

## Research



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## Reducing nonlinear dynamical systems to canonical forms

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A global framework for treating nonlinear differential dynamical systems is presented. It rests on the fact that most systems can be transformed into the quasi-polynomial format. Any system in this format belongs to an infinite equivalence class characterized by two canonical forms, the Lotka–Volterra (LV) and the monomial systems. Both forms allow for finding total or partial integrability conditions, invariants and dimension reductions of the original systems. The LV form also provides Lyapunov functions and systematic tools for stability analysis. An abstract Lie algebra is shown to underlie the whole formalism. This abstract algebra can be expressed in several realizations among which are the bosonic creation–destruction operators. One of these representations allows one to obtain the analytic form of the general coefficient of the Taylor series representing the solution of the original system. This generates a new class of special functions that are solutions of these nonlinear dynamical systems. From the monomial canonical form, one can prove an equivalence relationship between urn processes and dynamical systems. This establishes a new link between nonlinear dynamics and stochastic processes.

This article is part of the theme issue 'Dissipative structures in matter out of equilibrium: from chemistry, photonics and biology (part 1)'.

## 1. Preamble

Ilya Prigogine was among the first natural scientists in the twentieth century to draw the attention of the scientific community to the hitherto unsuspected complexity of the dynamics generated by systems of nonlinear ordinary differential equations. This complexity

was already partially known to mathematicians (H. Poincaré, A. N. Kolmogorov or G. D. Birkhoff), but was largely unfamiliar to the physicists, chemists and biologists.

Emerging phenomena like spatial and/or temporal structures, or, more mathematically, of limit cycles or chaotic regimes are all related to the nonlinearity of the laws that govern many physical systems. For spatially uniform systems, these laws are expressed as systems of nonlinear ordinary differential equations (ODEs). For inhomogeneous systems, diffusion and convection processes represented by spatial derivative operators must be added, thereby transforming the systems of ODEs into systems of nonlinear partial derivative equations.

The attitude of Ilya Prigogine faced with these systems was that of a thermodynamicist and a statistical physicist. He saw them as equations describing the evolution of macroscopic quantities fluctuating out of thermodynamical equilibrium. For small excursions out of that state, the equations are linear in the fluctuations of the macroscopic variables and nothing new appears apart from the equilibrium properties. However, for large fluctuations, these equations become nonlinear. Such large deviations can be sustained in a system by imposing on it external fluxes of matter, momentum and energy. Ilya Prigogine realized that when the values of these fluxes crossed some thresholds, qualitative changes emerged in the spatial and temporal organization of the systems. These transitions were in fact, from a mathematical standpoint, related to bifurcations. Very early, Ilya Prigogine foresaw that the emergence of primitive living organisms from inanimate matter was due to such transitions appearing when some thresholds in the intensities of external fluxes were crossed.

Inspired by this vision he attracted many researchers from a large spectrum of disciplines and engaged them in an extensive and multidisciplinary long-term research programme about the properties of nonlinear dynamical systems. This programme concerned domains as far apart as hydrodynamics, plasma physics or nonlinear optics, and molecular biology, insect societies, plant communities or the growth of cities. This led his group to many important breakthroughs in these fields [1].

Through many discussions, Ilya Prigogine transmitted the virus of nonlinearity to me. I mainly became interested in the mathematical aspects of nonlinear systems. While the main research orientation on these systems was essentially concerned with the topological properties of the phase space portrait, I engaged in a more constructivist approach.

In the following, I present a review of results, some of them quite recent, obtained in this field with several long-term co-workers. Most of these results have already been published in various articles except some of those appearing in §§4–6. In §2, the quasi-polynomial formalism is presented. Section 3 is a brief summary of results on the total or partial integrability of quasi-polynomial systems, their reducibility to lower dimensional systems and their stability properties. Section 4 presents an account of the Lie algebraic structure underlying the quasi-polynomial dynamical systems. Section 5 is dedicated to the solution of quasi-polynomial systems in terms of Taylor series with the analytic form of the general coefficient. Section 6 concerns a recently discovered equivalence between quasi-polynomial dynamical systems and urn stochastic processes.

## 2. Quasi-polynomial systems and canonical forms

In contrast to linear autonomous differential systems

$$\frac{dx_i}{dt} = \sum_{j=1}^n L_{ij}x_j; \quad i = 1, \dots, n, \quad (2.1)$$

in which the real independent variable  $t$  is time and  $L$  is any real and constant matrix, nonlinear differential systems

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n); \quad i = 1, \dots, n, \quad (2.2)$$

exhibit a great diversity of functional dependance of the functions  $f_i(x_1, \dots, x_n)$ . The vector constituted by these functions is called the vector field of the dynamical system (2.2). These functions, depending on the laws governing the phenomena to be modelled, may present any kind of nonlinearity. Thus, the vector field may have components that are combinations of linear functions, polynomials, elementary functions or even special functions. The only mathematical requirement comes from the conditions of the Cauchy–Lipschitz theorem for the existence and uniqueness of the solution of system (2.2). The drawback of this diversity is that it has prevented the construction of a theoretical apparatus as powerful as the one developed from linear algebra for linear dynamical systems of form (2.1).

However, despite their great variety, many nonlinear differential systems can be brought [2] to a unique format, the so-called quasi-polynomial (QP) format, also named Generalized Lotka–Volterra (LV) format [3]. A quasi-polynomial function is defined as a linear combination of monomials in the dependent real variables  $x_1, \dots, x_n$  in which the exponents are real numbers. Any QP system can be written in the following form:

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^N A_{ij} \prod_{k=1}^n x_k^{B_{jk}}, \quad i = 1, \dots, n, \quad (2.3)$$

where the numbers  $N$  and  $n$  are not necessarily equal.

As can be seen, in contrast with the linear systems that are characterized by only one square matrix,  $L$ , QP systems need two rectangular matrices,  $A$  and  $B$  to be characterized. The entries of the  $n \times N$  matrix  $A$  are real and constant, and denote the coefficients of the monomials  $\prod_{k=1}^n x_k^{B_{jk}}$ , while the entries of the  $N \times n$  matrix  $B$ , also real and constant, are the exponents of these monomials. To avoid divergences of certain monomials with negative exponents, the system must be such that its solutions stay in an open  $n$ -dimensional orthant for initial conditions that are already in it. Let us denote  $QP(A, B)$  as such a system (2.3).

Before going further, let us describe the contexts in which QP systems are found. They represent a large subset of the non-equilibrium dynamical systems that appear in various domains and can undergo self-organization transitions [4]. In nonlinear optics, the Lamb equations for a three-mode operating laser cavity are of that form. Electronic circuits obey systems of ODEs with a polynomial vector field. Chemical reactions are governed by rate equations that are systems of ODEs with polynomial nonlinearities. The dynamics of interacting populations also obeys the system of polynomial ODEs. In mechanics, systems with polynomial interaction potentials are represented by systems of ODEs with a polynomial vector field and, consequently, can be cast in QP form (2.3). Even in cosmology such systems appear under the name of generalized LV systems [5]. All these systems can be directly written in the QP form (2.3) without increasing the number of dependent variables.

Moreover, dynamical systems involving non-polynomial functions can sometimes be transformed to the QP format. For instance, biochemical kinetic equations for reactions involving enzymatic catalysis contain the so-called Michaelis–Menten terms. These terms are rational functions in dependent variables. In such cases, the QP format is recovered at the price of increasing the dimensions of the ODE system by introducing new variables that precisely are these terms. More generally, all ODE systems involving non-polynomial functions of the dependent variables in their vector field can be cast in the QP form if these functions obey ODEs with polynomial nonlinearities in terms of these functions [2]. The QP format (2.3) offers a unifying tool that turns out to be quite effective and used, among others, in the fields of biological networks [6] and of feedback control [7] of biological and mechanical systems.

The recognition of the quasi-polynomial notation (2.3) of polynomial vector fields with two matrices  $A$  and  $B$  paves the way to a novel approach of all the dynamical systems that can be cast in the form  $QP(A, B)$ . Indeed, the specific notation (2.3) lends itself to identify a group of mappings

of the phase-space that preserve the QP format. These transformations read

$$x_i = \prod_{k=1}^n \tilde{x}_k^{C_{ik}} \quad \text{for } i = 1, \dots, n, \quad (2.4)$$

where the matrix  $C$  can be any real and invertible  $n \times n$  matrix. This mapping transforms equation (2.3) into

$$\dot{\tilde{x}}_i = \tilde{x}_i \sum_{j=1}^N \tilde{A}_{ij} \prod_{k=1}^n \tilde{x}_k^{\tilde{B}_{jk}} \quad \text{for } i = 1, \dots, n, \quad (2.5)$$

with

$$\tilde{A} = C^{-1}A \quad (2.6)$$

and

$$\tilde{B} = BC. \quad (2.7)$$

A consequence of equations (2.6) and (2.7) is

$$\tilde{B}\tilde{A} = BA. \quad (2.8)$$

As can be seen, these transformations (2.4) form a group of diffeomorphisms that conserve the QP format (2.3). Moreover, the matrix product  $BA$  is an invariant of these mappings. This means that all systems  $QP(A, B)$  with the same matrix product  $BA$  belong to the same equivalence class under these transformations. Moreover, the phase portrait is preserved by these transformations.

These properties have been found independently several times [8–10] and led their authors to the discovery of two fundamental canonical forms for each equivalence class. The proof goes as follows. Let us replace the dependent variables  $x_1, \dots, x_n$  of the QP system (2.3) by the  $N$  monomials appearing in it

$$u_j = \prod_{k=1}^n x_k^{B_{jk}}, \quad (2.9)$$

with  $j = 1, \dots, N$ .

A simple calculation, using equation (2.3), of the time-derivative of these new variables gives

$$\frac{du_j}{dt} = u_j \sum_{k=1}^N M_{jk} u_k, \quad (2.10)$$

where  $j = 1, \dots, N$  and

$$M = BA. \quad (2.11)$$

System (2.10) is the first canonical form. These equations are well known to specialists in population dynamics: they are the LV equations about which there is an extensive number of results. The  $N \times N$  matrix  $M$  is called the population interaction matrix. This terminology is perhaps a bit far fetched for arbitrary matrices  $A$  and  $B$ ; the matrix  $M$  does not generally fulfil the conditions that are required by population dynamics. However, properties such as the existence of an explicit Lyapunov function for systems with an interior fixed point and with an admissible matrix  $M$  still exist [11]. Another fundamental property is the existence, in an appropriate Poisson structure, of Hamiltonian functions for the  $N$ -dimensional LV systems [12].

Equation (2.11) means that all the  $QP(A, B)$  systems such that the product  $BA$  of their matrices is equal to a given matrix  $M$  are equivalent to the system of LV equations (2.10) with that same interaction matrix. They, thus, constitute an equivalence class of label  $M$ .

For  $N > n$ , obviously the  $N$  variables  $u_j$  are not all independent. As a consequence, the trajectories of the original system (2.3) are confined to invariant subspaces corresponding to the relations between the variables  $u_j$  in the  $N$ -dimensional phase-space. In the case  $n > N$ , the passage from equation (2.3) to (2.10) corresponds to a reduction of dimensions of the system. More details on these questions can be found in references [3,7,11]. Note that transformation (2.9) is, in general, not bijective except for  $N = n$ , in which case it belongs to the group of monomial

transformations (2.4). Let us also stress that the LV canonical form is itself a particular  $N$ -dimensional QP system with  $A = M$  and  $B = I$ , where  $I$  is the identity matrix. In other words, it is a  $QP(M, I)$  system.

With regard to the second canonical form, it appears as follows. As the LV form (2.10) is an  $N$ -dimensional QP system, and provided the matrix  $M$  is invertible, one can apply to this system the following monomial transformation:

$$u_k = \prod_{j=1}^N \tilde{u}_j^{M_{kj}}, \quad (2.12)$$

for  $k = 1, \dots, N$ .

This is a transformation of type (2.4) with  $C = M$ . Now, using relations (2.6) and (2.7) one readily finds

$$\frac{d\tilde{u}_j}{dt} = \tilde{u}_j \prod_{k=1}^N \tilde{u}_k^{M_{jk}}, \quad (2.13)$$

with  $j = 1, \dots, N$ . This is the second canonical form. We shall call it the monomial canonical form. Unlike the first canonical form (2.10) which, in addition to population dynamics, is well known in several domains, such as evolution theory [13], plasma physics [14], cosmology [5], this second canonical form is almost unknown. Only recently and by accident, did we find an occurrence of this equation in relation to the so-called urn stochastic processes. This led us to find a deep equivalence relationship between the latter and the whole class of dynamical systems that are reducible to the QP format, as we shall see later.

Four more comments must be made on the second canonical form. First, in the case where the matrix  $M$  is not invertible but  $A$  is of maximal rank, it is possible to derive the second canonical form directly [3] from the  $QP(A, B)$  system (2.3). A sketch of the proof goes as follows. For  $N > n$ , one adds  $N - n$  auxiliary dependent variables obeying differential equations that contain only the monomials already present in the original system. This amounts to the original QP system being embedded into a new QP system in  $N$  dimensions for which the new matrix  $\hat{A}$  corresponds to the old rectangular matrix completed by  $N - n$  lines of arbitrary but non-vanishing entries. As for the new matrix  $\hat{B}$ , it is the matrix  $B$  completed by  $N - n$  columns of null entries. The two new matrices  $\hat{A}$  and  $\hat{B}$  are now square and if the original rectangular  $A$  was of maximal rank, that is  $n$ , then the square matrix  $\hat{A}$  is invertible. One, thus, can apply to the embedded system a monomial transformation of type (2.4) with an  $N \times N$  matrix  $C = \hat{A}$ . Applying rules (2.6) and (2.7), this leads to a new QP system with a  $\hat{A}' = I$  and  $\hat{B}' = \hat{B}\hat{A} = BA = M$  whose form is purely monomial. In the case  $N < n$ , one shows [15] that the original QP system can be reduced to a square QP system of  $N$  dimensions on which the same transformations as above can be performed.

Second, the same comments as those already discussed for the LV canonical form can be made about the cases when  $N > n$  and  $N < n$ . Third, the monomial canonical form is a particular  $N$ -dimensional QP system with  $A = I$  and  $B = M$ , that is it is a  $QP(I, M)$  system.

A fourth comment must be made for both the canonical forms. It is current in models to find a diagonal linear term in the vector field of the system. Hence, one must consider systems of the form

$$\frac{dx_i}{dt} = a_i x_i + x_i \sum_{j=1}^N A_{ij} \prod_{k=1}^n x_k^{B_{jk}}; \quad i = 1, \dots, n, \quad (2.14)$$

where the coefficients  $a_i$  are real. Now, this system can be written in the following QP form:

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^{N+1} \hat{A}_{ij} \prod_{k=1}^n x_k^{\hat{B}_{jk}}; \quad i = 1, \dots, n, \quad (2.15)$$

with an  $n \times (N + 1)$  matrix  $\hat{A}$  such that, for all  $i = 1, \dots, n$ , one has  $\hat{A}_{ij} = A_{ij}$  for  $j = 1, \dots, N$  and  $\hat{A}_{iN+1} = a_i$ , and a  $(N + 1) \times n$  matrix  $\hat{B}$  such that, for all  $k = 1, \dots, n$ , one has  $\hat{B}_{jk} = B_{jk}$  for  $j = 1, \dots, N$  and  $\hat{B}_{N+1k} = 0$ .

Let us summarize the above results in the following

**Statement 1.** The set of all the differential dynamical systems that can be cast into the QP format  $\dot{x}_i = x_j \sum_{j=1}^N A_{ij} \prod_{k=1}^n x_k^{B_{jk}}$ ;  $i = 1, \dots, n$ , for any couple of real constant matrices  $A$  and  $B$ , and whose solutions  $x_i(t)$  remain forever in the same open orthant, is divided into equivalence classes labelled by the matrix  $M = BA$ . In each equivalence class, there are two canonical forms  $\dot{u}_j = u_j \sum_{k=1}^N M_{jk} u_k$  and  $\dot{\tilde{u}}_j = \tilde{u}_j \prod_{k=1}^N \tilde{u}_k^{M_{jk}}$ ,  $j = 1, \dots, N$ . Moreover, the solutions to the QP systems that belong to a same equivalence class are exchanged under the group of monomial diffeomorphisms  $x_i = \prod_{k=1}^n \tilde{x}_k^{C_{ik}}$ ,  $i = 1, \dots, N$ , where  $C$  is any matrix belonging to  $Gl(n, \mathbb{R})$ , and can be retrieved from the solutions of each canonical form.

Before leaving this chapter, let me give a simple example of the above transformations. Consider the following three-dimensional system where the nonlinearity is not quadratic

$$\frac{dx_1}{dt} = -x_1 + x_1(x_1x_2x_3 + x_1^2x_2x_3^2), \quad (2.16)$$

$$\frac{dx_2}{dt} = 3x_2 + x_2(2x_1x_2x_3 + x_1^2x_2x_3^2) \quad (2.17)$$

$$\frac{dx_3}{dt} = 2x_3 + x_3(-x_1x_2x_3 + 2x_1^2x_2x_3^2). \quad (2.18)$$

The form (2.14) of this system has  $N = 2$  and  $n = 3$ . With the above reasoning, one thus expects a reduction of the dimension from 3 to 2 of the canonical forms. The matrices  $A$  and  $B$  of form (2.14) are

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \quad (2.19)$$

and

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad (2.20)$$

while the coefficients of the linear terms are  $a_1 = -1$ ,  $a_2 = 3$  and  $a_3 = 2$ .

The matrices  $\hat{A}$  and  $\hat{B}$  of the corresponding QP form (2.15) are

$$\hat{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ -1 & 2 & 2 \end{bmatrix} \quad (2.21)$$

and

$$\hat{B} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.22)$$

The matrix  $\hat{M} = \hat{B}\hat{A}$  that appears in the two canonical forms of the QP system is

$$\hat{M} = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.23)$$

The LV canonical form, hence, is

$$\frac{du_1}{dt} = u_1(2u_1 + 4u_2 + 4u_3), \quad (2.24)$$

$$\frac{du_2}{dt} = u_2(2u_1 + 7u_2 + 5u_3) \quad (2.25)$$

and

$$\frac{du_3}{dt} = 0. \quad (2.26)$$

The last equation has a constant solution  $u_3 = c$  that can be found in terms of the initial conditions on the  $x_i(0)$ ,  $i = 1, \dots, 3$ . The above system of equations, thus, becomes

$$\frac{du_1}{dt} = 4cu_1 + u_1(2u_1 + 4u_2) \quad (2.27)$$

and

$$\frac{du_2}{dt} = 5cu_2 + u_2(2u_1 + 7u_2). \quad (2.28)$$

This is a two-dimensional quadratic system of LV form.

The monomial canonical form of the QP system is

$$\frac{d\tilde{u}_1}{dt} = \tilde{u}_1^3 \tilde{u}_2^4 \tilde{u}_3^4, \quad (2.29)$$

$$\frac{d\tilde{u}_2}{dt} = \tilde{u}_1^2 \tilde{u}_2^8 \tilde{u}_3^5 \quad (2.30)$$

and

$$\frac{d\tilde{u}_3}{dt} = \tilde{u}_3. \quad (2.31)$$

The last equation can be solved yielding  $v_3 = c'e^t$ , where  $c'$  is a constant. This solution may be inserted in the two previous equations leading to a reduced system of two monomial ODEs.

The solution of the original system (2.16)–(2.18) can easily be found from the solution of either the LV form or the monomial form that are given above, if these equations can be solved.

The above results lead to many consequences that are exposed in the next chapter. Some of them cannot be detailed here by lack of place and we refer the reader to the publications. Others are presented in more details.

### 3. Applications

#### (a) Reduction properties, invariants and decoupling

As we saw in the previous example, the transcription of a  $n$ -dimensional dynamical system in the QP format immediately allows to detect whether that system can be reduced. Indeed, as the two canonical forms are of dimension  $N$ , it suffices to calculate the number of independent monomials  $N$  of the vector field. If  $N < n$ , then the canonical forms are reduced in dimension.

For the systems for which  $N \geq n$ , the situation is different and the rank of the matrices  $A$  and  $B$  must be inspected [15]. If matrix  $A$  has a rank, say  $r$ , smaller than  $n$  (the maximal rank), then there exists a particular quasi-monomial transformation that leads one to find  $n - r$  monomial invariants. If, on the contrary, it is  $B$  that is of non-maximal rank  $r$ , then, a quasi-monomial transformation exists that decouples the transformed system into a  $r$ -dimensional nonlinear sub-system of smaller dimension and a  $(n - r)$ -dimensional linear sub-system that depends on the solutions of the former.

Moreover, powerful criteria for finding and constructing more general invariants and first integrals, and invariant manifolds for QP systems have been systematically developed [16,17]. Most of these algorithms were implemented in the computer algebra software (QPSI) written in the computer algebra language MAPLE [18].

The QP approach is also particularly well adapted to the study of the integrability of differential systems via the Painlevé test as shown in [19].

#### (b) Stability properties

There exists an extensive literature on the stability properties of LV systems such as permanence and boundedness of their solutions. Lyapunov functions can systematically be constructed for LV systems which is not the case for most nonlinear differential systems.

Transferring these stability properties into stability properties of the QP systems that belong to the same equivalence class as these LV equations is possible and results in a large corpus of



results. The general procedure starts by checking the existence of a Lyapunov function for a given interior fixed point of the LV form. Then, the stability property of the corresponding fixed point of the QP systems is obtained by reverting to the original variables. It is impossible to summarize here these results and we refer the reader to the relevant articles [7,20–22].

## 4. Lie algebraic structure of quasi-polynomial systems

### (a) Carleman infinite embedding of Lotka–Volterra canonical form

Let us consider the LV canonical form

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^N M_{ij} x_j. \quad (4.1)$$

One should remember that this equation represents a whole class of equivalence of  $QP(A, B)$  systems for which  $BA = M$ .

The Carleman infinite embedding [23,24] amounts to a linearization of a finite-dimensional system of nonlinear ODEs by embedding it into an infinite system of linear ODEs. This is done by defining the following function:

$$\psi(r, t) \equiv x_1^{r_1} x_2^{r_2} \dots x_N^{r_N}, \quad (4.2)$$

where  $r$  is the vector  $(r_1, \dots, r_N)$  with  $r_i \geq 0$ , for  $i = 1, \dots, N$ . The  $x_i$ ,  $i = 1, \dots, N$ , are the  $N$  functions of time  $x_i(t)$  that are solutions of the LV system (4.1) for the initial condition  $(x_{01}, \dots, x_{0N})$ .

The time derivative of  $\psi(r, t)$  calculated from the LV system (4.1) yields

$$\frac{\partial}{\partial t} \psi(r, t) = \sum_{i=1}^N \sum_{j=1}^N r_i M_{ij} E_j \psi(r, t), \quad (4.3)$$

where the action of the translation operators  $E_j$  on any function  $f(r_1, \dots, r_N)$  is given by

$$E_j f(r_1, \dots, r_j, \dots, r_N) = f(r_1, \dots, r_j + 1, \dots, r_N). \quad (4.4)$$

Hence, the partial differential-difference equation (4.3) with the initial condition

$$\psi(r, 0) \equiv x_{01}^{r_1} \dots x_{0N}^{r_N} \quad (4.5)$$

is equivalent to the original LV system. The components  $x_i$ ,  $i = 1, \dots, N$ , of the solution of the LV system (4.1) are obtained from the function  $\psi(r, t)$  by

$$x_i(t) = \psi(0, \dots, r_i = 1, \dots, 0, t). \quad (4.6)$$

Defining the operator

$$H \equiv \sum_{i=1}^N \sum_{j=1}^N r_i M_{ij} E_j, \quad (4.7)$$

equation (4.3) reads

$$\frac{\partial}{\partial t} \psi(r, t) = H \psi(r, t). \quad (4.8)$$

As reported, this equation is linear in  $\psi$ . This was the objective of performing the Carleman embedding.

Let us have a more algebraic view of the situation. The basic operators here are  $r_i$  and  $E_i$ ,  $i = 1, \dots, N$ . Their commutators are

$$[r_i, r_j] = 0, [E_i, E_j] = 0, [E_i, r_j] = \delta_{ij} E_j, \quad (4.9)$$

for  $i, j \in \{1, \dots, N\}$ . One may view equation (4.8) as a Schrödinger-like equation expressed in the variables  $r_i$ ,  $i = 1, \dots, N$ , which are the eigenvalues of the multiplicative operators  $\hat{r}_i$ —the hat



superscript is suppressed to simplify the notation. In this picture, the operator  $H$  is a Hamiltonian-like operator, i.e. it is the infinitesimal generator of time transformations. We claim now that the LV equation (4.1) represents a Heisenberg-like picture of the Schrödinger-like evolution governed by (4.8). To prove this, let us define the time-dependent form of the basic operators  $r_i$  and  $E_i$ ,  $i = 1, \dots, N$  as follows:

$$r_i(t) \equiv e^{-tH} r_i e^{tH} \quad (4.10)$$

and

$$E_i(t) \equiv e^{-tH} E_i e^{tH} \quad (4.11)$$

Hence, the time derivative of  $E_i(t)$  is given by

$$\frac{dE_i(t)}{dt} = [E_i(t), H], \quad (4.12)$$

and using definitions (4.4), (4.7) and (4.11), it is easy to obtain

$$\frac{dE_i(t)}{dt} = E_i(t) \sum_{j=1}^N M_{ij} E_j(t). \quad (4.13)$$

As can be recognized at first glance, this equation is identical to the LV canonical form (4.1). This justifies the use of the term Heisenberg picture we used to qualify the LV equation. We can, therefore, state the following result.

**Statement 2.** The LV system and its Carleman embedding equation are, respectively, the Heisenberg-like and the Schrödinger-like pictures of the same dynamics.

A last remark: the use of the terms Hamiltonian-like, Schrödinger-like and Heisenberg-like reflects that the physics underlying these equations is not that of quantum mechanics. The operator  $H$  is not even hermitian. In the following the ending ‘like’ will be suppressed in order to simplify the text.

## (b) Abstract Lie algebraic structure of Lotka–Volterra systems

In the preceding subsection, we defined the Schrödinger picture as based on definition (4.2) of the  $\psi$  function. This function can be viewed as a member of a tensor product of one-particle Hilbert spaces. The numbers  $r_i$ ,  $i = 1, \dots, N$  represent the number of ‘particles’ in ‘state’  $i$ . This analogy is confirmed in §4c(ii), where a representation in terms of boson creation–destruction operators is given.

A more general version of equation (4.8) can be given by defining a state vector  $|\psi\rangle$  in an abstract Hilbert space. The particular representation introduced in the previous subsection corresponds to

$$\psi(r, t) = \langle r | \psi \rangle. \quad (4.14)$$

In other words,  $\psi(r, t)$  is the projection of the abstract state vector  $|\psi\rangle$  on the eigenvector basis of the vector operator  $r$ . Let us now distance ourselves from this particular basis. This is done by introducing two sets of abstract operators,  $\{A_i\}$  and  $\{B_i\}$ ,  $i = 1, \dots, N$  obeying the commutation rules

$$[A_i, A_j] = [B_i, B_j] = 0, [B_i, A_j] = \delta_{ij} B_j. \quad (4.15)$$

They generate a Lie algebra in which one can define the following Hamiltonian:

$$H \equiv \sum_{i=1}^N \sum_{j=1}^N A_i M_{ij} B_j. \quad (4.16)$$

In the Heisenberg picture, one, thus, has

$$\frac{dB_i(t)}{dt} = [B_i(t), H], \quad (4.17)$$

and a simple calculation leads to

$$\frac{dB_i(t)}{dt} = B_i(t) \sum_{j=1}^N M_{ij} B_j(t), \quad (4.18)$$

which, again, is an LV system for the operators  $B_i$ .

For curiosity, let us see what is the evolution equation of the  $A_i(t)$

$$\frac{dA_i(t)}{dt} = [A_i(t), H] = -B_i(t) \sum_{j=1}^N M_{ij}^T A_j(t). \quad (4.19)$$

This is a linear system for the  $A_i(t)$  with time-dependent coefficients that are the solution  $B_i(t)$  of the LV system (4.18).

In the Schrödinger picture, the state vector is time-dependent and obeys

$$\frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle. \quad (4.20)$$

Now, as a comparison, let us write down the corresponding abstract Lie algebra for linear systems of form (2.1). The algebra is now generated by two sets of abstract operators,  $\{A_i\}$  and  $\{B_i\}$ ,  $i = 1, \dots, N$ , verifying the commutation rules

$$[A_i, A_j] = [B_i, B_j] = 0, [B_i, A_j] = \delta_{ij}. \quad (4.21)$$

In this algebra, we define the Hamiltonian

$$H \equiv \sum_{i=1}^N \sum_{j=1}^N A_i L_{ij} B_j. \quad (4.22)$$

The time evolution of the operators is governed by the equations

$$\frac{dB_i(t)}{dt} = [B_i(t), H] = \sum_{j=1}^N L_{ij} B_j(t) \quad (4.23)$$

and

$$\frac{dA_i(t)}{dt} = [A_i(t), H] = - \sum_{j=1}^N L_{ij}^T A_j(t). \quad (4.24)$$

Equation (4.23) is a linear equation of matrix  $L$  identical to equation (2.1), while equation (4.24) is also linear but with a matrix  $-L^T$ . One could also write a Schrödinger equation but it is of no interest here as there is no necessity to embed an already linear system of finite dimension into another linear system of infinite dimension.

The difference between the abstract algebra associated with the LV canonical system and the algebra related to the linear system should be underlined. It consists in the difference between the commutators  $[B_i, A_j] = \delta_{ij} B_j$  appearing in the LV algebra, and the commutators  $[B_i, A_j] = \delta_{ij}$  for the linear algebra. That linear algebra is well known by physicists and corresponds to the Heisenberg–Weyl algebra that is fundamental in classical and quantum mechanics. The LV algebra has not been, up to now and to our knowledge, the object of a systematic study in nonlinear dynamics.

The following statement summarizes the main result of this section.

**Statement 3.** The abstract Lie algebra that underlies an LV dynamical system and, more generally, its equivalent QP systems is generated by two sets of operators  $\{A_i; i = 1, \dots, N\}$  and  $\{B_i; i = 1, \dots, N\}$  verifying the commutation rules  $[A_i, A_j] = [B_i, B_j] = 0$ ,  $[B_i, A_j] = \delta_{ij} B_j$ . The Hamiltonian corresponding to the LV dynamics in this algebra is  $H \equiv \sum_{i=1}^N \sum_{j=1}^N A_i M_{ij} B_j$ .

## (c) Some realizations of the abstract Lotka–Volterra Lie algebra

### (i) Carleman realization

We already introduced this realization in relation to the Carleman embedding method. The basic operators  $\{A_i\}$  and  $\{B_i\}$  are, respectively, represented by  $A_i = r_i$  and  $B_i = E_i$ . We already gave the Heisenberg and Schrödinger equations for this realization. In this realization, the Hamiltonian (4.16) has the form  $H = \sum_{i=1}^N \sum_{j=1}^N r_i M_{ij} E_j$ .

### (ii) Realization in terms of bosonic creation–destruction operators

This particular realization sheds some light on the difference between linear algebra and LV algebra. For the algebra of linear systems, the operators  $\{A_i\}$  and  $\{B_i\}$  are given in this realization by  $A_i = a_i^\dagger$  and  $B_i = a_i$ . Hence, one has  $[B_i, A_j] = [a_i, a_j^\dagger] = \delta_{ij}$  and  $[A_i, A_j] = [a_i^\dagger, a_j^\dagger] = [B_i, B_j] = [a_i, a_j] = 0$ , which are the commutation rules of the usual Heisenberg–Weyl algebra. Applying formula (4.22), the Hamiltonian for the linear systems is shown to be given by  $H \equiv \sum_{i=1}^N \sum_{j=1}^N a_i^\dagger L_{ij} a_j$ . Such Hamiltonians describe, for example, systems of oscillators in quantum mechanics. But they describe also all the linear dynamical systems that appear in any other context. Indeed, the Heisenberg dynamics of the operators  $a_i(t)$  generated by  $H$  and derived from equations (4.23), provides

$$\frac{da_i(t)}{dt} = \sum_{j=1}^N L_{ij} a_j(t). \quad (4.25)$$

Concerning the LV algebra, its realization in terms of the  $\{a_i\}$  and  $\{a_i^\dagger\}$  corresponds to  $A_i = a_i^\dagger a_i$  and  $B_i = a_i$ . That these operators satisfy the LV algebra (4.15) commutators can readily be checked by using the commutation relations between the bosonic creation–destruction operators. The abstract Hamiltonian (4.16) now takes the form  $H = \sum_{i=1}^N \sum_{j=1}^N a_i^\dagger a_i M_{ij} a_j$ . This Hamiltonian is manifestly not Hermitian. But this is not a necessity here as it does not represent a quantum mechanical process but, rather, an LV dynamical system. Using equation (4.18), one gets the Heisenberg picture of the dynamics of the operators  $a_i(t)$

$$\frac{da_i(t)}{dt} = a_i(t) \sum_{j=1}^N M_{ij} a_j(t), \quad (4.26)$$

which, again, is of the LV form.

### (iii) Liouville realization

Let us assume the following form of the  $\{A_i\}$  and  $\{B_i\}$  operators:  $A_i = -(\partial/\partial x_i)x_i$  and  $B_i = x_i$ . Using the properties of the derivation operators  $\partial/\partial x_i$ , it can be checked that this realization of the basic operators satisfy the LV commutation rules (4.15).

The LV Hamiltonian (4.16) reads now

$$H = - \sum_{i=1}^N \sum_{j=1}^N \frac{\partial}{\partial x_i} x_i M_{ij} x_j. \quad (4.27)$$

The dynamics in the Heisenberg picture is given by

$$\frac{dx_i(t)}{dt} = [x_i(t), H] = x_i(t) \sum_{j=1}^N M_{ij} x_j(t), \quad (4.28)$$

which is the LV system. The Schrödinger version is

$$\frac{\partial}{\partial t} \psi(x_1, \dots, x_N, t) = - \sum_{i=1}^N \sum_{j=1}^N \frac{\partial}{\partial x_i} x_i M_{ij} x_j \psi(x_1, \dots, x_N, t). \quad (4.29)$$

Obviously, the above equation corresponds to the Liouville equation associated with the LV system (4.28). In other words, it is the evolution equation for a probability density  $\psi(x_1, \dots, x_N, t)$ . This reflects a situation in which the initial conditions of a trajectory solution of the LV system is not given with infinite precision but, rather, in the form of an initial probability density

$$\psi(x_1, \dots, x_N, t=0) = \psi_0(x_1, \dots, x_N). \quad (4.30)$$

Obviously, equation (4.29) preserves the normalization of the initial probability density. Thus, it also conserves the non-negativity of the initial probability density in the time evolution of the solution.

To end this section, let us mention the extensive and profound work of Kowalski & Steeb [24] about Carleman linearization. It is among the very few works on nonlinear dynamics in that orientation. They developed, independently of the authors of the present article, the Carleman linearization of general nonlinear dynamical systems and its representation in terms of bosonic creation–destruction operators. They were, however, not aware of the QP systems formalism and, consequently, they did not know the property of reducibility of these systems to the two canonical forms that are at the focus of our work.

## 5. General solution as Taylor series

In this section, we deduce the Taylor series expansion of the solution to the LV canonical form (4.1) and obtain the analytic expression of its general coefficient.

The starting point is the Carleman embedding equation (4.3). Let us rewrite the action (4.4) of the displacement operators  $E_i$ ,  $i = 1, \dots, N$ , as follows:

$$E_i \psi(r, t) = \psi(r + e^i, t), \quad (5.1)$$

with  $r = (r_1, \dots, r_N)$  and where  $e^i$  is the unit vector along the coordinate axis  $i$ . The solution to equation (4.3) or (4.8) is

$$\psi(r, t) = e^{tH} \psi(r, 0). \quad (5.2)$$

with  $H$  given by formula (4.7).

We now express the exponential of the operator  $H$  by its Taylor series

$$\psi(r, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H^k \psi(r, 0). \quad (5.3)$$

The  $k$ th power of the Hamiltonian  $H$  on the initial function is readily obtained by successive iterations

$$H^k \psi(r, 0) = \sum_{i_1=1}^N \dots \sum_{i_k=1}^N [r.M]_{i_1} [(r + e^{i_1}).M]_{i_2} \dots [(r + e^{i_1} + \dots + e^{i_{k-1}}).M]_{i_k} \psi(r + e^{i_1} + \dots + e^{i_k}, 0). \quad (5.4)$$

Introducing in equation (5.4) the relations  $\psi(r, 0) \equiv x_{01}^{r_1} \dots x_{0N}^{r_N}$  and  $(e^i.M)_j = M_{ij}$ , and inserting it into the Taylor series (5.3), we get

$$\psi(r, t) = x_{01}^{r_1} \dots x_{0N}^{r_N} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N (r.M)_{i_1} [(r.M)_{i_2} + M_{i_1 i_2}] \dots [(r.M)_{i_k} + M_{i_{k-1} i_k}] x_{0i_1} \dots x_{0i_k}, \quad (5.5)$$

and using relation (4.6),  $x_i(t) = \psi(0, \dots, r_i = 1, \dots, 0, t) = \psi(e^i, t)$ , the Taylor series for the solution of the LV system is obtained

$$x_i(t) = x_{0i} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N M_{ii_1} [M_{ii_2} + M_{i_1 i_2}] \dots [M_{ii_k} + M_{i_{k-1} i_k}] x_{0i_1} \dots x_{0i_k} \quad (5.6)$$

This series in which the general coefficient is analytically known defines a new class of special functions. This class presents a certain analogy with the class of generalized hypergeometric functions. As for these functions, the differential equations they solve as well as their Taylor series, along with the analytic expression of their general coefficient, are known. The knowledge of the general term of series (5.6) allows for

1. evaluating the radius of convergence of the Taylor series
2. performing analytic continuation
3. evaluating the error at each truncation order of the series by calculating the remainder of the series
4. calculating Padé approximants
5. calculating the analytic expression of the solution at various approximation orders via computer algebra languages like MAPLE or Mathematica and construct a computer algebra software for solving QP systems [25].

One must now build the Taylor series that solves the original  $QP(A, B)$  systems that are equivalent to the LV canonical form (4.1). In order to do this, one has to resort to an arbitrary monomial transformation on the dependent variables  $x_i$  of the LV system

$$x_i = \prod_{k=1}^N \tilde{x}_k^{C_{ik}}; \quad i = 1, \dots, N. \quad (5.7)$$

Indeed, we know from relations (2.6) and (2.7) that this transforms the LV equation into

$$\dot{\tilde{x}}_i = \tilde{x}_i \sum_{j=1}^N \tilde{A}_{ij} \prod_{k=1}^N \tilde{x}_k^{\tilde{B}_{jk}}, \quad (5.8)$$

with

$$\tilde{A} = C^{-1}M; \quad \tilde{B} = C. \quad (5.9)$$

For systems with rectangular matrix  $\tilde{B}$ , the above transformation can be performed after an embedding of that matrix into a square matrix and can be performed only at the condition that  $\tilde{B}$  is of maximal rank.

All systems (5.8) belong to the same equivalence class with  $\tilde{B}\tilde{A} = CC^{-1}M = M$  corresponding to the LV canonical form.

This transformation induces the following transformation of the Carleman  $\psi$  function

$$\tilde{\psi}(r, t) = \tilde{x}_1^{r_1}(t) \dots \tilde{x}_N^{r_N}(t) = \psi(rC^{-1}, t), \quad (5.10)$$

which, using equations (5.5) and (5.9), yields

$$\begin{aligned} \tilde{\psi}(r, t) = & \tilde{x}_{01}^{r_1} \dots \tilde{x}_{0N}^{r_N} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N (r.A)_{i_1} [(r.A)_{i_2} + M_{i_1 i_2}] \dots \\ & [(r.A)_{i_k} + M_{i_1 i_k} + \dots + M_{i_{k-1} i_k}] \prod_{l=1}^N \tilde{x}_{0l}^{B_{i_1 l} + \dots + B_{i_N l}}. \end{aligned} \quad (5.11)$$

The solution  $\tilde{x}_i(t)$  of the QP systems of form (5.8) is, then, obtained by putting  $r = e^i$  in the above function

$$\tilde{x}_i(t) = \tilde{x}_{0i} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N A_{ii_1} [A_{ii_2} + M_{i_1 i_2}] \dots [A_{ii_k} + M_{i_1 i_k} + \dots + M_{i_{k-1} i_k}] \prod_{l=1}^N \tilde{x}_{0l}^{B_{i_1 l} + \dots + B_{i_N l}}, \quad (5.12)$$

for  $i = 1, \dots, N$  and with  $M = BA$ .

Here, again, the knowledge of the analytic form of the coefficient of the Taylor series brings the same benefits as enumerated above for the series solving the LV form. It defines also a new special function for any forms of the two rectangular matrices  $A$  and  $B$ .

One should stress here that the cases of the QP systems (2.3) with  $n \neq N$  are included in the above result. Indeed, for  $n < N$  as well as for  $n > N$ , the QP system can be, respectively, extended or reduced to an  $N$ -dimensional QP system that transforms into the LV form under a monomial transformation. More details on these question can be found in [3].

One can conclude this chapter by the following statement:

**Statement 4.** The Taylor series of the solution to the QP system  $\dot{x}_i = x_i \sum_{j=1}^N A_{ij} \prod_{k=1}^n x_k^{B_{jk}}$ ;  $i = 1, \dots, n$ , for the initial conditions  $x_i(0) = x_{0i}$ , is given by

$$x_i(t) = x_{0i} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N A_{ii_1} [A_{ii_2} + M_{i_1 i_2}] \cdots [A_{ii_k} + M_{i_1 i_k} + \cdots + M_{i_{k-1} i_k}] \prod_{l=1}^N x_{0l}^{B_{i_1 l} + \cdots + B_{i_k l}}.$$

This series defines an infinite class of special functions labelled by the two rectangular matrices  $A$  and  $B$ . These functions obey the above  $QP(A, B)$  system.

As said above, the knowledge of the general term of the Taylor series gives information on the convergence radius. But it also provides an evaluation of the rest of the series when the latter is truncated at a given order. Moreover, it offers the possibility of calculating the analytic continuation in order to extend the time domain on which the dynamics of the system is known.

## 6. Equivalence between quasi-polynomial systems and urn stochastic processes

We report here a recent and quite unforeseeable advance. It establishes an equivalence between stochastic processes called urn processes and a sub-class of the QP systems [26].

Urn processes are stochastic processes that offer powerful modelling tools in various scientific fields such as statistical physics, population genetics or epidemiology [27,28]. An urn process consists of a box, the urn, containing at the initial time a distribution of balls of  $N$  different colours, and an elementary process of replacement of these balls that we now describe. The initial composition of the urn is given by the vector  $U^0 = (u_{10}, \dots, u_{N0})$ , where  $u_{i0}$  is the initial number of balls of colour  $i$  in the urn.

The process starts as soon as a ball is picked from the urn with uniform probability, its colour is noted and the ball is re-introduced in the urn. If the colour of the drawn ball was  $i$ ,  $i = 1, \dots, N$ , then specified numbers  $M_{ij}$  of balls of the different colours  $j = 1, \dots, N$  are taken from an infinite reservoir of balls and introduced in the urn. If the integer number  $M_{ij}$  is negative, then that number of balls of colour  $j$  must be extracted from the urn and put in the reservoir.

We shall assume here that the process is balanced, that is at each step the total variation of the number of balls in the urn is a constant  $\sigma$ . This means that  $\sum_{j=1}^N M_{ij} = \sigma$ . At each step of the process, this procedure is repeated with the same replacement  $N \times N$  matrix  $M$ . After  $n$  successive replacement steps, the initial composition vector  $U_0$  evolves into a final vector  $U = (u_1, \dots, u_N)$ .

One would like to calculate the probability of such a transition in  $n$  steps between given vectors  $U_0$  and  $U$ . We adopt here the combinatorial approach developed in [29]. The trajectory of the  $n$ -step process in the  $N$ -dimensional composition space is called a history of length  $n$ . Let us introduce the number of possible histories of length  $n$  linking  $U_0$  to  $U$ ,  $H_n(u_{10}, \dots, u_{N0}, u_1, \dots, u_N)$ . We also define the number of  $n$ -steps histories starting from  $U_0$  irrespective of the final composition vector,  $H_n(u_{10}, \dots, u_{N0})$ .

As the process is balanced, all the histories in  $n$  steps are equiprobable. Hence, the probability we are looking for is given by

$$P(U_n = U \mid U_0 = U^0) = \frac{H_n(u_{10}, \dots, u_{N0}, u_1, \dots, u_N)}{H_n(u_{10}, \dots, u_{N0})}. \quad (6.1)$$

However, the most adapted tools for our purpose here are not the number of length  $n$  histories but their counting generating function defined as

$$H(x_1, \dots, x_N, u_{10}, \dots, u_{N0}, z) \equiv \sum_{n=0}^{\infty} \sum_{u_1=0}^{\infty} \cdots \sum_{u_N=0}^{\infty} H_n(u_{10}, \dots, u_{N0}, u_1, \dots, u_N) x_1^{u_1} \cdots x_N^{u_N} \frac{z^n}{n!}. \quad (6.2)$$

In terms of that function the searched probability reads

$$P(U_n = U \mid U_0 = U^0) = \frac{[x_1^{u_1} \cdots x_N^{u_N} z^n] H(x_1, \dots, x_N, u_{10}, \dots, u_{N0}, z)}{[z^n] H(1, \dots, 1, u_{10}, \dots, u_{N0}, z)}, \quad (6.3)$$

where the notation  $[x^m]S(x)$  for a power series  $S(x)$  represents, as usual, the coefficient of  $x^m$  in that series.

Now, there is a connection between the above generating function and the monomial canonical form (2.13) of an equivalence class of QP systems. Ph. Flajolet, Ph. Dumas and V. Puyhaubert established the following deep property [29]: any  $N$ -colours balanced urn process of replacement matrix  $M$  is isomorphic to the system of ordinary differential equation

$$\frac{dX_i}{dt} = X_i \prod_{j=1}^N X_j^{M_{ij}}, \quad (6.4)$$

with  $i = 1, \dots, N$ . This isomorphism amounts to the following equation relating the history counting generating function of the urn process and the solution of the above system

$$H(x_{10}, \dots, x_{N0}, u_{10}, \dots, u_{N0}, z) = X_1^{u_{10}}(z) \cdots X_N^{u_{N0}}(z), \quad (6.5)$$

where  $X_i(z)$ ,  $i = 1, \dots, N$ , is the solution at time  $t = z$  of system (6.4) with initial conditions  $X_i(0) = x_{i0}$ .

Apparently, the authors of this theorem did not know the theory of QP systems, otherwise they would immediately have recognized system (6.4) as one of the two canonical forms of that theory.

Using our statement 1, this identification allows us to make the following new statement:

**Statement 5.** To any balanced urn process of replacement matrix  $M$ , one can associate an infinite equivalence class of  $QP(A, B)$  systems,  $\dot{u}_i(t) = u_i \sum_{j=1}^N A_{ij} \prod_{k=1}^n u_k^{B_{jk}}$ ,  $i = 1, \dots, n$ , with  $BA = M$ . The counting generating function of this urn process is obtained via the relation  $H(x_{10}, \dots, x_{N0}, u_{10}, \dots, u_{N0}, z) = X_1^{u_{10}}(z) \cdots X_N^{u_{N0}}(z)$  where the  $X_i(z)$  are the solutions of the monomial canonical differential system associated with the above-mentioned equivalence class:  $\dot{X}_i(z) = X_i \prod_{j=1}^N X_j^{M_{ij}}$ ,  $i = 1, \dots, N$  for  $t = z$  and for the initial conditions  $X_i(0) = x_{i0}$ . Conversely, the solution of this monomial differential system for the initial conditions  $X_i(0) = x_{i0}$  can be calculated from the counting generating function of the associated urn process by:  $X_i(z) = H(x_{10}, \dots, x_{N0}, 0, \dots, u_{i0} = 1, \dots, 0, z)$ , for  $z = t$ . As a consequence, the solution of any  $QP(A, B)$  system with  $BA = M$  can be calculated from the previous solution by applying to it the monomial transformations  $u_i = \prod_{k=1}^N X_k^{C_{ik}^{-1}}$ ;  $i = 1, \dots, N$  for any invertible matrix  $C$  such that  $A = C^{-1}M$  and  $B = C$ .

A promising application of this result would be to use the above result to build an entirely new approach to computer simulation of dynamical systems. One could, indeed, compute the solutions of the monomial canonical form with initial conditions  $X_i(0) = x_{i0}$  by simulating the associated urn process and taking advantage of the relation  $X_i(z) = H(x_{10}, \dots, x_{N0}, 0, \dots, u_{i0} = 1, \dots, 0, z)$ . The coding in a computer simulation program of an elementary urn replacement process of coloured balls should be easy to implement and its computation, fast. The method would be limited, of course, to solving  $QP(A, B)$  systems such that the product matrix  $BA$  has only integer entries.



## 7. Conclusion

I hope to have shown that theory of quasi-polynomial systems is interesting. What started as merely a new notation of differential dynamical systems led to many applications. These were based on the property of systems with a large spectrum of nonlinearities to be transformed into two canonical forms, the LV and the monomial forms. The first form contains the simplest type of non-trivial quadratic nonlinearity and has been the object of a huge number of publications. The second, though, much less known, possesses many potentialities that should be envisaged. Future results on these two types of equations are waiting to be translated into new properties for QP systems belonging to their equivalence classes.

Prospectively, the study of the relation between urn processes and dynamical processes prefigures interesting and new interactions between nonlinear dynamics and the theory of stochastic processes. Moreover, the study of the special functions corresponding to the Taylor series derived in this article could be of interest for special functions theory. The study of the asymptotic behaviour in time of these functions could shed some new light on the properties of chaotic regimes, for instance.

From the computing viewpoint, the equivalence urn processes-dynamical systems as well as the availability of Taylor series solutions with analytic general coefficient offer new perspectives. They provide radically novel tools for the numerical and symbolical computation of the solutions of these systems.

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