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STATISTICAL

Stability Properties of Quasi-polynomial Systems: Convergence of Solutions and Structure of ω-Limit Sets

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Abstract In this paper, we analyze some properties of quasi-polynomial dynamical systems. This general class can be related to the class of quadratic Lotka-Volterra systems through a suitable mapping and change of variables. We analyze a criterion for convergence of solutions and the structure of ω-limit sets.

Keywords Lotka-Volterra · Quasi-polynomial · Convergence

1 Introduction: Lotka-Volterra Systems

In the years following the First World War, the mathematician Vito Volterra analyzed an intriguing problem involving the populations of two species of fish in the Adriatic Sea. Due the absence of fishing in the war period, it was supposed that the two species would present an increase in its population. Nevertheless, one of the fishes, which were prey to the other, end up by diminishing its population [1]. Volterra presented a simple model of differential equations to deal with the problem [2]. His equations, together with the works of Lotka [3], are known as Lotka-Volterra system and remain one of the most studied equations in the XX century. Let $U_1(t)$ be the density population of one of the fishes (the prey) at instant t, $U_2(t)$ the predator population density and $g > 0$ the growth rate of prey in the absence of predators. Its death rate is assumed proportional to the predator population with a rate $p > 0$, and thus $\dot{U}_1 = U_1\,(g - p\,U_2)$. For the predator dynamics, let $d > 0$ be the death rate. The growth rate is proportional to the prey population with $e > 0$: $\dot{U}_2 = U_2\,(-d + e\,U_1)$.

Note that if $U_2(0) = 0$ (no predators), the prey population grows exponentially following $U_2(t) = 0, \forall t$ and $U_1(t) = U_1(0)\,\exp(gt)$. We note that, in the prey absence, $U_1(0) = 0$, we have $U_1(t) = 0, \forall t$, and the predators will die since $U_2(t) = U_1(0)\exp(-dt)$. We also have the trivial solution $U_1(t) = U_2(t) = 0$ for the initial condition $U_1(0) = U_2(0) = 0$. Since trajectories do not intercept each other, the above rationale implies that, for an initial condition in the positive orthant: $U_1(0), U_2(0) \in \Re^+$, the associated solution remains in it.

The fixed points of the system are the origin $U_1^* = 0, U_2^* = 0$, and the interior fixed point $U_1^* = d/e, U_2^* = g/p$. It can be shown that the quantity $d\log U_1 - eU_1 + g\log U_2 - pU_2$ is constant along the trajectories of the system [1]. The model thus presents a neutral equilibrium: orbits starting in $\Re^+$ are periodic closed curves. Also, given the period T of the trajectory, we have [2]:

$$\frac{1}{T}\int_0^T U_2(t)dt = \frac{g}{p},$$
$$\frac{1}{T}\int_0^T U_1(t)dt = \frac{d}{e}. \qquad (1)$$

The above result shows what happened with the fishes in the Adriatic Sea. Fishing reduces the prey growth (from g

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to, say, $g-k$) as well as increases the death rate of the predators from d to $d+m$, letting p and e unaltered. Therefore, according to (1), in the presence of fishing, the mean density of predators will decrease to $(g-k)/p$ and the mean density of the prey will increase to $(d+m)/e$. Thus, with the reverse reasoning, an interruption in fishing will provoke an *increase* in the predator population and a decrease in the prey population, as observed in the Adriatic Sea.

The system can be generalized for m interacting species defining a set of parameters M_{ij} which can assume positive (symbiotic) or negative (competition) values to describe interaction between species i and j. Denoting the growth rate of species i by λ_i, we have the following generalized Lotka-Volterra (LV) system [1]:

$$\dot{U}_1 = \lambda_i U_i + U_i \sum_{j=1}^{m} M_{ij} U_j; \quad i = 1, \cdots, m. \tag{2}$$

Here, $U_i \in \Re^m$ and M is a square $m \times m$ matrix. The interior fixed points U^* of (2) are given by $\lambda_i + \sum_{j=1}^{m} M_{ij} U_j^* = 0; \; i = 1, \ldots, m, \; U_i^* > 0$.

In next section, we revisit the class of quasi-polynomial systems, presenting an important result due to L. Brenig and co-authors [4, 6] that show how to map this class of dynamical systems into quadratic systems of Lotka-Volterra type.

2 Quasi-polynomial Systems

The quasi-polynomial (QP) dynamical systems, extensively considered by the authors in previous references [4–14], are defined as:

$$\dot{x}_i = l_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}; \quad i = 1, \cdots, n, \tag{3}$$

with real A_{ij}, B_{ij}. Here, $m \geq n$ and rank of B is equal to n. Following [4, 5], let us introduce $m-n$ variables $x_{n+1}, x_{n+2}, \cdots, x_m$ such that

$$\dot{x}_i = \tilde{l}_i x_i + x_i \sum_{j=1}^{m} \tilde{A}_{ij} \prod_{k=1}^{m} x_k^{\tilde{B}_{jk}}; \quad i = 1, \cdots, m, \tag{4}$$

where

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{5}$$

and

$$\tilde{B} = \begin{bmatrix} B_{11} & \cdots & B_{1n} & b_{1,n+1} & \cdots & b_{1,m} \\ B_{21} & \cdots & B_{2n} & b_{2,n+1} & \cdots & b_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & \cdots & B_{mn} & b_{m,n+1} & \cdots & b_{m,m} \end{bmatrix}. \tag{6}$$

$\tilde{A}$ and $\tilde{B}$ are square matrices with arbitrary b_{jk} provided $\det \tilde{B} \neq 0$. To ensure equivalence between (3) and (4), we assume $x_k(t=0) = 1, \quad k = n+1, \cdots, m$. System (4) is form-invariant under transformations of type [4, 5]:

$$U_\alpha = \prod_{i=1}^{m} x_i^{F_{\alpha i}^{-1}}, \tag{7}$$

for arbitrary invertible F^{-1}. The key result here is that (4) is mapped onto a Lotka-Volterra (LV) system if $F = \tilde{B}^{-1}$ [5]:

$$\dot{U}_\alpha = (\tilde{B}\tilde{l})_\alpha U_\alpha + U_\alpha \sum_{\beta=1}^{m} (\tilde{B}\tilde{A})_{\alpha\beta} U_\beta; \quad \alpha = 1, \cdots, m. \tag{8}$$

With the identification $M \equiv BA = \tilde{B}\tilde{A}$ and $\lambda \equiv Bl = \tilde{B}\tilde{l}$, system (8) is a LV system. From (7) we have

$$\prod_{\beta=1}^{m} U_\beta^{\tilde{B}_{\alpha\beta}^{-1}} = 1; \quad \alpha = n+1, \cdots, m. \tag{9}$$

Equation (8), with $M = BA$ and $\lambda = Bl$, restricted by (9) is equivalent to system (3).

The Lotka-Volterra system was extensively studied in the literature [1, 15]. Several properties of QP systems can thus be assessed from results originally obtained for LV systems [11]. In [12, 13], the problem of conserved quantities and invariant surfaces in QP systems were considered by some of the authors. The problem of stability, in the sense of Lyapunov [16], was the subject of [7–10]. Stability is related to a property of the matrix M, which is said to be *admissible* if there exists $a_i > 0, \; i = 1, \ldots, m$; that satisfies: $\sum_{i,j=1}^{m} a_i M_{ij} w_i w_j \leq 0; \quad w \in \Re$. In [17], some of the authors shown that if the matrix $M = BA$ of system (3) is admissible and $U_i(0) U_i^* > 0$ then the QP system has a Lyapunov function $V = \sum_{i=1}^{m} a_i (U_i - U_i^* \ln \frac{U_i}{U_i^*} - U_i^*)$, U_i given by (7). Applications to ecological models were presented in [11]. More recently [14], the authors presented a connection between stable fixed points in square ($m = n$) QP systems and evolutionary stable states, a concept of evolutionary games [15].

Here, we extend the scope of our results presenting, in next sections, a convergence criteria and an analysis of ω-limit sets in QP systems.

3 A Convergence Criteria

Let a linear system such as

$$\dot{U}_i = \sum_{j=1}^{m} \hat{M}_{ij} U_j; \; i = 1, \cdots, m. \tag{10}$$

The following theorem holds [20]:

Theorem 1 *The origin $U = 0$ is an asymptotically stable fixed point for (10) iff $\hat{M}$ is a stable matrix, that is, all its eigenvalues have negative real part. Furthermore, $\hat{M}$ is stable iff there exists a definite positive matrix Q such that $Q\hat{M} + \hat{M}^T Q < 0$, and $\hat{M}$ is said to be negative definite.*

Consider polynomial systems of type:

$$\dot{x}_i = \sum_{j=1}^{q} c_{ij} v_j(x_1, \cdots, x_n); \; i = 1, \cdots, m. \tag{11}$$

with $c_{ij} \in \Re$, and v_j are monomials of type $x_1^{\beta_1} \cdots x_n^{\beta_n}$ with q the number of monomial. Let x restricted to $x \in \Re | x > 0$. Then (11) can be written as

$$\dot{x}_i = x_i \left(Ce^{D \ln x} \right)_i \equiv x \otimes Ce^{D \ln x}. \tag{12}$$

where the elements of D are β_j or $\beta_j - 1$. C is a $n \times p$ matrix and D a $p \times n$ matrix, with p the distinct number of monomials obtained after the factorization of x_i in the $i-th$ equation. We can rewrite (12) using

$$y = \ln x \Rightarrow \dot{y} = \dot{x}/x = Ce^{Dy} \tag{13}$$

the following theorems hold [18]:

Theorem 2 *Given C and D, let us assume that there exists N $p \times p$ non-negative non-diagonal matrices and two vectors $k > 0$ and ω such that $N^T \vec{1} = 0$ ($\vec{1}$ is a column vector whose elements are all equal to one), $Nk = 0$ and $\ln k = D\omega$. Furthermore, if these exists a symmetric $n \times n$ matrix P such that $PC = D^T N$, then the following function*

$$V(y) = \frac{1}{2}(y - \omega)^T P (y - \omega)$$

decreases along the trajectories i.e. $\dot{V}(y(t)) \leq 0$. Furthermore $\dot{V} = 0$ iff $e^{Dy} = \sum_{j=1}^{l} \lambda_j k_j$ with $\lambda_j \in \Re$

Theorem 3 *If C and D satisfy conditions of* theorem (2) *then the trajectories of (13) are either bounded or converge to the maximum invariant set such that $e^{Dy} = \sum_{j=1}^{l} \lambda_j k_j$.*

Note that (11), which is a particular case of a QP system, is invariant under monomial transformations of type (7), then theorem (3) applies when we write them in the LV format. The advantage here is that in the LV format we have $C \equiv M$ and $D \equiv I$ and condition $PC = D^T N$ gives $PM = N$. Since N is non-negative, we have

$$x^T (PM) x \geq 0 \tag{14}$$

Condition $N^T \vec{1}$ implies:

$$\sum_{j=1}^{n} N_{jk} = \sum_{j=1}^{n} \sum_{i=1}^{n} P_{ji} M_{ik} = 0 \tag{15}$$

which implies $\det N = 0$. In this case, we always have $k > 0$ such that $Nk = 0$.

As an example, consider the following particular case of a system describing an auto catalytic set of chemical reactions [15]:

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 - b_2 x_2^2), \\ \dot{x}_2 &= x_2(b_2 - b_1 x_1^2) \end{aligned} \tag{16}$$

with positive b_1, b_2. Matrices B and A are given by

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{17}$$

$$A = \begin{bmatrix} 0 & -b_2 \\ -b_1 & 0 \end{bmatrix}. \tag{18}$$

Condition (15) gives $-2b_2 p_{11} - 2b_2 p_{21} = 0$ and $-2b_1 p_{12} - 2b_1 p_{22} = 0$ which leads to

$$P = \begin{bmatrix} p & -p \\ -p & p \end{bmatrix}. \tag{19}$$

Thus, PM equals:

$$PM = \begin{bmatrix} 2pb_1 & -2pb_2 \\ -2pb_1 & 2pb_2 \end{bmatrix}. \tag{20}$$

Condition (14) implies:

$$\begin{aligned} x^T (PM) x \geq 0 \Rightarrow\; & x_1^2 (PM)_{11} + x_1 x_2 \left((PM)_{12} + (PM_{21})\right) \\ & + x_2^2 (PM)_{22} \geq 0 \end{aligned} \tag{21}$$

Equation (21) is a quadratic form in x_1 and is satisfied if $(PM)_{11} > 0 \Rightarrow b_1 > 0$ and $((PM)_{12} + (PM_{21}))^2 - 4(PM)_{11}(PM)_{22} \leq 0$ which in turn implies $(b_1 - b_2)^2 \leq 0 \Rightarrow b_1 = b_2$.

Under these conditions, system (16) is either bounded or converge to the maximum invariant set according theorem (3).

4 ω Limit Sets

Consider a dynamical system $\frac{dx}{dt} \equiv \dot{x}_i = f_i(x, t), \; i = 1, \cdots, n$. Let $\phi(x, t)$ the flux $\frac{d\phi(x,t)}{dt} = f(\phi(x, *))$, such that, given an initial condition $x(0) = x_0$ we have $\phi(x, 0) = x_0$ [19]. We assume $\phi(x, t)$ well defined $\forall x \in \Re^n$ and $\forall t \in \Re$.

Let $W \subset \Re^n$ an open subset of the euclidean real space and $f : W \to \Re^n$ of class C^0 with $x \in W$. Then, there is a constant $\beta > 0$ and an unique solution of $(-\beta, \beta) \to W$ which satisfies $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Field f must be locally Lipschitz, i.e., $|f(y) - f(x)| \leq K|y - x|$ for some real K where t_n stands for the sequence of t such that $\lim_{n\to\infty} t_n = \pm\infty$. W is an invariant set by the flux $\phi(x, t)$ iff

$$\forall x \in W \Rightarrow \phi(x, t) \in W, \forall t \in \Re.$$

Next, we define an $\omega(x)$ invariant set as the set $\omega(x)$ for which

$$\omega(x) = y \in \Re^n | \exists t_n, t_n \to +\infty \textit{ such that } \phi(x, t_n) \to \infty \textit{ when } n \to \infty$$

Information regarding the $\omega(x)$ limit set is useful when characterizing the trajectories of the system, e.g., when we are interested in characterizing the attractor set of the trajectories. The attractor is a set Y such that orbits which starts in a neighborhood of Y ends up in Y as $t \to \infty$. Thus, any attractor is an $\omega(x)$ set (the inverse is not true). The attractor set may be an asymptotically stable fixed point or a periodic orbit. Its *basin of attraction* is the set of initial conditions x_0 such that the $\omega(x)$ set of x_0 belongs to Y. An important theorem follows [20]:

Theorem 4 *Let $W : \Re^n \to \Re$ continuously differentiable and $\dot{x} = f(x)$ defined on $\Re^n$. If $\dot{W} \geq 0$ then the omega set $\omega(x)$ of $\dot{x} = f(x)$ is contained in the set $x \in \Re^n | \dot{W}(x) = 0$.*

In [7–10], we proved the following theorem:

Theorem 5 *Given the QP system (3) and its associated $M = BA$ matrix, if M is admissible and there is an interior positive fixed point $x^* \in \Re^n_+$, then the Lyapunov function $V = \sum_{i=1}^m a_i(U_i - U_i^* \ln \frac{U_i}{U_i^*} - U_i^*)$ can be used to obtain a Lyapunov function $\tilde{V}(x)$ for (3), namely $\tilde{V}(x) = V(U)|_{x_{n+1}=\cdots=x_m=1}$.*

It is worth to note that the above results are valid for negative fixed points provided the initial condition and the fixed point are in the same orthant. The fixed points U^* of (8) relates to those of (3), say x^*, through $U_\alpha^* = \prod_{i=1}^n (x_i^*)^{B_{i\alpha}}$.

Here, we may use the Lyapunov function $V = \sum_{i=1}^m a_i(U_i - U_i^* \ln \frac{U_i}{U_i^*} - U_i^*)$ as W, relaxing the condition $a_i > 0$ for admissibility, since the function does not need to be positive definite. We just let V such that $\dot{V} \leq 0$ and define $\dot{W} = -\dot{V}$. Depending on the structure of the region $\dot{W} = 0$ theorem (4) may not be useful, e.g., the case when $\dot{W} = 0$ lies in an open unlimited region of $\Re^n$ (one of its faces, for example). To use theorem (4) in the study of QP systems, we must connect the $\omega(x)$ set of a given QP system and the $\omega(U)$ set of the corresponding LV system. We state the following theorem:

Theorem 6 *Given a QP system and its equivalent LV system , if $V(U)$ is a Lyapunov function for the LV system such that (4) is satisfied, i.e.*

$$\omega(U) \in U \in \Re^m | \dot{V}(U) = 0$$

then it is possible to obtain, for the QP system, a function $\bar{V}(x)$ such that

$$\omega(x) \in x \in \Re^n | \dot{\bar{V}}(x) = 0$$

To proof the theorem, let V and $\bar{V}$ as in theorem (5). Consider

$$x_i(t_l) = \prod_{k=1}^n U_k^{\hat{B}_{ik}^{-1}(t_l)} \tag{22}$$

If the sequence t_l is such that $\lim_{l\to\infty} = \infty$ then:

$$x_i(t_l \to \infty) = \prod_{k=1}^n U_k(t_l \to \infty)^{\hat{B}_{ik}^{-1}}$$

And since $x_i(t_l \to \infty) \to x_\omega \in \omega(x)$ and $U_k(t_l \to \infty) \to U_\omega \in \omega(U)$ we conclude, from (22)

$$(x_i)_\omega = \prod_{k=1}^n (U_k)_\omega^{\hat{B}_{jk}^{-1}} \tag{23}$$

which connects $\omega(x)$ and $\omega(U)$. Furthermore, if $\dot{V}(U_\omega) = 0$ then we have

$$\begin{aligned} \frac{dV(U)}{dt} &= \sum_{\alpha=1}^m \frac{\partial V}{\partial U_\alpha}\frac{dU_\alpha}{dt} = \sum_{\alpha=1}^m \sum_{i=1}^n \frac{\partial V}{\partial U_\alpha}\frac{\partial U_\alpha}{\partial x_i}\frac{dx_i}{dt} \\ &= \sum_{i=1}^n \frac{\partial V(U(x))}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^n \frac{\partial \bar{V}(x)}{\partial x_i}\frac{dx_i}{dt} = \frac{d\bar{V}}{dt} = 0 \end{aligned} \tag{24}$$

which demonstrates the theorem.

When dealing with $\dot{V}(U) = 0$ and $\dot{\bar{V}}(x) = 0$ note that, if $\dot{V}(U) = 0$ defines a compact region in $\Re^m$, then $\dot{\bar{V}}(x) = 0$ will also define a compact region in $\Re^n$, since $\dot{\bar{V}}(x) = 0$ is a restriction of $V(U)$ to $x_{n+1} = \cdots = x_m = 1$. However, if $\dot{V}(U) = 0$ in an unbounded region in $\Re^m$, the same does not hold for $\bar{V}(x)$. Furthermore, analysis of $\dot{V}(U)$ is not a trivial task. For example, in three-dimensional matrices, one of the conditions for admissibility [7] is $M_{11} = 0$, $a_2 M_{22} < 0$, $a_1 M_{13} + a_3 M_{31} = 0$. In this way, the condition for $\dot{V} = 0$ gives

$$\begin{aligned} \dot{V}(U) &= 2M_{22}(U_2 - U_2^*)^2 + (a_2 M_{23} + a_3 M_{32})(U_2 - U_2^*) \\ &\quad \times (U_3 - U_3^*) + a_3 M_{33}(U_3 - U_3^*). \end{aligned}$$

Nevertheless, the theorem (4) remains useful if the region where $\dot{V}(U) = 0$ holds reduces to the fixed points of the system or a compact region of $\Re^m$.

5 Concluding Remarks

In this paper, we present results concerning analytical properties in a class of general non-linear systems known as quasi-polynomial systems. The possibility of relating these systems to Lotka-Volterra equations allows the development of methods to analyze several different non-linear dynamical systems. We analyzed a criteria for convergence and structure of ω-limit sets.

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