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Necessary conditions for the existence of quasi-polynomial invariants: the quasi-polynomial and Lotka–Volterra systems

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Abstract

We show that any quasi-polynomial invariant of a quasi-polynomial dynamical system can be transformed into a quasi-polynomial invariant of a homogeneous quadratic Lotka–Volterra dynamical system. We show how this quasi-polynomial invariant can be decomposed in a simple manner. This decomposition permits to conclude that the existence of polynomial semi-invariants in Lotka–Volterra systems is a necessary condition for the existence of quasi-polynomial invariants. We derive a method which allows to construct the necessary conditions for existence of semi-invariants on Lotka–Volterra dynamical systems. Applications are given.

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1. Introduction

One of the most important issues in the study of a dynamical system depending on parameters is to obtain the parameter values for which a first integral exists (a time-independent first integral is called an invariant). Some methods are used for this intent such as the Painlevé analysis [1–3], Lie symmetries [9,10], or a direct method where a functional form for the first-integral is assumed, as for instance a polynomial invariant, the coefficients in the polynomial being determined from the invariance conditions [4–8]. Recently a new method was developed based on Darboux polynomials [11] where the problem of integrability is related to the analysis of the existence of first integrals and invariants [12].

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Our analysis also delivers necessary conditions on the parameters for the existence of a quasi-polynomial invariant (QP-invariant), which must be fulfilled regardless of the exponents in the quasi-polynomial. This result greatly reduces the region in the parameter-space where to look for the existence of first-integrals.

In a previous article [13] we presented a new approach for the analysis of invariant manifolds of quasi-polynomial dynamical systems. In this paper we show, by the reduction of the equations belonging to the class of quasi-polynomial dynamical systems [14–16] to the Lotka–Volterra form [17–19], how to predict and compute QP-invariants. An algorithm for a characterization of irreducible invariants is given, which heavily relies on the structure of the Lotka–Volterra form associated to the quasi-polynomial class of dynamical systems. A simple relation between polynomial semi-invariants and the class of QP-invariants is obtained. They may be stated by the following fundamental result: *for the Lotka–Volterra canonical form the basic tenets of the QP-invariants are homogeneous polynomial semi-invariants. These homogeneous polynomials constitute an irreducible decomposition of the quasi-polynomial invariants.*

The second section of this paper presents the quasi-polynomial dynamical systems and the associated Lotka–Volterra canonical form. In Section 3 the QP-invariants for the quasi-polynomial dynamical systems (QP-systems) are characterized. The decomposition of invariants in irreducible homogeneous semi-invariants for the Lotka–Volterra form is demonstrated. In Section 4, the necessary conditions for the existence of homogeneous semi-invariants are studied for 2, 3 and 4-dimensional Lotka–Volterra canonical form. An explicit algorithm for computing these polynomial semi-invariants is given which takes advantages of a recursion property between Lotka–Volterra systems of successive dimensions. Applications are given in Section 5 for the (a,b,c) Lotka–Volterra and May–Leonard systems.

2. The set of quasi-polynomial dynamical systems and the Lotka–Volterra canonical form

Any QP-system defined on $\mathbb{R}^n$ can be put in the form:

$$\dot{x}_i = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \quad i = 1,\dots,n . \tag{1}$$

That class includes all polynomial differential equations. We may consider, without loss of generality, that $m \geqslant n$ and that $\mathrm{rank}(B) = n$ [20–22]. The QP-system (1) can be ordered such that the n first rows of the matrix B are linearly independent.

Let us consider the dynamics of the m quasi-monomial variables:

$$y_j = \prod_{k=1}^{m} x_k^{B_{jk}}, \quad j = 1,\dots,m . \tag{2}$$

Proposition 1. *For any QP-system* (1) *the dynamics of the quasi-monomials* (2) *is a Lotka–Volterra system* (*LV-system*) *given by*:

$$\dot{y}_j = y_j \sum_{k=1}^{m} M_{jk} y_k \tag{3}$$

with $M = BA$.

Naturally, we can associate to systems (1) and (3) the respective differential vector fields:

$$\delta_{(A,B)} = \sum_{i=1}^{n} (Ax^B)_i x_i \frac{\partial}{\partial x_i}, \qquad \delta_M = \sum_{i=1}^{m} (My)_i y_i \frac{\partial}{\partial y_i} \tag{4}$$

with

$$(Ax^B)_i = \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \qquad (My)_i = \sum_{j=1}^{m} M_{ij} y_j \ .$$

We finish this section defining the following quasi-monomial transformation (QMT):

$$y_i = \prod_{k=1}^{n} x_k^{\mathscr{B}_{ik}}, \quad i = 1, \dots, n \ , \tag{5}$$

where $\mathscr{B}$ is a $n \times n$ matrix with $\mathscr{B}_{ik} = B_{ik}$, $i, k = 1, \dots, n$. Since the n first rows of B are linearly independent, $\mathscr{B}$ is a non-singular matrix ($\det(\mathscr{B}) \neq 0$). Thus we define the inverse transformation of Eq. (5), which is a QMT given by the matrix $\mathscr{B}^{-1}$ [21].

3. Quasi-polynomials invariants

Let us consider a quasi-polynomial function on $\mathbb{R}^n$ given by

$$F = \sum_{i=1}^{N} F_i \prod_{k=1}^{n} x_k^{C_{ik}} \ , \tag{6}$$

where F_i and C_{ik} are real constants. The QMT in Eq. (5) transforms F into

$$F = \sum_{i=1}^{N} F_i \prod_{k=1}^{n} y_k^{[C\mathscr{B}^{-1}]_{ik}} \ . \tag{7}$$

Hence we can conclude that F is also a quasi-polynomial function in the new variables y_i. We summarize this property in the following proposition:

Proposition 2. *The QMT (5) transforms a quasi-polynomial function (6) into another quasi-polynomial function given by Eq. (7) with the same number of quasi-monomial terms.*

Now if we consider a QMT (5), we conclude that

$$\delta_M(F) = 0 \ \Leftrightarrow \ \delta_{(A,B)}(F) = 0 \ . \tag{8}$$

Proposition 2 and relation (8) permit to assert that in order to study the class of quasi-polynomials invariants (QP-invariants) of system (1) it is only necessary to analyze the invariants of the LV-system (3). Therefore the rest of this paper is devoted to the study of the QP-invariants of the LV-systems.

Let us consider a QP-invariant of the form:

$$G = \sum_{i=1}^{N} G_i y^{\beta_i} \tag{9}$$

with $y^{\beta_i} = \prod_{k=1}^{n} y_k^{\beta_{ik}}$ and $\beta_i = [\beta_{i1}, \ldots, \beta_{im}]$ and $G_i \in \mathbb{R}$. The function G satisfies the equation:

$$\delta_M(G(y)) = 0 \,, \tag{10}$$

which gives

$$\sum_{i=1}^{N} G_i y^{\beta_i} \sum_{j=1}^{m} (\beta_i M)_j y_j = 0 \,,$$

$$\sum_{i=1}^{N} \sum_{j=1}^{m} G_i [\beta_i M]_j y^{\beta_i + e_j} = 0 \,, \tag{11}$$

where $[\beta_i M]_j = \sum_{k=1}^{m} \beta_{ik} M_{kj}$ and e_j is a vector with the jth component equal to one and with the remaining components vanishing.

Let us consider two monomials y^{β_i} and y^{β_j} in the set $S = \{y^{\beta_1}, \ldots, y^{\beta_N}\}$. They are said connected ($y^{\beta_i} \leftrightarrow y^{\beta_j}$) if $\beta_i - \beta_j \in Z^m$, where

$$Z^m = \{(z_1, \ldots, z_m)/z_i \in Z\} \,.$$

We can see that the connection $\leftrightarrow$ is an equivalence relation on S, which implies that the set S is decomposed into the disjoint equivalence classes of this relation. Usually the number of equivalence classes in S is smaller than N. The different equivalent classes in this set are denoted by $[y^{\alpha_i}]$ where $i = 1, \ldots, r \leqslant N$ and $\alpha_i = \beta_{q_i}$ with: $q_i \in \{1, 2, \ldots, N\}$ and $q_i \neq q_j$ for $i \neq j$. We can rewrite G as follows:

$$G = G_{(1)} + G_{(2)} + \cdots + G_{(r)} \,, \tag{12}$$

where $G_{(i)}$ is a quasi-polynomial with all quasi-monomials in the same equivalence class $[y^{\alpha_i}]$. This set of equivalence classes is given by

$$[y^{\alpha_i}] = \{y^{\alpha_i + \eta_{ik}}/k = 1, \ldots, N_i \; \eta_{ik} \in Z^m\} \,, \tag{13}$$

where N_i is the number of elements in S equivalent to y^{α_i}. We can assert the following proposition:

Proposition 3. *Let us consider a QP-invariant G given by Eq. (12). Then the $G_{(i)}$ are also QP-invariants.*

Proof. Performing the derivation $\delta_M(G)=0$ we obtain

$$\delta_M(G_{(1)})+\cdots+\delta_M(G_{(r)})=0\,,$$
$$g_{(1)}+\cdots+g_{(r)}=0\,,$$

where $g_{(i)}=\delta_M(G_{(i)})$. We can see by (11) that all quasi-monomials in $g_{(i)}$ are of the form $y^{\alpha_i+\eta_{ik}+e_p}$ $(p=1,\dots,m)$. This implies that all monomials in $g_{(i)}$ are in the equivalence class $[y^{\alpha_i}]$.

Now for $i\neq j$ $([y^{\alpha_i}]\neq[y^{\alpha_j}])$ we can see that all quasi-monomials in $g_{(i)}$ are different from all quasi-monomials in $g_{(j)}$, or otherwise we should have $y^{\alpha_i+\eta_{ip}+e_r}=y^{\alpha_j+\eta_{jq}+e_s}$ with $\alpha_i-\alpha_j\in Z^m$ and consequently $y^{\alpha_i}\leftrightarrow y^{\alpha_j}$ and $[y^{\alpha_i}]=[y^{\alpha_j}]$ for $i\neq j$.

Finally, since all monomials in $g_{(i)}$ and $g_{(j)}$ $(i\neq j)$ are different we can conclude (see appendix A) that $g_{(i)}$ and $g_{(j)}$ are linearly independent. This implies that $g_{(i)}=\delta_M(G_{(i)})=0$. □

The true importance of this result can be seen if we rewrite $G_{(i)}$ considering Eq. (13):

$$G_{(i)}=y^{\alpha_i}\sum_{k=1}^{N_i}G_{ik}y^{\eta_{ik}},\qquad \eta_{ik}\in Z^m\,, \tag{14}$$

where $G_{ik}\in\mathbb{R}$. We factorize Eq. (14) by the highest common denominator y^{γ_i}, with $\gamma_i\in\mathbb{N}^m$, for the set of quasi-monomials $y^{\eta_{ik}}$ and obtain:

$$G_{(i)}=y^{(\alpha_i-\gamma_i)}\sum_{k=1}^{N_i}G_{ik}y^{(\eta_{ik}+\gamma_i)}\,. \tag{15}$$

Such that $\eta_{ik}+\gamma_k$ is a positive integer. In fact we have proved that $G_{(i)}=y^{\theta_i}P_i(y)$, where $P_i(y)$ is a polynomial function and $\theta_i\in\mathbb{R}^m$. Finally we can conclude by Proposition 3 that a QP-invariant (9) can be decomposed in the form

$$G(y)=\sum_{i=1}^{r}y^{\theta_i}P_i(y)\,, \tag{16}$$

where $P_i(y)$ is a homogeneous polynomial function. Here we remember that if the number N_i of monomials in the equivalence class $[y^{\alpha_i}]$ is one, then $G_{(i)}$ is a quasi-monomial invariant (QM-invariant). If $G_{(i)}$ has at least two quasi-monomials then $P_i(y)$ is a non-constant polynomial. The QM-invariants has been studied exhaustively by Brenig and Goriely [11] and Gouzé [22].

From the relation $\delta_M(y^{\theta_i}P_i(y))=0$, we obtain

$$P_i\delta_M(y^{\theta_i})+y^{\theta_i}\delta_M(P_i)=0\,,$$

$$y^{\theta_i}\left[P_i\sum_{j=1}^{m}(\theta_iM)_jy_j+\delta_M(P_i)\right]=0\,,$$

and finally

$$\delta_M(P_i) = \left(-\sum_{j=1}^{m} (\theta_i M)_j y_j \right) P_i \, . \tag{17}$$

A polynomial P satisfying $\delta_M(P) = \lambda P$ with $\lambda = \sum_{i=1}^{m} \lambda_i y_i$ is called a *semi-invariant* or *Darboux's polynomial* of the derivative δ_M [23]. In this work λ is called the *eigenvalue* of the semi-invariant. The following properties about semi-invariants of homogeneous derivatives are known in the literature [23]:

Proposition 4. *Let us consider a semi-invariant P and its respective eigenvalue λ. P can be written as $P = \prod_k P_k^{r_k}$ ($r_k \in \mathbb{N}$) factorized in its irreducible factors P_k such that P_k is a semi-invariant with eigenvalue λ_k and $\lambda = \sum_k r_k \lambda_k$.*

Proposition 5. *Let us suppose a semi-invariant P with eigenvalue λ written as follows: $P = P_1 + \cdots + P_s$, where P_i is a homogeneous polynomial of degree n_i and $n_i \neq n_j$ for $i \neq j$. Thus P_i is a semi-invariant with eigenvalue λ.*

To end this section we observe that by Eqs. (16) and (17) and Propositions 4 and 5 we can state the most important result of this paper:

Proposition 6 (Decomposition Theorem). *Any QP-invariant G given by Eq. (12) can be decomposed into the form:*

$$G(y) = \sum_{i=1}^{r} y^{\theta_i} P_i(y) \, , \tag{18}$$

where $y^{\theta_i} P_i(y)$ is a QP-invariant, $P_i(y)$ is a homogeneous semi-invariant with eigenvalue given in Eq. (17) and $P_i(y)$ can be decomposed as a product of irreducible homogeneous semi-invariants.

4. Homogeneous semi-invariants for the Lotka–Volterra system in 2, 3 and 4 dimensions

We now establish the necessary conditions for the existence of irreducible semi-invariants, the basic blocks for building the quasi-polynomial invariants in LV-systems. An important property of LV-system (3) used in our analysis is that the restriction of a LV-system to any face ($y_i = 0$) of the phase space is again a LV-system of lower dimension. This allows for constructing the necessary conditions for the existence of semi-invariants by analyzing the conditions for the lower dimensional LV-systems obtained by successive projections into the phase-space faces. These steps may be iterated and allows for finding in a simple and systematic way the general necessary conditions for n-dimensional systems, as explained below. Finally, before entering into more details, let us stress that all the results obtained below are automatically translated into properties of the QP-invariants by the Decomposition Theorem.

4.1. Two-dimensional LV-systems

Let us write explicitly the 2-dimensional LV-system:

$$\dot{y}_1 = y_1(M_{11}y_1 + M_{12}y_2) ,$$
$$\dot{y}_2 = y_2(M_{21}y_1 + M_{22}y_2) . \tag{19}$$

We obtain first its linear semi-invariants. It is immediate to see that the monomials y_1 and y_2 are linear semi-invariants. Now let us consider a semi-invariant of the form:

$$I_1 = a_1 y_1 + a_2 y_2 , \tag{20}$$

such that $a_1, a_2 \neq 0$. We know [13,24] that the semi-invariant of the form given in Eq. (20) is given by

$$I_1 = (M_{21} - M_{11})y_1 + (M_{22} - M_{12})y_2 , \tag{21}$$

and I_1 satisfies the equation:

$$\delta_M(I_1) = (M_{11}y_1 + M_{22}y_2)I_1 . \tag{22}$$

When $M_{11} = M_{21}$ and $M_{22} = M_{12}$ we have a degenerate case as I_1 given in Eq. (20) is a semi-invariant with eigenvalue $\lambda = M_{11}y_1 + M_{22}y_2$ for any value of a_1 and a_2.

A homogeneous irreducible semi-invariant of degree n can be written as

$$I_n = \sum_{\alpha=0}^{n} a_\alpha y_1^\alpha y_2^{n-\alpha} = \sum_{\alpha=0}^{n} a_\alpha y^{\hat{\alpha}(n)} , \tag{23}$$

with $y^{\hat{\alpha}(n)} \equiv y_1^\alpha y_2^{n-\alpha}$. The action of the derivative δ_M on I_n yields:

$$\delta_M(I_n) = \sum_{\alpha=0}^{n} a_\alpha y_1^\alpha y_2^{n-\alpha}([\hat{\alpha}(n)M]_1 y_1 + [\hat{\alpha}(n)M]_2 y_2) . \tag{24}$$

We also have that $\delta_M(I_n) = (\lambda_1 y_1 + \lambda_2 y_2)I_n$ or more explicitly:

$$\delta_M(I_n) = \sum_{\alpha=0}^{n} a_\alpha y_1^\alpha y_2^{n-\alpha}(\lambda_1 y_1 + \lambda_2 y_2) . \tag{25}$$

Using expressions (24) and (25) of $\delta_M(I_n)$ we obtain:

$$\sum_{\alpha=0}^{n} a_\alpha y_1^{\alpha+1} y_2^{n-\alpha}([\hat{\alpha}(n)M]_1 - \lambda_1) + \sum_{\alpha=0}^{n} a_\alpha y_1^\alpha y_2^{n-\alpha+1}([\hat{\alpha}(n)M]_2 - \lambda_2) = 0 . \tag{26}$$

The coefficients of the different monomials in Eq. (26) must vanish separately, i.e.,

$$([\hat{0}(n)M]_2 - \lambda_2)a_0 = 0$$

$$\vdots$$

$$([\hat{j}(n)M]_2 - \lambda_2)a_j + ([\widehat{j-1}(n)M]_1 - \lambda_1)a_{j-1} = 0$$

$$\vdots$$

$$([\hat{n}(n)M]_1 - \lambda_1)a_n = 0\ . \tag{27}$$

From Eq. (23) we see that for the cases $a_0 = 0$ and $a_n = 0$ the variables y_1 and y_n can be factored, and correspond consequently to a reducible semi-invariant which is not possible from our starting assumption, and will not be considered in the remaining of our analysis.

Now let us study Eqs. (27) more thoroughly. First one should remember that $\hat{\alpha}(n) = [\alpha, n-\alpha]$ and $[\hat{\alpha}(n)M]$ denote the multiplication of a vector $\hat{\alpha}(n)$ by the matrix M. Thus we have

$$\hat{\alpha}(n)M = [\alpha M_{11} + (n-\alpha)M_{21}, \alpha M_{12} + (n-\alpha)M_{22}]\ . \tag{28}$$

From Eq. (27) we obtain:

$$\lambda_1 = [\hat{n}(n)M]_1 = nM_{11}, \qquad \lambda_2 = [\ \hat{0}\,(n)M]_1 = nM_{22}\ . \tag{29}$$

After expanding the coefficients of a_j and a_{j-1} in Eq. (27), for $i \leqslant j \leqslant n$, we obtain:

$$j(M_{12} - M_{22})a_j + (n-j+1)(M_{21} - M_{11})a_{j-1} = 0\ . \tag{30}$$

From Eq. (30) we have that $M_{12} - M_{22} = 0$ implies $M_{21} - M_{11} = 0$. The converse is also true. This can be seen from the cases $j = 1$ and $j = n$ and using the fact that $a_0, a_n \neq 0$. Therefore we have two different cases: (i) $M_{12} = M_{22}$; $M_{21} = M_{11}$ and (ii) $(M_{12} - M_{22})(M_{21} - M_{11}) \neq 0$. Case (i) corresponds to the degenerate linear case. Case (ii) yields after solving Eq. (30) for the semi-invariant:

$$I_n = a_0((M_{21} - M_{11})y_1 + (M_{22} - M_{12})y_2)^n\ . \tag{31}$$

Therefore we conclude that the only irreducible polynomial semi-invariants are the linear semi-invariants. Hence, any semi-invariant of degree n is written as

$$I_n = y_1^{n_1} y_2^{n_2}((M_{21} - M_{11})y_1 + (M_{22} - M_{12})y_2)^{(n-n_1-n_2)}\ . \tag{32}$$

4.2. Three-dimensional Lotka–Volterra system

In the 3-dimensional case, the property of the LV system, that its restrictions to faces (coordinate planes or axes) are still LV-systems will be systematically used. Qualitative new features will appear in contrast with the 2-dimensional case. The n-dimensional structure of our interative process will not appear until the 4-dimensional case is considered.

Let us now write down the explicit form of the 3-dimensional LV-system:

$$\begin{aligned}\dot{y}_1 &= y_1(M_{11}y_1 + M_{12}y_2 + M_{13}y_3)\ ,\\ \dot{y}_2 &= y_2(M_{21}y_1 + M_{22}y_2 + M_{23}y_3)\ ,\\ \dot{y}_3 &= y_3(M_{31}y_1 + M_{32}y_2 + M_{33}y_3)\ .\end{aligned} \tag{33}$$

Now we write the same irreducible homogeneous polynomial semi-invariant for this system in the three following equivalent ways, without any loss of generality:

$$\begin{aligned}I &= y_1A_1(y_1, y_2, y_3) + c_1I_1(y_2, y_3)\ ,\\ I &= y_2A_2(y_1, y_2, y_3) + c_2I_2(y_1, y_3)\ ,\\ I &= y_3A_3(y_1, y_2, y_3) + c_3I_3(y_1, y_2)\ ,\end{aligned} \tag{34}$$

where A_1, A_2, A_3, I_1, I_2 and I_3 are real polynomial functions to be determined using our procedure. The fact that I is an irreducible polynomial of degree n imposes that the coefficients c_i and the functions I_i are non vanishing for $i = 1, 2, 3$.

The homogeneous polynomials I_1, I_2 and I_3 are respectively semi-invariants of the LV-system restricted to faces $y_1 = 0$, $y_2 = 0$ and $y_3 = 0$. Using the results established above for the 2-dimensional LV-system we can write the following form for these polynomials:

$$\begin{aligned}I_1 &= y_2^{\gamma_1} y_3^{\gamma_2}((M_{32} - M_{22})y_2 + (M_{33} - M_{23})y_3)^{\gamma_3}\ ,\\ I_2 &= y_1^{\beta_1} y_3^{\beta_2}((M_{31} - M_{11})y_1 + (M_{33} - M_{13})y_3)^{\beta_3}\ ,\\ I_3 &= y_1^{\alpha_1} y_2^{\alpha_2}((M_{21} - M_{11})y_1 + (M_{22} - M_{12})y_2)^{\alpha_3}\ ,\end{aligned} \tag{35}$$

with

$$\gamma_1 + \gamma_2 + \gamma_3 = n, \qquad \beta_1 + \beta_2 + \beta_3 = n, \qquad \alpha_1 + \alpha_2 + \alpha_3 = n\ , \tag{36}$$

respectively.

Let us introduce the restrictions of the 3-dimensional LV-derivatives to the faces by $\delta_M\,|_{y_i=0} \equiv \delta_{M^{(i)}}$ where the 2×2 matrix $M^{(i)}$ is obtained by eliminating the ith row and column from matrix M. Since I is a semi-invariant for the 3-dimensional system we have

$$\delta_M I = (\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3)I\ . \tag{37}$$

The restriction of Eq. (37) to the face $y_1 = 0$ then yields

$$\delta_{M^{(1)}}(I_1) = (\lambda_2 y_2 + \lambda_3 y_3)I_1(y_2, y_3)\ , \tag{38}$$

where I_1 is given in Eq. (35). After some simple manipulations one obtains:

$$\begin{aligned}\delta_{M^{(1)}}(I_1) = ((&\gamma_1M_{22} + \gamma_2M_{32} + \gamma_3M_{22})y_2\\ &+ (\gamma_1M_{23} + \gamma_2M_{33} + \gamma_3M_{33})y_3)I_1(y_2, y_3)\ .\end{aligned} \tag{39}$$

Combining Eqs. (38) and (39), we obtain:

$$\lambda_2 = \gamma_1M_{22} + \gamma_2M_{32} + \gamma_3M_{22}, \qquad \lambda_3 = \gamma_1M_{23} + \gamma_2M_{33} + \gamma_3M_{33}\ . \tag{40}$$

In face $y_2 = 0$ and $y_3 = 0$ the same algebraic manipulations lead respectively to

$$\lambda_1 = \beta_1 M_{11} + \beta_2 M_{31} + \beta_3 M_{11}, \qquad \lambda_3 = \beta_1 M_{13} + \beta_2 M_{33} + \beta_3 M_{33} , \tag{41}$$

$$\lambda_1 = \alpha_1 M_{11} + \alpha_2 M_{21} + \alpha_3 M_{11}, \qquad \lambda_2 = \alpha_1 M_{12} + \alpha_2 M_{22} + \alpha_3 M_{22} . \tag{42}$$

Clearly compatibility conditions between Eqs. (40), (41) and (42) are necessary. If we remember that $\alpha_3 = n-(\alpha_1+\alpha_2)$, $\beta_3 = n-(\beta_1+\beta_2)$ and $\gamma_3 = n-(\gamma_1+\gamma_2)$, these conditions are easily obtained and read:

$$\begin{aligned}
&(M_{31} - M_{11})\beta_2 + (M_{11} - M_{21})\alpha_2 = 0 ,\\
&(M_{32} - M_{22})\gamma_2 + (M_{22} - M_{12})\alpha_1 = 0 ,\\
&(M_{23} - M_{33})\gamma_1 + (M_{33} - M_{13})\beta_1 = 0 .
\end{aligned} \tag{43}$$

These conditions constitute a linear system of equations for the exponents α_1, α_2, β_1, β_2, γ_1 and γ_3, which are all integers and satisfy the relations $\alpha_1 + \alpha_2 \leqslant n$, $\beta_1 + \beta_2 \leqslant n$ and $\gamma_1 + \gamma_2 \leqslant n$.

Remarkably this system is expressed in terms of a matrix derived from Lotka–Volterra matrix M. There are 3 equations for 6 unknowns which must be non-negative integers. Thus the problem of finding the necessary conditions for the existence of the irreducible homogeneous semi-invariants and consequently for the quasi-polynomial invariants of the 3-dimensional LV-system is reduced to the solution of an incompletely determined linear system of equations on the set $\mathbb{N}$.

Now we are going to analyse two families of solutions of Eq. (43): those leading to linear semi-invariants and those leading to non-linear semi-invariants. In the former case the same condition were obtained by the authors using an approach based on the Markus algebras, and the conditions so obtained are not only necessary but also sufficient [13,24]. In the non-linear case we are only able to provide necessary conditions as will be shown in the following.

Let us consider the condition $\beta_2 \neq 0$, $\alpha_2 = 0$, $\gamma_2 = \alpha_1 = 0$ and $\gamma_1 = \beta_1 = 0$. That implies from Eq. (43) that $M_{31} - M_{11} = 0$. Using the relations in Eqs. (34)–(36) we obtain:

$$\begin{aligned}
&y_2 A_2 + c_2 (M_{33} - M_{13})^n y_3^n\\
&\quad = y_3 A_3 + c_3[(M_{21} - M_{11})y_1 + (M_{22} - M_{12})y_2]^n .
\end{aligned} \tag{44}$$

Equating the coefficients of each monomial in both sides and using $c_3 \neq 0$ implies that $M_{21} = M_{11}$. The conditions $\beta_2 = 0$, $\alpha_2 \neq 0$, $\gamma_2 = \alpha_1 = 0$ and $\gamma_1 = \beta_1 = 0$ leads to the same result. Finally considering the condition $\beta_2 = \alpha_2 \neq 0$ it is easily seen from Eq. (43) that $M_{31} - M_{21} = 0$. Consequently, all conditions analysed above in this paragraph, which are sufficient conditions for the existence of linear semi-invariants [13], lead to the condition $M_{31} - M_{21} = 0$.

In the same way we analysed all others solutions of Eq. (43) leading to the existence of linear semi-invariants. We summarize all these conditions in the following items:

- ($\beta_2 \neq 0$, $\alpha_2 = 0$, $\gamma_2 = 0$, $\alpha_1 = 0$, $\gamma_1 = 0$, $\beta_1 = 0$) or ($\beta_2 = 0$, $\alpha_2 \neq 0$, $\gamma_2 = 0$, $\alpha_1 = 0$, $\gamma_1 = 0$, $\beta_1 = 0$) or ($\beta_2 = \alpha_2 \neq 0$) $\Rightarrow$ $M_{31} - M_{21} = 0$.

- ($\beta_2 = 0,\ \alpha_2 = 0,\ \gamma_2 \neq 0,\ \alpha_1 = 0,\ \gamma_1 = 0,\ \beta_1 = 0$) or ($\beta_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = 0,\ \alpha_1 \neq 0, \gamma_1 = 0,\ \beta_1 = 0$) or ($\gamma_2 = \alpha_1 \neq 0$) $\Rightarrow M_{32} - M_{12} = 0$.
- ($\beta_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = 0,\ \alpha_1 = 0,\ \gamma_1 \neq 0,\ \beta_1 = 0$) or ($\beta_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = 0,\ \alpha_1 = 0, \gamma_1 = 0,\ \beta_1 \neq 0$) or ($\gamma_1 = \beta_1 \neq 0$) $\Rightarrow M_{32} - M_{13} = 0$.

Finally the trivial solution for the system (43) is $\beta_2 = \alpha_2 = 0$, $\gamma_2 = \alpha_1 = 0$ and $\gamma_1 = \beta_1 = 0$. Thus we expand I_1, I_2 and I_3 in Eq. (35) and rewrite Eq. (34) as follows:

$$\begin{aligned}
I &= y_1 A_1 + c_1[(M_{32} - M_{22})y_2 + (M_{33} - M_{23})y_3]^n\,,\\
I &= y_2 A_2 + c_2[(M_{31} - M_{11})y_1 + (M_{33} - M_{13})y_3]^n\,,\\
I &= y_3 A_3 + c_3[(M_{21} - M_{11})y_1 + (M_{22} - M_{12})y_2]^n\,.
\end{aligned}\tag{45}$$

Equating the coefficients of the same monomials on both sides yields:

$$\begin{aligned}
&(M_{32} - M_{22})^n c_1 - c_3(M_{22} - M_{12})^n = 0\,,\\
&(M_{33} - M_{23})^n c_1 - c_2(M_{33} - M_{13})^n = 0\,,\\
&(M_{31} - M_{11})^n c_2 - c_3(M_{21} - M_{11})^n = 0\,.
\end{aligned}\tag{46}$$

System (46) is a homogeneous linear system for the $c_i's$, with $c_i \neq 0$, and therefore the determinant of the coefficients matrix must vanish:

$$\begin{aligned}
&[(M_{32} - M_{22})(M_{13} - M_{33})(M_{21} - M_{11})]^n\\
&\quad -(-1)^n[(M_{12} - M_{22})(M_{23} - M_{33})(M_{31} - M_{11})]^n = 0\,.
\end{aligned}\tag{47}$$

System (47) has two solutions: the first one is given by the conditions:

- ($\beta_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = 0,\ \alpha_1 = 0\ \gamma_1 = 0,\ \beta_1 = 0$) and

$$(M_{32} - M_{22})(M_{13} - M_{33})(M_{21} - M_{11}) + (M_{12} - M_{22})(M_{23} - M_{33})(M_{31} - M_{11}) = 0.$$

which is a necessary and sufficient condition for the existence of a linear semi-invariant [13]. Another possible solution of Eq. (47) exists when

- ($\beta_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = 0,\ \alpha_1 = 0$ and $\gamma_1 = 0,\ \beta_1 = 0$) and $(M_{32} - M_{22})(M_{13} - M_{33})(M_{21} - M_{11}) - (M_{12} - M_{22})(M_{23} - M_{33})(M_{31} - M_{11}) = 0.$

If this solution exists then the corresponding semi-invariant must be non-linear.

Finally let us suppose that the LV-system (33) admits an irreducible non-linear semi-invariant and that the last condition • above is not obeyed. Then, by exclusion of the former possibilities in this section, at least one of the conditions below must be satisfied:

- ($\alpha_2, \beta_2 \neq 0$ and $\alpha_2 \neq \beta_2$) $(M_{31} - M_{11})\beta_2 + (M_{11} - M_{21})\alpha_2 = 0\,.$
- ($\gamma_2, \alpha_1 \neq 0$ and $\gamma_2 \neq \alpha_1$) $(M_{32} - M_{22})\gamma_2 + (M_{22} - M_{12})\alpha_1 = 0.$
- ($\gamma_1, \beta_1 \neq 0$ and $\gamma_1 \neq \beta_1$) $(M_{23} - M_{33})\gamma_1 + (M_{33} - M_{13})\beta_1 = 0\,.$

To conclude we can assert the following proposition:

Proposition 7 (Semi-invariants of the 3-D LV-system). *If the LV-system* (33) *has a non-monomial semi-invariant then at least one of the conditions • must be satisfied.*

4.3. 4-dimensional Lotka–Volterra system

In fact, it will be obvious after reading this paragraph that the n-dimensional case is obtained in complete analogy with the 4-dimensional case. Let us first write explicitly the 4-dimensional LV-system:

$$\begin{aligned}
\dot{y}_1 &= y_1(M_{11}y_1 + M_{12}y_2 + M_{13}y_3 + M_{14}y_4)\,,\\
\dot{y}_2 &= y_2(M_{21}y_1 + M_{22}y_2 + M_{23}y_3 + M_{24}y_4)\,,\\
\dot{y}_3 &= y_3(M_{31}y_1 + M_{32}y_2 + M_{33}y_3 + M_{34}y_4)\,,\\
\dot{y}_4 &= y_1(M_{41}y_1 + M_{42}y_2 + M_{43}y_3 + M_{44}y_4)\,.
\end{aligned} \tag{48}$$

An irreducible homogeneous semi-invariant of degree n may be written in the four equivalent ways:

$$I = y_iA_i(y_1, y_2, y_3, y_4) + c_iI_i, \quad i = 1,2,3,4\,; \tag{49}$$

where $c_iI_i = I\,|_{y_i=0}$ is the restriction of I to the face $y_i = 0$. Again here and for the same reason as in the 3-dimensional case c_i and I_i must be non-vanishing or otherwise I would not be irreducible.

Let us define the 3-dimensional restrictions of the Lotka–Volterra derivatives in the four faces $y_i = 0$, $i = 1,2,3,4$ as $\delta_M\,|_{y_i=0} = \delta_{M^{(i)}}$, where $M^{(i)}$ is the matrix obtained from M by eliminating the ith column and row. Since I is a semi-invariant, we have $\delta_M(I) = (\lambda_1y_1 + \lambda_2y_2 + \lambda_3y_3 + \lambda_4y_4)I$. We then get

$$\delta_{M^{(i)}}(I_i) = \lambda\,|_{y_i=0}I_i, \quad i = 1,2,3,4\,. \tag{50}$$

As seen above, a monomial is always a semi-invariant. As a consequence, the semi-invariants I_i may either be a monomial or a polynomial, i.e. with at least two monomials. In the later I_i is a semi-invariant in the face $y_i = 0$. However, this situation has been exhaustively studied in the preceding section for the 3-dimensional system and we only have to analyse the case where the I_i are monomials, that is

$$\begin{aligned}
I_1 &= y_2^{\alpha_1}y_3^{\alpha_2}y_4^{\alpha_3}, \qquad I_2 = y_1^{\beta_1}y_3^{\beta_2}y_4^{\beta_3}\,,\\
I_3 &= y_1^{\gamma_1}y_2^{\gamma_2}y_4^{\gamma_3}, \qquad I_4 = y_1^{\delta_1}y_2^{\delta_2}y_3^{\delta_3}
\end{aligned} \tag{51}$$

with

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \alpha_3 = n, \qquad \beta_1 + \beta_2 + \beta_3 = n\,,\\
&\gamma_1 + \gamma_2 + \gamma_3 = n, \qquad \delta_1 + \delta_2 + \delta_3 = n\,.
\end{aligned} \tag{52}$$

The action of the Lotka–Volterra derivative in the face $y_1 = 0$ gives

$$\begin{aligned}
\delta_{M^{(1)}}(y_2^{\alpha_1}y_3^{\alpha_2}y_4^{\alpha_3}) &= y_2^{\alpha_1}y_3^{\alpha_2}y_4^{\alpha_3}\\
&\quad\times([\hat{\alpha}(n)M^{(1)}]_1y_2 + [\hat{\alpha}(n)M^{(1)}]_2y_3 + [\hat{\alpha}(n)M^{(1)}]_3y_4)\,.
\end{aligned} \tag{53}$$

By Eq. (50) we have $\delta_{M^{(1)}}(I_1) = (\lambda_2y_2 + \lambda_3y_3 + \lambda_4y_4)I_1(y_2, y_3, y_4)$, leading to

$$\lambda_2 = [\alpha M^{(1)}]_1, \qquad \lambda_3 = [\alpha M^{(1)}]_2, \qquad \lambda_4 = [\alpha M^{(1)}]_3\,, \tag{54}$$

with $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3)$. In the same way, we obtain for the faces $y_2 = 0$, $y_3 = 0$ and $y_4 = 0$, respectively:

$$\lambda_1 = [\beta M^{(2)}]_1, \qquad \lambda_3 = [\beta M^{(2)}]_2, \qquad \lambda_4 = [\beta M^{(2)}]_3 , \tag{55}$$

$$\lambda_1 = [\gamma M^{(3)}]_1, \qquad \lambda_2 = [\gamma M^{(3)}]_2, \qquad \lambda_4 = [\gamma M^{(3)}]_3 , \tag{56}$$

$$\lambda_1 = [\delta M^{(4)}]_1, \qquad \lambda_2 = [\delta M^{(4)}]_2, \qquad \lambda_3 = [\delta M^{(4)}]_3 . \tag{57}$$

The compatibility between Eqs. (54)–(57) leads to the following necessary conditions:

$$\begin{aligned}
\lambda_1 &= [\beta M^{(2)}]_1 = [\gamma M^{(3)}]_1 = [\delta M^{(4)}]_1 , \\
\lambda_2 &= [\alpha M^{(1)}]_1 = [\gamma M^{(3)}]_2 = [\delta M^{(4)}]_2 , \\
\lambda_3 &= [\alpha M^{(1)}]_2 = [\beta M^{(2)}]_2 = [\delta M^{(4)}]_3 , \\
\lambda_4 &= [\alpha M^{(1)}]_3 = [\beta M^{(2)}]_3 = [\gamma M^{(3)}]_3 .
\end{aligned} \tag{58}$$

Here, a distinction must be carefully made between the three and four-dimensional cases. Indeed, the linear system (43) giving the exponents α, β and γ is not completely determined. For the 4-dimensional case, the linear system (58) with (52) has 12 equations and is complete for the 12 integer unknowns in α, β, γ and δ. This structure will be similar for dimensions higher than four as a consequence of the monomial form for I_i. In the n-dimensional LV-system ($n \geqslant 4$) the linear system equivalent to Eq. (58) has $n(n-1)$ equations and integer unknowns.

Let us finally remark that semi-invariants for the Lotka–Volterra system lead to quasi-polynomial invariants for the general quasi-polynomial system (1), which can be readily computed in the corresponding LV-systems by using the quasi-monomial transformations. In the present work we investigated the consequences of these general results on the semi-invariants of the LV-systems. Our approach will be more clarified in the next section where it is applied to the (a,b,c) Lotka–Volterra and May–Leonard systems.

5. Applications

In order to understand the results presented in this section, we introduce the following denominations for the different types of semi-invariants: a semi-invariant I is called polynomial decomposed semi-invariant (PDSI) if there is a face $y_i = 0$ such that $I_i = I|_{y_i=0}$ is a polynomial (with at least two monomials). A semi-invariant I is called monomial decomposed semi-invariant (MDSI) if $I_i = I|_{y_i=0}$ is a monomial for any face $y_i = 0$.

We apply our results for following systems: (A) (a,b,c) Lotka–Volterra system; (B) (a,b,c) Lotka–Volterra system with linear terms and (C) May–Leonard system.

5.1. (a,b,c) Lotka–Volterra system

The (a,b,c) Lotka–Volterra system is given by [7,11,13]

$$\dot{y}_1 = y_1(cy_2 + y_3), \qquad \dot{y}_2 = y_2(y_1 + ay_3), \qquad \dot{y}_3 = y_3(by_1 + y_2) . \tag{59}$$

In agreement with Eqs. (34) and (35), the semi-invariant I can be decomposed as follows:

$$I = y_1 A_1 + c_1 y_2^{\gamma_1} y_3^{\gamma_2} (y_2 - a y_3)^{\gamma_3} ,$$
$$I = y_2 A_2 + c_2 y_1^{\beta_1} y_3^{\beta_2} (b y_2 - y_3)^{\beta_3} ,$$
$$I = y_3 A_3 + c_3 y_1^{\alpha_1} y_2^{\alpha_2} (y_2 - c y_3)^{\alpha_3} . \tag{60}$$

The compatibility conditions (43) give here

$$b\beta_2 - \alpha_2 = 0, \qquad \gamma_2 - c\alpha_1 = 0, \qquad a\gamma_1 - \beta_1 = 0 . \tag{61}$$

Proposition 7 allows to state the following result:

Proposition 8 (Semi-invariants of the (a,b,c) Lotka–Volterra system). *If the system* (59) *has a non-monomial semi-invariant, then at least one of the conditions • below must be satisfied.*

- $abc = 1$;
- $abc = -1$;
- $(a-1)(b-1)(c-1) = 0$;
- $c = \frac{q}{p}$, $q,\ p \in \mathbb{N}^+$, $q \neq p$;
- $b = \frac{q}{p}$, $q,\ p \in \mathbb{N}^+$, $q \neq p$;
- $a = \frac{q}{p}$, $q,\ p \in \mathbb{N}^+$, $q \neq p$.

The condition $abc = 1$ is equivalent to the vanishing of the determinant of the Lotka–Volterra matrix M and is associated with the existence of a quasi-monomial invariant [18]. The second and third conditions correspond to the existence of linear semi-invariants. The remaining conditions correspond to the possible existence of non-linear semi-invariants.

In particular Proposition 8 allows to affirm: let us suppose that a, b and c are non-rational or negative numbers such that $abc \neq \pm 1$. Thus there exists no semi-invariant for the (a,b,c) Lotka–Volterra system and consequently there is no QP-invariant, which is a general result for this system.

5.2. (a,b,c) Lotka–Volterra with linear terms

The (a,b,c) Lotka–Volterra with linear terms [7,17,18] is given by

$$\dot{x}_1 = x_1(l_1 + c x_2 + x_3) ,$$
$$\dot{x}_2 = x_2(l_2 + x_1 + a x_3) ,$$
$$\dot{x}_3 = x_3(l_3 + b x_1 + x_2) . \tag{62}$$

We recast this system into the Lotka–Volterra variables $y_1 = 1,\ y_2 = x_1,\ y_3 = x_2, y_4 = x_3$ and obtain the 4-dimensional homogeneous associated LV-system with the 4×4 matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ l_1 & 0 & c & 1 \\ l_2 & 1 & 0 & a \\ l_3 & b & 1 & 0 \end{pmatrix} . \tag{63}$$

We will analyse the two types of semi-invariants: (i) PDSI and (ii) MDSI. In order to analyse (i) and considering Propositions 4 and 5, we only have to study the homogeneous irreducible non-monomial semi-invariants of degree $m \leqslant n$ for the four 3-dimensional LV-systems associated with the faces $y_i = 0,\ i = 1,2,3,4$. We first note that:

- The face $y_1 = 0$ corresponds to the (a,b,c) Lotka–Volterra (59) analysed in the last section.

Now let us consider the LV-system associated with the face $y_2 = 0$, i.e., the LV-system given by the 3×3 matrix obtained eliminating the second row and column in Eq. (63). The compatibility relations (43) reads:

$$l_3\beta_2 - l_2\alpha_2 = 0, \qquad \gamma_2 = 0, \qquad a\gamma_1 = 0 \tag{64}$$

with: $\alpha_1 + \alpha_2 + \alpha_3 = m$, $\beta_1 + \beta_2 + \beta_3 = m$ and $\gamma_1 + \gamma_2 + \gamma_3 = m$. Considering $\gamma_2 = 0$ the semi-invariants in Eqs. (34) and (35) can be written as follows:

$$\begin{aligned} I &= y_1A_1 + c_1 y_2^{\gamma_1}(y_2 - ay_3)^{\gamma_3} \\ &= y_1A_1 + c_1 y_2 y_3 G_1(y_2, y_3) + c_1(-a)^m y_3^m, \\ I &= y_2A_2 + c_2' y_1^{\beta_1+\beta_3} y_3^{\beta_2}, \\ I &= y_3A_3 + c_3' y_1^{\alpha_1+\alpha_3} y_2^{\alpha_2} , \end{aligned} \tag{65}$$

where $G_1(y_2, y_3)$ is defined as given.[1] Since I is an irreducible semi-invariant, then $c_1 y_2^m = c_3' y_1^{\alpha_1+\alpha_3} y_2^{\alpha_2}$ and consequently $\alpha_1 = 0$, $\alpha_2 = m$ and $\alpha_3 = 0$. Since $a\gamma_1 = 0$ in Eq. (64), then either (A) $\gamma_1 = 0$ or (B) $a = 0$. In case (A) the decompositions of I in Eq. (65) impose:

$$y_1A_1 + c_1 y_2 y_3 G_1(y_2, y_3) + c(-a)^m y_3^m = y_2A_2 + c_2' y_1^{\beta_1+\beta_3} y_3^{\beta_2} . \tag{66}$$

Since I is irreducible, which implies from Eq. (65) that $c_2' \neq 0$, we conclude that $\beta_1 = 0$, $\beta_2 = m$ and $\beta_3 = 0$. Finally, from the first condition in Eq. (64) we have $l_3 - l_2 = 0$.

Now we consider the possibility (B). In this case the decompositions (34) imply:

$$I = y_1A_1 + c_1' y_2^m = y_2A_2 + c_2' y_1^{\beta_1+\beta_3} y_3^{\beta_2} = y_3A_3 + c_3' y_2^m . \tag{67}$$

From Eq. (67) we conclude that $\beta_2 \neq 0$. In another way we should have $c_1' = 0$, which is not possible as I is irreducible. The first condition in Eq. (64) then implies

[1] The explicit form of $G_1(y_2, y_3)$ can be obtained from the Newton binomial expansion.

that $l_3=(m/\beta_2)l_2$. We can see that if the system has a non-monomial semi-invariant in the face $y_2=0$, then at least one condition below must be satisfied:

- $l_3=l_2$;
- $a=0,\ l_3=\frac{m}{q}l_2,\ 1\leqslant q\leqslant m\leqslant n$.

In a similar way we analyse the face $y_3=0$ and conclude that for the existence of a non-monomial semi-invariant at least one of the conditions below must hold:

- $l_1=l_3$;
- $b=0,\ l_1=\frac{m}{q}l_3,\ 1\leqslant q\leqslant m\leqslant n$.

For the face $y_4=0$ the existence of the non-monomial semi-invariant implies that at least one of the two conditions is satisfied:

- $l_1=l_2$;
- $c=0,\ l_2=\frac{m}{q}l_1,\ 1\leqslant q\leqslant m\leqslant n$.

If system (62) has a PDSI and $l_1\neq l_2$, $l_1\neq l_3$, $l_2\neq l_3$ and $abc\neq 0$, then the only remaining possibility is that there exists a non-monomial semi-invariant for the (a,b,c) Lotka–Volterra system (without a linear term) obtained by the restriction $y_1=0$.

In order to analyse the necessary conditions for the existence of a MDSI we write the decompositions given in (49) and (51) explicitly:

$$\begin{aligned}
I&=y_1A_1(y_1,y_2,y_3,y_4)+c_1y_2^{\alpha_1}y_3^{\alpha_2}y_4^{\alpha_3}\,,\\
I&=y_2A_2(y_1,y_2,y_3,y_4)+c_2y_1^{\beta_1}y_3^{\beta_2}y_4^{\beta_3}\,,\\
I&=y_3A_3(y_1,y_2,y_3,y_4)+c_3y_1^{\gamma_1}y_2^{\gamma_2}y_4^{\gamma_3}\,,\\
I&=y_4A_4(y_1,y_2,y_3,y_4)+c_4y_1^{\delta_1}y_2^{\delta_2}y_3^{\delta_3}
\end{aligned}\tag{68}$$

with $\alpha_i,\beta_j,\gamma_k,\delta_p\in\mathbb{N}$ for $i,j,k,p=1,2,3$.

The linear system of compatibility equations (58) is studied and solved in Appendix B. Here we present a summary of the conclusions obtained. If system (63) has a MDSI then at least one of the eighth conditions below must hold:

- $(\alpha_1,\alpha_2,\alpha_3\in\mathbb{N}^+)\quad a=-\frac{\alpha_1}{\alpha_2},\ b=-\frac{\alpha_2}{\alpha_3},\ c=-\frac{\alpha_3}{\alpha_1}$.
- $(\delta_1,\delta_2,\delta_3\in\mathbb{N}^+)\quad b=\frac{\delta_3}{n},\ c=\frac{n}{\delta_2},\ l_3=\frac{\delta_3}{n}l_2+\frac{\delta_2}{n}l_1$.
- $(\gamma_1,\gamma_2,\gamma_3\in\mathbb{N}^+)\quad b=\frac{\gamma_2}{n},\ b=\frac{n}{\gamma_3},\ l_2=\frac{\gamma_2}{n}l_1+\frac{\gamma_3}{n}l_3$.
- $(\beta_1,\beta_2,\beta_3\in\mathbb{N}^+)\quad a=\frac{n}{\beta_2},\ c=\frac{\beta_3}{n},\ l_1=\frac{\beta_2}{n}l_2+\frac{\beta_3}{n}l_3$.
- $(\alpha_2,\beta_3,\delta_2\in\mathbb{N}^+)\quad a=-\frac{\alpha_1}{\alpha_2},\ b=\frac{\delta_3}{\beta_3},\ c=\frac{\beta_3}{\delta_2},\ l_3=\frac{\delta_2}{\beta_3}l_1+\frac{\delta_3}{\beta_3}l_2$.
- $(\alpha_1,\beta_2,\gamma_3\in\mathbb{N}^+)\quad a=\frac{\gamma_2}{\beta_2},\ b=\frac{\beta_2}{\gamma_3},\ c=-\frac{\alpha_3}{\alpha_1},\ l_2=\frac{\gamma_2}{\beta_2}l_1+\frac{\gamma_3}{\beta_2}l_3$.
- $(\alpha_3,\beta_2,\gamma_2\in\mathbb{N}^+)\quad a=\frac{\gamma_2}{\beta_2},\ b=-\frac{\alpha_2}{\alpha_3},\ c=\frac{\beta_3}{\gamma_2},\ l_1=\frac{\beta_2}{\gamma_2}l_2+\frac{\beta_3}{\gamma_2}l_3$.
- $(\beta_2,\gamma_3,\delta_2\in\mathbb{N}^+)\quad a=\frac{\gamma_2}{\beta_2},\ b=\frac{\delta_3}{\gamma_3},\ c=\frac{\beta_3}{\delta_2}$.

Thus we can assert that:

Proposition 9 (Semi-invariants of the (a,b,c) LV system with linear terms). *Let us suppose that system (63) has a non-monomial semi-invariant* (*or a QP-invariant*), *then at least one of the conditions • in this subsection must be satisfied.*

5.3. May–Leonard system

The May–Leonard system [13,25] is

$$\begin{aligned}
\dot{x}_1 &= l_1 x_1 - x_1(x_1 + a x_2 + b x_3)\,,\\
\dot{x}_2 &= l_2 x_2 - x_2(b x_1 + x_2 + a x_3)\,,\\
\dot{x}_3 &= l_3 x_3 - x_3(a x_1 + b x_2 + x_3)\,,
\end{aligned}$$

and the matrix M of the associated LV-system, with monomial variables $y_1 = 1,\ y_2 = x_1,\ y_3 = x_2$ and $y_4 = x_3$, is given by

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ l_1 & -1 & -a & -b \\ l_2 & -b & -1 & -a \\ l_3 & -a & -b & -1 \end{pmatrix}. \tag{69}$$

We see that the corresponding LV-system is 4-dimensional. We will consider the two kinds of nonmonomial irreducible semi-invariants: (i) the PDSI and (ii) the MDSI.

(i) PDSI: the Propositions 4 and 5 assure that any semi-invariant is decomposed into irreducible semi-invariants. The results of Proposition 7 are used to obtain the possible conditions for existence of a PDSI for the May–Leonard systems and yield:

- $(l_1 - l_2)(l_1 - l_3)(l_2 - l_3) = 0.$
- $\alpha\beta(\alpha - \beta)(\alpha + \beta - 2) = 0.$
- $l_2 = \frac{q}{p} l_3;\ p,\ q \in \mathbb{N}^+,\ p \neq q.$
- $l_1 = \frac{q}{p} l_3;\ p,\ q \in \mathbb{N}^+,\ p \neq q.$
- $l_1 = \frac{q}{p} l_2;\ p,\ q \in \mathbb{N}^+,\ p \neq q.$
- $a = 1 - \frac{q}{p};\ p,\ q \in \mathbb{N}^+,\ p \neq q.$
- $b = 1 - \frac{q}{p};\ p,\ q \in \mathbb{N}^+,\ p \neq q.$
- $aq + bp = q + p;\ p,\ q \in \mathbb{N}^+,\ p \neq q.$

The first two conditions correspond to the existence of a linear semi-invariant for the 3-dimensional LV-system associated to the faces $y_i = 0$.

(ii) MDSI: In this case we consider, as in the previous section, the semi-invariant decomposition given by Eq. (68). The computations for the May–Leonard system proceeds analogously as in the preceeding cases. For the sake of brevity they will not be presented here and will be published separately. We show below the final results. Using the compatibility equations (58) the solutions for the parameters a and b gives

$$* \quad a = \frac{p}{q},\ b = \frac{r}{s};\ p,q,r,s \in Z,\ q,s \neq 0\,. \tag{70}$$

The solutions for the parameters l_1, l_2 and l_3 are given by

$$** \quad l_1 = \frac{p'}{q'} l_3,\ l_2 = \frac{r'}{s'} l_3;\ p',q',r',s' \in Z,\ q',s' \neq 0\,. \tag{71}$$

$$** \quad l_1 = \frac{p''}{q''} l_2 + \frac{r''}{s''} l_3;\ p'',q'',r'',s'' \in Z,\ q'',s'' \neq 0\,. \tag{72}$$

Thus we conclude this section by the following result:

Proposition 10 (Semi-invariants of the May–Leonard system). (a) *If the May–Leonard system has a semi-invariant* (*PDSI*) *then at least one condition* • *in this section must be satisfied.*

(b) *If the May–Leonard system has a semi-invariant* (*MDSI*) *then the condition* $*$ *and one condition* $**$ *must be satisfied simultaneously.*

6. Concluding remarks

The decomposition theorem obtained in Section 3 is fundamental for a systematic search of QP-invariants for quasi-polynomial systems, which is otherwise not possible without this result. Computations performed using our method can be found in Ref. [13] and all agree, for the systems considered here, with the necessary conditions we obtained in the last section. The decomposition theorem strongly generalizes a similar study by Feix and Cairo [5].

Our approach also delivers necessary conditions on the parameters for the existence of QP-invariants, without having to explicitly obtain it. Our analysis can be applied to any quasi-polynomial dynamical system and greatly simplifies the search for such invariants. It also extends the field of functions which can be used for a systematic search of first integrals and is well suited for symbolic computations. A package in a symbolic computation language has been written by the authors and will be published elsewhere.

Appendix A. Linearly independent quasi-monomials

In this appendix we prove that N different quasi-monomials are linearly independent. Let us consider N quasi-monomials $y^{\alpha_1},\dots,y^{\alpha_N}$; where $y^{\alpha_i} = \prod_{k=1}^{m} y_k^{\alpha_{ik}}$ and $\alpha_i \neq \alpha_j$ for all $i \neq j$. Let us consider a linear combination such that

$$c_1 y^{\alpha_1} + c_2 y^{\alpha_2} + \cdots + c_N y^{\alpha_N} = 0\,, \tag{73}$$

and the following minimum:

$$\mathscr{A}_i = \min\{\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{Ni}\} \; ; \tag{74}$$

i.e. $\alpha_{ji} \geqslant \mathscr{A}_i$ for all $j = 1,\dots,N$. We multiply Eq. (73) by the quasi-monomial $\prod_{k=1}^{m} y_k^{-\mathscr{A}_k}$ and obtain

$$f = \prod_{k=1}^{m} y_k^{-\mathscr{A}_k}(c_1 y^{\alpha_1} + \cdots + c_N y^{\alpha_N}) = 0 \; , \tag{75}$$

which satisfies:

$$\lim_{y_m \to 0} \lim_{y_{m-1} \to 0} \cdots \lim_{y_2 \to 0} \lim_{y_1 \to 0} (f) = 0 \; . \tag{76}$$

Since $\mathscr{A}_k \geqslant \alpha_{ik}$ $i = 1,\dots,N$, then $\lim_{y_k \to 0}(y_k^{\alpha_{ik} - \mathscr{A}_k}) = 0$ when $\alpha_{ik} > \mathscr{A}_k$ and $\lim_{y_k \to 0} \times (y_k^{\alpha_{ik} - \mathscr{A}_k}) = 1$ when $\alpha_{ik} = \mathscr{A}_k$. Thus, the fact that $\alpha_i \neq \alpha_j$ for $i \neq j$ implies that

$$\lim_{y_m \to 0} \lim_{y_{m-1} \to 0} \cdots \lim_{y_2 \to 0} \lim_{y_1 \to 0} (f) = c_p \; , \tag{77}$$

where $p \in \{1,\dots,N\}$, or otherwise we should have two identical quasi-monomials in $y^{\alpha_1},\dots,y^{\alpha_N}$, which is not the case by our initial assumption. Hence, we write (73) as follows:

$$c_1 y^{\alpha_1} + \cdots + c_{p-1} y^{\alpha_{p-1}} + c_{p+1} y^{\alpha_{p+1}} + \cdots + c_N y^{\alpha_N} = 0 \; . \tag{78}$$

Reasoning as above we finally conclude that $c_j = 0$ for all $j = 1,\dots,N$.

Appendix B. Lotka system with linear terms

The semi-invariant of the Lotka system with linear terms according to Eq. (68) can be decomposed in the following four ways:

$$\begin{aligned}
I &= y_1 A_1 + w_1 \, y_2^{\alpha_1} y_3^{\alpha_2} \, y_4^{\alpha_3} \; , \\
I &= y_2 A_2 + w_2 y_1^{\beta_1} y_3^{\beta_2} y_4^{\beta_3} \; , \\
I &= y_3 A_3 + w_3 y_1^{\gamma_1} y_2^{\gamma_2} y_4^{\gamma_3} \; , \\
I &= y_4 A_4 + w_4 y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3} \; .
\end{aligned} \tag{79}$$

The compatibility equations (58) become:

$$\begin{aligned}
&(\beta_3 - \gamma_3) l_3 + l_2 \beta_2 - l_1 \gamma_2 = 0, \qquad l_3 \gamma_3 - l_2 \delta_3 + (\gamma_2 - \delta_2) l_1 = 0 \; , \\
&(\alpha_3 - \gamma_3) b + \alpha_2 = 0, \qquad b\gamma_3 - \delta_3 = 0 \; , \\
&c\alpha_1 + \alpha_3 - \beta_3 = 0, \qquad \beta_3 - c\delta_2 = 0 \; , \\
&(\alpha_2 - \beta_2) a + \alpha_1 = 0, \qquad a\beta_2 - \gamma_2 = 0 \; .
\end{aligned} \tag{80}$$

In order to solve Eq. (80) we consider the eight possibilities:

$$
\begin{aligned}
&AA = \{\gamma_3 = 0, \delta_2 = 0, \beta_2 = 0\}, \qquad BB = \{\delta_2 = 0, \beta_2 = 0, \gamma_3 \neq 0\}\ ,\\
&CC = \{\gamma_3 = 0, \beta_2 = 0, \delta_2 \neq 0\}, \qquad DD = \{\gamma_3 = 0, \delta_2 = 0, \beta_2 \neq 0\}\ ,\\
&EE = \{\beta_2 = 0, \gamma_3 \neq 0, \delta_2 \neq 0\}, \qquad FF = \{\delta_2 = 0, \gamma_3 \neq 0, \beta_2 \neq 0\}\ ,\\
&GG = \{\gamma_3 = 0, \delta_2 \neq 0, \beta_2 \neq 0\}, \qquad HH = \{\gamma_3 \neq 0, \delta_2 \neq 0, b_2 \neq 0\}\ .
\end{aligned} \tag{81}
$$

(AA): In this case Eq. (80) allows to conclude that $\beta_3 = 0$, $\gamma_2 = 0$ and $\delta_3 = 0$, consequently Eq. (79) is rewritten as

$$
\begin{aligned}
&I = y_1 A_1 + w_1 y_2^{\alpha_1} y_3^{\alpha_2} y_4^{\alpha_3}, \qquad I = y_2 A_2 + w_2 y_1^n\ ,\\
&I = y_3 A_3 + w_3 y_1^n, \qquad I = y_4 A_4 + w_4 y_1^n\ .
\end{aligned} \tag{82}
$$

Finally, Eq. (80) amounts to

$$
\begin{aligned}
&c = -\frac{\alpha_3}{\alpha_1}, \qquad a = -\frac{\alpha_1}{\alpha_2}, \qquad b = -\frac{\alpha_2}{\alpha_3}\ ,\\
&\gamma_3 = 0, \quad \delta_2 = 0, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \delta_1 = n, \quad \gamma_1 = n, \quad \beta_1 = n\ ,\\
&\quad \gamma_2 = 0, \quad \delta_3 = 0\ ,\\
&\alpha_1\alpha_2\alpha_3 \neq 0\ .
\end{aligned} \tag{83}
$$

(BB): We use BB in Eq. (80) and obtain $\beta_3 = 0$ and $\gamma_2 = 0$, and consequently $b_1 = n$. The semi-invariant in Eq. (79) becomes

$$
\begin{aligned}
&I = y_1 A_1 + w_1 y_2^{\alpha_1} y_3^{\alpha_2} y_4^{\alpha_3}, \qquad I = y_2 A_2 + w_2 y_1^n\ ,\\
&I = y_3 A_3 + w_3 y_1^{\gamma_1} y_4^{\gamma_3}, \qquad I = y_4 A_4 + w_4 y_1^{\delta_1} y_3^{\delta_3}\ .
\end{aligned} \tag{84}
$$

As we supposed that $\gamma_3 \neq 0$ then $w_2 = 0$ and the semi-invariant is not irreducible. However, we are interested only an irreducible MDSI.

(CC) and (DD) amounts to the same conclusions as (BB).

(EE): We put $\beta_2 = 0$ in Eq. (80) and obtain $\gamma_2 = 0$. The semi-invariant in Eq. (79) becomes:

$$
\begin{aligned}
&I = y_1 A_1 + w_1 y_2^{\alpha_1} y_3^{\alpha_2} y_4^{\alpha_3}\ ,\\
&I = y_2 A_2 + w_2 y_1^{\beta_1} y_4^{\beta_3}\ ,\\
&I = y_2 A_2 + w_2 y_1^{\beta_1} y_4^{\beta_3}\ ,\\
&I = y_4 A_4 + w_4 y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}\ .
\end{aligned} \tag{85}
$$

Since I is irreducible, we conclude that $\gamma_1 = \beta_1$ and $\gamma_3 = \beta_3$. Finally, Eq. (80) becomes

$$
\begin{aligned}
&\beta_3 l_3 - l_2\delta_3 - l_1\delta_2 = 0, \qquad (\alpha_3 - \beta_3)b + \alpha_2 = 0, \qquad b\beta_3 - \delta_3 = 0\ ,\\
&c\alpha_1 + \alpha_3 - \beta_3 = 0, \qquad \beta_3 - c\delta_2 = 0, \qquad a\alpha_2 + \alpha_1 = 0\ .
\end{aligned} \tag{86}
$$

We consider two possibilities: (a) $\alpha_2 \neq 0$ and (b) $\alpha_2 = 0$. In case (a) we have the solution:

$$a = -\frac{\alpha_1}{\alpha_2}, \quad l_3 = \frac{l_2\delta_3}{\beta_3} + \frac{\delta_2 l_1}{\beta_3}, \quad c = \frac{\beta_3}{\delta_2}, \quad b = \frac{\delta_3}{\beta_3},$$
$$\beta_2 = 0, \quad \gamma_2 = 0, \quad \gamma_1 = \beta_1, \quad \gamma_3 = \beta_3, \quad \alpha_1 = -\frac{\delta_2(\alpha_3 - \beta_3)}{\beta_3} ,$$
$$\alpha_2 = -\frac{(\alpha_3 - \beta_3)\delta_3}{\beta_3} , \quad \beta_3\delta_2\alpha_2 \neq 0 . \tag{87}$$

In case (b) we have the following solution:

$$l_3 = \frac{l_2\delta_3}{n} + \frac{\delta_2 l_1}{n}, \quad c = \frac{n}{\delta_2}, \quad b = \frac{\delta_3}{n} ,$$
$$\beta_2 = 0, \quad \gamma_2 = 0, \quad \alpha_2 = 0, \quad \alpha_1 = 0, \quad \alpha_3 = n, \quad \beta_1 = 0, \quad \beta_3 = n, \quad \gamma_1 = 0 ,$$
$$\gamma_3 = n , \quad \delta_1\delta_2\delta_3 \neq 0 . \tag{88}$$

(FF): This case is analogous to the case (EE). We obtain the two possible solutions:

$$c = -\frac{\alpha_3}{\alpha_1}, \quad l_2 = \frac{l_3\gamma_3}{\beta_2} + \frac{l_1\gamma_2}{\beta_2}, \quad a = \frac{\gamma_2}{\beta_2}, \quad b = \frac{\beta_2}{\gamma_3} ,$$
$$\delta_2 = 0, \quad \beta_3 = 0, \quad \delta_1 = \beta_1, \quad \delta_3 = \beta_2, \quad \alpha_1 = \frac{\alpha_3\gamma_2}{\gamma_3}, \quad \alpha_2 = -\frac{(\alpha_3 - \gamma_3)\beta_2}{\gamma_3} ,$$
$$\gamma_3 \neq 0, \quad \alpha_1 \neq 0, \quad \beta_2 \neq 0 , \tag{89}$$

and

$$b = \frac{n}{\gamma_3}, \quad a = \frac{\gamma_2}{n}, \quad l_2 = \frac{l_3\gamma_3}{n} + \frac{l_1\gamma_2}{n} ,$$
$$\delta_2 = 0, \quad \beta_3 = 0, \quad \alpha_1 = 0, \quad \beta_1 = 0, \quad \alpha_3 = 0, \quad \alpha_2 = n, \quad \beta_2 = n, \quad \delta_1 = 0 ,$$
$$\delta_3 = n , \quad \gamma_1\gamma_2\gamma_3 \neq 0 . \tag{90}$$

(GG): This case is similar to (EE) and (FF). We have two solutions:

$$l_1 = \frac{\beta_3 l_3}{\gamma_2} + \frac{l_2\beta_2}{\gamma_2}, \quad c = \frac{\beta_3}{\gamma_2}, \quad b = -\frac{\alpha_2}{\alpha_3}, \quad a = \frac{\gamma_2}{\beta_2} ,$$
$$\gamma_3 = 0, \quad \delta_1 = \gamma_1, \quad \delta_2 = \gamma_2, \quad \delta_3 = 0, \quad \alpha_2 = \frac{\alpha_3\beta_2}{\beta_3}, \quad \alpha_1 = -\frac{\gamma_2(\alpha_3 - \beta_3)}{\beta_3} ,$$
$$\gamma_2 \neq 0, \quad \beta_2 \neq 0, \quad \alpha_3 \neq 0 , \tag{91}$$

and

$$l_1 = \frac{\beta_3 l_3}{n} + \frac{l_2\beta_2}{n}, \quad a = \frac{n}{\beta_2}, \quad c = \frac{\beta_3}{n} ,$$
$$\alpha_1 = n, \quad \gamma_1 = 0, \quad \gamma_2 = n, \quad \delta_2 = n, \quad \alpha_2 = 0, \quad \gamma_3 = 0, \quad \alpha_3 = 0, \quad \delta_1 = 0 ,$$
$$\delta_3 = 0 , \quad \beta_1\beta_2\beta_3 \neq 0 . \tag{92}$$

(HH): In this case we solve Eq. (80) with respect to the parameters a, b, c, l_1, l_2 and obtain:

$$
\begin{aligned}
&c=\frac{\beta_3}{\delta_2}, \qquad b=\frac{\delta_3}{\gamma_3}, \qquad a=\frac{\gamma_2}{\beta_2}\,,\\
&l_1=-\frac{l_3(\beta_2\gamma_3+\delta_3\beta_3-\delta_3\gamma_3)}{\gamma_2\beta_2-\beta_2\delta_2-\delta_3\gamma_2}, \qquad l_2=-\frac{l_3(\beta_3\gamma_2-\beta_3\delta_2+c_3\delta_2)}{\gamma_2\beta_2-\beta_2\delta_2-\delta_3\gamma_2}\,,\\
&\alpha_1=\frac{\gamma_2\delta_2(\beta_2\gamma_3+\delta_3\beta_3-\delta_3\gamma_3)}{\beta_3\delta_3\gamma_2+\gamma_3 b_2\delta_2}, \qquad \alpha_2=\frac{\beta_2\delta_3(\beta_3\gamma_2-\beta_3\delta_2+\gamma_3\delta_2)}{\beta_3\delta_3\gamma_2+\gamma_3 b_2\delta_2}\,,\\
&\alpha_3=-\frac{\gamma_3\beta_3(\gamma_2\beta_2-\beta_2\delta_2-\delta_3\gamma_2)}{\beta_3\delta_3\gamma_2+\gamma_3 b_2\delta_2}, \qquad \gamma_3\neq 0,\ \delta_2\neq 0,\ \beta_2\neq 0\,. \qquad (93)
\end{aligned}
$$

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