17 April 2000

PHYSICS LETTERS A

Physics Letters A 268 (2000) 335–341

www.elsevier.nl/locate/physleta

Boundedness of solutions and Lyapunov functions in quasi-polynomial systems

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Received 4 October 1999; received in revised form 22 February 2000; accepted 1 March 2000
Communicated by A.P. Fordy

Abstract

In this Letter we establish sufficient conditions for the existence of a Lyapunov function for a large class of non-linear systems, the Quasi-Polynomial systems [Figueredo et al., J. Math. Phys. 39 (1998) 2929; Figueredo et al., Phys. A 262 (1999) 158; Brenig, Phys. Lett. A 133 (1988) 378] . We also present sufficient conditions such that the solutions are bounded and bounded away from zero componentwise. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 47.20.Ky
Keywords: Differential equations; Stability; Lyapunov function

1. Introduction

Given a set of ODEs describing a physical system, the study of the global stability of its solutions is often an important issue. The stability analysis is usually performed by the determination of a Lyapunov function associated with a fixed point of the system [4]. Unfortunately a general recipe to find such function is not available, with few exceptions as for Hamiltonian systems, for which there is a natural Lyapunov Function, the energy of the system, or dissipative systems where a monotonously decreasing energy can be identified. For Lotka–Volterra systems a sufficient condition for its stability were first obtained by Volterra [5], and more recently by Redheffer et al. [6–8] and Takeuchi et al. [9–11].

The purpose of this Letter is twofold. First to establish sufficient conditions for the boundedness of the solutions of a Quasi-Polynomial (QP) system (see Eq. (1)) and for the invariance of the orthants of its associated phase space. Second to determine a Lyapunov function associated with a interior fixed point and establish its stability. The scope of these results is extended by the introduction of a special type of reparametrization of the time variable.

The structure of the Letter is as follows: in Section 2 we state and proof two theorems on the boundedness of solutions, on the invariance of the orthants of the phase space and on the existence of a Lyapunov function, which are the main results of this Letter. In Section 3 we develop some algebraic conditions related to the theorems in Section 2. In Section 4 we apply our approach to a Generalized

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PII: S0375-9601(00)00175-4

Mass Action System, and close with some concluding remarks in Section 5.

2. Lyapunov functions in Quasi-Polynomial systems

A Quasi-Polynomial system of ODEs has the generic form

$$\dot{x}_i = l_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \quad i = 1,\dots,n, \tag{1}$$

where the dot stands for the time derivative, A and B are real constant rectangular matrices and m is the number of quasi-monomials appearing in the vector field of Eq. (1). The denomination *quasi-monomial* (QM) for $\prod_k (x_k)^{B_{jk}}$ is justified by the fact that the B_{jk} are real and not necessarily integer numbers.

In all that follows we assume that the rank r of B is maximal, i.e. $r = n$. This represents no restriction at all for the present approach since if $r < n$ then the system can be reduced to a new QP system with a new exponent matrix B' whose rank is maximal (see Ref. [3]). Let us define the matrix $M = BA$. We say that M is *admissible* if there are $a_i > 0, i = 1,\dots,m$; such that [5]

$$\sum_{i,j=1}^{m} a_i M_{ij} w_i w_j \le 0, \quad \forall w_i \in \Re. \tag{2}$$

We define the positive orthant of the phase space associated to system (1) by the subset

$$\Re^n_+ = \{(x_1,\dots,x_n) \in \Re^n \mid x_i > 0\}, \quad i = 1,\dots,n.$$

With these definitions we state the following theorem:

Theorem 1. *If the matrix $M = BA$ is admissible and the QP system in Eq. (1) has a fixed point $x^* = (x_1^*,\dots,x_n^*) \in \Re^n_+$ then there exists a Lyapunov function V(x) valid in the positive orthant of the phase space of Eq. (1).*

Proof. Let us consider the following monomial functions:

$$U_j = \prod_{k=1}^{n} x_k^{B_{jk}}. \tag{3}$$

Using Eq. (3) the time derivative of U_i is given by

$$\dot{U}_i = \lambda_i U_i + U_i \sum_{i=1}^{m} M_{ij} U_j, \quad j = 1,\dots,m; \tag{4}$$

where $\lambda = Bl$ and $M = BA$. This is a quadratic equation of Lotka–Volterra type [1–3].

Now consider a fixed point of Eq. (4) with coordinates q_i given by

$$q_i = \prod_{k=1}^{m} (x_k^*)^{B_{ik}} > 0, \quad i = 1,\dots,m. \tag{5}$$

The function $V: \Re^n_+ \to \Re$ defined by

$$V(x) = \sum_{i=1}^{m} a_i \left(U_i - q_i \ln \frac{U_i}{q_i} - q_i \right), \tag{6}$$

with U_i given by Eq. (3) and a_i's are a solution of Eq. (2), satisfies:

a) $V(x^*) = 0$;

b) $V(x) > 0$ if $x \neq x^*$;

c) $\dfrac{dV(x)}{dt} \le 0$.

Properties (a) and (b) follow from the fact that $a_i > 0$ and each function $U_i - q_i \ln U_i/q_i - q_i$ equals zero for $U_i = q_i$ and is positive for any $U_i > 0$, $U_i \neq q_i$. For property (c) we compute the time derivative of $V(x)$:

$$\dot{V}(x) = \sum_{i=1}^{m} \frac{\partial V}{\partial U_i} \frac{dU_i}{dt} = \sum_{i,j=1}^{m} a_i M_{ij} w_i w_j \le 0, \tag{7}$$

from the hypothesis that M is admissible and where $w_i = U_i - q_i$. So $V(x)$ is a Lyapunov function associated with the fixed point x^* valid in all the positive orthant. It is relevant to note that this theorem holds for both singular and non-singular M matrices.

A closer inspection on the form of the Lyapunov function in Eq. (6) leads to the following lemma and theorem:

Lemma 1. *If M is admissible and $U_i(0) > 0$, $i = 1,\dots,m$; then there are 2m positive numbers ϵ_i, ν_i such that:*

$$\epsilon_i < U_i(t) < \nu_i, \quad \forall t; \tag{8}$$

or equivalently

$$\frac{1}{\nu_i} < \frac{1}{U_i(t)} < \frac{1}{\epsilon_i}. \tag{9}$$

Proof. The Lyapunov function in Eq. (6) can be seen as a function of the variables U_i, and has the following properties:

$$\lim_{U_i \to \infty} V(U_1,\dots,U_m) = \infty, \quad \forall U_j, \quad j \neq i;$$

$$\lim_{U_i \to 0} V(U_1,\dots,U_m) = \infty, \quad \forall U_j, \quad j \neq i. \tag{10}$$

Since V is monotonously decreasing with time, if we consider an initial condition $U_i(0)$ in the positive orthant it defines the initial value for the Lyapunov function and therefore an upper bound for $V(U_1,\dots,U_m)$ in all subsequent times. It becomes clear from Eq. (10) that the solution cannot approach arbitrarily the hyperplanes $U_i = 0$ and also that it is necessarily bounded, i.e., that there are $2m$ numbers $\epsilon_i, \nu_i \in \Re$ such that (8) and (9) hold. This closes the proof of the lemma.

Theorem 2. *If the matrix $M = BA$ is admissible and the initial condition is in the positive orthant, the corresponding solution is bounded and component-wise bounded away from zero, i.e., if $(x_1(0),\dots,x_n(0)) \in \Re^n_+$ then:*

$$\exists \epsilon_i, \delta_i \in \Re \mid 0 < \epsilon_i < x_i(t) < \delta_i,$$
$$i = 1,\dots,n \ \forall t.$$

Proof. We reorder the quasi-monomials in (3) such that the first n lines are linearly independent. This is possible as B is of rank n. Let us define a new $n \times n$ matrix $\tilde{B}$ by

$$\tilde{B}_{ij} = B_{ij}, \quad i,j = 1,\dots,n. \tag{11}$$

The n first quasi-monomials can then be written as

$$U_i = \prod_{k=1}^{n} x_k^{\tilde{B}_{ik}}. \tag{12}$$

This transformation of variables can be inverted as $\tilde{B}$ is not singular:

$$x_i = \prod_{k=1}^{n} U_k^{\tilde{B}^{-1}_{ik}}. \tag{13}$$

From lemma 1 each term in the product in the right hand side of Eq. (13) is bounded from above and from below. This closes the proof.

As a consequence of theorem 2 we also have the following corollary:

Corollary 1. *If for the ODE system in Eq. 1 the matrix $M = BA$ is admissible and if there is a fixed point $x^* \in \Re^n_+$, then this fixed point is stable, i. e. for any neighborhood $U \subset \Re^n_+$ of x^* there is a neighbourhood $W \subset \Re^n_+$, such that $U \subset W$ and for any solution x(t) starting in U we have $x(t) \in W$, for all t.*

Proof. for any neighbourhood U of x^* let us consider $V_U = max\{V(x) | x \in U\}$. As the Lyapunov function $V(x)$ is decreasing for any trajectory, then for all solution $x(t)$ starting in U we have that $V(x(t)) \leq V_U$, hence from theorem 2 there are $2m$ positive numbers ϵ_i and δ_i $(i = 1,\dots,n)$ such that $\epsilon_i \leq x_i(t) \leq \delta_i$ for all solution starting in U. Let us consider the following set $W = \{x \in \Re^n_+ / \epsilon_i \leq x_i \leq \delta_i\}$. Then all solutions starting in U stay in W.

Theorem 2 states sufficient conditions on A and B for the invariance of the positive orthant, which is far from being a trivial result for QP systems. In fact the theorem ensures that the solutions are away from the coordinate planes $x_i = 0$ and bounded. If the QP system in Eq. (1) is defined in the other orthants, the above results are still valid provided the initial condition and the fixed point are in the same orthant.

The conditions on the statements of theorem 1 and 2 can be somewhat relaxed as follows. Let us suppose once again that the quasi-monomial functions U_i in Eq. (3) are ordered in such a way that the first lines of B are linearly independent. This ensures the inversibility of the transformation in Eq. (12) as explained above. Then the condition $a_i > 0$ in Eq. (2) can be replaced by $a_i \geq 0$ for $i = n+1,\dots,m$ and $a_i > 0$ for $i = 1,\dots,n$. The proof follows the same steps as in the proofs of theorems 1 and 2.

The present approach can be extended by considering a reparametrization of the time variable of the form

$$dt = \prod_{k=1}^{n} x_k^{\Omega_k} dt', \tag{14}$$

for $\Omega_k \in \Re$. The original QP system in Eq. (1) is thus rewritten as

$$\frac{dx_i}{dt'} = x_i \sum_{j=1}^{m+1} A'_{ij} \prod_{k=1}^{n} x_k^{B'_{jk}},$$

$$i = 1,\dots,n, \tag{15}$$

where

$$A'_{ij} = A_{ij}, \quad i = 1,\dots,n;\ j = 1,\dots,m;$$

$$A'_{i,m+1} = l_i, \quad i = 1,\dots,n;$$

and

$$B'_{ij} = B_{ij} + \Omega_j, \quad i = 1,\dots,m;\ j = 1,\dots,n;$$

$$B'_{m+1,j} = \Omega_j, \quad j = 1,\dots,n.$$

Note that there are no linear terms in Eq. (15), except for the particular case where B' has at least one zero line. Theorem 2 can be applied using the new matrix $M' = B'A'$ instead of $M = BA$. Using the Ω_k in Eq. (14) as free parameters gives much more freedom to obtain an admissible matrix. This is an important and original result of the present approach that enlarges even the original results obtained for purely Lotka–Volterra systems [5–11].

3. Conditions on an admissible matrix

The conditions for a given matrix M to be admissible can be obtained in a systematic way as we proceed to show. Consider the following inequality:

$$\alpha y^2 + \beta y + \gamma \le 0, \quad \forall y. \tag{16}$$

The necessary and sufficient conditions for Eq. (16) to be satisfied are

$$\alpha < 0 \text{ and } \beta^2 - 4\alpha\gamma \le 0, \tag{17}$$

or

$$\alpha = \beta = 0 \text{ and } \gamma \le 0. \tag{18}$$

However Eq. (2) is a quadratic inequality on the w_i's, so we group different terms according to their order in one of the variables, say w_1

$$\sum_{i,j=1}^{m} a_i M_{ij} w_i w_j = a_1 M_{11} w_1^2 + \beta(w_2,\dots,w_m) w_1 + \gamma(w_2,\dots,w_m) \le 0. \tag{19}$$

Eq. (19) is a quadratic algebraic inequation in w_1 and therefore one of the following sets of conditions must hold. From (17):

$$a_1 M_{11} < 0, \quad \text{and}$$

$$\beta(w_2,\dots,w_m)^2 - 4 a_1 \gamma(w_2,\dots,w_m) \le 0; \tag{20}$$

or, from (18):

$$a_1 M_{11} = 0 \Rightarrow M_{11} = 0, \quad \text{and } \beta(w_2,\dots,w_m) = 0$$

$$\text{and } \gamma(w_2,\dots,w_m) \le 0. \tag{21}$$

The conditions in Eqs. (17) and (18) are independent of w_1 and quadratic in $w_2,\dots,w_m$. The procedure can be iterated for these inequalities, now in the variables $w_2,\dots,w_m$. At each step one set of conditions split in two, corresponding to the different possibilities to satisfy Eq. (16). Therefore for a m-dimensional matrix M we obtain 2^{m-1} independent sets of conditions, each set yielding different solutions for the parameters and the numbers a_i's such that M is admissible. As commented in Section 2, this procedure can be slightly altered to take into consideration the possibilities for some $a_i \ge 0$.

4. Application: the generalized mass action system

In this section we apply our approach to the following system:

$$\dot{x}_1 = l_1 x_1 - \alpha_1 x_1^{b_3} x_2^{b_1};$$

$$\dot{x}_2 = -l_2 x_2 + \alpha_2 x_1^{b_1} x_3^{b_2};$$

$$\dot{x}_3 = -l_3 x_3 + \alpha_3 x_2^{b_1}. \tag{22}$$

This system is an example of the Generalized Mass Action (GMA) model for biochemical reactions [12,13]. The functions x_i are concentrations (e.g., of a hormone or a protein), the elements b_i are of rates of different processes and the other parameters describe interactions between the elements. For this system we have the following matrices B and A:

$$B = \begin{bmatrix} b_3 - 1 & b_1 & 0 \\ b_1 & -1 & b_2 \\ 0 & b_1 & -1 \end{bmatrix}; \tag{23}$$

Table 1
Conditions for the GMA system

Conditions 1	$(1-b_3)\alpha_1<0$
	$a_2=a_1\alpha_2\left[\frac{2(b_3-1)+b_1^2\pm\sqrt{(b_3-1)(b_3-1+b_1^2)}}{b_1^2\alpha_1}\right]>0$
	$a_3=-a_1a_2\frac{b_2\alpha_3}{b_1\alpha_2}>0$
	$\frac{b_2}{b_1\alpha_2}\leq 0$
Conditions 2	$b_3=1$
	$a_2=a_1\frac{\alpha_2}{\alpha_1}>0$
	$a_3>\left[a_1\frac{\alpha_3(-b_2b_1+2-2\sqrt{-b_2b_1+1}}{b_1^2\alpha_1}\right]>0$
	$a_3<\left[a_1\frac{\alpha_3(-b_2b_1+2+2\sqrt{-b_2b_1+1}}{b_1^2\alpha_1}\right]>0$
	$\alpha_2>0$

and

$$A=\begin{bmatrix}-\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{bmatrix}. \tag{24}$$

The matrix $M=BA$ is then

$$M=\begin{bmatrix}(1-b_3)\alpha_1 & b_1\alpha_2 & 0\\ -\alpha_1 b_1 & -\alpha_2 & b_2\alpha_3\\ 0 & b_1\alpha_2 & -\alpha_3\end{bmatrix}, \tag{25}$$

corresponding to the quasi-monomials $U_1=x_1^{b_3-1}x_2^{b_1}$, $U_2=x_1^{b_1}x_2^{-1}x_3^{b_2}$ e $U_3=x_2^{b_1}x_3^{-1}$.

Applying our approach and solving the conditions in Section 3 such that M is an admissible matrix, implies $2^{3-1}=4$ sets of conditions. Two are reduced to trivial cases and will not be presented here. The others two sets of conditions are presented in Table 1.

The corresponding Lyapunov function is explicitly given by

$$\begin{aligned}V=&\,a_1\left(x_1^{b_3-1}x_2^{b_1}-q_1\ln\left(\frac{x_1^{b_3-1}x_2^{b_1}}{q_1}\right)-q_1\right)\\ &+a_2\left(\frac{x_1^{b_1}x_3^{b_2}}{x_2}-q_2\ln\left(\frac{x_1^{b_1}x_3^{b_2}}{x_2q_2}\right)-q_2\right)\\ &+a_3\left(\frac{x_2^{b_1}}{x_3}-q_3\ln\left(\frac{x_2^{b_1}}{x_3q_3}\right)-q_3\right).\end{aligned} \tag{26}$$

With a_1,a_2 and a_3 given in Table 1, and the fixed points q_i by

$$q_1=\frac{l_1}{\alpha_1},\quad q_2=\frac{l_2}{\alpha_2},\quad q_3=\frac{l_3}{\alpha_3}. \tag{27}$$

The use of the time reparametrization defined in Eq. (14) can lead to some interesting results in this case. For simplicity we consider the following transformation:

$$dt=x_2^{-b_1}x_3\,dt'. \tag{28}$$

The transformed QP system is

$$\begin{aligned}\frac{dx_1}{dt'}&=l_1x_1x_2^{-b_1}x_3-\alpha_1x_1^{b_3}x_3;\\ \frac{dx_2}{dt'}&=-l_2x_2^{-b_1+1}x_3+\alpha_2x_1^{b_1}x_2^{-b_1}x_3^{b_2+1};\\ \frac{dx_3}{dt'}&=-l_3x_2^{-b_1}x_3^2+\alpha_3x_3.\end{aligned} \tag{29}$$

The new quasi-monomials associated to system (29) are

$$\begin{aligned}&V_1=x_2^{-b_1}x_3,\quad V_2=x_1^{b_3-1}x_3,\\ &V_3=x_1^{b_1}x_2^{-1-b_1}x_3^{b_2+1}.\end{aligned} \tag{30}$$

and the corresponding matrices B' and A'

$$B'=\begin{bmatrix}0 & -b_1 & 1\\ b_3-1 & 0 & 1\\ b_1 & -b_1-1 & b_2+1\end{bmatrix}; \tag{31}$$

Table 2
Conditions for the GMA system with $dt = x_2^{-b_1} x_3 dt'$

Conditions 1	$(1-b_3)\alpha_1 < 0$ $l_3 = b_1 l_2 = l_1(b_3 - 1)$ $a_3 = a_1\left[\frac{b_1\alpha_2}{b_1(l_1+l_2)+l_2+l_3(-1-b_2)}\right] > 0$ $\frac{\alpha_1 b_1}{b_1(l_1+l_2)+l_2-b_2 l_3 - l_3}\left[4(1+b_1-b_3-b_1b_3)a_2 + \frac{\alpha_1 b_1^3}{b_1(l_1+l_2)+l_2-b_2l_3-l_3}\right] \le 0$
Conditions 2	$b_1 l_2 - l_3 < 0$ $a_2 = -4a_1\frac{\alpha_1(b_3-1)(b_1l_2-l_3)}{(l_3+l_1-l_1b_3)^2} > 0$ $a_3 = 2a_1b_1\alpha_2(b_3-1)/\left[2\left(l_2+l_3(-1-b_2+b_2b_3)\right)+(l_1+2l_2-l_3)b_1+2b_3(l_3-l_2)+(-2l_2-l_1)b_1b_3\right] > 0$ $a_1^2\alpha_2^2b_1^2 + 2a_1a_3\alpha_2(b_1^2(l_2-l_1)+(l_3b_2+l_2-l_3)b_1-2l_3)+a_3^2(-b_1(l_1+l_2)+l_3-l_2+b_2l_3)^2 \le 0$

and

$$A' = \begin{bmatrix} l_1 & -\alpha_1 & 0 \\ -l_2 & 0 & \alpha_2 \\ -l_3 & 0 & 0 \end{bmatrix}. \tag{32}$$

The matrix $M' = B'A'$ is

$$M' = \begin{bmatrix} b_1 l_2 - l_3 & 0 & -b_1\alpha_2 \\ l_1 b_3 - l_1 - l_3 & \alpha_1(-b_3+1) & 0 \\ b_1(l_1+l_2)+l_2-l_3(b_2+1) & -b_1\alpha_1 & -\alpha_2(b_1+1) \end{bmatrix}. \tag{33}$$

We note that this matrix involves the parameters l_i, which do not appear in Eq. (25), while α_3 is absent (33). This yields completely new conditions for stability shown in Table 2, which can be extended with more general time reparametrizations.

Lemma 1 holds for the quasi-monomial variables V_1, V_2 and V_3 and since the transformation $x_1, x_2, x_3 \to V_1, V_2, V_3$ in Eq. (30) is invertible, theorem 2 holds for x_1, x_2 and x_3 whenever M' is admissible.

The corresponding Lyapunov function is explicitly given by

$$\begin{aligned} V = a_1&\left(x_2^{-b_1}x_3 - q_1^*\ln\left(\frac{x_2^{-b_1}x_3}{q_1^*}\right) - q_1^*\right) \\ &+ a_2\left(x_1^{b_3-1}x_3 - q_2^*\ln\left(\frac{x_1^{b_3-1}x_3}{q_2^*}\right) - q_2^*\right) \\ &+ a_3\Bigg(x_1^{b_1}x_2^{-1-b_1}x_3^{b_2+1} - q_3^* \\ &\times\ln\left(\frac{x_1^{b_1}x_2^{-1-b_1}x_3^{b_2+1}}{q_3^*}\right) - q_3^*\Bigg). \end{aligned} \tag{34}$$

With a_1, a_2 and a_3 given in Table 2, and the fixed points q_i^* given by

$$q_1^* = q_1/q_3, \quad q_2^* = q_2/q_3, \quad q_3^* = 1/q_3. \tag{35}$$

5. Conclusions

In this work we presented a general method to study the boundedness of solutions, the invariance of orthants, and the existence of a Lyapunov function associated to a fixed point of a QP system. Even though theorems 1 and 2 involves only sufficient conditions, the present approach is based purely on algebraic conditions developed in Sections 2 and 3 and, as far as the authors are aware, it is more general than other approaches in the literature. Also the introduction of the time reparametrization in Eq. (14) greatly extends its scope, as shown in the application for the GMA system in Section 5. This approach is therefore an attempt to develop a general theory of global stability of fixed points and boundedness of solutions in general QP systems.

Acknowledgements

We are indebted to Prof. L. Brenig from Université Libre de Bruxelles for fruitful suggestions and discussions.

References

[1] A. Figueiredo, T.M. Rocha Filho, L. Brenig, J. Math. Phys. 39 (1998) 2929.
[2] A. Figueiredo, T.M. Rocha Filho, L. Brenig, Physica A 262 (1999) 158.
[3] L. Brenig, Phys. Lett. A 133 (1988) 378.
[4] J.P. LaSalle, S. Lefshets Stability by Lyapunov Direct Method, Academic Press, New York, 1961.
[5] V. Volterra, Leçons sur la Théorie Mathématique de la Lutte pour la Vie, Gauthier Villar, Paris, 1931.
[6] R. Redheffer, W. Walter, J. Diff. Eqs. 5 (1984) 245.
[7] R. Redheffer, Z. Zhiming, SIAM J. Alg. Disc. Meth. 3 (1982) 122.
[8] R. Redheffer, SIAM J. Alg. Disc. Meth. 6 (1985) 592.
[9] Y. Takeuchi, N. Adachi, H. Tokumaru, Math. Biosc. 42 (1978) 119.
[10] Y. Takeuchi, N. Adachi, J. Math. Biol. 10 (1980) 401.
[11] Y. Takeuchi, N. Adachi, J. Math. Anal. Apl. 88 (1982) 157.
[12] D.H. Irving, E.O. Voit, M.A. Savageau Analysis of Complex Dynamic Networks with ESSYNS, in: E.O. Voit (Ed.), Canonical Non-Linear Modelling S-systems Approach to Understanding Complexity, Von Nostrand, Reinhold, 1991.
[13] R. Curto, E.O. Voit, A. Sorribas, M. Cascantes, J. Biochem 324 (1997) 761.