of form (1) with $m < n$ can be decomposed into an equivalent system with $n' \leqslant m$ equations [10], thus we assume $m \geqslant n$ in Eq. (1). Without loss of generality, the rank of B is assumed to be maximal [10].

Define matrix $M \equiv BA$. We say that M is *admissible* if there exists a set of numbers $a_i > 0$, $i = 1,\dots,m$ such that

$$\sum_{i,j=1}^{m} a_i M_{ij} w_i w_j \leqslant 0, \quad \forall w \in \mathfrak{R} \tag{2}$$

or

$$\exists P > 0 \quad | PM + M^{\mathrm{T}} P \leqslant 0, \quad P \equiv diag(a_1,\dots,a_m), \tag{3}$$

where $diag(a_1,\dots,a_m)$ denotes a positive diagonal matrix of elements $a_1,\dots,a_m$. A core result of this approach [5,7] is the following theorem.

Theorem 1. *If the QP system* (1) *has a fixed point x^* in the interior of the positive orthant and its characteristic matrix $M = BA$ is admissible*, *then there exists a Lyapunov function for this system. Furthermore*, *its trajectories have upper and lower bounds*, *i.e.*

$$\exists \varepsilon_i, \delta_i \in \mathfrak{R} \,|\, 0 < \varepsilon_i < x_i < \delta_i, \quad \forall i \in \{1,\dots,n\}. \tag{4}$$

It is worth noticing that matrix M is *singular* in all cases when $m > n$, *i.e.* in most of the cases [7] of interest. Expression (4) also implies that trajectories departuring from the interior of the positive orthant never cross its boundaries and remain bounded.

The existence of a Lyapunov function ensures stability to the interior fixed point. It is given by

$$V(x) = \sum_{i=1}^{m} a_i \left(\prod_{k=1}^{n} x_k^{B_{ik}} - q_i \ln \left(\frac{\prod_{k=1}^{n} x_k^{B_{ik}}}{q_i} \right) - q_i \right), \quad q_i \equiv \prod_{k=1}^{n} (x_k^*)^{B_{ik}}. \tag{5}$$

Such a Lyapunov function satisfies all the properties required within the entire positive orthant [5,7]. Under the conditions imposed by Theorem 1, several constraints on the dynamics of the system are present. Interior equilibria such as saddles and sources, for instance, are excluded. This in turn implies that homoclinic and heteroclinic orbits cannot exist, hence chaos. If (1) is well defined in other orthants, Theorem 1 can be used as long as both the initial condition and fixed points belong to the same orthant. An algebraic procedure has been presented to analyze (3), which is suitable when parameters M_{ij} are not numerically specified. For an m-dimensional matrix (3) is equivalent to 2^{m+1} independent sets of conditions involving equations and inequalities. For admissibility to occur, at least one of these conditions should be satisfied.

For practical purposes, it is of interest to determine the regions in parameter space where the system is stable. This is equivalent to solve (3). Here, generalizing the method presented in [5,7] could not be appropriate, because it usually solves a non-trivial set of inequalities. Next section presents a more general numerical procedure to solve (3).

3. Analysis of $PM + M^{\mathrm{T}}P \leqslant 0$

Strict version of (3), namely

$$PM + M^{\mathrm{T}}P < 0, \quad P > 0 \ diag \tag{6}$$

appears in problems of stability of non-linear systems as quadratic equations of Lotka–Volterra type [12], economic problems [8], and interconnected systems [11]. It is also related to the existence of a diagonal quadratic Lyapunov function for a given linear system [2]. Necessary and sufficient conditions for Eq. (6) be satisfied exist only for some particular cases. A well-known necessary condition is $det(M) \neq 0$ since (6) implies that M is a stable matrix (i.e. all eigenvalues of M have a negative real part). Then (6) turns out to be unimportant, since we are mainly dealing with singular matrices.

A numerical solution to (6) has been presented already [9]. This algorithm has been extended to encompass more general cases [6]. Here we extend these results for

$$PM + M^{\mathrm{T}}P \leqslant 0, \quad P > 0 \ diag \tag{7}$$

to be considered. A major difference between (6) and (7) is that the latter allows for $det(M) = 0$, which often will be the case in our method, that is why we need an algorithm to solve (7). For $P > 0$ not necessarily diagonal, (7) is feasible if and only if the eigenvalues of M have non-positive real parts, and those with zero real part correspond to Jordan blocks of size one [2].

3.1. Basic definitions

Let $a=(a_1,\ldots,a_n)$ be a point in $\Re^n$ and $P\in\Re^{n\times n}$ be a square diagonal matrix with components a_i, i.e. $P=diag(a_1,\ldots,a_n)$. Given an $n\times n$ square real matrix M, define the convex scalar function $\phi(a)=\phi(a_1,\ldots,a_n)$ as

$$\phi(a)\equiv\max_{v\in V}\{v^{\mathrm{T}}(PM+M^{\mathrm{T}}P)v\},\quad V\equiv\{v\in\Re^n\colon \|v\|=1\}.$$

Define also the set χ as

$$\chi=\{a\in\Re^n\colon 0\leqslant a_i\leqslant 1\}. \tag{8}$$

From the above definitions we observe that there exists a $P>0$ diagonal such that $PM+M^{\mathrm{T}}P\leqslant 0$ if and only if there exists $a^*\in int\,\chi$ such that $\phi(a^*)=0$. Likewise $PM+M^{\mathrm{T}}P<0$ if and only if there exists $a^\dagger\in int\,\chi$ such that $\phi(a^\dagger)<0$.

Our method is based on a special property of function $\phi(a)$. Define the vector $\hat{g}$

$$\hat{g}\equiv\nabla\{v^{\mathrm{T}}(PM+M^{\mathrm{T}}P)v\}|_{a=\hat{a}}=2\hat{\Lambda}M\hat{v}, \tag{9}$$

where $\hat{v}$ is the normalized eigenvector corresponding to the maximum eigenvalue of $PM+M^{\mathrm{T}}P$ at $\hat{a}$, and $\hat{\Lambda}$ is a diagonal matrix defined by $\hat{\Lambda}=diag(\hat{v}_1,\ldots,\hat{v}_n)$. The following theorem holds [9]:

Theorem 2. *Given $\hat{g}$ as in* (9) *then*

(i) $\phi(\hat{a})=\hat{a}\hat{g}$,
(ii) $\phi(a)\geqslant a\hat{g},\quad \forall a\in\chi$.

3.2. Search procedure

Here we present a search procedure based on Theorem 2.

Step 1: Pick an arbitrary point a^1 in the interior of χ. Determine $\phi(a^1)$. If $\phi(a^1)\leqslant 0$ then *stop* because a^1 solves (7) (or (6) provided that $\phi(a^1)<0$). Otherwise go to the next step.

Step 2: Denote by χ_1 the region where $(a-a^1)g^1<0$ holds. Choose a new point a^2 in the interior of χ_1. For a^* and $(a^*-a^1)g^1\geqslant 0$ we have from Theorem 1

$$\phi(a^*)\geqslant a^*g^1\geqslant a^1g^1=\phi(a^1)>0$$

and all such points can be discarded.

Step 3: Go back to *step* 1 with a^1 replaced with a^2.

In the kth step we construct g^k and $\phi(a^k)$. If $\phi(a^k)\leqslant 0$ then *stop* (if (7) is the problem being considered). Otherwise determine a new point a^{k+1} in the region χ_k defined by

$$\chi_k\equiv\{a\in\chi\,|\,(a^{k+1}-a^i)g^i<0\},\quad i=1,\ldots,k. \tag{10}$$

a^{k+1} can be determined by the *minimax* problem

$$s_k \equiv \min_{a\in\chi} \max_{1\leqslant i\leqslant k} (a - a^i)g^i. \tag{11}$$

Given that $\hat{a}^k$ is the solution to (11), the new point a^{k+1} can be taken as

$$a^{k+1} = \frac{a^k + \hat{a}^k}{2}, \tag{12}$$

which satisfies all restrictions (10), i.e.

$$a^{k+1} - a^i = \frac{a^k + \hat{a}^k}{2} - \frac{a^i}{2} - \frac{a^i}{2}$$

$$\Rightarrow (a^{k+1} - a^i)g^i = \frac{(a^k - a^i)g^i}{2} + \frac{(\hat{a}^k - a^i)g^i}{2}. \tag{13}$$

The second term on the right-hand side of (13) is negative because $\hat{a}$ is the solution to (11). For $k = i = 1$ the first term on the right-hand side is null. Then the left-hand side of (13) is negative. Supposing (13) to hold for k

$$(a^{k+1} - a^i)g^i < 0, \quad \forall i = 1,\dots,k$$

follows by induction. Appendix A shows how a solution to (11) can be obtained using linear programming.

The result below will be of critical importance to our methodology. Since

$$\phi(a) \geqslant ag^i, \quad \forall i = 1,\dots,k,$$

we have

$$\phi(a) - a^i g^i \geqslant ag^i - a^i g^i$$

$$\Rightarrow \max_{1\leqslant i\leqslant k} \{\phi(a) - a^i g^i\} \geqslant \max_{1\leqslant i\leqslant k} \{ag^i - a^i g^i\}$$

$$\Rightarrow \max_{1\leqslant i\leqslant k} \{\phi(a) - a^i g^i\} \geqslant s_k.$$

Since $\phi(a^i) = a^i g^i > 0$, $\forall i = 1,\dots,l$, the left-hand side of the equation above will be satisfied at $\min_{1\leqslant i\leqslant k} \{a^i g^i\}$ i.e.

$$\max_{1\leqslant i\leqslant k} \{\phi(a) - a^i g^i\} = \phi(a) - \min_{1\leqslant i\leqslant k} \{a^i g^i\}$$

$$\Rightarrow \phi(a) \geqslant s_k + \min_{1\leqslant i\leqslant k} \{a^i g^i\}. \tag{14}$$

The above result suggests that if $s_k = 0$ we must stop because there is no a to which $\phi(a) \leqslant 0$. Otherwise the procedure should continue until a solution point is reached. The procedure converge as well, as shown below.

3.3. Algorithm convergence

If at any step k $\phi(a^k) \leqslant 0$ obtains, then the algorithm terminates. It turns out that a^k is the point which solves (7) (or (6) if $\phi(a^k) < 0$). Similarly if $s_k = 0$ at any step k, the procedure should halt because (14) would imply $\phi(a) > 0$, $\forall a \in \chi$. Neither (7) nor (6) could be satisfied. Otherwise if $\phi(a^k) > 0$ and $s_k \neq 0$, $\forall k$, the algorithm converge according to the theorem

Theorem 3. *If at any interaction k we have $\phi(a^k) > 0$ and $s_k \neq 0$ then* $\lim_{k\to\infty} s_k = 0$.

Proof. We first show that the sequence s_k converges as $k \to \infty$:

$$\max_{1\leqslant i\leqslant k+1} \{(a - a^i)g^i\} \geqslant \max_{1\leqslant i\leqslant k} \{(a - a^i)g^i\}$$

$$\Rightarrow \min_{a\in\chi} \max_{1\leqslant i\leqslant k+1} \{(a - a^i)g^i\} \geqslant \min_{a\in\chi} \max_{1\leqslant i\leqslant k} \{(a - a^i)g^i\}$$

$$\Rightarrow s_{k+1} \geqslant s_k \cdots \geqslant s_0.$$

But $s_k \leqslant 0$, hence it converges as $k \to \infty$. To show that $\lim_{k\to\infty} s_k = 0$, we assume the opposite holds such that $s_k \to -\delta < 0$. From (9) we have

$$\|g\| \leqslant 2\|\Lambda\|\,\|M\|\,\|v\| \Rightarrow \|g\| \leqslant 2\|M\|$$

because $\|\Lambda\| = 1$ and $\|v\| = 1$. Moreover, we obtain from (13) that

$$\max_{1\leqslant i\leqslant k} \{(a^{k+1} - a^i)\}g^i \leqslant \max_{1\leqslant i\leqslant k} \left\{\frac{(a^k - a^i)g^i}{2}\right\} - \frac{\delta}{2}.$$

The first term on the right-hand side is negative for $i < k$ (by construction). For $i = k$, it equals zero. Thus

$$\max_{1\leqslant i\leqslant k} \{(a^{k+1} - a^k)g^k\} \leqslant -\frac{\delta}{2}, \quad \forall k \Rightarrow \{(a^j - a^k)g^k\} \leqslant -\frac{\delta}{2}, \quad \forall j > k$$

$$\Rightarrow \|a^j - a^k\|\,\|g^k\| \geqslant \frac{\delta}{2}, \quad \forall j > k \Rightarrow \|a^j - a^k\| \geqslant \frac{\delta}{4\|M\|}, \quad \forall j > k. \tag{15}$$

Since $\{a^k\}$ is a bounded sequence, it has at least one accumulation point; this contradicts (15). Then $s_k \to 0$ as $k \to \infty$. This proves the theorem. □

One consequence of Theorem 3 is that (6) has no solution. Such a result follows straightforwardly from (14), i.e.

$$\phi(a) \geqslant s_k + \min_{1\leqslant i\leqslant k} \{a^i g^i\}$$

$$\Rightarrow \phi(a) \geqslant \lim_{k\to\infty} s_k + \lim_{k\to\infty} \min_{1\leqslant i\leqslant k} \{a^i g^i\}$$

$$\Rightarrow \phi(a) \geqslant \lim_{k\to\infty} \min_{1\leqslant i\leqslant k} \{a^i g^i\} \geqslant 0 \Rightarrow \phi(a) \geqslant 0; \quad \forall a \in \chi. \tag{16}$$

Expression (16) shows that there is no $a \in \chi$ such that $\phi(a) < 0$.

Regarding inequality (7) two possibilities should be considered, depending on the value of (16), that is

$$\alpha \equiv \lim_{k\to\infty} \min_{1\leqslant i\leqslant k} \{a^i g^i\}. \tag{17}$$

Let us first consider the case $\alpha \neq 0$. From Eq. (16)

$$\phi(a) \geqslant \alpha > 0, \quad \forall a \in \chi$$

follows. Thus (7) has no solution.

As far as the case $\alpha = 0$ is concerned, a more careful analysis is required. First recall that

$$\lim_{k\to\infty} \min_{1\leqslant i\leqslant k} \phi(a^i) = \lim_{k\to\infty} \min_{1\leqslant i\leqslant k} \{a^i g^i\} = 0. \tag{18}$$

Under the assumption $\phi(a^i) > 0$, $\forall 1 \leqslant i \leqslant k$, from Eq. (18)

$$\lim_{k\to\infty} \phi(a^k) = 0 \tag{19}$$

follows. Expression (19) says that the sequence $\{\phi(a^k)\}$ converges regardless of convergence of $\{a^k\}$. Given a set of accumulation points of $\{a^k\}$ (say X^{ac}) both (19) and $\phi(a^i) > 0$ imply that

$$\phi(a^0) \equiv a^0 g^0 = 0, \quad \forall a^0 \in X^{\mathrm{ac}}. \tag{20}$$

At this point, let us return to the basic definitions of our search procedure. We get started with a convex polytope χ and a point $a^1 \in int\,\chi$. The polytope χ is divided in two disjoint regions χ_1, χ_1^* by the plane $(a - a^1)g^1 = 0$, i.e.

$$\chi_1 \equiv \{a \in \chi \,|\, (a - a^1)g^1 < 0\}, \quad \chi_1^* \equiv \{a \in \chi \,|\, (a - a^1)g^1 \geqslant 0\}.$$

These are two convex sets with $(a - a^1)g^1 = 0$ as a supporting plane, and a^1 is on the boundary of both χ_1 and χ_1^*. We then pick a new trial point a^2 from the interior of χ_1. The point a^k is picked out from the interior of χ_k defined by (10), and a^k belongs to the boundary of χ_{k+1}. The point a^{k+1} is picked out from the interior of χ_{k+1}.

The remaining region where it is possible to find solution points as $k \to \infty$ is denoted by χ_∞

$$\lim_{k\to\infty} \chi_k \equiv \chi_\infty. \tag{21}$$

The convexity of set χ_∞ is assured since the intersections of the convex sets χ_k are also convex.

At the limit $k \to \infty$, the algorithm must necessarily pick out trial points within the set $X^{\mathrm{ac}} \subset \chi_\infty$. As discussed above, all points in X^{ac} must lie on the boundary of χ_∞. Given any two convex sets A and B such that $A \subset B$ then $V(A) < V(B)$, where $V(\cdot)$ denotes the volume function of convex sets [1]. Let us analyze the volume of χ_∞. We start out with the cube χ, whose volume is denoted $V(\chi)$. At the second step convex sets χ_1 and χ_1^* satisfy

$$int(\chi_1) \cap int(\chi_1^*) = \emptyset, \quad \chi_1 \cup \chi_1^* = \chi.$$

Thus $\chi_1 \subset \chi$, $\chi_1^* \subset \chi$ and

$$V(\chi^1) < V(\chi), \quad V(\chi^2) < V(\chi).$$

Analogously,

$$\lim_{k\to\infty} [V(\chi_k)] \equiv V(\chi_\infty) < \cdots < V(\chi_{k+1}) < V(\chi_k) < \cdots < V(\chi). \tag{22}$$

Suppose that sequence (22) has a finite limit, i.e. $V(\chi_\infty) = \varepsilon \neq 0.$: By considering the convexity of χ_∞, it is always possible to find $a^{01} \in X^{\mathrm{ac}}$ and $a^{02} \in X^{\mathrm{ac}}$. These lie on the boundary of χ_∞ such that

$$\tilde{a} = \alpha a^{01} + \beta a^{02}, \quad \alpha + \beta = 1$$

is an *interior* point of χ_∞. It is then possible to carry on with our procedure using $\tilde{a}$ as a trial point rather than a point of X^{ac}. Here plane $(a - \tilde{a})\tilde{g} = 0$ is a supporting plane of two sets $\tilde{\chi}_\infty$ and $\bar{\chi}_\infty$ such that

$$V(\tilde{\chi}_\infty) < V(\chi_\infty), \quad V(\bar{\chi}_\infty) < V(\chi_\infty).$$

The latter expression contradicts (22). The only way to prevent this contradiction to occur is set $\varepsilon = 0 \Rightarrow V(\chi_\infty) = 0$. Doing that, χ_∞ turns out to have dimension $n - 1$ at most. This means it does not have any interior points (with respect to the topology of $\mathfrak{R}^n$).

An important property of set X^{ac} is now presented. According to (20), one of two possibilities must be satisfied by a^0. Either

(i) $\exists a^0 \in X^{\mathrm{ac}} : a_i^0 \neq 0, \ \forall 1, \ldots, n$ or
(ii) $\forall a^0 \in X^{\mathrm{ac}} : \exists i = 1, \ldots, n : a_i^0 = 0.$

Regarding possibility (i), a^0 is an interior point of χ and therefore a solution to (7). Case (ii) requires a more detailed analysis. We know that $a^0 \in \chi_\infty$—which is the region where it is possible to find out solutions for $k \to \infty$—and that χ_∞ lies on a hyperplane of χ. If (ii) holds, thus χ_∞ can lie or not on a face of χ. To decide if this is the case, we simply check for the direction of vector g^k at the limit $k \to \infty$. Here,

$$g^0 = \lim_{k \to \infty} g^k$$

follows from (19). Suppose first that g^0 is normal to some face of χ. Now take into account that $a^0 g^0 = 0$. Then a^0 (and therefore χ_∞) lies in that face. Since χ_∞ is the only region where it is possible to find out feasible points, we conclude that (7) has no solution, as there are no interior points of χ satisfying it.

The other possibility occurs whenever the vector g^0 is not perpendicular to any face of χ. In this case χ_∞ has points which are interior points of χ and one of them can occasionally satisfy $\phi = 0$. To provide a definitive answer regarding the existence or not of solutions, some other test is needed. Next section presents an alternative approach to deal with such a problem.

3.4. Alternative approach

Let us consider the case of the last paragraph. To determine a new trial point we solve the *simplex* (11). However, we can alternatively pick out a^{k+1} as the geometrical center of region χ_k defined in (10). The convergence of our procedure is analyzed by taking into account this modification. Relation $V(\chi_\infty) = 0$ clearly does not change. At each step, region χ_k is intersected with plane $(a - a^k)g^k = 0$. χ_{k+1} is the region defined by $\chi_{k+1} \equiv (a - a^k)g^k < 0$. The new trial point a^{k+1} is picked out as the geometrical center of χ_{k+1}.

At the limit $k \to \infty$ we are left in region χ_∞, whose geometrical center is denoted by $a^{\rm gc}$. Since in χ_∞ the point $a^{\rm gc}$ is the trial point we must have

$$\lim_{k\to\infty} a^k = a^{\rm gc}.$$

Otherwise a contradiction would emerge. Namely, that the geometrical center of χ_∞ is not unique.

In short, we must have in χ_∞

$$V(\chi_\infty) = 0, \quad (a - a^{\rm gc})g^{\rm gc} = 0, \quad \forall a \in \chi_\infty. \tag{23}$$

Eq. (23) must hold since otherwise we could be able to pick out a new trial point as the geometrical center of $(a - a^{\rm gc})g^{\rm gc} < 0$ and carry on with the procedure until condition (23) is reached.

The origin belongs to χ_∞, since $\phi(0) = 0$ and the procedure never excludes such points by construction. From Eq. (23)

$$(0 - a^{\rm gc})g^{\rm gc} = 0 \Rightarrow \phi(a^{\rm gc}) = 0$$

obtains. This in turn implies that $a^{\rm gc}$ is a solution to (7) provided that none of its components is zero. Otherwise $a^{\rm gc}$ would lie on a face of χ. However, we can conclude that χ_∞ also lies on this face in the latter situation because $V(\chi_\infty) = 0$. Since this is the only region in which solutions can be found, we conclude that (7) has no solution in this case.

One drawback of this alternative approach is that it is a little bit cumbersome for programming. The present procedure should be used only in cases where it is not possible to evaluate conclusively the feasibility or not of (7) by using the procedure presented in Section 3.

3.5. Examples

We illustrate our methodology by recurring to two examples. An application to an actual physical system will be presented in the next section.

Example 1. A singular matrix M given by

$$M = \begin{bmatrix} -1 & -0.5 \\ -2 & -1 \end{bmatrix}. \tag{24}$$

Table 1 displays the respective x^k and $\phi(x^k)$ for a convergence of 25 steps. The algorithm converges to the exact solution $x = (k, k/4)$; $\forall 0 \leqslant k \leqslant 1$.

Example 2. A matrix M given by

$$M = \begin{bmatrix} -1 & -0.5 \\ -2.1 & 1 \end{bmatrix}. \tag{25}$$

This matrix does not satisfy (7). This is shown in Table 2, for a convergence of 35 steps. Using the geometrical center approach, the algorithm converges to origin. This implies that (7) cannot be satisfied. For sake of briefness, only the first and last seven steps of the procedure are presented.

Table 1
Example 1

Step	a^k	$\phi(a^k)$
1	$a^1 = [0.5000000000, 0.5000000000]$	0.2500000000
2	$a^2 = [0.7500000000, 0.2500000000]$	0.0077822185
3	$a^3 = [0.8750000000, 0.1250000000]$	0.0174262873
4	$a^4 = [0.9350000000, 0.1704078300]$	0.0073620721
5	$a^5 = [0.9687500000, 0.2121156100]$	0.0015306278
6	$a^6 = [0.9843750000, 0.2518603530]$	0.0000537971
7	$a^7 = [0.9921875000, 0.2384319910]$	0.0001502338
8	$a^8 = [0.9960937500, 0.2422691580]$	0.0000736761
9	$a^9 = [0.9980468750, 0.2456424820]$	0.0000240749
10	$a^{10} = [0.9990234380, 0.2487852650]$	0.0000015090
11	$a^{11} = [0.9995117190, 0.2518128777]$	0.0000059807
12	$a^{12} = [0.9997558590, 0.2513874060]$	0.0000033536
13	$a^{13} = [0.9998779300, 0.2509315260]$	0.0000014798
14	$a^{14} = [0.9999389650, 0.2504606160]$	0.0000003622
15	$a^{15} = [0.9999694820, 0.2499823600]$	0.0000000014
16	$a^{16} = [0.9999847410, 0.2502241250]$	0.0000000831
17	$a^{17} = [0.9999923710, 0.2502210310]$	0.0000000795
18	$a^{18} = [0.9999961850, 0.2502169830]$	0.0000000759
19	$a^{19} = [0.9999980930, 0.2502124570]$	0.0000000725
20	$a^{20} = [0.9999990460, 0.2502076930]$	0.0000000691
21	$a^{21} = [0.9999995230, 0.2502028000]$	0.0000000658
22	$a^{22} = [0.9999997620, 0.2501978670]$	0.0000000626
23	$a^{23} = [0.9999998810, 0.2501928050]$	0.0000000595
24	$a^{24} = [0.9999999400, 0.2501879070]$	0.0000000564
25	$a^{25} = [0.9999999700, 0.2501829130]$	0.0000000535

Table 2
Example 2

Step	a^k	$\phi(a^k)$
1	$a^1 = [0.5000000000, 0.5000000000]$	1.2806248480
2	$a^2 = [0.5000000000, 0.2500000000]$	0.5488272656
3	$a^3 = [0.5000000000, 0.1250000000]$	0.2501249877
4	$a^4 = [0.5000000000, 0.06250000000]$	0.1373980888
5	$a^5 = [0.5000000000, 0.03125000000]$	0.0935850452
6	$a^6 = [0.2870923975, 0.02483311321]$	0.0627805192
7	$a^7 = [0.2087542354, 0.01528440853]$	0.0419397993
$\vdots$	$\vdots$	$\vdots$
29	$a^{29} = [0.0000270909, 0.0000021107]$	0.0000056103
30	$a^{30} = [0.0000180606, 0.0000014071]$	0.0000037402
31	$a^{31} = [0.0000120404, 0.0000009381]$	0.0000024934
32	$a^{32} = [0.0000080269, 0.0000006250]$	0.0000016623
33	$a^{33} = [0.0000053512, 0.0000004169]$	0.0000011082
34	$a^{34} = [0.0000035675, 0.0000002779]$	0.0000007388
35	$a^{35} = [0.0000023783, 0.0000001853]$	0.0000004925

4. Application

Here we consider the problem of a non-linear interaction between three waves which is an approximation for a general description of coupling in various fields of physics [13]. The set of equations are given by

$$\dot{x}_1 = \lambda_1 x_1 + x_1 \left[\sum_{j=1}^{3} N_{1j} x_j^2 \right] + \gamma x_2 x_3;$$

$$\dot{x}_2 = \lambda_2 x_2 + x_2 \left[\sum_{j=1}^{3} N_{2j} x_j^2 \right];$$

$$\dot{x}x_3 = \lambda_3 x_3 + x_3 \left[\sum_{j=1}^{3} N_{3j} x_j^2 \right]. \tag{26}$$

where γ, N_{ij} and λ_i are real parameters. Matrix M is given by

$$M = \begin{bmatrix} 2N_{11} & 2N_{12} & 2N_{13} & 2\gamma \\ 2N_{21} & 2N_{22} & 2N_{23} & 0 \\ 2N_{31} & 2N_{32} & 2N_{33} & 0 \\ -N_{11}+N_{21}+N_{31} & -N_{12}+N_{22}+N_{32} & -N_{13}+N_{23}+N_{33} & -\gamma \end{bmatrix}. \tag{27}$$

We present below some particular values for the system parameters such that M given by (27) is admissible. Each example provides the elements $\{a_1, \ldots, a_4\}$ necessary to the construction of the Lyapunov function as well as the tolerance in determining $\phi = 0$.

Using Theorem 1, for the cases presented above is valid:

- the interior fixed points of the system are stable, and
- the trajectories of the system are bounded and componentwise bounded away from zero, regardless of the initial conditions.

$$\begin{cases} N_{11} = -1 \\ N_{12} = 1 \\ N_{13} = 7 \\ N_{21} = -10 \\ N_{22} = -10 \\ N_{23} = 7 \\ N_{31} = -10 \\ N_{32} = -1 \\ N_{33} = -4 \\ \gamma = 1 \end{cases} \rightarrow \begin{bmatrix} a_1 = 1 \\ a_2 = 0.4273093560 \\ a_3 = 0.5222357800 \\ a_4 = 0.1194451320 \\ \phi \approx 10^{-10} \end{bmatrix}, \tag{28}$$

$$\left\{\begin{array}{l} N_{11}=-1\\ N_{12}=1\\ N_{13}=10\\ N_{21}=-7\\ N_{22}=-4\\ N_{23}=1\\ N_{31}=-7\\ N_{32}=-1\\ N_{33}=-10\\ \gamma=1 \end{array}\right. \rightarrow \left[\begin{array}{l} a_1=0.5121904950\\ a_2=0.2727203710\\ a_3=1\\ a_4=0.1280474540\\ \phi\approx 10^{-11} \end{array}\right., \tag{29}$$

$$\left\{\begin{array}{l} N_{11}=-1\\ N_{12}=4\\ N_{13}=1\\ N_{21}=-1\\ N_{22}=-10\\ N_{23}=7\\ N_{31}=-1\\ N_{32}=-1\\ N_{33}=-4\\ \gamma=1 \end{array}\right. \rightarrow \left[\begin{array}{l} a_1=0.1914997010\\ a_2=0.8181970680\\ a_3=1\\ a_4=0.1800067230\\ \phi\approx 10^{-10} \end{array}\right., \tag{30}$$

$$\left\{\begin{array}{l} N_{11}=-1\\ N_{12}=4\\ N_{13}=4\\ N_{21}=-4\\ N_{22}=-4\\ N_{23}=1\\ N_{31}=-7\\ N_{32}=-4\\ N_{33}=-1\\ \gamma=1 \end{array}\right. \rightarrow \left[\begin{array}{l} a_1=1\\ a_2=0.7272898500\\ a_3=0.6666536130\\ a_4=0.1599999660\\ \phi\approx 10^{-10} \end{array}\right.. \tag{31}$$

5. Conclusions

This paper presents a numerical methodology for determining the sufficient conditions for stability of equilibria and boundedness of trajectories in a broad class of non-linear systems. Our method is based on a particular candidate for a Lyapunov function, valid for a wide class of systems, and works well in both algebraic and numerical cases.

The strength of the method is that it can be applied to all systems of type (1), ubiquitous in non-linear models. However, since the methodology deals with sufficient conditions, nothing can be said about the stability of the system if its associated matrix

M is not admissible. This is the only limitation of the procedure and it is characteristic of other methods as well [3].

In [5,7] the authors show how a special reparametrization in the time variable enlarges the scope of application of the methodology, associating to each system (1) a multitude of matrices M. The stability of the interior fixed points is granted provided one of these matrices is admissible.

Appendix A. Solution of the minimax problem

Each iteration of the search algorithm has to solve the minimax problem (11). We now prove that a simple solution can be obtained using linear programming. This shows that the procedure is very effective, even in the case of systems of considerable dimension size.

Theorem 4. *s_k is solution to the following linear programming*:

$$\begin{aligned} & s_k = -\max_{a,s'} \quad s' \\ & \text{subject to} \\ & \qquad 0 \leqslant a_i \leqslant 1, \quad i = 1, \ldots, n; \\ & \qquad 0 \leqslant s' \leqslant 2\sqrt{n}\|M\|; \\ & \qquad s' + (a - a^i)g^i \leqslant 0, \quad i = 1, \ldots, k. \end{aligned} \tag{A.1}$$

If (a^*, s'^*) is a solution to the linear programming problem (A.1), *then a^* is a solution of the minimax problem* (11).

Proof. We first note that

$$\begin{aligned} & \|s_k\| \leqslant \max\|(a - a^i)g^i\| \Rightarrow \|s_k\| \leqslant \|(a - a^i)\| \ \|g^i\| \\ & \quad \Rightarrow \|s_k\| \leqslant 2\sqrt{n}\|M\| \Rightarrow s_k \geqslant -2\sqrt{n}\|M\|. \end{aligned}$$

Let

$$\begin{aligned} & S_k = \min_{a \in \chi} \quad S' \\ & \text{subject to} \\ & \qquad 0 \leqslant a_i \leqslant 1, \quad i = 1, \ldots, n; \\ & \qquad -2\sqrt{n}\|M\| \leqslant S' \leqslant 0; \\ & \qquad S' \geqslant (a - a^i)g^i, \quad i = 1, \ldots, k. \end{aligned} \tag{A.2}$$

Let $\hat{a}^k$ be a solution to (11). Then $(\hat{a}^k, s_k)$ satisfies all constraints (A.2), which imply that $S_k \leqslant s_k$. Let (a^*, S'^*) be a solution to (A.2). Constraint $S' \geqslant (a - a^i)g^i$ implies that

$$S'^* = \max\{(a^* - a^i)g^i\} \geqslant s_k \Rightarrow S_k \geqslant s_k.$$

To avoid contradiction we must have $S_k = s_k$. Then a^* is a solution to (11). The theorem is proved by substituting $s' = -S'$.

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