PERGAMON

Applied Mathematics Letters 14 (2001) 697–699

Applied Mathematics Letters

www.elsevier.nl/locate/aml

A Lyapunov Function for Leslie-Gower Predator-Prey Models

A. KOROBEINIKOV
Department of Mathematics, University of Auckland
Private Bag 92019, Auckland, New Zealand

(Received and accepted July 2000)

Communicated by G. C. Wake

Abstract—A Lyapunov function for continuous time Leslie-Gower predator-prey models is introduced. Global stability of the unique coexisting equilibrium state is thereby established. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Leslie-Gower systems, Lyapunov functions, Stability.

In his papers [1,2], Leslie introduced a predator-prey model where the "carrying capacity" of the predator's environment is proportional to the number of prey. Leslie stresses the fact that there are upper limits to the rates of increase of both prey H and predator P, which are not recognised in the Lotka-Volterra model. These upper limits can be approached under favourable conditions: for the predator, when the number of prey per predator is large; for the prey, when the number of predators (and perhaps the number of prey also) is small.

In the case of continuous time, these considerations lead to the differential equation models

$$\frac{dH}{dt} = (r_1 - a_1 P) H, \qquad \frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{H}\right) P \tag{1}$$

and

$$\frac{dH}{dt} = (r_1 - a_1 P - b_1 H) H, \qquad \frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{H}\right) P, \tag{2}$$

which are known, respectively, as the first and second Leslie-Gower predator-prey models [3, p. 91]. (All the constants in systems (1) and (2) are positive.) System (1) is a simplification of system (2) in which within-species competition has negligible influence on prey population growth (i.e., ($b_1 = 0$)). System (2) is one of the simplest having maximum growth rates which each population approaches under favourable conditions.

Both systems (1) and (2) have the unique coexisting fixed point

$$H^* = \frac{r_1 a_2}{a_1 r_2 + a_2 b_1}, \qquad P^* = \frac{r_1 r_2}{a_1 r_2 + a_2 b_1}. \tag{3}$$

PII: S0893-9659(01)00031-3

It follows from equations (3) that

$$r_2 H^* = a_2 P^*, \qquad a_1 P^* + b_1 H^* = r_1. \tag{4}$$

Linear analysis of models (1) and (2) shows that their coexisting fixed point is stable. Numerical computations [3, p. 91] suggest that the fixed point is globally stable.

In this paper, we introduce a Lyapunov function for both models (1) and (2) and use it to prove their global stability.

THEOREM. *The coexisting fixed point (H^*, P^*) of the first and the second Leslie-Gower predator-prey models is globally stable.*

PROOF. A Lyapunov function

$$V(H,P) = \ln\frac{H}{H^*} + \frac{H^*}{H} + \frac{a_1 H^*}{a_2}\left(\ln\frac{P}{P^*} + \frac{P^*}{P}\right)$$

is defined and continuous for all $H, P > 0$. The function $V(H,P)$ satisfies

$$\frac{\partial V}{\partial H} = \frac{1}{H}\left(1 - \frac{H^*}{H}\right), \qquad \frac{\partial V}{\partial P} = \frac{a_1 H^*}{a_2 P}\left(1 - \frac{P^*}{P}\right),$$

hence, the fixed point (H^*, P^*) is the only extremum of the function $V(H,P)$ in the positive quadrant. It is easy to see that the point (H^*, P^*) is a minimum. Since

$$\lim_{H\to 0} V(H,P) = \lim_{P\to 0} V(H,P) = \lim_{H\to\infty} V(H,P) = \lim_{P\to\infty} V(H,P) = +\infty,$$

the point (H^*, P^*) is the global minimum, i.e.,

$$V(H,P) > V(H^*,P^*) = \ln H^* + 1 + \frac{a_1 H^*}{a_2}(\ln P^* + 1) > 0$$

holds for all $H, P > 0$.

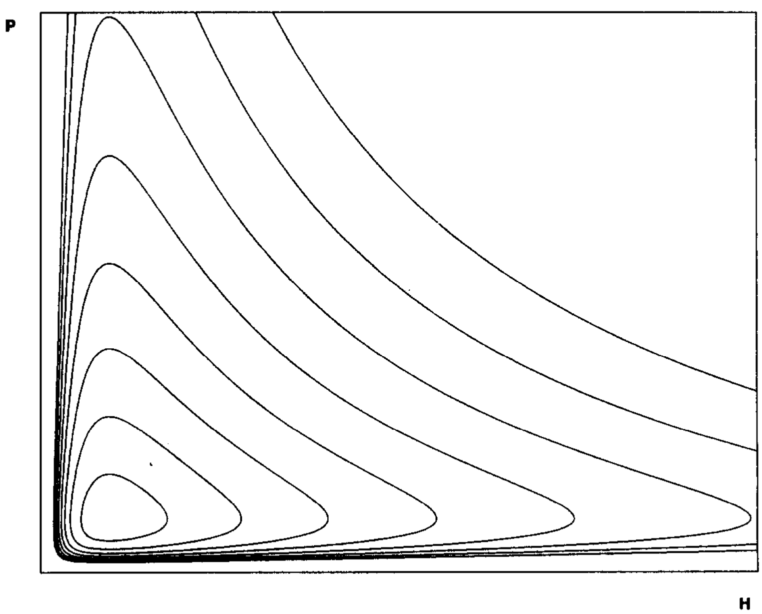

Figure 1. Level curves of the Lyapunov function $V(H,P)$.

Using equalities (4), we obtain that the derivative of the function $V(H,P)$ satisfies

$$\frac{dV}{dt} = r_1 - a_1P - b_1H - \frac{r_1H^*}{H} + \frac{a_1H^*P}{H} + b_1H^* + \frac{a_1H^*}{a_2}r_2 - \frac{a_1H^*P}{H} - \frac{a_1H^*P^*}{a_2P}r_2 + \frac{a_1H^*P^*}{H}$$
$$= -\frac{a_1}{P}\left(P^* - P\right)^2 - \frac{b_1}{H}\left(H^* - H\right)^2.$$

For the second model, $\frac{dV}{dt} < 0$ strictly for all $H, P > 0$ except the fixed point (H^*, P^*) where $\frac{dV}{dt} = 0$. Hence, the function $V(H,P)$ satisfies Lyapunov's asymptotic stability theorem, and the fixed point (H^*, P^*) of system (2) is globally stable.

For the first model, $b_1 = 0$, hence,

$$\frac{dV}{dt} = -\frac{a_1}{P}\left(P^* - P\right)^2.$$

In this case, the equality $\frac{dV}{dt} = 0$ holds on the set (the straight line)

$$M = \{(H,P) \mid P = P^*,\ H \in \mathbf{R}\}$$

and $\frac{dV}{dt} < 0$ off M. The fixed point (H^*, P^*) is the only invariant set of system (1) contained entirely in M. Consequently, the function $V(H,P)$ satisfies the asymptotic stability theorem (see [4, p. 28; 5, p. 58]), and by the theorem, the fixed point (H^*, P^*) of system (1) is globally stable as well. This completes the proof. ∎

REFERENCES

1. P.H. Leslie, Some further notes on the use of matrices in population mathematics, *Biometrika* **35**, 213–245, (1948).
2. P.H. Leslie, A stochastic model for studying the properties of certain biological systems by numerical methods, *Biometrika* **45**, 16–31, (1958).
3. E.C. Pielou, *Mathematical Ecology*, John Wiley & Sons, New York, (1977).
4. E.A. Barbashin, *Introduction to the Theory of Stability*, Wolters-Noordhoff, Groningen, (1970).
5. J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method*, Academic Press, New York, (1961).