ELSEVIER

2 October 1995

Physics Letters A 206 (1995) 31-37

PHYSICS LETTERS A

Nonpolynomial vector fields under the Lotka-Volterra normal form

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Received 15 March 1995; revised manuscript received 19 July 1995; accepted for publication 26 July 1995
Communicated by C.R. Doering

Abstract

We carry out the generalization of the Lotka-Volterra embedding to flows not explicitly recognizable under the generalized Lotka-Volterra format. The procedure introduces appropriate auxiliary variables, and it is shown how, to a great extent, the final Lotka-Volterra system is independent of their specific definition. Conservation of the topological equivalence during the process is also demonstrated.

1. Introduction

The concept of normal form is frequently used in mathematical physics to describe peculiar forms of mathematical objects for which the properties of a large class of them are particularly evident and easy to analyze. They are thus especially useful in the context of classification of those mathematical objects, and found in matrix theory (Jordan form) [1], catastrophe theory [2], bifurcation theory [3], etc.

Contrarily to the theory of linear ODEs there is no normative approach in its nonlinear counterpart through which we could relate the properties of the structure of the equations themselves to the properties of their corresponding solutions. The problem lies in the lack of unifying structures for nonlinear representations. There has been, however, a great deal of fruitful work devoted to this subject, directed towards the elaboration of recasting procedures of general nonlinear systems into standard simple forms, which are amenable to systematic analysis. In the early thirties Carleman [4] showed how a nonlinear system of ODEs could be transformed into an infinite-dimensional linear system. Kowalski and Steeb [5] have reviewed the theory and applications of this embedding, while Cairó and Feix [6] have recently shown the interest of the technique in the study of invariants of motion. An alternative approach has been suggested by Kerner [7], in terms of an embedding into a Riccati format. This scheme can be plugged into the classification and analysis of quadratic ODEs by means of nonassociative algebras [8]. The Riccati embedding has also been successfully used in time-saving, high precision numerical codes [9,10] and in nonlinear model building and prediction in chaos [11]. Finally, we can mention the S-system formalism, originally devised in the context of a representation of nonlinear biochemical schemes, but which has proved to be a good framework for the recasting and solution of nonlinear differential systems [12].

The central idea behind these unifying representations is that they should constitute the framework for

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SSDI 0375-9601(95)00623-0

building a method of classification based on powerful algebraic techniques, differing from the more traditional geometric methods customarily used in the qualitative theory of differential equations, especially in high dimensional systems, in which we can lose the intuitive benefits of the geometric approach. The gain in structural simplicity is bought at the price of an increase in the dimension of the system, but this cost will become negligible as soon as further progress is made on the subject.

In this context, Brenig and Goriely [13,14] have recently introduced the concept of generalized Lotka–Volterra (GLV) form,

$$\dot{x}_i = x_i\left(\lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}\right),$$
$$i = 1,\dots,n, \quad m \geqslant n, \tag{1}$$

where A and B are $n \times m$ and $m \times n$ matrices, respectively. The interesting point here lies in the fact that the properties of system (1) are associated to those of algebraic objects as are matrices λ, A and B: the family of systems (1) is split in classes of equivalence according to the values of the products $B \cdot A$ and $B \cdot \lambda$. Accordingly, certain prescribed forms of those products define representative systems of the class for which an integrability analysis can be systematically carried out [15]. The classical Lotka–Volterra (LV) system is one of those forms. If matrix B is of rank n, (1) can be embedded into the following m-dimensional system,

$$\dot{z}_\alpha = \lambda'_\alpha z_\alpha + z_\alpha \sum_{\beta=1}^{m} A'_{\alpha\beta} z_\beta, \quad \alpha = 1,\dots,m, \tag{2}$$

where each one of the Lotka–Volterra variables z_j stands for any one of the quasimonomials

$$\prod_{k=1}^{n} x_k^{B_{jk}}, \quad j = 1,\dots,m, \tag{3}$$

in (1), while $A' = B \cdot A$ and $\lambda' = B \cdot \lambda$.

The simplicity and ubiquity of the Lotka–Volterra equations (2) has made them especially attractive and amenable to systematic analysis by algebraic techniques [16]. They are algebraically related to the replicator equations [17], of primary importance in mathematical biology and for which we have fairly general results, and also to the rapidly growing theory of neural nets [18]. Moreover, the Lotka–Volterra equations can be straightforwardly embedded within a connectionist representation of dynamical systems, in terms of digraphs and nodes, an approach which opens exciting prospects for a near future [19].

The purpose of the present Letter is to show how a fairly general class of differential systems, apparently not covered by the GLV form, can be easily rewritten in terms of (1) by means of the introduction of suitable auxiliary variables. We may recall all vector fields containing elementary functions: rational, exponential, etc. Those systems can then be recast in terms of an equivalent Lotka–Volterra system, thus providing, as mentioned before, a possible route from the traditional view of the field – as a potpourri of structureless and apparently unrelated systems – to a new emerging unifying one.

2. Embedding into the GLV form

We shall consider a system of the general form

$$\dot{x}_s = \sum_{i_{s1},\dots,i_{sn},j_s} a_{i_{s1}\dots i_{sn} j_s} x_1^{i_{s1}} \dots x_n^{i_{sn}} f(\bar{x})^{j_s},$$
$$x_s(t_0) = x_s^0, \quad s = 1,\dots,n, \tag{4}$$

where $f(\bar{x})$ is some scalar function not reducible to the quasimonomial form (3) in terms of the $\bar{x}$ variables. All constants in (4) are assumed to be real. There is no loss of generality in what follows in elaborating on systems with a single non-quasimonomial function, as in (4). The addition to the scheme of other appropriate functions, if eventually needed, does not alter the method we are going to develop.

Now, we additionally assume that $f(\bar{x})$ is such that its partial derivatives can be expressed in the following form,

$$\frac{\partial f}{\partial x_s} = \sum_{e_{s1},\dots,e_{sn},e_s} b_{e_{s1}\dots e_{sn} e_s} x_1^{e_{s1}} \dots x_n^{e_{sn}} f(\bar{x})^{e_s}. \tag{5}$$

All constants are again real numbers. The possibility of dealing with an $f(\bar{x})$ whose derivatives do not verify (5) does not actually affect the generality of the following considerations. This problem has already been analyzed [7]; it can be shown that any function satisfying a finite order differential equation can be reduced to a first order polynomial system by means of a method that consists in assigning new variables to the nonpolynomial terms and differentiate them to find their differential equations. The process is repeated successively until we finally reach an expanded polynomial differential first order system equivalent to the initial equation (suitable initial conditions must be taken into account). For example, if $f(x) = \sin x$, the first derivative is not polynomial in x and f,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \sqrt{1-f^2}, \tag{6}$$

but the sine function is also the solution of

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f = 0. \tag{7}$$

The above method would introduce a new variable $q = \mathrm{d}f/\mathrm{d}x$ reducing (7) to

$$\frac{\mathrm{d}f}{\mathrm{d}x} = q, \quad \frac{\mathrm{d}q}{\mathrm{d}x} = -f.$$

There is then no loss of generality in assuming (5).

The choice of $f(\bar{x})$ in (4) is certainly ambiguous given the fact that any other function of the form

$$f_{k,l}(x) = f^k \prod_{s=1}^{n} x_s^{l_s},$$

with $k \neq 0$, will also preserve the format (4). Moreover, expression (5) may not be unique for a given $f(\bar{x})$. For example, to

$$f(x) = \frac{x^2}{1+x^2} \quad (n=1)$$

we may associate a countable family of possible quasipolynomial representations of the derivative

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2x^{-2i-3} f^{i+2} (1+x^2)^i, \quad i \in \mathbb{N}.$$

In the following development, we shall henceforth proceed for one given selection in (4) and (5) of both $f(\bar{x})$ and the form of its derivatives.

The procedure to transform (4) and (5) into a GLV system is then straightforward. It is carried out by introducing an additional variable in the form

$$y = f^q \prod_{s=1}^{n} x_s^{p_s}, \quad q \neq 0, \tag{8}$$

with real exponents q, p_s. The set (8) of all possible new variables includes $y = f(\bar{x})$ as a special (and simplest) element. The transformations which map $f(\bar{x})$ onto any other element of the set,

$$\xi(p,q) : f \rightarrow f^q \prod_{s=1}^{n} x_s^{p_s},$$

constitute a $n+1$ parameter noncommutative Lie group, with the composition

$$\xi(p_1,q_1) \circ \xi(p_2,q_2) = \xi(p_1 + q_1 p_2, q_1 q_2)$$

as inner operation. Thus, the set of functions (8) generated from f make up an equivalence class, where for every two elements f_i and f_j there is a group element ξ_{ij} which transforms one into the other,

$$\xi_{ij} : f_i \rightarrow f_j.$$

The introduction of the auxiliary variable (8) leads to the following system for the original variables,

$$\dot{x}_s = \left(x_s \sum_{i_{s1},\ldots,i_{sn},j_s} a_{i_{s1}\ldots i_{sn} j_s} y^{j_s/q} \prod_{k=1}^{n} x_k^{i_{sk} - \delta_{sk} - j_s p_k/q} \right), \tag{9}$$

for $s = 1,\ldots,n$. As usual, $\delta_{sk} = 1$ if $s = k$, and 0 otherwise. For the new variable (8) we obtain

$$\dot{y} = \sum_{s=1}^{n} \frac{\partial y}{\partial x_s} \dot{x}_s = y \left[\sum_{s=1}^{n} \left(p_s x_s^{-1} \dot{x}_s + \sum_{i_{s\alpha}, j_s, e_{s\alpha}, e_s} a_{i_{s\alpha}, j_s} b_{e_{s\alpha} e_s} q y^{(e_s + j_s - 1)/q} \times \prod_{k=1}^{n} x_k^{i_{sk} + e_{sk} + (1 - e_s - j_s) p_k / q} \right) \right], \tag{10}$$

where $\alpha = 1,\ldots,n$. An appropriate initial condition $y(t_0)$ must also be included (this will be assumed whenever a new variable is introduced). With (9) and (10) the reduction of system (4) to the GLV format is achieved.

3. Embedding into the LV form

In order to prove our assertions on (9), (10) we shall find of interest to use some known results from

Brenig and Goriely [13,14], which we briefly recall. A general GLV system (1) is formally invariant under quasimonomial transformations

$$x_i = \prod_{k=1}^{n} \hat{x}_k^{C_{ik}}, \quad i = 1, \ldots, n, \tag{11}$$

for any invertible matrix C. The matrices B, A and λ change to $\hat{B} = B \cdot C$, $\hat{A} = C^{-1} \cdot A$ and $\hat{\lambda} = C^{-1} \cdot \lambda$, respectively, but the GLV format is preserved. These transformations define the important concept of Brenig's equivalence classes (BEC), which consist of all GLV systems related through transformations (11). In such BECs, the products $B \cdot A$ and $B \cdot \lambda$ are invariants of the class. As mentioned in the introduction these invariants define the LV matrices, A' and λ', in (2). Then, the LV matrices are unique for a given BEC. This summarizes earlier results from Brenig and Goriely.

We are now in a position to introduce our two preliminary propositions concerning general properties of the BECs.

Lemma 1. The quasimonomials of a GLV system are also invariants of the BEC to which the GLV system belongs.

Proof. If we call $D = C^{-1}$ the ith new quasimonomial $1 \leqslant i \leqslant m$ will be

$$\prod_{j=1}^{N} \hat{x}_j^{\hat{B}_{ij}} = \prod_{j=1}^{N} \left(\prod_{k=1}^{N} x_k^{D_{jk}} \right)^{\hat{B}_{ij}}$$

$$= \prod_{k=1}^{N} x_k^{\left(\sum_{j=1}^{N} \hat{B}_{ij} D_{jk}\right)} = \prod_{k=1}^{N} x_k^{B_{ik}}.$$

The quasimonomials are, then, conserved in the class of equivalence (q.e.d.).

As was mentioned in Section 1, the quasimonomials of the GLV system are precisely the LV variables. Thus, since both the LV matrices and variables are the same for the whole BEC, we have proved the following:

Corollary 1. All GLV systems belonging to the same BEC embed into a single initial-value problem, defined by a unique LV system and initial condition on the quasimonomials (3).

We are then ready to focus our attention on the generic GLV system (9), (10). It is clear that different systems are obtained for distinct choices of the auxiliary variable (8). We shall first demonstrate that all these systems are part of one BEC, that is:

Theorem 1. For any auxiliary variable (8), all resulting GLV systems belong to the same BEC.

Proof. Given two different choices of auxiliary variables

$$y_i = f^{q_i} \prod_{s=1}^{n} x_s^{p_{si}}, \quad q_i \neq 0, \quad i = 1, 2,$$

the resulting sets of GLV variables will be, respectively, $(x_1, \ldots, x_n, y_1)$ and $(x_1, \ldots, x_n, y_2)$. A straightforward calculation shows that both sets of variables are connected through a transformation of the kind (11), with C given by

$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n & \beta \end{pmatrix},$$

where

$$\alpha_s = p_{s2} - \beta p_{s1}, \quad \beta = q_2/q_1.$$

Consequently, both systems are members of the same BEC (q.e.d.).

All systems complying the format (9), (10) are in the same BEC: according to Lemma 1 they must thus possess identical quasimonomials. This can be easily checked if we rewrite such quasimonomials in terms of the original variables $\bar{x}$ and $f(\bar{x})$. The corresponding equations for the x_s are

$$\dot{x}_s = x_s \left(\sum_{i_{s1}, \ldots, i_{sn}, j_s} a_{i_{s1} \ldots i_{sn} j_s} f^{j_s} \prod_{k=1}^{n} x_k^{i_{sk} - \delta_{sk}} \right), \tag{12}$$

with $s = 1, \ldots, n$. For the y we obtain

$$\dot{y} = y\left[\sum_{s=1}^{n}\left(p_s x_s^{-1}\dot{x}_s + \sum_{i_{s\alpha}j_s e_{s\alpha}e_s} a_{i_{s\alpha}j_s} b_{e_{s\alpha}e_s} q f^{e_s+j_s-1}\prod_{k=1}^{n} x_k^{i_{sk}+e_{sk}}\right)\right], \tag{13}$$

where $\alpha = 1,\dots,n$. The quasimonomials, as functions of $\bar{x}$, do not depend in any way on the definition of the auxiliary variable (8), but only on constants from (4), (5).

From the previous line of argument the following conclusion holds:

Theorem 2. The LV system (2) generated from (4) is completely determined from the choices for $f(\bar{x})$ and the representation of its derivatives.

In order to finalize the analysis it is necessary to verify the equivalence between the solutions of the initial and final systems. The conservation of the topology through the whole process which carries (4) into (2) is a necessary condition for ensuring the equivalence between the initial and final systems. A sufficient condition for this property (see Ref. [7], p. 22) is the existence of a diffeomorphism connecting the initial and final phase spaces. Since the dimension of the LV system is greater than that of (4) due to the succesive embeddings, such a diffeomorphism should connect the phase space of (4) and the manifold of $\mathbb{R}^m$ into which it is mapped.

The transformation embedding (9), (10) into (2) can be written as [14]

$$z_\alpha = \prod_{\beta=1}^{m} x_\beta^{\mathcal{B}_{\alpha\beta}}, \quad \alpha = 1,\dots,m, \tag{14}$$

where $x_1,\dots,x_n$ are the variables in (4), $x_{n+1} = y$ and $x_\alpha = 1$, for $\alpha = n+2,\dots,m$. $\mathcal{B}$ is the matrix of the expanded GLV system (see Ref. [14] for details) which is finally mapped onto the LV system. Eq. (14) is mathematically a diffeomorphism: it is obviously differentiable and onto map. Thus, we only need to prove that it is one to one. If we take logarithms in both sides of (14),

$$\begin{pmatrix} \ln(z_1) \\ \vdots \\ \ln(z_m) \end{pmatrix} = \mathcal{B}\begin{pmatrix} \ln(x_1) \\ \vdots \\ \ln(x_{n+1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$\mathrm{rank}(\mathcal{B}) = m$ by construction. Then, for any two vectors $\bar{x}_1$ and $\bar{x}_2$, $\mathcal{B}\ln(\bar{x}_1) \neq \mathcal{B}\ln(\bar{x}_2)$, unless $\bar{x}_1 = \bar{x}_2$. Thus the map is one to one and the topology is preserved by the successive transformations.

A necessary requirement for the previous results to hold is that the vectors $\bar{x}$ and $\bar{z}$ in (14) must have strictly positive entries. A necessary and sufficient condition for this is that both variables $x_1,\dots,x_n$ and the selected function $f(\bar{x})$ in (4) are positive. If this is the case, the LV variables $z_1,\dots,z_m$ and the intermediate variables defined along the process will be also strictly positive.

When this condition is not a priori satisfied, a phase-space translation is to be performed,

$$\bar{x} \longrightarrow \bar{x}' - \bar{c}, \quad f(\bar{x}) \longrightarrow f'(\bar{x}' - \bar{c}) - k.$$

In most cases arising in practice (for example with integer exponents), the translation preserves the format (4).

4. Examples

A first illustration is provided by an equation modelling the concentration of an allosteric enzyme (see Ref. [8], p. 137),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x\frac{a+bx}{c+x+dx^2}, \tag{15}$$

where a,b,c,d are positive real constants. If we exclude the asymptotic state $x = 0$, then, for any transient, we can have the following definition,

$$f(x) = \frac{1}{c+x+dx^2}, \quad \frac{\mathrm{d}f}{\mathrm{d}x} = -f^2 - 2dxf^2.$$

We now consider an auxiliary variable of form (8),

$$y = x^p f(x)^q, \quad q \neq 0,$$

from which we shall obtain a GLV system. Independently of the concrete values of p and q, such GLV systems will always embed into an unique LV system, as can be inferred from Theorem 2. To see this we introduce y as an explicit function of the parameters p and q. The resulting family of (p, q)-dependent GLV systems is

$$\dot{x} = x(-ax^{-p/q}y^{1/q} - bx^{1-p/q}y^{1/q}),$$
$$\dot{y} = y[-pax^{-p/q}y^{1/q} - pbx^{1-p/q}y^{1/q} + qax^{1-2p/q}y^{2/q} + q(b+2da)x^{2-2p/q}y^{2/q} + 2bdqx^{3-2p/q}y^{2/q}].$$

There are then five different quasimonomials,

$$x^{-p/q}y^{1/q} = f, \quad x^{1-p/q}y^{1/q} = xf,$$
$$x^{1-2p/q}y^{2/q} = xf^2, \quad x^{2-2p/q}y^{2/q} = x^2f^2,$$
$$x^{3-2p/q}y^{2/q} = x^3f^2.$$

Thus we have no dependence on p and q as far as the quasimonomials are concerned. The GLV matrices are

$$A = \begin{pmatrix} -a & -b & 0 & 0 & 0 \\ -pa & -pb & qa & q(b+2da) & 2bdq \end{pmatrix},$$

$$B = \begin{pmatrix} -p/q & 1/q \\ 1-p/q & 1/q \\ 1-2p/q & 2/q \\ 2-2p/q & 2/q \\ 3-2p/q & 2/q \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

And the LV matrices are given by

$$\lambda' = B\cdot\lambda = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A' = B\cdot A = \begin{pmatrix} 0 & 0 & a & b+2da & 2bd \\ -a & -b & a & b+2da & 2bd \\ -a & -b & 2a & 2(b+2da) & 4bd \\ -2a & -2b & 2a & 2(b+2da) & 4bd \\ -3a & -3b & 2a & 2(b+2da) & 4bd \end{pmatrix}.$$

Thus the LV matrices do not depend on p and q, just like the quasimonomials. The LV system is (p,q)-independent.

As a second example we shall mention the Morse oscillator, of relevance in the field of molecular structure [22],

$$\ddot{x} = -2d\alpha e^{-\alpha x}(1 - e^{-\alpha x}). \qquad (16)$$

Setting $y = \dot{x}$ we arrive at a first order differential system,

$$\dot{x} = y, \quad \dot{y} = -2d\alpha e^{-\alpha x}(1 - e^{-\alpha x}).$$

This system has form (4). The obvious choices at this stage are

$$f(x) = e^{-\alpha x}, \quad \frac{df}{dx} = -\alpha f.$$

Although $f(x) > 0$ for all x, this is not the case for x and y. Thus we must perform a phase-space translation of magnitude c, with c sufficiently large to ensure the positiveness of both x and y. When this is done, followed by the introduction of a new variable $z = x^p y^{p'} f^q$, the result is a GLV system of strictly positive variables,

$$\dot{x} = x(x^{-1}y - cx^{-1}),$$
$$\dot{y} = y(ax^{-p/q}y^{-1-p'/q}z^{1/q} - abx^{-2p/q}y^{-1-2p'/q}z^{2/q}),$$
$$\dot{z} = z(\alpha cq + px^{-1}y - cpx^{-1} + ap'x^{-p/q}y^{-1-p'/q}z^{1/q} - abp'x^{-2p/q}y^{-1-2p'/q}z^{2/q} - \alpha qy),$$

where $a = -2db\alpha$ and $b = e^{\alpha c}$. There are five quasimonomials, which will be the variables of the resulting 5×5 LV system,

$$x^{-1}y, \quad x^{-1}, \quad x^{-p/q}y^{-1-p'/q}z^{1/q} = y^{-1}f,$$
$$x^{-2p/q}y^{-1-2p'/q}z^{2/q} = y^{-1}f^2, \quad y.$$

The resultant GLV matrices are

$$A = \begin{pmatrix} 1 & -c & 0 & 0 & 0 \\ 0 & 0 & a & -ab & 0 \\ p & -cp & ap' & -abp' & -\alpha q \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ -p/q & -1-p'/q & 1/q \\ -2p/q & -1-2p'/q & 2/q \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 \\ 0 \\ \alpha cq \end{pmatrix}.$$

And the LV matrices will be now

$$\lambda' = B\cdot\lambda = \begin{pmatrix} 0 \\ 0 \\ \alpha c \\ 2\alpha c \\ 0 \end{pmatrix},$$

$$A' = B \cdot A = \begin{pmatrix} -1 & c & a & -ab & 0 \\ -1 & c & 0 & 0 & 0 \\ 0 & 0 & -a & ab & -\alpha \\ 0 & 0 & -a & ab & -2\alpha \\ 0 & 0 & a & -ab & 0 \end{pmatrix}.$$

Again, both the LV variables and matrices are independent of p, p' and q.

Acknowledgement

We would like to thank Dr. L. Brenig for fruitful discussions and suggestions.

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