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On the time-reparametrization of quasi-polynomial systems

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Abstract

In this Letter we show that an important type of time-reparametrization problem in quasi-polynomial systems is equivalent to the feasibility of a suitably constructed bilinear matrix inequality where the unknowns are the coefficients of the Lyapunov function and the exponents in the time-reparametrization transformation.

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1. Introduction

The class of quasi-polynomial (QP) systems has gained an important role in the modelling of nonlinear dynamical systems since the majority of smooth nonlinear systems occurring in practice can be algorithmically transformed to QP form [3,11,13].

Unlike the general nonlinear case, there are computationally effective methods for analyzing local and global stability of QP systems that are based on system invariants and a suitable Lyapunov function candidate [5,10].

As it is shown in [5], the time-reparametrization transformation significantly enlarges the possibilities to find a Lyapunov function for the investigated QP system to prove its global (asymptotic) stability. However, according to the best of the authors' knowledge, no systematic and numerically tractable method has been proposed in the literature to decide whether an appropriate time-reparametrization exists and to compute at least one feasible solution. Our aim here is to give a solution to this problem.

This Letter is organized as follows. Section 2 contains the basic notions (i.e., the general form and sufficient global stability conditions of QP systems,

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and the notion of bilinear matrix inequalities) that are needed to derive the main results. The investigated time reparametrization transformation and its main properties are described in Section 3. The main contribution of the Letter can be found in Section 4, while Section 5 contains two simple numerical examples that illustrate the proposed method.

2. Basic notions

2.1. Quasi-polynomial systems

Let us denote the element of an arbitrary matrix W with row index i and column index j by $[W]_{i,j}$. Quasi-polynomial systems are systems of ODEs of the following form

$$\dot{y}_i = y_i\left(L_i + \sum_{j=1}^{m}[A]_{i,j}\prod_{k=1}^{n} y_k^{[B]_{j,k}}\right), \quad i = 1,\dots,n, \tag{1}$$

where $y \in \operatorname{int}(\mathbb{R}^n_+)$, $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$, $L_i \in \mathbb{R}$, $i = 1,\dots,n$. Furthermore, $L = [\,L_1 \quad \dots \quad L_n\,]^T$. Let us denote the equilibrium point of interest of (1) as $y^* = [\,y_1^* \quad y_2^* \quad \dots \quad y_n^*\,]^T$. Without the loss of generality we can assume that $\operatorname{Rank}(B) = n$ and $m \geqslant n$ (see [13]).

2.2. Linear and bilinear matrix inequalities

A (nonstrict) linear matrix inequality (LMI) is an inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \leqslant 0, \tag{2}$$

where $x \in \mathbb{R}^m$ is the variable and $F_i \in \mathbb{R}^{n\times n}$, $i = 0,\dots,m$ are given symmetric matrices. The inequality symbol in (2) stands for the negative semidefiniteness of $F(x)$.

One of the most important properties of LMIs is the fact, that they form a convex constraint on the variables, i.e., the set $\{x \mid F(x) \leqslant 0\}$ is convex and thus many different kinds of convex constraints can be expressed in this way [2,17].

A bilinear matrix inequality (BMI) is a diagonal block composed of q matrix inequalities of the following form

$$G_0^i + \sum_{k=1}^{p} x_k G_k^i + \sum_{k=1}^{p}\sum_{j=1}^{p} x_k x_j K_{kj}^i \leqslant 0, \quad i = 1,\dots,q, \tag{3}$$

where $x \in \mathbb{R}^p$ is the decision variable to be determined and G_k^i, $k = 0,\dots,p$, $i = 1,\dots,q$ and K_{kj}^i, $k, j = 1,\dots,p$, $i = 1,\dots,q$ are symmetric, quadratic matrices.

The main properties of BMIs are that they are nonconvex in x (which makes their solution numerically much more complicated than that of linear matrix inequalities), and their solution is NP-hard [14]. However, there exist practically applicable and effective algorithms for BMI solution [15,19].

2.3. Global stability of quasi-polynomial systems

For the convenience of notation, let us introduce new variables, often called *monomials* of (1) as

$$z_j = \prod_{k=1}^{n} y_k^{[B]_{j,k}}, \quad j = 1,\dots,m. \tag{4}$$

Then the monomials corresponding to the equilibrium point of (1) are

$$z_j^* = \prod_{k=1}^{n} (y_k^*)^{[B]_{j,k}}, \quad j = 1,\dots,m. \tag{5}$$

Most often, the following, so-called "entropy-like" Lyapunov function candidate is used for examining the global stability of QP systems

$$V(z) = \sum_{i=1}^{m} c_i\left(z_i - z_i^* - z_i^* \ln\frac{z_i}{z_i^*}\right), \tag{6}$$

where $c_i > 0$, $i = 1,\dots,m$.

Let us introduce the following notation

$$M = B \cdot A, \tag{7}$$

where M is a parameter matrix of the system that is invariant to quasi-monomial transformations (see, e.g., [12]).

It can be easily shown that V is nonincreasing (i.e., the equilibrium y^* is globally stable) if and only if the following linear matrix inequality

$$M^T C + CM \leqslant 0 \tag{8}$$

can be solved for a positive definite diagonal matrix $C \in \mathbb{R}^{m\times m}$. In this case, the c_i coefficients in (6) are the diagonal elements of C (see, e.g., [7] or [18]) and the matrix M is called *diagonally stabilizable* [1] or *admissible* [5].

We note that the solvability of the LMI (8) and thus the existence and nonincreasing nature of the Lyapunov function (6) is not a necessary condition for the global stability of (1). However, this Lyapunov function is used most frequently for QP systems because it is existence can be tested numerically or even symbolically [16].

We remark that the numerical solution of (8) can be of different difficulty depending on the rank of M. If M is of full rank, then there are several possibilities to check the feasibility and solve (8). One of the most popular tools is the LMI control toolbox for the Matlab computing software environment [6]. If M is rank deficient (which is the general case if the number of monomials is greater than the number of QP variables) then the only applicable numerical method known by the authors is described in [8].

3. Time-reparametrization in QP systems

The significance of time-reparametrization is that it largely extends the possibilities to prove the global stability of a QP system (see, e.g., [5]). As we will see on the examples in Section 5, there are cases when the invariant matrix M of the system itself is not diagonally stabilizable, but with an appropriate time-reparametrization, it is possible to find a Lyapunov function of the form (6) for the transformed (reparametrized) model.

3.1. The time-reparametrization transformation

3.1.1. The generic case

Let $\Omega = [\,\Omega_1 \quad \ldots \quad \Omega_n\,]^T \in \mathbb{R}^n$. It is shown, e.g., in [5] that the following reparametrization of time

$$dt = \prod_{k=1}^{n} y_k^{\Omega_k}\, dt' \tag{9}$$

transforms the original QP system into the following (also QP) form

$$\frac{dy_i}{dt'} = y_i \sum_{j=1}^{m+1} [\tilde{A}]_{i,j} \prod_{k=1}^{n} y_k^{[\tilde{B}]_{j,k}}, \quad i = 1,\ldots,n, \tag{10}$$

where $\tilde{A} \in \mathbb{R}^{n\times(m+1)}$, $\tilde{B} \in \mathbb{R}^{(m+1)\times n}$ and

$$[\tilde{A}]_{i,j} = [A]_{i,j}, \quad i = 1,\ldots,n, \quad j = 1,\ldots,m, \tag{11}$$

$$[\tilde{A}]_{i,m+1} = L_i, \quad i = 1,\ldots,n \tag{12}$$

and

$$[\tilde{B}]_{i,j} = [B]_{i,j} + \Omega_j, \quad i = 1,\ldots,m, \quad j = 1,\ldots,n \tag{13}$$

$$[\tilde{B}]_{m+1,j} = \Omega_j, \quad j = 1,\ldots,n. \tag{14}$$

It can be seen that the number of monomials is increased by one and vector $\tilde{L}$ is zero in the transformed system.

3.1.2. A special (nongeneric) case

A special case of the time-reparametrization or new time transformation occurs when the following relation holds:

$$\Omega^T = -b_j, \quad 1 \leqslant j \leqslant m, \tag{15}$$

where b_j is an arbitrary row of the B matrix of the original system (1). From Eqs. (13) and (14) we can see that in this case the jth row of $\tilde{B}$ is a zero vector. This means that the number of monomials in the transformed system (10) remains the same as in the original QP system (1) and a nonzero $\tilde{L}$ vector that is equal to the jth column of A appears in the transformed system (for an example, see [5]).

In this case, the Ω vector can take only m possible different values (see Eq. (15)), therefore the stability analysis of the transformed system reduces to the feasibility check of m different LMIs of the form (8). However, our approach treats the Ω vector as part of the unknowns to be determined, therefore from now on we will only consider the generic case discussed in Section 3.1.1.

3.2. Properties of the time-reparametrization transformation

The most important properties of the time-reparametrization transformation that are used for analyzing local and global stability are as follows.

3.2.1. Monomials

The set of monomials $p_1, \ldots, p_{m+1}$ for the reparametrized system can be written up in terms of the original monomials:

$$p_j = \prod_{k=1}^{n} y_k^{\Omega_k} \cdot \prod_{k=1}^{n} y_k^{[B]_{j,k}} = \prod_{k=1}^{n} y_k^{[B]_{j,k}+\Omega_k}, \quad j = 1, \ldots, m$$

and

$$p_{m+1} = \prod_{k=1}^{n} y_k^{\Omega_k}$$

or using a shorter notation:

$$p_j = r \cdot z_j, \quad j = 1, \ldots, m, \qquad p_{m+1} = r,$$

where z_j is given in (4) and

$$r = \prod_{k=1}^{n} y_k^{\Omega_k}.$$

3.2.2. Equilibrium points

Since the equations of the reparametrized system (10) can be written as

$$\frac{dy_i}{dt'} = y_i \left(L_i + \sum_{j=1}^{m} [A]_{i,j} \prod_{k=1}^{n} y_k^{[B]_{j,k}} \right) \prod_{k=1}^{n} y_k^{\Omega_k}, \quad i = 1, \ldots, n \tag{16}$$

and we assume that $y_i > 0$, $i = 1, \ldots, n$, it is clear that the equilibrium point y^* of the original QP system (1) is also an equilibrium point of the reparametrized system (16).

3.2.3. Local stability

Let us denote the Jacobian matrix of the original QP system (2) at the equilibrium point by $J_{\mathrm{QP}}(y^*)$. Then the Jacobian matrix of the time reparametrized QP system at the equilibrium point can be computed by using the formula described in [4]:

$$\begin{aligned} \tilde{J}_{\mathrm{QP}}(y^*) &= Y^* \cdot \tilde{A} \cdot \tilde{Z}^* \cdot \tilde{B} \cdot (Y^*)^{-1} \\ &= r^* \cdot J_{\mathrm{QP}}(y^*) = \prod_{k=1}^{n} y_k^{*\Omega_k} \cdot J_{\mathrm{QP}}(y^*), \end{aligned} \tag{17}$$

where

$$\tilde{Z}^* = \mathrm{diag}(p_1^*, \ldots, p_m^*, p_{m+1}^*),$$

$$Y^* = \mathrm{diag}(y_1^*, \ldots, y_n^*),$$

are the quasi-monomials of the time-reparametrized system and the system variables in the equilibrium point. From Eq. (17) one can see that (as we naturally expect) local stability is not affected by the time-reparametrization, because this transformation just multiplies the eigenvalues of the Jacobian by a positive constant r^*.

3.2.4. Global stability

Rewriting (9) gives

$$\frac{dt}{dt'} = \prod_{k=1}^{n} \left(y_k(t') \right)^{\Omega_k}, \tag{18}$$

from which we can see that t is a strictly monotonously increasing continuous and invertible function of t'. This means that global stability of the QP system in the reparametrized time t' is equivalent to global stability in the original time scale t.

4. The time-reparametrization problem as a bilinear matrix inequality

We denote an $n \times m$ matrix containing zero elements by $0^{n \times m}$. Let us define two auxiliary matrices by extending A with a zero column and B with a zero row, i.e.,

$$\bar{A} = \left[A | 0^{n\times 1} \right] \in \mathbb{R}^{n\times(m+1)}, \tag{19}$$

and

$$\bar{B} = \left[\frac{B}{0^{1\times n}} \right] \in \mathbb{R}^{(m+1)\times n}. \tag{20}$$

Then $\tilde{A}$ and $\tilde{B}$ can be written as

$$\tilde{A} = [A|L] = \bar{A} + \left[0^{n\times m} | L \right], \tag{21}$$

and

$$\tilde{B} = \begin{bmatrix} b_1 + \Omega^T \\ b_2 + \Omega^T \\ \vdots \\ b_m + \Omega^T \\ \Omega^T \end{bmatrix} = \bar{B} + S \cdot \Omega^*, \tag{22}$$

where

$$\Omega^* = \mathrm{diag}(\Omega) \in \mathbb{R}^{n\times n} \tag{23}$$

and

$$S=\begin{bmatrix}1&1&\dots&1\\1&1&\dots&1\\\vdots&&&\\1&1&\dots&1\end{bmatrix}\in\mathbb{R}^{(m+1)\times n}. \tag{24}$$

It can be seen from Eqs. (21) and (22) that the invariant matrix of the reparametrized system is

$$\tilde{M}=\tilde{B}\cdot\tilde{A}=\left(\bar{B}+S\cdot\Omega^*\right)\cdot\tilde{A}. \tag{25}$$

Therefore the matrix inequality for examining the global stability of the reparametrized system is the following

$$-C<0, \tag{26}$$

$$\tilde{M}^T\cdot C+C\cdot\tilde{M}\leqslant 0, \tag{27}$$

i.e.,

$$-C<0, \tag{28}$$

$$\tilde{A}^T\left(\bar{B}^T+\Omega^*S^T\right)C+C\left(\bar{B}+S\Omega^*\right)\tilde{A}\leqslant 0, \tag{29}$$

which clearly has the same form as (3) with the following set of unknowns:

$$x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_{m+1}\\x_{m+2}\\\vdots\\x_{m+n+1}\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\\vdots\\c_{m+1}\\\Omega_1\\\vdots\\\Omega_n\end{bmatrix}. \tag{30}$$

Now we are ready to construct the matrices in the BMI (3) starting with

$$G_0^1=G_0^2=0^{(m+1)\times(m+1)}, \tag{31}$$

$$[G_k^1]_{i,j}=\begin{cases}-1, & i=j=k,\\ 0, & \text{otherwise},\end{cases}$$
$$i,j,k=1,\dots,m+1, \tag{32}$$

$$G_k^1=0^{(m+1)\times(m+1)},$$
$$k=m+2,\dots,m+n+1, \tag{33}$$

and

$$K_{kl}^1=0^{(m+1)\times(m+1)},$$
$$k,l=1,\dots,m+n+1. \tag{34}$$

Furthermore, let us introduce the following notations

$$P_k\in\mathbb{R}^{(m+1)\times(m+1)},$$
$$[P_k]_{i,j}=\begin{cases}[\bar{B}\cdot\tilde{A}]_{i,j}, & i=k,\\ 0, & i\neq k,\end{cases}$$
$$i,j,k=1,\dots,m+1, \tag{35}$$

and

$$Q_{kl}\in\mathbb{R}^{(m+1)\times(m+1)},$$
$$[Q_{kl}]_{i,j}=\begin{cases}[\tilde{A}]_{l-m-1,j}, & i=k,\\ 0, & i\neq k,\end{cases}$$
$$i,j,k=1,\dots,m+1,\quad l=m+2,\dots,m+n+1. \tag{36}$$

Then

$$G_k^2=\begin{cases}P_k+P_k^T, & k=1,\dots,m+1,\\ 0^{(m+1)\times(m+1)}, & k=m+2,\dots,m+n+1,\end{cases} \tag{37}$$

and

$$K_{kl}=\begin{cases}Q_{kl}+Q_{kl}^T, & k=1,\dots,m+1,\\ \quad l=m+2,\dots,m+n+1, & \\ 0^{(m+1)\times(m+1)}, & \text{otherwise},\end{cases}$$
$$k,l=1,\dots,m+n+1. \tag{38}$$

We note that in certain cases the feasibility of a BMI can be traced back to the feasibility of equivalent LMIs (see [2] or [17]), but in our case it is not possible because of the structural (diagonality) constraint on both of the unknown matrices Ω^* and C in (29).

5. Examples

In order to illustrate the above proposed method of finding time-reparametrization transformations for global stability analysis, two simple examples are presented.

5.1. Example with a full rank M matrix

Consider a QP system with the following matrices

$$A=\begin{bmatrix}\frac{2}{3}&-\frac{8}{3}\\\frac{2}{3}&-\frac{7}{3}\end{bmatrix}\approx\begin{bmatrix}0.6667&-2.6667\\0.6667&-2.3333\end{bmatrix}, \tag{39}$$

$$B=\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}\\-\frac{8}{3}&\frac{16}{3}\end{bmatrix}\approx\begin{bmatrix}0.6667&-0.3333\\-2.6667&5.3333\end{bmatrix}, \tag{40}$$

$$L = \begin{bmatrix} 2 \\ \frac{5}{3} \end{bmatrix} \approx \begin{bmatrix} 2 \\ 1.6667 \end{bmatrix}. \tag{41}$$

Its equilibrium point of interest is:

$$y^* = [1 \quad 1]^T. \tag{42}$$

The Jacobian matrix of the locally linearized system in y^* has the following eigenvalues: -0.1187, -4.9924. This shows that the investigated equilibrium point is at least locally asymptotically stable.

Using an appropriate LMI solver (e.g., Matlab's LMI control toolbox) it can be checked that the LMI (8) cannot be solved for $M = B \cdot A$. However, using the algorithm [15] for solving the corresponding BMI we find that a feasible solution of (29) is, e.g.,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} \frac{2}{3} & -\frac{5}{3} \end{bmatrix}^T \tag{43}$$

The eigenvalues of $\tilde{M}^T \cdot C + C \cdot \tilde{M}$ are

$$\lambda_1 = 0, \qquad \lambda_2 \approx -0.2374, \qquad \lambda_3 \approx -9.9848, \tag{44}$$

which shows that the examined system is globally stable.

5.2. Example with a rank deficient M matrix

Consider the following open generalized mass-action law system

$$\begin{aligned} \dot{y}_1 &= 0.5y_1 - y_1^{2.25} - 0.5y_1^{1.5}y_2^{0.25} + u_1, \\ \dot{y}_2 &= y_2 - 0.5y_2^{1.75} + u_2, \end{aligned} \tag{45}$$

where y_1 and y_2 are the concentrations of chemical species $\mathcal{A}_1$ and $\mathcal{A}_2$ ($[\frac{\mathrm{mol}}{\mathrm{m}^3}]$), while u_1 and u_2 (the manipulable inputs) are their volume-specific component mass inflow rates ($[\frac{\mathrm{mol}}{\mathrm{m}^3\,\mathrm{s}}]$). The above two differential equations originate from the component mass conservation equations constructed for a perfectly stirred balance volume [9] under the following modelling assumptions:

(1) constant temperature and overall mass,
(2) constant physico-chemical properties (e.g., density),
(3) presence of an inert solvent in a great excess,
(4) presence of the following reaction network:

- autocatalytic generation of the species $\mathcal{A}_1$ and $\mathcal{A}_2$ (e.g., by polymer degradation when they are the monomers and the polymers are present in a great excess) giving rise to the reaction rates $0.5y_1$ and y_2 (the first terms in the right-hand sides) respectively,
- a self-degradation of these species described by the reaction rates $-y_1^{2.25}$ and $-0.5y_2^{1.75}$ (the second terms on the right-hand sides) respectively,
- a catalytic degradation of the specie $\mathcal{A}_1$ catalyzed by specie $\mathcal{A}_2$ that corresponds to $-0.5y_1^{1.5}y_2^{0.25}$ in the first equation only (the third term).

The control aim is to drive the system to a positive equilibrium

$$y_1^* = 2.4082\,\frac{\mathrm{mol}}{\mathrm{m}^3}, \qquad y_2^* = 16.3181\,\frac{\mathrm{mol}}{\mathrm{m}^3}.$$

This goal can be achieved, e.g., by the following nonlinear feedback:

$$\begin{aligned} u_1 &= 0.5y_1y_2^{0.75}, \\ u_2 &= 0.5y_1^{1.25}y_2 + 0.5y_1^{0.5}y_2^{1.25}. \end{aligned} \tag{46}$$

The above inputs being the component mass flow rates fed to the system (they are both positive) are needed for compensating for the degradation of the specie $\mathcal{A}_1$ and $\mathcal{A}_2$.

By substituting Eq. (46) into Eq. (45), we obtain the controlled system that is a QP system with the following matrices

$$A = \begin{bmatrix} -1 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}, \tag{47}$$

$$B = \begin{bmatrix} 1.25 & 0 \\ 0 & 0.75 \\ 0.5 & 0.25 \end{bmatrix}, \qquad L = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}. \tag{48}$$

The eigenvalues of the Jacobian matrix of the system at the equilibrium point are -6.4076 and -0.7768.

Since the rank of $M = B \cdot A$ in this case is only 2, we can only use the algorithm described in [8] to prove that the LMI (8) is not feasible in this case.

However, by solving (29) using again the algorithm described in [15] we find that we can use the following time-reparametrization:

$$\Omega = [-0.25 \quad -0.5]^T \tag{49}$$

and the diagonal matrix containing the coefficients of the Lyapunov function is:

$$C = \mathrm{diag}\big([\,1 \quad 2 \quad 2 \quad 2\,]\big). \tag{50}$$

The eigenvalues of $\tilde{M}^T \cdot C + C \cdot \tilde{M}$ in this case are

$$\lambda_1 = 0, \qquad \lambda_2 = 0, \qquad \lambda_3 = -4.5, \qquad \lambda_4 = -2.5 \tag{51}$$

which again proves the global stability of the system.

The above example demonstrates how time-reparametrization can be used in the design of suitable globally stabilizing static feedbacks for nonlinear process systems.

6. Conclusions

In this Letter it was shown that the solvability of a class of time-reparametrization problem used in global stability analysis of QP systems is equivalent to the feasibility of a bilinear matrix inequality, where the unknowns to be determined are the coefficients of the Lyapunov function candidate and the parameters of the time-reparametrization transformation. Using the proposed method, it is possible to decide whether an appropriate time-reparametrization exists and to determine the time-reparametrizing transformation and the Lyapunov function together, using the same algorithm.

Although the solution of BMIs is known to be an NP-hard problem, there exist effective numerical algorithms for handling them.

Two simple examples were given, where global stability could not be proven in the original time-scale but after solving the corresponding BMI and finding the right time-reparametrization parameters, it turned out that the systems are globally stable. One of the examples demonstrated the use of time-reparametrization in designing a globally stabilizing static nonlinear controller for an open generalized mass action kinetic system.

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