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Equivalence between nonlinear dynamical systems and urn processes

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Abstract

An equivalence is shown between a large class of deterministic dynamical systems and a class of stochastic processes, the balanced urn processes. These dynamical systems are governed by quasi-polynomial differential systems that are widely used in mathematical modeling while urn processes are actively studied in combinatorics and probability theory. The presented equivalence extends a theorem by Flajolet *et al* (2006 *Discrete Mathematics and Theoretical Computer Science, AG (DMTCS Proc.)* pp 59–118) already establishing an isomorphism between urn processes and a particular class of differential systems with monomial vector fields. The present result is based on the fact that such monomial differential systems are canonical forms for more general dynamical systems.

Keywords: nonlinear dynamical systems, urn processes, Lotka–Volterra systems, canonical forms

1. Introduction

In this work we establish a quite general equivalence between two apparently unrelated fields of mathematics, nonlinear differential dynamical systems and balanced urn processes. As will be shown in the sequel, to any balanced urn process corresponds an infinite equivalence class of nonlinear differential systems. The urn models are stochastic processes while the differential dynamical systems are deterministic. Although at first sight such a bridge between stochastic and deterministic processes could seem unlikely, the clue, however, is that while the time evolution of a stochastic process is random by definition, the time evolution of its probability density is deterministic. More precisely, as shown in the sequel, the evolution equation for the probability density of a balanced urn process is related to a wide class of differential dynamical systems that are used in many fields of mathematical modeling. These results opens the way to knowledge transfer between both fields of research.

Urn processes have first been introduced by Laplace [2] and later re-introduced and systematically studied for two-colours models by Pólya [3]. Nowadays, they are the object of an intense theoretical activity involving combinatorics and probability theory [4]. They provide a powerful modeling tool in many scientific domains such as statistical physics, population genetics, epidemiology, economy and some social phenomena such as innovation diffusion [5].

An urn process consists of three items: a box, the so-called urn, containing objects that can differ by some distinctive features, an infinite reservoir of such objects and a given set of replacement rules of the objects in the urn. The paradigm of such a process is an urn containing balls that only differ by their colours, with N possible colours. The replacement rules are the prescriptions that make the composition of the urn evolve at each discrete step. Thus, at each step a ball is randomly picked from the urn, with equal chance for all balls present in the urn. Its colour is noted and the ball is reintroduced in the urn. Depending on the colour of the picked ball, fixed numbers of balls of each colour are then transferred to the urn from the reservoir. These integer numbers, M_{ij}, with $i, j \in \{1, ..., N\}$ form a matrix, the replacement matrix. An entry M_{ij} means that if the ball drawn from the urn is of colour i, then one has to transfer M_{ij} balls of colour j from the reservoir into the urn, for each j. Some entries can be negative, in which case the balls of the corresponding colours are transferred from the urn to the reservoir at each step. Frequently, the entries are non-negative but if some of the diagonal entries are negative, conditions must be imposed in order to avoid blocking the process. The collection of the numbers of balls of each colour in the urn at a given step n forms its composition vector at that step, U_n and the sequence $(U_n; n \geqslant 0)$, where the initial composition U_0 of the urn is given, represents the urn process up to the nth step.

The equivalence property presented in this work is limited to the so-called balanced urn processes. For such processes, the total number of balls replaced at each step in the urn is independent of the colour i of the picked ball, that is, $\sum_{j=1}^{N} M_{ij} = \sigma$ for all i, where σ is called the balance. Flajolet and co-workers [1] showed an isomorphism between balanced urn processes and certain systems of autonomous ordinary differential equations with monomial vector fields.

Differential dynamical systems are systems of autonomous first order ordinary differential equations (ODEs) of the type

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f_i(x_1(t), ..., x_N(t)); i = 1, ..., n$$

where the x_i are real functions of the time t and the functions f_i can be quite general. The only constraint on the functions f_i are dictated by the fundamental laws of the phenomena to be modeled and by the existence and unicity of the solutions to the Cauchy problem. In most cases they are nonlinear functions. Such systems are ubiquitous in the mathematical modeling in physics, chemistry, biology, ecology, economical and social sciences [6–8]. Generally, due to their nonlinearity, these systems are non-integrable and present a rich diversity of behaviours. It is well known that their solutions are very sensitive to the functional form of the nonlinear functions f_i. Nevertheless, it has been shown that a large class of such systems can be brought to two canonical forms. This is shown in two steps.

The first step uses the property that many dynamical systems can be brought to the quasi-polynomial, also called generalized Lotka–Volterra, form [9, 10]:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = u_i \sum_{j=1}^{M} A_{ij} \prod_{k=1}^{N} u_k^{B_{jk}}$$

with $i \in \{1, .., N\}$ and where we omitted the dependence in t of the dependent real variables u_i. The matrices A and B are any rectangular matrices with real and constant entries.

The second step rests on the fact, independently discovered at least three times [11–13], that systems in the quasi-polynomial format and for which the variables u_i remain positive, can be transformed into two canonical forms. One of these forms corresponds to the well-known Lotka–Volterra systems commonly used in population dynamics. The other canonical form is a system of ODEs with monomial functions of the dependent variables in the right-hand-side that we called the monomial canonical form. This type of differential systems is much less known and used in models. It has, thus, been a surprise to us to discover a work [1] showing that it is related to the balanced urn stochastic processes. In the present work we extend the result of [1] showing the equivalence of these urn processes to a more general class of systems, the so called quasi-polynomial systems, and not less importantly, we show the equivalence to Lotka–Volterra systems, the relevance of this result lying in the vast litterature on the latter systems.

In the next section, we briefly summarize the derivation of the two canonical forms for quasi-polynomial systems. In section 3, we recall the proof of the isomorphism between balanced urn processes and the monomial differential systems. Section 4 is devoted to the statement and proof of our main result and to some of its consequences. Conclusions and perspectives are discussed in section 5.

2. Canonical forms

Let us consider the set of all dynamical systems that can be brought to the quasi-polynomial (QP) form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \qquad \text{for } \ i = 1, \ldots, n, \tag{2.1}$$

and such that all the functions $x_i(t)$ remain positive for all t. The rectangular matrices A and B have real and constant entries. N and n do not need to be equal. Let us call the above form $QP(A, B)$. Under the action of the following monomial transformations

$$x_i = \prod_{k=1}^{n} \tilde{x}_k^{C_{ik}} \qquad \text{for } \ i = 1, \ldots, n, \tag{2.2}$$

where C is any real, constant and invertible $n \times n$ matrix, the system (2.1) becomes

$$\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}t} = \tilde{x}_i \sum_{j=1}^{N} \tilde{A}_{ij} \prod_{k=1}^{n} \tilde{x}_k^{\tilde{B}_{jk}} \qquad \text{for } \ i = 1, \ldots, n, \tag{2.3}$$

with

$$\tilde{A} = C^{-1} A, \tag{2.4}$$

and

$$\tilde{B} = BC, \tag{2.5}$$

where the product is the matrix product. A transformation (2.2), thus, sends the system $QP(A,B)$ to the system $QP(C^{-1}A,BC)$.

From equations (2.4) and (2.5) it is obvious that

$$\tilde{B}\tilde{A} = BA \tag{2.6}$$

so that the $N \times N$ matrix $M = BA$ is an invariant of transformations (2.2). In other words, all the $QP(A,B)$ having the same product matrix $BA = M$ belong to a same equivalence class represented by this matrix. Moreover, the transformations (2.2) are diffeomorphisms and form a group.

Now, let us take as new variables the N monomials that appear in the right-hand-side of equation (2.1)

$$u_j = \prod_{k=1}^{n} x_k^{B_{jk}}, \tag{2.7}$$

for $j \in \{1, ..., N\}$ and calculate the time-derivative of these variables using equation (2.1). This yields

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = u_j \sum_{k=1}^{N} M_{jk} u_k, \tag{2.8}$$

with $j \in \{1, ..., N\}$ and $M = BA$. This is the first canonical form, the Lotka–Volterra (LV) system already known from population dynamics. In the latter field, the $N \times N$ matrix M is named the interaction matrix of the system. The meaning of the equivalence class related to the invariant matrix BA becomes now evident: the systems $QP(A,B)$ belonging to a given equivalence class are all equivalent to the same LV system (2.8) with the interaction matrix M equal to BA. Let us also notice that the above system corresponds to the $QP(M,I)$ system, where I is the $N \times N$ identity matrix

Clearly, if $N > n$ the phase-space associated to the LV system (2.8) is higher dimensional than the phase-space of the original QP system (2.1). This means that the variables u_j are not all independent and that the trajectories corresponding to the original system are confined to invariant subspaces in the N-dimensional positive orthant of the phase-space of equation (2.8). Conversely, for $n > N$ the transformation to the LV system corresponds to a reduction of the number of variables of the original $QP(A,B)$. More results on the transformation to the LV canonical form can be found in [14–16]. Let us just remark that for $N \neq n$, the transformation (2.7) is not bijective. In the case $N = n$, the transformation is a bijection and belongs to the group of monomial transformations (2.2).

The second canonical form is obtained in the following way. As the canonical form (2.8) is a N-dimensional QP system of the form $QP(M,I)$, we can apply on it a monomial transformation of type (2.2) . Provided M is invertible we can choose the form of the matrix C as $C = M$ yielding:

$$u_k = \prod_{j=1}^{N} \tilde{u}_j^{M_{kj}}. \tag{2.9}$$

Using relations (2.4) and (2.5) for $A = M$ and $B = I$ one, then, obtains

$$\frac{\mathrm{d}\tilde{u}_j}{\mathrm{d}t} = \tilde{u}_j \prod_{k=1}^{N} \tilde{u}_k^{M_{jk}}, \tag{2.10}$$

with $j \in \{1, ..., N\}$.

System (2.10) represents the second canonical form of interest here. Again, this means that all the $QP(A, B)$ systems such that their matrix product $BA = M$ are equivalent to this canonical form which can be noted $QP(I, M)$.

Let us summarize the above results in the following theorem, with $\dot{v} \equiv \mathrm{d}v(t)/\mathrm{d}t$:

Theorem 1. *All the $QP(A, B)$ systems with the same $N \times N$ product matrix $BA = M$ are equivalent to two canonical forms, $\dot{u}_j = u_j \sum_{k=1}^{N} M_{jk} u_k$, $j = 1, ..., N$, and $\dot{\tilde{u}}_j = \tilde{u}_j \prod_{k=1}^{N} \tilde{u}_k^{M_{jk}}$, $j = 1, ..., N$. The trajectories of all n-dimensional $QP(A, B)$ systems with the same product matrix BA are mapped one into the other by the transformations $x_i = \prod_{k=1}^{n} \tilde{x}_k^{C_{ik}}$, $i = 1, ..., n$ for any invertible real matrix C. Moreover, these transformations are diffeomorphisms and constitute a group.*

A direct derivation of equation (2.10) from the QP system (2.1) is also possible [14] with the condition that the $n \times N$ matrix A be of maximal rank [10]. In the case $N > n$, the trajectories of the original $QP(A, B)$ system are, as in the case of the first canonical form, restricted to invariants subspaces in the positive orthant of the N-dimensional phase-space associated to the second canonical form (2.10). We stress here the purely monomial dependence of the vector field of this canonical form. For a review on the properties of QP systems, we refer the reader to [20] .

3. Urn processes and monomial differential systems

In this chapter, we report the proof of a theorem obtained by Flajolet, Dumas and Puyhaubert [1] mentioned in the Introduction. We directly present the proof for N-colours urns as these authors already gave it for two-colours urns processes and assumed the proof for N colours to be obvious to the reader.

Let us consider an urn with balls of N possible colours. The process is characterized by a $N \times N$ replacement matrix M with integer entries. We assume an initial composition vector $U^0 = (u_{10}, ..., u_{N0})$ of the urn. A history of length n of the urn process is a succession of n replacement steps of the urn's content starting from the initial composition vector. It can be viewed as a trajectory of the urn's content in the N-dimensional space of compositions. Since the urn process is balanced, all histories of length n are equiprobable. Hence, the probability of finding in the urn at step n a composition vector $U = (u_1, ..., u_N)$ is given by the ratio of the number of histories of length n starting at U^0 and ending at U over the total number of possible histories of length n starting at U_0. This statement can be written as follows

$$P(U_n = U \mid U_0 = U^0) = \frac{[x_1^{u_1} ....x_N^{u_N} z^n]\, H(x_1, ..., x_N, u_{10}, ..., u_{N0}, z)}{[z^n]\, H(1, ..., 1, u_{10}, ..., u_{N0,}z)}, \tag{3.1}$$

with the following definitions. The central tool here is the counting generating function H defined as

$$H(x_1, ..., x_N, u_{10}, ..., u_{N0}, z) \equiv \sum_{n:=0}^{\infty} \sum_{u_{1:}=0}^{\infty} ... \sum_{u_N=0}^{\infty} H_n(u_{10}, ..., u_{N0}, u_1, ..., u_N) x_1^{u_1} ....x_N^{u_N} \frac{z^n}{n!}, \tag{3.2}$$

where $H_n(u_{10},...,u_{N0},u_1,...,u_N)$ denotes the number of histories that connect in n steps the initial urn's composition vector U^0 to an arbitrary composition vector U. The notation $[x^m]\,S(x)$ represents as usual the coefficient of x^m in the power series $S(x)$.

The association of a differential system with an urn process originates from the analogy between, on one side, the actions of differentiation and multiplication on monomials and, on the other, the replacement procedures in an elementary step of the urn process. As simple examples, consider the actions of the operators $\partial/\partial x$ and $x^{p+1}\partial/\partial x$ over the monomial x^m:

$$\frac{\partial}{\partial x}x^m = mx^{m-1} = x^0x...x + xx^0...x + ... + xx...x^0, \tag{3.3}$$

and

$$x^{p+1}\frac{\partial}{\partial x}x^m = mx^{m+p} = (x^1x...x + xx^1...x + ... + xx...x^1)x^p. \tag{3.4}$$

The action (3.3) can be interpreted in a combinatoric way as choosing an object x among m similar objects in any possible order and remove it. Whereas, equation (3.4) corresponds to choose an object x among a set of m similar objects in any order, keep it in the set and add to it p similar objects.

Assuming that the variable x_i, $i \in \{1,...,N\}$, represents balls of colour i, one can associate to a replacement matrix M the partial differential operator

$$D \equiv \sum_{i=1}^{N} x_1^{M_{i1}}x_2^{M_{i2}}...x_i^{M_{ii}+1}...x_N^{M_{iN}}\frac{\partial}{\partial x_i}. \tag{3.5}$$

In the same spirit, the monomial $x_1^{u_1}...x_N^{u_N}$ is associated to the composition vector of the urn U. Obviously, the action of D on that monomial

$$Dx_1^{u_1}...x_N^{u_N} = \sum_{i=1}^{N} u_i x_1^{u_1+M_{i1}}...x_N^{u_N+M_{iN}}, \tag{3.6}$$

gives all possible transitions in one step of the urn's composition that respect the prescriptions encoded in the replacement matrix M. As a consequence, the n-iterated action of D on the initial composition monomial $x_1^{u_{10}}...x_N^{u_{N0}}$ generates all the possible compositions of the urn after n steps starting from that initial composition. This can be expressed in terms of $H_n(u_{10},...,u_{N0},u_1,...,u_N)$, the numbers of histories in n steps linking the initial configuration vector U_0 with a given final composition U, as follows:

$$D^n x_1^{u_{10}}...x_N^{u_{N0}} = \sum_{u_1=0}^{\infty}...\sum_{u_N=0}^{\infty} H_n(u_{10},...,u_{N0},u_1,...,u_N)x_1^{u_1}....x_N^{u_N}. \tag{3.7}$$

From equations (3.2) and (3.7), we obtain the counting generating function:

$$H(x_1,...,x_N,u_{10},...,u_{N0},z) = \sum_{n=0}^{\infty} D^n x_1^{u_{10}}...x_N^{u_{N0}}\frac{z^n}{n!}, \tag{3.8}$$

which can be summed as

$$H(x_1,...,x_N,u_{10},...,u_{N0},z) = e^{zD}x_1^{u_{10}}...x_N^{u_{N0}}. \tag{3.9}$$

With these results at hand, one can now prove the isomorphism found by Flajolet and coworkers. Let us define the following system of ODEs:

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = X_i \prod_{j=1}^{N} X_j^{M_{ij}}, \tag{3.10}$$

with $i \in \{1, ..., N\}$ and where t is a real variable. M is a constant matrix with integer entries and with balance σ. As can be seen, the functions in the right-hand-side of this system are monomials. The total degrees of each of these monomials are all identical and equal to $\sum_{j=1}^{N} M_{ij} + 1 = \sigma + 1$.

Let us consider now any solution $X(t) = (X_1(t), ..., X_N(t))$ of system (3.10) and compute the derivative with respect to t of the monomial $X_1^{u_1}...X_N^{u_N}$ using equation (3.10):

$$\frac{\partial}{\partial t}(X_1^{u_1}...X_N^{u_N}) = \sum_{i=1}^{N} u_i X_1^{u_1+M_{i1}}...X_N^{u_N+M_{iN}}. \tag{3.11}$$

Obviously, this result correspond to the action (3.6) of the operator D defined in (3.5)

$$\frac{\partial}{\partial t}(X_1^{u_1}...X_N^{u_N}) = [Dx_1^{u_1}...x_N^{u_N}]_{\{x_i=X_i;i=1,...,N\}}, \tag{3.12}$$

and the nth iteration of the operator D gives

$$\frac{\partial^n}{\partial t^n}(X_1^{u_1}...X_N^{u_N}) = [D^n x_1^{u_1}...x_N^{u_N}]_{\{x_i=X_i;i=1,...,N\}}. \tag{3.13}$$

As our aim is to make the connection between the differential system (3.10) and the counting generating function H of an urn process, let us combine formula (3.8) and (3.13) to get

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}\frac{\partial^n}{\partial t^n}[X_1^{u_{10}}(t)...X_N^{u_{N0}}(t)] = X_1^{u_{10}}(t+z)...X_N^{u_{N0}}(t+z) = H(X_1(t), ..., X_N(t), u_{10}, ..., u_{N0}, z). \tag{3.14}$$

In the left hand side of the above equation, for t and z small enough, the Taylor series can be applied thanks to the Cauchy–Kovalevskaya theorem for the solutions of system (3.10) which ensures their existence and analyticity in the neighborhood of the origin. Now, taking $t = 0$ and z small enough for convergence in the above formula, along with $(X_1(0) = x_{10}, ..., X_N(0) = x_{N0})$, we obtain:

$$H(x_{10}, ..., x_{N0}, u_{10}, ..., u_{N0}, z) = X_1^{u_{10}}(z)...X_N^{u_{N0}}(z). \tag{3.15}$$

We thus state the theorem:

Theorem 2 (Flajolet, Dumas, Puyhaubert). *To any N-colour balanced urn process of replacement matrix M one can associate one and only one differential system of the form* $\dot{X}_i = X_i \prod_{j=1}^{N} X_j^{M_{ij}},\ j = 1, ..., N$. *Moreover, the counting generating function of the histories of the urn process with initial composition vector* $(u_{10}, ..., u_{N0})$ *is obtained in terms of the solution with initial condition* $(X_1(0) = x_{10}, ..., X_N(0) = x_{N0})$ *of the above differential system by the relation* $H(x_{10}, ..., x_{N0}, u_{10}, ..., u_{N0}, z) = X_1^{u_{10}}(z)...X_N^{u_{N0}}(z)$.

Note that, in order to avoid divergences in case of negative entries in M, the initial condition vector $(x_{10}, ..., x_{N0})$ of the solution to the differential system must be in the open positive orthant. The isomorphism between the monomial differential system (3.10) and the urn process with the same matrix M as replacement matrix is, thus, established.

4. General dynamical systems, urn processes and applications

4.1. Main result

A simple glance at equations (2.10) and (3.10) shows that they are strictly of the same form. The only restriction being the fact that while the matrix M can be quite general in (2.10), in the case of equation (3.10) its entries must be integers as they represent numbers of balls, positive when they are introduced and negative when they are extracted from the urn. Moreover, the matrix must be balanced. Nevertheless, the formal identity between both differential systems and the use of theorems 1 and 2 allows us to state the following theorem:

Theorem 3. *To any N-colour balanced urn process of replacement matrix M one can associate an infinite equivalence class of quasi-polynomial systems of form $QP(A,B)$ given by equation $\dot{x}_i = x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}$, $i = 1, ..., n$, restricted to the open N-dimensional orthant and with matrices A and B such that their matrix product $BA = M$. This association is given by $H(x_{10}, ..., x_{N0}, u_{10}, ..., u_{N0}, z) = X_1^{u_{10}}(z)...X_N^{u_{N0}}(z)$ and using the relations between the variables X_i, $i = 1, ..., N$, of the monomial canonical form and the variables x_i, $i = 1, ..., n$, of any $QP(A,B)$ system belonging to the equivalence class.*

As reported above, a wide class of dynamical systems can be brought to the QP form [9]. This permits to extend the above theorem to that whole class of dynamical systems.

The above theorem paves the way to new techniques for studying both urn processes and dynamical systems. As for finding solutions to an urn process, one can try to solve any $QP(A,B)$ dynamical system belonging to the equivalence class labeled by the replacement matrix M of this urn process. The solution of such a QP system can be transformed into a solution of the monomial canonical form and, thence, by applying equation (3.15) the generating function of the urn process is obtained. Even though most QP systems cannot be exactly solved due to their nonlinearity, many stability properties of their solutions are known [15, 17, 18]. This is due to the fact that being reducible to the Lotka–Volterra canonical form, there is a systematic method to construct Lyapunov functions for them [21]. More generally, the monomial transformation (2.9) connects the LV system to the monomial one and, furthermore, is a diffeomorphism in the open positive orthant. One can, thus, transfer all the mathematical properties that are preserved by that diffeomorphism from the first to the the second canonical form. Among these properties, chaotic regime have been systematically studied for the LV systems [19] and should be of interest for the associated urn process. Another result of interest for urn processes is the fact that the solutions of the LV systems can be expressed in terms of a Taylor series whose general coefficient is analytically known [20]. These series define new special functions. All these results can be transferred via transformation (2.9) to the solutions of the monomial differential system (3.10) and, afterwards, to the associated urn process via formula (3.15).

As a corollary one can state:

Corollary. *The solution to the N-colour balanced urn process with replacement matrix M and initial composition $\{u_{j0} = 0, j \neq i; u_{i0} = 1\}$ can be used to solve any $QP(A,B)$ system belonging to the equivalence class labeled by the matrix M. This solution is obtained via the relation*

$$X_i(z) = H(x_{10}, ..., x_{N0}, 0, ..., 1, ...0, z), \tag{4.1}$$

with $i = 1, ..., N$.

This provides the solution at time z of the canonical system $\dot{X}_i = X_i \prod_{j=1}^{N} X_j^{M_{ij}}$, $i = 1, ..., N$, with initial point $(x_{10}, ..., x_{N0})$. Next, using the relation between the variables X_i, $i = 1, ..., N$, of the monomial canonical system and the variables x_i, $i = 1, ..., n$, of the original $QP(A, B)$ system, one gets the solution of the latter.

4.2. Isomorphism between balanced urn processes and Lotka–Volterra dynamical systems

As an immediate consequence of theorems 1 and 2, we can state the following result:

Theorem 4. *Any N-colour balanced urn process of replacement matrix M is isomorphic to a Lotka–Volterra system of equations* $\dot{Y}_i = Y_i \sum_{j=1}^{N} M_{ij} Y_j$, $i = 1, ..., N$, *provided M is invertible.*

Proof. The inverse of transformation (2.9), written here as $Y_k = \prod_{j=1}^{N} X_j^{M_{kj}}$, $k = 1, ..., N$, provided the matrix M is invertible, maps the monomial canonical form (3.10) to the Lotka–Volterra form $\dot{Y}_i = Y_i \sum_{j=1}^{N} M_{ij} Y_j$, $i = 1, ..., N$. This proves the proposition. □

This theorem allows for transferring the abundant mathematical results attached to the LV systems to the theory of urn processes [21].

4.3. Example

Consider the three-colours urn process with balance $\sigma = 3$ and replacement matrix

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}. \tag{4.2}$$

By theorem 2, the monomial differential system to which it is equivalent is given by

$$\begin{aligned} \dot{X}_1 &= X_1^2 X_3^2, \\ \dot{X}_2 &= X_1 X_2^2 X_3, \\ \dot{X}_3 &= X_1^2 X_2 X_3. \end{aligned} \tag{4.3}$$

The above urn process is also equivalent, by theorem 4, to the following Lotka–Volterra differential system

$$\begin{aligned} \dot{X}_1 &= X_1(X_1 + 2X_3), \\ \dot{X}_2 &= X_2(X_1 + X_2 + X_3), \\ \dot{X}_3 &= X_3(2X_1 + X_2). \end{aligned} \tag{4.4}$$

Moreover, theorem 3 tells us that this process is also equivalent to an infinity of QP systems

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \sum_{j=1}^{3} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \qquad \text{for } i = 1, \ldots, n, \tag{4.5}$$

with A and B satisfying the constraint

$$BA = M. \tag{4.6}$$

The natural number n must satisfy the inequality $n \geqslant [N/2]$, where $[r]$ for a real number r is the smallest natural number that is larger than r. In the present case of a 3-colours urn process we have $n \geqslant 2$. This inequality is due to the fact that the matrix equation (4.6) corresponds to a system of N^2 multivariate quadratic polynomial equations while the number of unknowns, i.e. the entries of the $n \times N$ matrix A and those of the $N \times n$ matrix B, is $2nN$. The above inequality ensures that this system is not overdetermined.

4.4. Some consequences of theorems 2–4

4.4.1. Equivalence between a urn process with matrix M and a urn process with matrix $M - \mathrm{diag}(M)$. For that purpose, we have to introduce time reparametrizations. These transformations play an important role in QP theory. The general form (2.1) of a QP system is covariant under a time reparametrization of the form:

$$\mathrm{d}t = \prod_{i=1}^{n} x_i^{\beta_i} \mathrm{d}\tau, \quad \beta_i \in \mathbb{R}. \tag{4.7}$$

Under such a transformation the equation (2.1) becomes

$$\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}\tau} = \tilde{x}_i \sum_{j=1}^{N} \tilde{A}_{ij} \prod_{k=1}^{n} \tilde{x}_k^{\tilde{B}_{jk}} \qquad \text{for } i = 1, \ldots, n \tag{4.8}$$

with

$$\tilde{A} = A, \tag{4.9}$$

and

$$\tilde{B_{jk}} = B_{jk} + \beta_k, \qquad j = 1, ..., N;\ k = 1, ..., n. \tag{4.10}$$

Let us now apply this transformation with $\beta_i = -M_{ii}$, $i = 1, ..., N$, to the monomial system (3.10) associated to the urn process with replacement matrix M and balance σ. This yields

$$\frac{\mathrm{d}X_i}{\mathrm{d}\tau} = X_i \prod_{j=1}^{N} X_j^{\tilde{M}_{ij}} \quad ; i = 1, ..., N, \tag{4.11}$$

with $\tilde{M_{ij}} = M_{ij} - M_{jj}$. This differential system is associated to an urn process with replacement matrix $M - \mathrm{diag}(M)$ and with balance $\tilde{\sigma} = \sigma - \mathrm{Tr}(M)$. This allows us to state the following theorem:

Theorem 5. *Any urn process with matrix M and balance σ is isomorphic to a urn process with matrix $M - \mathrm{diag}(M)$ and balance $\sigma - \mathrm{Tr}(M)$.*

4.4.2. Dimensional reduction of the N-dimensional monomial differential system associated to a urn process with matrix M to a N − 1 dimensional differential system.

Theorem 6. *Any monomial differential system associated to a urn process with matrix M and with balance σ:*

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = u_i \prod_{j=1}^{N} u_j^{M_{ij}}\ , \quad i = 1, \ldots, N, \tag{4.12}$$

can be reduced to the $(N-1)$*-dimensional QP system*

$$\frac{\mathrm{d}y_i}{\mathrm{d}\tau} = y_i \left(\prod_{j=1}^{N-1} y_j^{M^*_{ij}} - 1 \right) \ , \quad i = 1, \ldots, N-1, \tag{4.13}$$

where τ *arises from a time reparametrization,* $M^*_{ij} = M_{ij} - M_{Nj}$*,* $j = 1, ..., N$*, and the* y_j *are defined in terms of the* u_j *by means of a monomial transformation.*

Proof. We first perform the time reparametrization $\mathrm{d}t = \left(\prod_{i=1}^{N} u_i^{-M_{Ni}}\right) \mathrm{d}\tau$ on the monomial system associated to the urn process of matrix M. The outcome is a system of the same form, now with exponent matrix:

$$\tilde{M} = \begin{pmatrix} M^*_{(N-1)\times N} \\ \hline O_{1\times N} \end{pmatrix} \tag{4.14}$$

where $M^*_{ij} = M_{ij} - M_{Nj},\ j = 1, ..., N$, and where the indices denote the sizes of each submatrix, while O means the null matrix. By construction, $\tilde{M}$ is a matrix of null balance, namely

$$\sum_{j=1}^{N} \tilde{M}_{ij} = 0 \ , \quad i = 1, \ldots, N. \tag{4.15}$$

This zero-balance property leads naturally to the introduction of the monomial transformation defined by the $N \times N$ regular matrix:

$$C = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 1 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}. \tag{4.16}$$

The result is a $QP(A^*, B^*)$ system with matrices

$$B^* = B \cdot C = \tilde{M} \cdot C = \left(\begin{array}{c|c} M^*_{(N-1)\times(N-1)} & O_{(N-1)\times 1} \\ \hline O_{1\times(N-1)} & 0 \end{array} \right) \tag{4.17}$$

and

$$A^* = C^{-1} \cdot A = C^{-1} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & -1 \\ 0 & 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & 0 & -1 \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}. \tag{4.18}$$

Accordingly, we are led to the reduced system

$$\frac{\mathrm{d}y_i}{\mathrm{d}\tau} = y_i\left(\prod_{j=1}^{N-1} y_j^{M^*_{ij}} - 1\right) , \quad i = 1, \ldots, N-1, \tag{4.19}$$

and the trivial quadrature

$$\frac{\mathrm{d}y_N}{\mathrm{d}\tau} = y_N. \tag{4.20}$$

This proves the theorem. □

The interpretation of the above result is that the constant balance is actually a parametric constraint that can be exploited within the operational framework of QP systems, leading to this dimensional reduction.

5. Conclusions and perspectives The equivalence between urn processes and QP dynamical systems leads to a wealth of consequences. The most evident among them are the transfer of knowledge between the extensive literature on Lotka–Volterra systems and the urn processes. As an example, one should exploit the fact that there exists a systematic method for constructing Lyapunov functions for fixed points of LV systems. Theorem 4, hence, allows us to apply these Lyapunov functions to the study of the stability of fixed points for urn processes. Conversely, results in balanced urn theory can be brought to the realm of LV and, more generally, QP systems. Urns for which the history counting generating function can be analytically found may be used to find the solutions to the associated QP systems.

A more speculative application would be the numerical simulation of dynamical systems via urn processes. Indeed, the computer simulation of drawing balls at random from an urn and removing or adding others could be quite fast. The probability of reaching a given composition of the urn after a given number of steps should easily be computed from the data and, in turn, the counting generating function H could be obtained by using relation (3.1). The latter provides the solution of the monomial canonical form (3.10) through formula (4.1). From that solution, the solution of any QP systems belonging to the same equivalence class could be computed. The difficulty, though, could appear in relation with the calculation of the generating function as the latter is given by an infinite series in term of the probability (see equations (3.1) and (3.2)).

To conclude, let us mention two open questions and a generalization concerning the equivalence between urn processes and differential systems. The first question is the generalization of the above equivalence to the case of unbalanced urn processes. In this case the fundamental relation (3.1) looses its validity and the combinatorial approach adopted here can no longer be used. The second question concerns the $QP(A, B)$ systems whose matrices A and B have non-integer entries and are such that their product $BA = M$ has non-integer entries. The question amounts to the possibility of generalizing the urn processes, up to now characterized by integer random variables, to processes that are similar but with real random variables.

As for the generalization mentioned above, after reading article [22] recommended by one of the referees, we realized that the equivalence between quasi-polynomial systems and balanced urns processes can be extended to urns with random entries in the replacement matrix M. This important generalization will be exposed in a future publication.

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