Stability of ecological and epidemiological models via representation as Generalized Lotka-Volterra dynamics

A Preprint

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Abstract

Enter the text of your abstract here.

***K*eywords** Lotka-Volterra · Stability · Quasi-polynomial · Lyapunov function

1 Introduction

Blah, blah

2 Generalized Lotka-Volterra model

$$\dot{x}_i = x_i \left(r_i + \sum_{j=1}^{m} A_{ij} x_j \right) \tag{1}$$

3 Quasi-Polynomial systems

$$\dot{y}_i = y_i \left(\lambda_i + \sum_{j=1}^{m} M_{ij} \prod_{k=1}^{n} y_k^{B_{jk}} \right) \tag{2}$$

where λ is a vector of length n; M is a matrix of size $n \times m$ containing real coefficients, and B a matrix of size $m \times n$, also containing real coefficients. If $n = m$, and thus M is a square matrix, and $B = I_n$ the model reduces to the Generalized Lotka-Volterra model in Eq. 1 with $r = \lambda$ and $A = M$. If B contains only integers, Eq. 2 defines a *polynomial* system of differential equations; relaxing this condition to admit B composed of real numbers, we obtain a *quasi-polynomial* (QP) system.

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QP-representation of Leslie-Gower predator-prey model

$$\begin{cases} \dot{y}_1 = y_1(\rho_1 - y_1 - \alpha_1 y_2) \\ \dot{y}_2 = y_2 \left(\rho_2 - \alpha_2 \frac{y_2}{y_1}\right) \end{cases} \tag{3}$$

with y_1 being the prey, y_2 the predator, and all coefficients positive. The system differs from GLV in that we have a ratio between the predator and prey in the equation for the predator. The system is however in QP form, as seen by defining:

$$\lambda = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \quad M = \begin{pmatrix} -1 & -\alpha_1 & 0 \\ 0 & 0 & -\alpha_2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \tag{4}$$

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4 From Quasi-Polynomial to Generalized Lotka-Volterra

We define a set of m *quasi-monomials*:

$$x_j = \prod_{k=1}^{n} y_k^{B_{jk}} \tag{5}$$

Quasi-monomials for the Leslie-Gower model

For the Leslie-Gower model in Eq. 3 we identify three quasi-monomials:

$$\begin{cases} x_1 = y_1^1 \, y_2^0 = y_1 \\ x_2 = y_1^0 \, y_2^1 = y_2 \\ x_3 = y_1^{-1} \, y_2^1 = \frac{y_2}{y_1} \end{cases} \tag{6}$$

By chain rule, we have:

$$\begin{aligned} \dot{x}_j &= \sum_k B_{jk} \, \dot{y}_k \, y_k^{(B_{jk}-1)} \prod_{l \neq k} y_l^{B_{jl}} \\ &= \sum_k B_{jk} \, \frac{\dot{y}_k}{y_k} \prod_l y_l^{B_{jl}} \\ &= \sum_k B_{jk} \, \frac{\dot{y}_k}{y_k} x_j \\ &= x_j \sum_k B_{jk} \, \frac{\dot{y}_k}{y_k} \\ &= x_j \left(\sum_k B_{jk} \lambda_k + \sum_k B_{jk} \sum_l M_{kl} \prod_l y_l^{B_{kl}} \right) \\ &= x_j \left(\sum_k B_{jk} \lambda_k + \sum_k B_{jk} \sum_l M_{kl} x_l \right) \\ &= x_j \left((B\lambda)_j + \sum_l (BM)_{jl} x_l \right) \\ &= x_j \left(r_j + \sum_l A_{jl} x_l \right) \end{aligned} \tag{7}$$

where we have defined $A = BM$ and $r = B\lambda$, $(B\lambda)_j$ is the j^{th} element of the vector $B\lambda$ and $(BM)_{jl}$ is the coefficient in row j and column l of the matrix BM.

Via the quasi-monomial transformation in Eq. 5 we have turned the $n-$dimensional QP-system in Eq. 2 into an $m-$dimensional GLV system in Eq. 1.

GLV representation of the Leslie-Gower model

We can represent the Leslie-Gower model in Eq. 3 as a three-dimensional GLV model defined by the quasi-monomials in Eq. 6 and:

$$r = B\lambda = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_2 - \rho_1 \end{pmatrix} \quad A = BM = \begin{pmatrix} -1 & -\alpha_1 & 0 \\ 0 & 0 & -\alpha_2 \\ 1 & \alpha_1 & -\alpha_2 \end{pmatrix} \tag{8}$$

Note that A is rank deficient, given that the third row can be written as the difference between the second and first row. Rank-deficiency of A is expected whenever $m > n$ (as here, where we went from two to three equations).

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5 Examples of citations, figures, tables, references

You can insert references. Here is some text (Kour and Saabne 2014b, 2014a) and see Hadash et al. (2018).

The documentation for `natbib` may be found at

You can use custom blocks with LaTeX support from **rmarkdown** to create environment.

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf%7D`

Of note is the command `\citet`, which produces citations appropriate for use in inline text.

You can insert LaTeX environment directly too.

```
\citet{hasselmo} investigated\dots
```

produces

Hasselmo, et al. (1995) investigated...

`https://www.ctan.org/pkg/booktabs`

5.1 Figures

You can insert figure using LaTeX directly.

See Figure 1. Here is how you add footnotes. [^Sample of the first footnote.]

But you can also do that using R.

```
plot(mtcars$mpg)
```

You can use **bookdown** to allow references for Tables and Figures.

5.2 Tables

Below we can see how to use tables.

See awesome Table~1 which is written directly in LaTeX in source Rmd file.

You can also use R code for that.

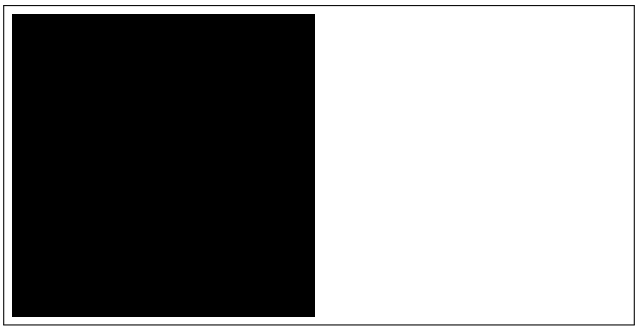

Figure 1: Sample figure caption.

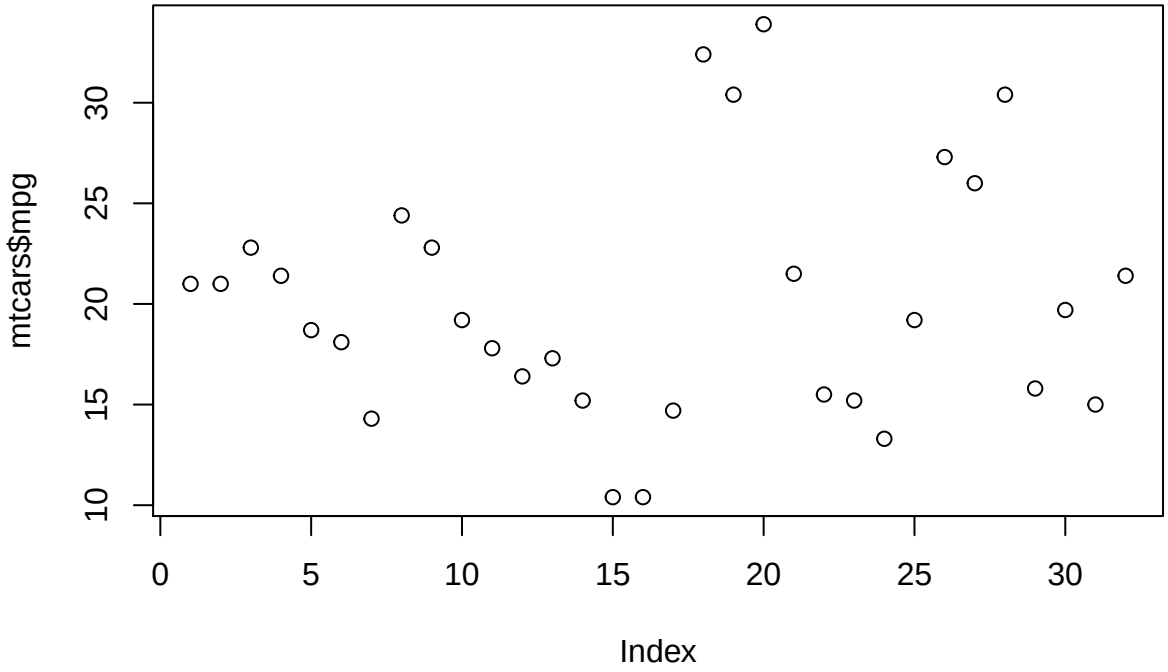

Figure 2: Another sample figure

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

```
knitr::kable(head(mtcars), caption = "Head of mtcars table")
```

Table 2: Head of mtcars table

	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
Mazda RX4	21.0	6	160	110	3.90	2.620	16.46	0	1	4	4
Mazda RX4 Wag	21.0	6	160	110	3.90	2.875	17.02	0	1	4	4
Datsun 710	22.8	4	108	93	3.85	2.320	18.61	1	1	4	1
Hornet 4 Drive	21.4	6	258	110	3.08	3.215	19.44	1	0	3	1
Hornet Sportabout	18.7	8	360	175	3.15	3.440	17.02	0	0	3	2
Valiant	18.1	6	225	105	2.76	3.460	20.22	1	0	3	1

5.3 Lists

- Item 1
- Item 2
- Item 3

Hadash, Guy, Einat Kermany, Boaz Carmeli, Ofer Lavi, George Kour, and Alon Jacovi. 2018. "Estimate and Replace: A Novel Approach to Integrating Deep Neural Networks with Existing Applications." *arXiv Preprint arXiv:1804.09028*.

Kour, George, and Raid Saabne. 2014a. "Fast Classification of Handwritten on-Line Arabic Characters." In *Soft Computing and Pattern Recognition (SoCPaR), 2014 6th International Conference of*, 312–18. IEEE.

———. 2014b. "Real-Time Segmentation of on-Line Handwritten Arabic Script." In *Frontiers in Handwriting Recognition (ICFHR), 2014 14th International Conference on*, 417–22. IEEE.