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Algebraic structures and invariant manifolds of differential systems

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Algebraic tools are applied to find integrability properties of ODEs. Bilinear nonassociative algebras are associated to a large class of polynomial and nonpolynomial systems of differential equations, since all equations in this class are related to a canonical quadratic differential system: the Lotka–Volterra system. These algebras are classified up to dimension 3 and examples for dimension 4 and 5 are given. Their subalgebras are associated to nonlinear invariant manifolds in the phase space. These manifolds are calculated explicitly. More general algebraic invariant surfaces are also obtained by combining a theorem of Walcher and the Lotka–Volterra canonical form. Applications are given for Lorenz model, Lotka, May–Leonard, and Rikitake systems. © *1998 American Institute of Physics.*
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I. INTRODUCTION

Given a dynamical system it is relevant to determine if the system is integrable. This can be achieved by obtaining constants of motion. If the system is not integrable, the existence of a constant of motion gives substantial information about its solutions. If the system depends on parameters, it is useful to obtain the parameter values for which such constants of motion exist. Even though this is a difficult problem some methods offer a partial answer. We can cite the Painlevé analysis,[1–4] the Lie symmetry method,[5–8] and the compatibility analysis.[9]

In this paper we present a new approach based on an algebraic theory of differential equations initiated by L. Markus[10] and allow for a systematic study of quasi-polynomial dynamical systems. The fundamental contribution of Markus has been the discovery of a link between systems of differential equations with homogeneous quadratic polynomial vector fields and nonassociative commutative algebras.[11] To such a differential system $\dot{x} = p(x)$ defined on an *n-dimensional* vector space V one can associate an algebra with a bilinear multiplication operation as defined below.

Markus showed that two quadratic differential systems are linearly equivalent if and only if their respective associated algebras are isomorphic. He also established a direct connection between geometrical objects such as invariant hyperplanes in V and subalgebras belonging to the above described algebras. Thus, certain properties of the flows generated by differential equations can be studied by investigating the structure of the associated algebras.[12–14]

These concepts have been generalized to polynomial differential systems with degrees higher than 2.[12–14] To a differential system with homogeneous polynomial vector field of degree $m > 1$ may be associated a nonassociative commutative m-ary algebra. Their subalgebras are also related to invariant hyperplanes in the phase space of the differential system.

Nonhomogeneous polynomial differential systems[13,14] may also be related to such nonassociative algebras by a straightforward embedding. However, these more general nonassociative algebras are less popular than the bilinear algebras since calculations may become very complex when $m > 2$.

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In this article we present a different generalization of the ideas of Markus. We show that bilinear algebras may be associated to any polynomial (even of degree larger than 2 and nonhomogeneous) differential systems. This property is even deeper since it applies also to a larger class of differential systems including ''monomials'' with noninteger exponents in their vector fields, the so-called quasi-polynomial systems for which the previous algebras do not exist. This association may be established thanks to the fact that any differential system in the quasi-polynomial class of equations may be related to a universal quadratic canonical form, the Lotka–Volterra system.[15–19] This canonical form is also combined with a theorem due to Walcher relating ideals of polynomials and invariant surfaces. These concepts lead to a constructive theory of very general invariant surfaces and invariants for the quasi-polynomial class of ordinary differential equations (ODEs).

The article is divided into four sections plus four appendices. In Sec. II we present the association of bilinear Lotka–Volterra algebras to general quasi-polynomial differential systems. In Sec. III, the general properties of the Lotka-Volterra nonassociative algebras are studied. A general algorithm is constructed for analyzing the subalgebras. It is illustrated by classifying the 2.3-dimensional Lotka–Volterra algebras. The method presented in Sec. III is applied on two examples: May–Leonard and Lorenz. In Sec. IV the conditions for the existence of algebraic invariant surfaces are obtained and applied on four systems: (α, β, δ) Lotka–Volterra, May–Leonard, Lorenz, and Rikitake. New integrability cases are found. The results of this method are given in the four tables of Appendix D.

II. DIFFERENTIAL EQUATIONS AND NONASSOCIATIVE ALGEBRAS

Let us define the class of quasi-polynomial differential equations as follows:

$$\dot{x}_i = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}, \quad i = 1, \dots, n, \tag{1}$$

which is defined on R^n or C^n and where the dot means time derivative.

In Eq. (1), A and B are real or complex, constant and rectangular matrices. Moreover, the number m is related to the number of monomials appearing in the vector field of Eq. (1). Thus, m is generally different from n. The use of the term ''quasi-monomial'' for $\prod_{k=1}^{n} (x_k)^{B_{jk}}$ is, of course, more accurate since the exponents B_{jk} may be real or complex. This explains the denomination of quasi-polynomial differential equation. These equations are ubiquitous in physics, chemical physics, and biomathematics.

The class (1) includes all polynomial differential equations. Indeed, the factor x_i in front of the right side in Eq. (1) may disappear by cancellation with some factors included in $\prod_{k=1}^{n} (x_k)^{B_{jk}}$. The factor x_i is extracted in Eq. (1) in order to yield certain useful properties for the matrices A and B. Moreover, linear terms may be hidden in the expression (1) depending on the values of certain entries of the B matrix. In order to give a more systematic idea of the domain of applications of the equations of type (1), let us remark that many equations with right-hand sides more general than quasi-polynomials [like sin (x), exp (x),...] may be brought to form (1) by an appropriate embedding.[20–22]

A first step in establishing a connection between Eq. (1) and bilinear nonassociative commutative algebra (NACA) is based on the following lemma:

Lemma 1: Any differential system of class (1) is associated to a homogeneous Lotka–Volterra system.

Proof: One way to exhibit this property is to consider the time evolution of all the quasi-monomials in Eq. (1). Thus, let us define new variables $U_j = \prod_{k=1}^{n} (x_k)^{B_{jk}}$, $j = 1, \dots, m$. The differential equations obeyed by these variables can easily be derived from Eq. (1) and read

$$\dot{U}_i = U_i \sum_{l=1}^{m} M_{jl} U_l, \quad j = 1, \dots, m, \tag{2}$$

where $M = BA$ (matrix product). The same result has been shown earlier by using quasi-monomials transformations and an appropriate embedding.[15,16]

Thus, Eq. (1) can be recast in the following system:

$$\dot{x}_i = x_i \sum_{j=1}^{n} A_{ij} U_j, \quad \dot{U}_j = U_j \sum_{l=1}^{m} M_{jl} U_l, \quad i=1,...,n, \quad j=1,...,m. \tag{3}$$

Since the first equation in this system is linear in x, the essential properties of the solutions of Eq. (1) are already embedded in Eq. (2) which is closed in U.

We want to stress here that Eq. (2) is a homogeneous quadratic equation but not the most general one which would be given by

$$\dot{V}_i = \sum_{j,k=1}^{m} C_{ijk} V_j V_k, \quad i=1,...,m. \tag{4}$$

In fact, Eq. (2), the so-called homogeneous Lotka–Volterra equation,[23–25] is a particular case of (4) which amounts to choosing the tensor C as follows:

$$C_{ijk} = \tfrac{1}{2} (\delta_{ik} M_{kj} + \delta_{ij} M_{jk}). \tag{5}$$

We are now ready for the second step, i.e., the connection of (1) with a nonassociative commutative algebra (NACA).[26,27] This is made along the definition given by Markus in 1960[10] for homogeneous quadratic differential equations:

Definition 1: To a homogeneous quadratic differential system of type (4), where C_{ijk} is symmetric in j,k, one can associate an algebra based on the vector space R^n or C^n and whose multiplication is defined for a basis $e_1,...,e_m$ as follows:

$$e_i \cdot e_j = \sum_{k=1}^{m} C_{kij} e_k. \tag{6}$$

We may thus apply this definition to Eq. (2) which is homogeneous and quadratic. This leads to the following theorem.

Theorem 1: *Differential equations with quasi-polynomial vector fields (1) are associated to bilinear NACAs, whose structure tensor is given by (5), where $M=BA$.*

We would like to stress at this level that in previous works differential systems with homogeneous polynomial vector fields of degree k were only associated to a k-ary NACA.[12–14] Furthermore, such algebras are not defined whenever noninteger exponents appear in the vector field.

In contrast, we have shown here the following: *any quasi-polynomial differential system is associated to a bilinear NACA.*

To end this section, let us remark that the bilinear NACAs we found have a universal structure based on an object which is a matrix: the Lotka–Volterra interaction matrix M. All differential systems of type (1) with matrices A and B such that the product BA is equal to the same matrix M have the same bilinear algebra. They belong to the same equivalence class.

III. STRUCTURE OF LOTKA–VOLTERRA ALGEBRAS

A. Definitions and theorems

The qualitative behavior of the solutions to the systems belonging to the class (1) will now be analyzed through their associated Lotka–Volterra algebras. Indeed, Markus[10] proved two basic theorems for quadratic differential systems. They are the following:

Theorem 2: *Two quadratic differential systems are equivalent under a nonsingular linear transformations iff their related NACAs are isomorphic.*

Theorem 3: *A quadratic m-dimensional differential system has an invariant r-plane $(1 \leqslant r \leqslant m)$ through the origin iff the related NACA has an r-dimensional subalgebra.*

These results may be directly applied to the Lotka–Volterra equation associated to a given quasi-polynomial system. We consequently may find invariant r-planes for this Lotka–Volterra

equation. Now, how are these r-planes related to the geometry of the phase portrait of the original equation (1)? To clarify this, let us first write the equation of an r-plane passing through the origin in the m-dimensional phase space of the Lotka–Volterra system (2):

$$\sum_{j=1}^{m} l_{ij}U_j = 0, \quad i=1,...,m-r.$$

It is an invariant r-plane if the coefficients l_{ij} satisfy the existence conditions of a r-dimensional subalgebra of the Lotka–Volterra NACA. Next, we know that the U_i are quasi-monomials in the variable x_i of the equation (1):

$$U_i = \prod_{k=1}^{n} x_k^{B_{ik}}, \quad i=1,...,m.$$

We have thus shown the following result: *the invariant surface for the original system (1) associated to the existence of an r-dimensional subalgebra of the Lotka–Volterra NACA is an r-dimensional manifold:*

$$\sum_{j=1}^{m} l_{ij}\prod_{k=1}^{n} x_k^{B_{jk}} = 0, \quad i=1,...,m-r. \tag{7}$$

The geometric properties of the phase portrait for (1) can thus be classified along the structures of the Lotka–Volterra NACAs. We propose now to carry on this algebraic classification. First of all we must introduce some useful definitions:

Definition 2: Let us define a basis of the Lotka–Volterra algebra—LV-NACA—as the set of linearly independent vectors $e_1,...,e_m$ for which the multiplication is defined by Eq. (6) and tensor (5). We shall call axial algebras the one-dimensional subalgebras generated by each of the e_i. $\langle e_i \rangle$.

Definition 3: The algebras $\langle e_{i_1},...,e_{i_p} \rangle$ will be called faces or facial algebras. Here the indices $i_1,...,i_p$ are all different and take the values $1,...,m$.

Definition 4: We define $v_{(i_1,...,i_p)}$ as the generator of a one-dimensional subalgebra which belongs to the face $\langle e_{i_1},...,e_{i_p} \rangle$ and does not belong to any subface belonging to the face $\langle e_{i_1},...,e_{i_p} \rangle$.

Since $v_{(i_1,...,i_p)}$ generates a one-dimensional algebra, it must satisfy

$$v_{(i_1,...,i_p)} \cdot u_{(i_1,...,i_p)} = \lambda v_{(i_1,...,i_p)}, \tag{8}$$

where $\lambda \in C$. When $\lambda = 0$, $v_{(i_1,...,i_p)}$ is called a *nilpotent* element; if $\lambda = 1$, $v_{(i_1,...,i_p)}$ is an *idempotent* element.

Definition 5: The $p \times p$ submatrix of M obtained by erasing the lines and columns of indices not in $\{i_1,...,i_p\}$ will be denoted $M^{(i_1,...,i_p)}$.

Definition 6: The matrix obtained by replacing the column i of a matrix N by a column where all entries are 1 is denoted $N^{(i)}$.

Another useful property concerns the multiplication rule. Let us compute the product of two elements of the algebra

$$w_1 \cdot w_2 = \left(\sum_{i=1}^{m} w_{1i}e_i \right) \cdot \left(\sum_{j=1}^{m} w_{2j}e_j \right).$$

Hence

$$w_1 \cdot w_2 = \sum_{i,j=1}^{m} w_{1i}w_{2j}e_i \cdot e_j$$

and finally

$$w_1 \cdot w_2 = \sum_{k=1}^{m} \left[w_{1k}(Mw_2)_k + w_{2k}(Mw_1)_k \right] e_k, \tag{9}$$

where Mw_i means the action of matrix M on vector w_i. We now state two fundamental theorems:

Theorem 4: *Let us consider* $N = M^{(i_1,...,i_p)}$. *Two cases are possible: (a) If* det $(N) \neq 0$ *and* det $(N^{(i)}) \neq 0$, *then a one-dimensional subalgebra generated by* $v_{(i_1,...,i_p)}$ *exists. (b) If* det $(N) = 0$, *then the kernel of N generates a nilpotent subalgebra (called in the following nul-algebra).*

Proof: Let us expand $v_{(i_1,...,i_p)}$ in the basis $e_{i_1},...,e_{i_p}$:

$$v_{(i_1,...,i_p)} = \sum_{k=1}^{p} \gamma_k e_{i_k}, \tag{10}$$

where by definition $\gamma_1,...,\gamma_p$ are all different of zero. We now insert Eq. (10) into Eq. (8), yielding

$$\sum_{k,l=1}^{p} \gamma_l \gamma_k (e_{i_k} \cdot e_{i_l}) = \lambda \sum_{r=1}^{p} \gamma_r e_{i_r}. \tag{11}$$

We use formula (9) for the left-hand side of (11) and obtain

$$(M^{(i_1,...,i_p)} \gamma)_i = \frac{\lambda}{2}, \quad i = 1,...,p, \tag{12}$$

where γ is the vector $(\gamma_1,...,\gamma_p)$. From (12) it is clear that if det $(M^{(i_1,...,i_p)}) \neq 0$, then necessarily $\lambda \neq 0$ and the vector γ is uniquely determined. The solution of (12) is given by

$$\gamma_i = \frac{\det (N^{(i)})}{\det (N)} \frac{\lambda}{2}. \tag{13}$$

Otherwise, if det $(M^{(i_1,...,i_p)}) = 0$, the kernel of $M^{(i_1,...,i_p)}$ generates a nilpotent algebra. Some comments on these results are here necessary:

(1) The property of idempotence or nilpotence depends entirely on the rank of the corresponding submatrix $M^{(i_1,...,i_p)}$.
(2) When $M^{(i_1,...,i_p)}$ is not of maximal rank, then its kernel is itself a nul-algebra.
(3) In some exceptional but evident cases the condition det $(M^{(i_1,...,i_p)}) = 0$ may not necessarily imply $\lambda = 0$.

This result exhausts all the possible structures of the one-dimensional subalgebras of a LV-NACA. The second fundamental theorem is the following:

Theorem 5: *A p-dimensional subalgebra of the LV-NACA is always generated by a set of one-dimensional subalgebras.*

Proof: The proposition is clearly true for Lotka–Volterra algebras of dimension 1. Let us suppose the theorem to be true for a Lotka–Volterra with dimension $m-1$. Now, let us consider a p-dimensional subalgebra $B_{(p)}$ which belongs to an m-dimensional LV-NACA. It is easy to see that the intersection between $B_{(p)}$ and any face in this LV-NACA is a $(p-1)$-dimensional subalgebra belonging to the face. However, any face is a LV-NACA of dimension $m-1$, hence this intersection is generated by $(p-1)$ one-dimensional subalgebras. This shows that there exists a set of one-dimensional subalgebras which generates $B_{(p)}$.

Those two theorems provide an algorithm for constructing the existence conditions of any $(m-r)$-dimensional subalgebra of a LV-NACA of dimension m.

Indeed, let us consider an $(m-r)$-dimensional subalgebra $B_{(m-r)}$ and the facial algebra $\langle e_1,...e_{m-1} \rangle$. Then

$$B_{(m-r)} \cap \langle e_1,...,e_{m-1} \rangle = A_{(m-r-1)},$$

which is an $(m-r-1)$-dimensional subalgebra of the face.

By construction, there always exist $r+1$ axial vectors which are transversal to $A_{(m-r-1)}$. These vectors constitute a subface $\langle e_{i_1},\ldots,e_{i_r},e_m \rangle$ with $i_1,\ldots,i_r \in \{1,\ldots,m-1\}$. Now, the intersection

$$B_{(m-r)} \cap \langle e_{i_1},\ldots,e_{i_r},e_m \rangle$$

is a one-dimensional subalgebra generated by a vector $v_{(j_1,\ldots,j_p)}$ (with $p \leq r+1$ and $j_1,\ldots,j_p \in \{i_1,\ldots,i_r,m\}$). Thus

$$B_{(m-r)} = \langle A_{(m-r-1)}, v_{(j_1,\ldots,j_p)} \rangle.$$

If $A_{(m-r-1)}$ is known, this yields $B_{(m-r)}$.

We now illustrate this algorithm on 2-dimensional and 3-dimensional algebras.

B. Two-dimensional LV-NACAs

Let us construct the conditions for the existence of one-*dimensional* nonaxial algebras for a two-dimensional LV-NACA. By definition, the generator satisfies

$$v_{(1,2)} \cdot v_{(1,2)} = \lambda v_{(1,2)}.$$

Here $v_{(1,2)} = w_1 e_1 + w_2 e_2$.

Using formula (9) we obtain

$$w_1(Mw)_1 = \frac{\lambda}{2} w_1, \quad w_2(Mw)_2 = \frac{\lambda}{2} w_2, \tag{14}$$

where M is a 2×2 matrix and w_1 and w_2 are nonvanishing (if this would not be the case, the one-dimensional subalgebra would be axial). Now, there are two possibilities:

(1) $\det(M) \neq 0$: In this case the solution of (14) is the generator of a one-dimensional subalgebra:

$$v_{(1,2)} = (M_{12} - M_{22})e_1 + (M_{21} - M_{11})e_2, \tag{15}$$

where $(M_{12} - M_{22})(M_{21} - M_{11}) \neq 0$.

(2) $\det(M) = 0$: Here again, there are two possibilities:

(a) $(M_{12} - M_{22})(M_{21} - M_{11}) \neq 0$. Here, formula (8) is still true but $v_{(1,2)} \cdot v_{(1,2)} = 0$.

(b) $M_{12} = M_{22}$ and $M_{21} = M_{11}$. Then w_1 and w_2 are arbitrary and $v_{(1,2)} = w_1 e_1 + w_2 e_2$ is any vector. There is an infinite family of one-dimensional subalgebras.

C. Three-dimensional LV-NACAs

We first analyze the one-dimensional subalgebras. The possible generators are, in this case, e_1, e_2, e_3, $v_{(1,2)}$, $v_{(1,3)}$, $v_{(2,3)}$, and $v_{(1,2,3)}$. The algebras generated respectively by e_1, e_2, e_3 are obvious. As to the generators $v_{(1,2)}, v_{(1,3)}, v_{(2,3)}$, the analysis is the same as for the two-dimensional LV-NACA with the matrix M successively replaced by $M^{(1,2)}$, $M^{(2,3)}$, and $M^{(1,3)}$. Finally, concerning the one-dimensional algebras generated by $v_{(1,2,3)} = w_1 e_1 + w_2 e_2 + w_3 e_3$ (with $w_1, w_2, w_3 \neq 0$), we have

$$Mv_{(1,2,3)} = \frac{\lambda}{2}. \tag{16}$$

Hence, there are again two cases:

(a) $\det(M) \neq 0$: thus, $v_{(1,2,3)} = M^{-1} \frac{\lambda}{2}$. This yields a one-dimensional non-nul-algebra.

(b) $\det(M) = 0$: in this case $\lambda = 0$. If rank $(M) = 2$, there exists a one-dimensional nul-algebra, and if rank $(M) = 1$, there exists a two-dimensional nul-algebra. This closes the analysis of the one-dimensional subalgebras.

Next, we study the two-dimensional subalgebras of a three-dimensional LV-NACA. Clearly, we may limit the existence analysis to the nonfacial two-dimensional subalgebras. We now apply the second fundamental theorem to this case: let B_2 be a two-dimensional subalgebra. Then, the intersection $B_2 \cap \langle e_1, e_2 \rangle = A_1$ may be generated by three possible generators: e_1, e_2, and $v_{(1,2)}$. For each of these generators we may proceed as follows: we consider the two-dimensional facial subalgebra $\langle e_i, e_3 \rangle$ $(i=1,2)$ which excludes A_1. The last step in building the two-dimensional subalgebras amounts to taking $B_2 \cap \langle e_i, e_3 \rangle$, which yields as possible generators e_3 or $v_{(i,3)}$. Thus, the admitted two-dimensional subalgebras are $B_2 = \langle A_1, e_3 \rangle$ or $B_2 = \langle A_1, v_{(i,3)} \rangle$, $i=1,2$.

Let us now explicitly list these B_2 subalgebras (excluding from it the facial subalgebras) and their respective existence conditions:

$$B_2 = \langle e_1, v_{(2,3)} \rangle, \quad M_{21} = M_{31},$$

$$B_2 = \langle e_2, v_{(1,3)} \rangle, \quad M_{12} = M_{32}, \tag{17}$$

$$B_2 = \langle v_{(1,2)}, e_3 \rangle, \quad M_{13} = M_{23},$$

$$B_2 = \langle v_{(1,2)}, v_{(2,3)} \rangle,$$

$$(M_{12} - M_{22})(M_{23} - M_{33})(M_{31} - M_{11}) + (M_{13} - M_{33})(M_{21} - M_{11})(M_{32} - M_{22}) = 0.$$

The above considerations yield the existence conditions for all the possible subalgebras of a three-dimensional LV-NACA. It should be stressed that these conditions are all expressed in terms of the matrix $M = BA$ of the quasi-polynomial system (1). We end this chapter with two applications.

D. Applications

We apply the above method to the May–Leonard and Lorenz systems. For both examples the Lotka–Volterra matrix $M = BA$ is easily computed. For the May–Leonard system the M matrix is 4×4, whereas for the Lorenz system M is 5×5. The algebraic structures of both systems are to be obtained as follows.

1. May–Leonard system

The May–Leonard system[28] is given by

$$\dot{x}_1 = l_1 x_1 - x_1(x_1 + \alpha x_2 + \beta x_3),$$

$$\dot{x}_2 = l_2 x_2 - x_2(\beta x_1 + x_2 + \alpha x_3), \tag{18}$$

$$\dot{x}_3 = l_3 x_3 - x_3(\alpha x_1 + \beta x_2 + x_3).$$

It can be cast in the Lotka–Volterra form using the monomials $U_1 = 1$, $U_2 = x_1$, $U_3 = x_2$, and $U_4 = x_3$. The corresponding M matrix as defined in Eq. (2) is given by

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ l_1 & -1 & -\alpha & -\beta \\ l_2 & -\beta & -1 & -\alpha \\ l_3 & -\alpha & -\beta & -1 \end{bmatrix}. \tag{19}$$

The generators of one-dimensional sub-algebras are obtained solving equation (12) and using Theorem 3. The results are summarized in Table I (Appendix A) for the case (1) and (2) (a) in Sec. III B corresponding to face $U_1 = 0$. The degenerate case $(\alpha = \beta = 1)$, which corresponds to case (2)(b) in Sec. III B for the face $U_1 = 0$, has a different structure as shown in Table II (Appendix A).

Two- and three-dimensional subalgebras existence conditions are given in Tables IV and VI (Appendix B) for the general and degenerate cases, respectively. For the degenerate case the two- and three-dimensional subalgebras are given in Table V (Appendix B). We obtained one- and

two-dimensional invariant manifolds for the May–Leonard system for the parameters values given in the tables of Appendix B. This is, to our knowledge, a result not found in the literature.

For the three-dimensional subalgebras the existence conditions reported in tables of Appendix B correspond to similar result by Cairo and Feix[29] who obtained algebraic equations for the components of M. In contrast to the method of Cairo and Feix our approach yields all the solutions of these equations.

2. Lorenz system

The Lorenz system[30,31] reads

$$\dot{x}_1 = \sigma(x_2 - x_1),$$

$$\dot{x}_2 = \rho x_1 - x_2 - x_1 x_3, \tag{20}$$

$$\dot{x}_3 = -\beta x_3 + x_1 x_2.$$

The Lotka–Volterra form is obtained using the quasi-monomials $U_1 = 1$, $U_2 = x_1^{-1} x_2$, $U_3 = x_1 x_2^{-1}$, $U_4 = x_1 x_2^{-1} x_3$, and $U_5 = x_1 x_2 x_3^{-1}$. Its form is given by Eq. (2) with the following matrix:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sigma - 1 & -\sigma & \rho & -1 & 0 \\ 1 - \sigma & \sigma & -\rho & 1 & 0 \\ 1 - \sigma - \beta & \sigma & -\rho & 1 & 1 \\ \beta - 1 - \sigma & \sigma & \rho & -1 & -1 \end{bmatrix}. \tag{21}$$

In our analysis the parameters σ, ρ, and β are considered as nonvanishing.

The generator of one-dimensional subalgebras are given in Table III (Appendix A). The higher dimensional subalgebras are given in Tables VII (Appendix C). The subalgebras existence conditions are given in Table VIII (Appendix C).

IV. ALGEBRAIC INVARIANT SURFACES

As seen above, the LV-NACA's subalgebras yield invariant manifolds whose equations may only contain quasi-monomials already appearing in the vector field of Eq. (1). In this section, we briefly present a further extension of the algebraic method in order to study a larger class of invariant manifolds including more general quasi-monomials.

We define the flow of the Lotka–Volterra system (2) by the function $U(U_0, t)$ which satisfies (2) and such that $U(U_0, 0) = U_0$. In fact, $U(U_0, t)$ is the solution curve to the system (2) with initial conditions U_0. We say that a surface $S \subset R^m$ is an invariant surface of the Lotka–Volterra system (2) if, for any $U_0 \in S$, $U(U_0, t) \in S$ for all t.

Now, we denote the ring of all polynomial functions by Pol $[U]$. We consider a surface $S \subset \mathbb{R}^m$, an ideal $I \subset$ Pol $[U]$, and define two special sets: (a) the ideal $J(S) = \{f \in$ Pol $[U]/f(S) = 0\}$ and (b) the surface $S(I) = \{U \in \mathbb{R}^m / \forall f \in I, f(U) = 0\}$.

According to the above definitions and using results due to Walcher[14] we can state the following theorem.

Theorem 6: *For any autonomous polynomial dynamical system we have the following properties:*

(a)   *If $(d/dt)(I) \subset I$, then $S(I)$ is an invariant surface.*
(b)   *If $S \subset \mathbb{R}^m$ is an invariant surface, then $(d/dt)(J(S)) \subset J(S)$.*

Here, we remember that

$$\frac{d}{dt} = \sum_{i=1}^{m} \left(U_i \sum_{j=1}^{m} M_{ij} U_j \right) \frac{\partial}{\partial U_i}$$

is a differential operator and $\dot{f}=(d/dt)(f)$ is the derivative of f along the flow of the Lotka–Volterra system (2).

We know that any ideal $I \subset \text{Pol}\,[U]$ is finitely generated. Hence, if $I=\langle P_1,...P_q\rangle$, then the surface $S(I)$ is defined by the system of q polynomials equations

$$P_i(U_1,...,U_m)=0, \quad i=1,...,q. \tag{22}$$

Here $S(I)$ is a surface of dimension $m-q$. In fact, the polynomials invariant surfaces S given by Eqs. (22) generalize the invariant hyperplanes presented in Sec. IV. Indeed, the invariant hyperplanes correspond to the case where the P_i are linear.

If there exists a polynomial $P(U)$ satisfying the equation $\dot{P}=\lambda P$, with $\lambda \in \text{Pol}\,[U]$, then, by Theorem 6, we conclude that the surface, given by $P(U)=0$, is an $(m-1)$-dimensional invariant surface for the system (2). In this case $P(U)$ is called a semi-invariant or a Darboux polynomial[32,33] and we denote the set of all semi-invariants of a given Lotka–Volterra system by $SI[U]$. Moreover, it can be shown that λ (see Ref. 14) is necessarily a linear function.

Let us show now the link between semi-invariants and a particular class of invariants for the system (2). Consider a class of invariants of the form

$$K=\prod_{i=1}^{m} U_i^{\alpha_i} P(U_1,...,U_m), \tag{23}$$

where the α_i are real numbers and $P(U)$ is a polynomial function. Cairo, Feix, and Hua[29,33] have studied invariants of this form. The equation for an invariant is

$$\dot{K}=\sum_{j=1}^{m} \frac{\partial K}{\partial U_j} \dot{U}_j=0. \tag{24}$$

Then, by inserting (2) and (23) in (24) we obtain

$$\prod_{i=1}^{m} U_i^{\alpha_i}(\dot{P}-\lambda P)=0, \tag{25}$$

where $\lambda=-\Sigma_{j=1}^{m}(\alpha M)_j U_j$ and $(\alpha M)_j=\Sigma_{k=1}^{m}\alpha_k M_{kj}$. The condition (25) is equivalent to

$$\dot{P}=\lambda P. \tag{26}$$

Thus, P is a semi-invariant.

Let us define the set $L[U]$ of all linear functions on U, and the set $SI[U]$ of all semi-invariants of the system (3). We also define the function $\Lambda:SI[U]\to L[U]$ associating to any $P \in SI[U]$ its corresponding eigenvalue given in Eq. (26), and the mapping $\hat{M}:R^m\to L[U]$ which associates the vector $\alpha \in R^m$ to the linear function $\Sigma_{i=1}^{m}(\alpha M)_i U_i$. With these definitions, we conclude that a function of the type (23) is an invariant for the system (2) if two conditions are satisfied:

(a) There is a semi-invariant for the system.
(b) Image $(\bar{M})\cap$ Image $(\Lambda)\neq\varnothing$.

To end this section, we observe that the invariant surface given by (22) and the invariant of the form (23) in the Lotka–Volterra variables correspond respectively to the following invariant surface and invariant in the original variable x of system (1):

$$P_i\left(\prod_{k=1}^{n} x^{B_{1k}},...,\prod_{k=1}^{n} x^{B_{mk}}\right)=0, \quad i=1,...,q,$$

and

$$K = \prod_{k=1}^{n} x_i^{(\alpha B)_i} P \left(\prod_{k=1}^{n} x^{B_{1k}}, \ldots, \prod_{k=1}^{n} x^{B_{mk}} \right).$$

Such invariants are clearly more general than those furnished by the NACA approach.

A. Applications

In this section we apply our method to the well studied (α, β, δ) Lotka–Volterra. May–Leonard, Lorenz, and Rikitake systems. To obtain the solutions for Eq. (25) we used a computer algebra in MAPLE written by the authors.[34] All computations for the semi-invariants are performed to polynomials of degree up to 5. The results are shown in Appendix D. We observe that since any quasi-monomial is a semi-invariant, they are not reported in the tables.

1. (α, β, δ) Lotka–Volterra system

The (α, β, δ) Lotka–Volterra system with vanishing eigenvalues[35] is given by

$$\dot{x}_1 = x_1(\delta x_2 + x_3),$$

$$\dot{x}_2 = x_2(x_1 + \alpha x_3), \tag{27}$$

$$\dot{x}_3 = x_3(\beta x_1 + x_2).$$

The monomials are trivially obtained: $U_1 = x_1$, $U_2 = x_2$, and $U_3 = x_3$. The matrix M is given by

$$M = \begin{bmatrix} 0 & \delta & 1 \\ 1 & 0 & \alpha \\ \beta & 1 & 0 \end{bmatrix}. \tag{28}$$

The semi-invariants and the corresponding quasi-polynomials invariants (QP-invariants) obtained for this system are given in Table IX (Appendix D). We obtained more cases than in a similar study performed by Labrunie and Conte,[32] who looked for rational invariants obtained from semi-invariants with the same eigenvalue.

Our analysis yields some interesting results: cases 4 and 10–20 are new integrable cases not found in the literature to the authors' knowledge (an exhaustive study of all known cases can be found in Ref. 35). Cases 3, 5, 8, and 9 are particular cases for more general classes of integrable Lotka–Volterra systems for which only one invariant was known[35] and the invariant obtained in this work is a new one not yet obtained. Case 2 is a known integrable class with a new invariant obtained here. Cases 6 and 7 are particular cases of integrable classes for which our approach does not yield any invariant. Nonetheless we obtained an invariant surface that, together with the known invariant, determines the orbit of the system. All know quasi-polynomial invariants for this system plus some new cases were obtained.

2. May–Leonard system

This system, the corresponding monomials, and the matrix M are given in Sec. III D by Eqs. (18) and (19). The semi-invariants and the corresponding QP-invariants are given in Table IX (Appendix D). The linear semi-invariants correspond to cases already obtained by the three-dimensional subalgebras in Table I (Appendix B).

3. Lorenz system

This system is described in Sec. III D by Eqs. (20) and (21). The results obtained are given in Table XI (Appendix C). Most known integrable cases[31] are obtained here while Labrunie and Conte[32] obtained only the cases ($\beta = 2\sigma$) and ($\beta = 1$, $\rho = 0$).

4. Rikitake system

The Rikitake system[36] is given by

$$\dot{x}_1 = -\mu x_1 + x_2(x_3 + \beta),$$

$$\dot{x}_2 = -\mu x_2 + x_1(x_3 - \beta), \tag{29}$$

$$\dot{x}_3 = \alpha - x_1 x_2.$$

The monomials are $U_1 = 1$, $U_2 = x_1^{-1} x_2 x_3$, $U_3 = x_1^{-1} x_2$, $U_4 = x_1 x_2^{-1} x_3$, $U_5 = x_1 x_2^{-1}$, $U_6 = x_3^{-1}$, and $U_7 = x_1 x_2 x_3^{-1}$. The matrix M is given by

$$
M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -\beta & 1 & -\beta & 1 & -1 \\
0 & -1 & -\beta & 1 & -\beta & 0 & 0 \\
0 & 1 & \beta & -1 & \beta & 1 & -1 \\
0 & 1 & \beta & -1 & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
-2\mu & 1 & \beta & 1 & -\beta & -1 & 1
\end{bmatrix}. \tag{30}
$$

Our results are shown in Table XII (Appendix D) and coincide with those obtained by Labrunie and Conte.[32]

V. CONCLUSION

We have shown how some algebraic concepts such as nonassociative algebras and polynomial ideals associated to systems of ODEs generate a systematic theory for finding invariant surfaces and invariants for the quasi-polynomial ODEs systems. These results are both exhaustive and constructive in a given functional class. Our next step will be, using the same tools, the generalization of these results to more general functional classes of invariants and the exploration of nonintegrability conditions.

ACKNOWLEDGMENTS

APPENDIX A: ONE-DIMENSIONAL SUBALGEBRAS

TABLE I. May–Leonard generators—general case.

	Belong to two-dim faces	
	$v_{(12)}=[1,l_1,0,0],\quad v_{(13)}=[1,0,l_2,0],\quad v_{(14)}=[1,0,0,l_3]$	
	Belong to three-dim faces	
Nilpotents	$v_{(123)}=[1-\alpha\beta,l_1-\alpha l_2,l_2-\beta l_1,0],\quad v_{(124)}=[1-\alpha\beta,l_1-\beta l_3,0,l_3-\alpha l_1]$	
	$v_{(134)}=[1-\alpha\beta,0,l_2-\alpha l_3,l_3-\beta l_2]$	
	Nonfacial	
	$v_{(1234)}=[\alpha^3+\beta^3-3\alpha\beta+1,(\alpha\beta-1)l_1+(\alpha-\beta^2)l_2+(\beta-\alpha^2)l_3,$	
	$(a^2-\beta)l_1+(1-\alpha\beta)l_2+(\beta^2-\alpha)l_3,(\alpha-\beta^2)l_1+(\beta-\alpha^2)l_2+(\alpha\beta-1)l_3]$	
	Belong to two-dim faces	
Idempotents	$v_{(23)}=[0,\alpha-1,\beta-1.0]$	$v_{(23)}^2=(1-\alpha\beta)v_{(23)}$
	$v_{(24)}=[0,\beta-1,0,\alpha-1]$	$v_{(24)}^2=(1-\alpha\beta)v_{(24)}$
	$v_{(34)}=[0,0,\alpha-1,\beta-1]$	$v_{(34)}^2=(1-\alpha\beta)v_{(34)}$
	Belong to three-dim faces	
	$v_{(234)}=[0,1,1,1],\quad v_{(234)}^2=(\alpha^3+\beta^3-3\alpha\beta+1)v_{(234)}$	

TABLE II. May–Leonard generators—degenerate case (c_1, c_2, c_3, x_1, x_2, y_1, y_2, z_1, and z_2 are nonvanishing real numbers).

	Belong to two-dim faces	
Nilpotents	$v_{(12)}=[1,l_1,0,0],\quad v_{(13)}=[1,0,l_2,0],\quad v_{(14)}=[1,0,0,l_3]$	
	$v_{(23)}=[0,-1,1,0],\quad v_{(24)}=[0,-1,0,1],\quad v_{(34)}=[0,0,-1,1]$	
	Belong to three-dim faces	
	$v_{(234)}=[0,-c_1,c_1-c_2,c_2]$	
	Belong to two-dim faces	
Idempotents	$v_{(23)}=[0,0,-x_1,x_2]$	$v_{(23)}^2=-(x_1+x_2)v_{(23)}$
	$v_{(24)}=[0,-y_1,0,y_2]$	$v_{(24)}^2=-(y_1+y_2)v_{(23)}$
	$v_{(34)}=[0,0,z_1,z_2]$	$v_{(34)}^2=-(z_1+z_2)v_{(34)}$

TABLE III. Lorenz generators (w_1, w_2, y_1, y_2, z_1, z_2, c_1, and c_2 are nonvanishing real numbers).

	Belong to two-dim faces	
	$v_{(12)}=[\sigma,\sigma-1,0,0,0],\quad v_{(13)}=[\rho,0,1-\sigma,0,0]$	
	$v_{(14)}=[1,0,0,\sigma+\beta-1,0],\quad v_{(15)}=[1,0,0,0,\beta-\sigma-1]$	
	$v_{(23)}=[0,\rho,\sigma,0,0],\quad v_{(24)}=[0,1,0,-\sigma,0]$	
	$v_{(45)}=[0,0,0,-1,-1]$	
	Belong to three-dim faces	
Nilpotents	$v_{(125)}=[\sigma,\sigma-1,0,0,\sigma(\beta-2)],\quad v_{(135)}=[\rho,0,1-\sigma,0,\rho(\beta-2\sigma)]$	
	$v_{(235)}=[0,\rho,\sigma,0,2\rho\sigma],\quad v_{(123)}=[w_1\sigma,w_1(\sigma-1)+w_2\rho,w_2\rho,0]$	
	$v_{(234)}=[0,y_1\rho,y_1\rho+y_2,y_2\rho,0]$	
	Belong to four-dim faces	
	$v_{(1245)}=[1,1,0,-1,\beta]$	
	$v_{(1235)}=[z_1\rho,z_2\rho,z_1(1-\sigma)+z_2\sigma,0,z_1\rho(\beta-2\sigma)+2z_2\rho\sigma]$	
	Belong to two-dim faces	
Idempotents	$v_{(25)}=[0,-1,0,0,-2\sigma]$	$v_{(25)}^2=\sigma v_{(25)}$
	$v_{(35)}=[0,0,-1,0,-2\rho]$	$v_{(35)}^2=\rho v_{(35)}$
	$v_{(34)}=[0,0,c_1,c_2,0]$	$v_{(34)}^2=(c_2-\rho c_1)v_{(34)}$
	Belong to three-dim faces	
	$v_{(245)}=[0,-2,0,4\sigma,-4\sigma],$	$v_{(245)}^2=-2\sigma v_{(245)}$

APPENDIX B: MAY–LEONARD SUBALGEBRAS

TABLE IV. Subalgebra existence conditions—May–Leonard general case. (Terms of the form $\langle e_i, v_{(jk)} \rangle$ on the third column correspond to the existence conditions for these two-dimensional subalgebras.)

Faces	Two-dim subalgebras	Existence conditions
$U_1 = 0$	$\langle e_2, v_{(34)} \rangle,\ \langle e_3, v_{(24)} \rangle,\ \langle e_4, v_{(23)} \rangle$ $\langle v_{(23)}, v_{(34)} \rangle$	$\alpha = \beta$ $(\alpha + \beta - 2) \times (\alpha(\alpha - 1) + \beta(\beta - 1) - \alpha\beta + 1) = 0$
$U_2 = 0$	$\langle e_1, v_{(34)} \rangle$ $\langle e_3, v_{(14)} \rangle$ $\langle e_4, v_{(13)} \rangle$ $\langle v_{(23)}, v_{(34)} \rangle$	$l_2 = l_3$ $\beta = 0$ $\alpha = 0$ $l_2(1 - \beta) = l_3(\alpha - 1)$
$U_3 = 0$	$\langle e_1, v_{(24)} \rangle$ $\langle e_2, v_{(14)} \rangle$ $\langle e_4, v_{(12)} \rangle$ $\langle v_{(12)}, v_{(24)} \rangle$	$l_1 = l_3$ $\alpha = 0$ $\beta = 0$ $l_3(1 - \beta) = l_1(\alpha - 1)$
$U_4 = 0$	$\langle e_1, v_{(23)} \rangle$ $\langle e_2, v_{(13)} \rangle$ $\langle e_3, v_{(12)} \rangle$ $\langle v_{(12)}, v_{(23)} \rangle$	$l_1 = l_2$ $\beta = 0$ $\alpha = 0$ $l_1(1 - \beta) = l_2(\alpha - 1)]$
	$\langle v_{(12)}, v_{(34)} \rangle$	$\alpha(\alpha - 1) + \beta(\beta - 1) = 0$ $l_3 = l_2 + l_1(\alpha - \beta)$
	$\langle v_{(13)}, v_{(24)} \rangle$	$\alpha(\alpha - 1) + \beta(\beta - 1) = 0$ $l_3 = l_1 + l_2(\beta - \alpha)$
	$\langle v_{(23)}, v_{(14)} \rangle$	$\alpha(\alpha - 1) + \beta(\beta - 1) = 0$ $l_2 = l_1 + l_3(\alpha - \beta)$
	$\langle v_{(23)}, v_{(134)} \rangle$	$\alpha(\alpha - 1) + \beta(\beta - 1) = 0$ $l_1(1 - \beta) = l_2(\alpha - 1)$
	$\langle v_{(24)}, v_{(134)} \rangle$	$\alpha(\alpha - 1) + \beta(\beta - 1) = 0$ $l_3(1 - \beta) = l_1(\alpha - 1)$
	$\langle v_{(34)}, v_{(124)} \rangle$	$\alpha(\alpha - 1) + \beta(\beta - 1) = 0$ $l_2(1 - \beta) = l_3(\alpha - 1)$
Nonfacial subalgebras	$\langle v_{(12)}, v_{(234)} \rangle$ $\langle v_{(13)}, v_{(234)} \rangle$ $\langle v_{(14)}, v_{(234)} \rangle$ $\langle v_{(123)}, v_{(234)} \rangle$	$l_2 = -l_1(\alpha + 1),\ l_3 = -l_1(\beta + 1)$ $l_1 = -l_2(\beta + 1),\ l_3 = -l_2(\alpha + 1)$ $l_2 = -l_3(\beta + 1),\ l_1 = -l_3(\alpha + 1)$ $l_2(1 + \alpha) = l_3(1 + \beta)$ $l_3(\alpha + 1)(\alpha\beta - 1) = (1 - 2\beta - 3\alpha\beta)$

Three-dim subalgebras	Existence conditions
$\langle e_1, e_2, v_{(34)} \rangle$	$\langle e_1, v_{(34)} \rangle,\ \langle e_2, v_{(34)} \rangle$
$\langle e_1, e_3, v_{(24)} \rangle$	$\langle e_1, v_{(24)} \rangle,\ \langle e_3, v_{(24)} \rangle$
$\langle e_2, e_3, v_{(14)} \rangle$	$\langle e_2, v_{(14)} \rangle,\ \langle e_3, v_{(14)} \rangle$
$\langle e_1, e_4, v_{(23)} \rangle$	$\langle e_1, v_{(23)} \rangle\ \langle e_4, v_{(23)} \rangle$
$\langle e_2, e_4, v_{(13)} \rangle$	$\langle e_2, v_{(13)} \rangle,\ \langle e_4, v_{(13)} \rangle$
$\langle e_3, e_4, v_{(12)} \rangle$	$\langle e_3, v_{(12)} \rangle,\ \langle e_4, v_{(12)} \rangle$
$\langle e_1, v_{(23)}, v_{(34)} \rangle$	$\langle e_1, v_{(23)} \rangle,\ \langle e_1, v_{(34)} \rangle,\ \langle v_{(23)}, v_{(34)} \rangle$
$\langle e_2, v_{(13)}, v_{(34)} \rangle$	$\langle e_2, v_{(13)} \rangle,\ \langle e_2, v_{(34)} \rangle,\ \langle v_{(13)}, v_{(34)} \rangle$
$\langle e_3, v_{(12)}, v_{(24)} \rangle$	$\langle e_3, v_{(12)} \rangle,\ \langle e_3, v_{(24)} \rangle,\ \langle v_{(12)}, v_{(24)} \rangle$
$\langle e_4, v_{(12)}, v_{(23)} \rangle$	$\langle e_4, v_{(12)} \rangle,\ \langle e_4, v_{(23)} \rangle,\ \langle v_{(12)}, v_{(23)} \rangle$
$\langle v_{(12)}, v_{(23)}, v_{(34)} \rangle$	$\langle v_{(12)}, v_{(23)} \rangle,\ \langle v_{(23)}, v_{(34)} \rangle$ $2(\alpha + \beta + 2)(\alpha^2 + \beta^2 - \alpha\beta - \alpha)l_1 - (\alpha - 1)^2(l_3 - l_2) = 0$

TABLE V. Subalgebra structure—May–Leonard degenerate case.

Faces	Two-dim subalgebras	Multiplicative structure
$U_1 = 0$	$\langle e_2, v_{(34)} \rangle$ $\langle e_3, v_{(24)} \rangle$ $\langle e_4, v_{(23)} \rangle$ $\langle v_{(23)}, v_{(34)} \rangle$	$e_2 \cdot v_{(34)} = -1/2(z_1 + z_2)e_2 - 1/2 v_{(34)}$ $e_3 \cdot v_{(24)} = -1/2(y_1 + y_2)e_3 - 1/2 v_{(24)}$ $e_4 \cdot v_{(23)} = -1/2(x_1 + x_2)e_4 - 1/2 v_{(23)}$ $v_{(23)} \cdot v_{(34)} = -1/2(z_1 + z_2)v_{(23)} - 1/2(x_1 + x_2)v_{(34)}$
$U_2 = 0$ $U_3 = 0$ $U_4 = 0$ Nonfacial	$\langle v_{(13)}, v_{(34)} \rangle\ (z_1 = l_2, z_2 = -l_3)$ $\langle v_{(12)}, v_{24} \rangle\ (y_1 = l_1, y_2 = -l_3)$ $\langle v_{(12)}, v_{(23)} \rangle\ (x_1 = l_1, x_2 = -l_2)$ $\langle v_{(12)}, v_{(23)}, v_{(34)} \rangle$ $(x_1 = l_1, x_2 = -l_2)\ (z_1 = l_2, z_2 = -l_2)$	$v_{(13)} \cdot v_{(34)} = 1/2(l_3 - l_2)v_{(34)}$ $v_{(12)} \cdot v_{(24)} = 1/2(l_3 - l_1)v_{(24)}$ $v_{(12)} \cdot v_{(23)} = 1/2(l_2 - l_1)v_{(23)}$ $v_{(12)} \cdot v_{(34)} = 1/2(l_3 - l_2)v_{(23)} + 1/2(l_3 - l_1)v_{(34)}$

TABLE VI. Subalgebra existence conditions—May–Leonard degenerate case. (Terms of the form $\langle e_i, v_{(jk)} \rangle$ on the third column correspond to the existence conditions for these two-dimensional subalgebras. The * corresponds to the subalgebras in Table V.)

Faces	Two-dim subalgebras	Existence conditions
$U_2=0$	$\langle e_1, v_{(34)} \rangle$	$l_2=l_3$
$U_3=0$	$\langle e_1, v_{(24)} \rangle$	$l_1=l_3$
$U_4=0$	$\langle e_1, v_{(23)} \rangle$	$l_1=l_2$
	$\langle e_1, v_{(234)} \rangle$	$l_1=l_2=l_3$
	$\langle v_{(13)}, v_{(24)} \rangle\ (y_2=-y_1)$	$l_1=l_3$
	$\langle v_{(12)}, v_{(34)} \rangle\ (z_2=-z_1)$	$l_2=l_3$
	$\langle v_{(23)}, v_{(14)} \rangle\ (x_2=-x_1)$	$l_1=l_2$
Nonfacial	$\langle v_{(12)}, v_{(234)} \rangle$	$l_1=l_2=l_3$
subalgebras	$\langle v_{(14)}, v_{(234)} \rangle$	$l_1=l_2=l_3$
	$\langle v_{(13)}, v_{(234)} \rangle$	$l_1=l_2=l_3$

	Three-dim subalgebras	Existence conditions
	$\langle e_1, e_2, v_{(34)} \rangle$	$\langle e_2, v_{(34)} \rangle^*\ \langle e_1, v_{(34)} \rangle$
	$\langle e_1, e_3, v_{(24)} \rangle$	$\langle e_3, v_{(24)} \rangle^*\ \langle e_1, v_{(24)} \rangle$
	$\langle e_1, e_4, v_{(23)} \rangle$	$\langle e_4, v_{(23)} \rangle^*\ \langle e_1, v_{(23)} \rangle$
	$\langle e_1, v_{(23)}, v_{(34)} \rangle$	$\langle v_{(23)}, v_{(34)} \rangle^*\ \langle e_1, v_{(23)} \rangle\ \langle e_1, v_{(34)} \rangle$

APPENDIX C: LORENZ SUBALGEBRAS

TABLE VII. Subalgebra structure—Lorenz system.

	Faces	Two-dim subalgebras
Nil subalgebras	$U_1=U_5=0$	$\langle v_{(12)}, v_{(23)} \rangle$
	$U_4=U_5=0$	$\langle v_{(23)}, v_{(34)} \rangle\ (c_1=1, c_2=\rho)$
	$U_4=0$	$\langle v_{(123)}, v_{(235)} \rangle\ (w_1=2\rho, w_2=2-\beta)$
	Nonfacial	$\langle v_{(34)}, v_{(1245)} \rangle\ (c_1=1, c_2=\rho)$

Faces	Two-dim subalgebras	Multiplicative structure
$U_3=U_4=0$	$\langle e_5, v_{(12)} \rangle$	$e_5 \cdot v_{(12)} = 1/2\sigma(\beta-2)e_5$
$U_2=U_5=0$	$\langle v_{(13)}, v_{(34)} \rangle$	$v_{(13)} \cdot v_{(34)} = 1/2\rho(2\sigma-\beta)v_{(34)}$
	$[c_1=\sigma-1, c_2=\rho(\beta-\sigma-1)]$	
$U_2=U_4=0$	$\langle e_5, v_{(13)} \rangle$	$e_5 \cdot v_{(13)} = 1/2\rho(\beta-2\sigma)e_5$
$U_1=U_5=0$	$\langle e_5, v_{(34)} \rangle$	$e_2 \cdot v_{(34)} = 1/2(\rho c_1-c_2)e_2 + 1/2\sigma v_{(34)}$
$U_1=U_4=0$	$\langle e_2, v_{(35)} \rangle$	$e_2 \cdot v_{(35)} = -1/2\rho e_2 + 1/2\sigma v_{(35)}$
	$\langle e_3, v_{(25)} \rangle$	$e_3 \cdot v_{(25)} = -1/2\sigma e_3 + 1/2\rho v_{(25)}$
	$\langle e_5, v_{(23)} \rangle$	$e_5 \cdot v_{(23)} = \rho\sigma e_5$
$U_1=U_3=0$	$\langle e_2, v_{(45)} \rangle$	$e_2 \cdot v_{(45)} = -1/2e_2 + 1/2\sigma v_{(45)}$
	$\langle e_4, v_{(25)} \rangle$	$e_4 \cdot v_{(25)} = -3/2\sigma e_4 - 1/2v_{(25)}$
$U_1=U_2=0$	$\langle v_{(34)}, v_{(45)} \rangle\ (c_1=1, c_2=2\rho)$	$v_{(34)} \cdot v_{(45)} = -1/2v_{(34)} - 1/2\rho v_{(45)}$
$U_1=0$	$\langle v_{(34)}, v_{(245)} \rangle\ (c_1=1, c_2=2\rho)$	$v_{(34)} \cdot v_{(245)} = \sigma v_{(34)} - \rho v_{(245)}$
	$\langle v_{(35)}, v_{(234)} \rangle\ (y_1=1, y_2=-2\sigma)$	$v_{(35)} \cdot v_{(234)} = \sigma\rho v_{(35)} - 1/2\rho v_{(234)}$
$U_4=0$	$\langle e_5, v_{(123)} \rangle$	$e_5 \cdot v_{(123)} = 1/2\sigma(x_1(\beta-2) + 2x_2\rho)e_5$

Faces	Three-dim subalgebras	Multiplicative structure
$U_1=0$	$\langle e_3, e_4, v_{(25)} \rangle$	$\langle e_3, v_{(25)} \rangle\ \langle e_4, v_{(25)} \rangle$
	$\langle e_2, v_{(34)}, v_{(45)} \rangle\ (c_1=1, c_2=2\rho)$	$\langle e_2, v_{(34)} \rangle\ \langle e_2, v_{(45)} \rangle\ \langle v_{(34)}, v_{(45)} \rangle$
$U_4=0$	$\langle e_5, v_{(12)}, v_{(23)} \rangle$	$\langle e_5, v_{(12)} \rangle\ \langle e_5, v_{(23)} \rangle\ \langle v_{(12)}, v_{(23)} \rangle$

Non-nil subalgebras (leftmost label spanning the non-nil rows)

TABLE VIII. Subalgebra existence conditions—Lorenz system. (The * correspond to the two-dimensional subalgebras in Table VII.)

Faces	Two-dim subalgebras	Existence conditions
$U_4=U_5=0$	$\langle e_1, v_{(23)}\rangle$	$\sigma+1=0$
$U_3=U_5=0$	$\langle e_1, v_{(24)}\rangle$	$\beta+2\sigma-2=0$
$U_3=U_4=0$	$\langle e_1, v_{(25)}\rangle$	$\beta-2\sigma=0$
	$\langle v_{(12)}, v_{(25)}\rangle$	$\sigma+\beta-3=0$
$U_2=U_4=0$	$\langle e_1, v_{(35)}\rangle$	$\beta-2=0$
	$\langle v_{(13)}, v_{(35)}\rangle$	$3\sigma-\beta-1=0$
$U_2=U_3=0$	$\langle e_1, v_{(45)}\rangle$	$\beta-1=0$
$U_2=0$	$\langle v_{(34)}, v_{(15)}\rangle \, [c_1=1, c_2=\rho]$	$\sigma+1=0$
$U_3=0$	$\langle e_1, v_{(245)}\rangle$	$\sigma=1/2 \beta=1$
	$\langle v_{(14)}, v_{(245)}\rangle$	$\sigma=-1 \beta=-2$
$U_4=0$	$\langle e_1, v_{(235)}\rangle$	$\sigma=1 \beta=2$
	$\langle v_{(13)}, v_{(235)}\rangle$	$\beta-2\sigma=0$

	Three-dim subalgebras	Existence conditions
$U_3=0$	$\langle e_1, e_2, v_{(45)}\rangle$	$\langle e_2, v_{(45)}\rangle^* \; \langle e_1, v_{(45)}\rangle$
	$\langle e_1, e_4, v_{(25)}\rangle$	$\langle e_4, v_{(25)}\rangle^* \; \langle e_1, v_{(25)}\rangle$
$U_4=0$	$\langle e_1, e_2, v_{(35)}\rangle$	$\langle e_2, v_{(35)}\rangle^* \; \langle e_1, v_{(35)}\rangle$
	$\langle e_1, e_3, v_{(25)}\rangle$	$\langle e_3, v_{(25)}\rangle^* \; \langle e_1, v_{(25)}\rangle$
	$\langle e_1, e_5, v_{(23)}\rangle$	$\langle e_5, v_{(23)}\rangle^* \; \langle e_1, v_{(23)}\rangle$

	Two-dim subalgebras	Existence conditions
Nonfacial	$\langle v_{(34)}, v_{(125)}\rangle \, [c_1=1, c_2=\rho]$	$\sigma\beta-\beta+2=0$

	Four-dim subalgebras	Existence conditions
	$\langle e_1, e_3, e_4, v_{(25)}\rangle$	$\langle e_3, v_{(25)}\rangle^* \; \langle e_4, v_{(25)}\rangle^* \; \langle e_1, v_{(25)}\rangle$

APPENDIX D: POLYNOMIAL SEMI-INVARIANTS (SI) AND QUASI-POLYNOMIAL INVARIANTS (QP-INVARIANTS)

TABLE IX. Semi-invariants and invariants of the Lotka system.

No.	Parameter conditions	Semi-invariant	Eigenvalue	QP-invariant
1	$\alpha\beta\delta=-1$	$\alpha\beta U_1+U_2-\alpha U_3$	0	SI
2	$3=1$	αU_3-U_2	U_1	$U_1^{\alpha/1+\alpha\delta}U_2^{-1/1+\alpha\delta}U_3^{-\alpha\delta/1+\alpha\delta}$
3	$\beta=1,\ \delta=2$	$(1+\alpha)(U_1U_3-U_3^2)-U_1U_2$	U_1+2U_2	$U_1^{-\delta/1+2\alpha}U_2^{1/1+2\alpha}U_3^{-2\,1+\alpha/1+2\alpha}$
4	$\beta=2,\ \delta=-1+\alpha/\alpha$	$\alpha^2U_3(2U_1-U_3)+U_2(2\alpha U_3-U_2)$	$2U_1$	$U_1^{-2\alpha/1+2\alpha}U_2^{2/1+2\alpha}U_3^{-2(1+\alpha)/1+2\alpha}$
5	$\alpha=1/2,\ \beta=\delta=1$	$-U_2^2-U_3^2+U_2U_3$	$2U_1$	$U_1^{1/3}U_2^{4/3}U_3^{2/3}$
6	$\alpha=1,\ \beta=1/3,\ \delta=-3$	$U_1^2U_2+9U_2^3+6U_1U_2^2-9U_1U_2U_3+27U_2^2U_3+27U_2U_3^2-9U_3^3$	U_1	Not found
7	$\alpha=-1/3,\ \beta=3,\ \delta=1$	$9U_1^2U_3-6U_1U_3^2+27U_2^3-27U_1U_2U_3+27U_2^2U_3+9U_2U_3^2+U_3^3$	$3U_1$	Not found
8	$\beta=1,\ \alpha=-2,\ \delta=-3/2$	$4U_1^2U_2+72U_1^2U_3-144U_1U_3^2+9U_2^3+12U_1U_2^2+144U_1U_2U_3+54U_2^2U_3+108U_2U_3^2+72U_3^3$	U_1	$U_1^{-1/2}U_2^{-1/4}U_3^{-3/4}$
9	$\beta=1,\ \alpha=-2/3,\ \delta=1/2$	$108U_1^2U_2+8U_1U_3^2-16U_1U_3^2+27U_2^3+108U_1U_2^2-104U_1U_2U_3+54U_2^2U_3+36U_2U_3^2+8U_3^3$	U_1	$U_1^{1/2}U_2^{-3/4}U_3^{-1/4}$
10	$\beta=2,\ \alpha=-3/2,\ \delta=-1/3$	$4U_1^2U_2+72U_1^2U_3-144U_1U_3^2+9U_2^3+12U_1U_2^2+144U_1U_2U_3+54U_2^2U_3+108U_2U_3^2+72U_3^3$	U_1	$U_1^{-1/2}U_2^{-1/4}U_3^{-3/4}$
11	$\beta=1/3,\ \alpha=-4,\ \delta=-3/4$	$16U_1^2U_2+9U_2^3+24U_1U_2^2+96U_1U_2U_3+108U_2^2U_3+432U_2U_3^2+576U_3^3$	U_1	$U_1^{-2}U_2^{-1/2}U_3^{-3/2}$
12	$\beta=1/2,\ \alpha=6,\ \delta=-2/3$	$9U_1^2U_2+432U_1U_3^2+4U_2^3+12U_1U_2^2+432U_2U_3^2-144U_1U_2U_3-72U_2^2U_3-864U_3^3$	U_1	$U_1^{-6}U_2U_3^{-4}$
13	$\beta=2,\ \alpha=-1/6,\ \delta=4$	$4U_1^2U_3-4U_1U_3^2+216U_2^3-54U_1U_2^2U_3^3-36U_1U_2U_3+108U_2^2U_3+18U_2U_3^2$	$2U_1$	$U_1U_2^6U_3^{-4}$
14	$\beta=1/2,\ \alpha=1/4,\ \delta=-6$	$32U_1^2U_2+9U_1U_3^2+1152U_2^3+384U_1U_2^2+216U_2U_3^2-144U_1U_2U3-864U_2^2U_3-18U_3^3$	U_1	$U_1U_2^{-4}U_3^6$
15	$\beta=1/3,\ \alpha=-4/5,\ \delta=-3/4$	$2000U_1^2U_2+1125U_2^3+3000U_1U_2^2+2700U_2^2U_3-7200U_1U_2U_3+2160U_2U_3^2+576U_3^3$	U_1	$U_1^{-2/3}U_2^{-5/6}U_3^{-1/2}$
16	$\beta=1/2,\ \alpha=-3/4,\ \delta=-2/3$	$288U_1^2U_2-27U_1U_3^2+128U_2^3+384U_1U_2^2+54U_3^3-720U_1U_2U_3+288U_2^2U_3+216U_2U_3^2$	U_1	$U_1^{-3/5}U_2^{-4/5}U_3^{-2/5}$
17	$\beta=2,\ \alpha=-3/2,\ \delta=1/6$	$108U_1^2U_3-108U_1U_3^2+8U_2^3-48U_1U_2^2\,36U_2^2U_3-144U_1U_2U_3+54U_2U_3^2+27U_3^3$	$2U_1$	$U_1^{-6}U_2^{-4}U_3$
18	$\beta=1/2,\ \alpha=-3,\ \delta=-2/3$	$9U_1^2U_2-54U_1U_3^2+4U_2^3+12U_1U_2^2+18U_1U_2U_3+36U_2^2U_3+108U_2U_3^2+108U_3^3$	U_1	$U_1^{-3/2}U_2^{-1/2}U_3^{-1}$
19	$\beta=3,\ \alpha=-4/3,\ \delta=-1/4$	$576U_1^2U_3-384U_1U_3^2+27U_3^3-288U_1U_2U_3+108U_2^2U_3+144U_2U_3^2+64U_3^3$	$3U_1$	$U_1^{-2}U_2^{3/2}U_3^{-1/2}$
20	$\beta=3,\ \alpha=-4/3,\ \delta=-5/4$	$576U_1^2U_3-384U_1U_3^2+27U_3^3+64U_3^3+864U_1U_2U_3+108U_2^2U_3+144U_2U_3^2$	$3U_1$	$U_1^{-2/3}U_2^{1/2}U_3^{-5/6}$
21	$\alpha=\beta=\delta=-1$	$1/3(U_1^3+U_2^3+U_3^3)+U_1^2U_2+U_1^2U_3+U_2^2U_1+U_2^2U_3+U_3^2U_1+U_3^2U_2$	0	SI
		$U_1U_2U_3$	0	SI

TABLE X. Semi-invariants and invariants of the May–Leonard system.

No.	Parameter conditions	Semi-invariant	Eigenvalue	QP-invariant
1	$\alpha=2-\beta$	$U_2+U_3+U_4$	$l_1U_1-U_2-U_3-U_4$	$(U_2U_3U_4)^{-1/3}$
	$l_1=l_2=l_3$	$l_1U_1-U_2-U_3-U_4$	$-(U_2+U_3+U_4)$	Not found
2	$\alpha=\beta$			
	$l_1=l_2$	U_3-U_2	$l_2U_1-U_2-U_3-\alpha U_4$	Not found
3	$\alpha=\beta=1$	$l_1l_2l_3-l_2l_3U_2-l_1l_3U_3-l_1l_2U_4$	$-U_2-U_3-U_4$	$U_2^a U_3^{[a(l_1-l_3)-l_3]/(l_3-l_2)}$
				$\times U_4^{[a(l_1-l_2)-l_2]/(l_2-l_3)}$
		$l_3U_1-U_4$	$-U_4$	Not found
4	$\alpha=\beta=0$	$l_2U_1-U_3$	$-U_3$	Not found
		$l_1U_1-U_2$	$-U_2$	Not found
5	$\alpha=\beta=0$			
	$l_1=-l_2$	$l_2U_1+U_2-U_3$	$-(U_2+U_3)$	$(U_2U_3)^{-1}$
6	$\alpha=\beta=0$	$l_1(-4U_1U_2+U_1U_3+2l_1U_1^2)+$		
	$l_3=-2l_1$	$+2U_2^2-2U_2U_4$	$-(2U_2+U_4)$	$U_2^{-2}U_4^{-1}$
7	$\alpha=\beta=0$			
	$l_1=2l_2$	$-2U_2U_3+l_2U_1U_2+2U_3^2$	$2l_2U_1-U_2-2U_3$	Not found
8	$\alpha=\beta=0$	$l_3(l_3U_1^2-U_1U_4+U_1U_3+U_1U_2)+$		
	$l_1=-l_3l_2=-l_3$	$+U_2U_3-U_3U_4$	$-(U_2+U_3+U_4)$	Not found
9	$\alpha=\beta=1$	$1/2(l_3^2U_1+U_2^2+U_3^2+U_4^2)+$		
	$l_1=l_3l_2=-l_3$	$+l_3U_1U_3-l_3U_1U_2$	$-2(U_2+U_3+U_4)$	$(U_2U_3)^{-1}$

TABLE XI. Semi-invariants and invariants of the Lorenz system.

No.	Parameter conditions	Semi-invariant	Eigenvalue	QP-invariant
1	$\sigma=0,\ \beta=-1$	$2U_1-2\rho U_3+U_4+U_5$	$-\rho U_3+U_4-U_5$	$U_1^a U_2^{(2+c)}\times U_3^c U_4^{(b+1)}U_5^b$
2	$\sigma=0,\ \rho=0$	$(\beta-1)U_1-U_4-U_5$	U_4-U_5	Not found
3	$\beta=1,\ \rho=0$	U_4+U_5	$-\sigma U_1+\sigma U_2+U_4-U_5$	Not found
4	$\beta=2\sigma$	$-2\sigma U_2+U_5$	$(\sigma-1)U_1+\sigma U_2$	Not found
			$+\rho U_3-U_4-U_5$	
5	$\beta=1,\ \sigma=1/2,\ \rho=0$	$-U_2^2+U_4U_5-2U_4U_2$	$-(U_1+U_2)$	Not found
6	$\beta=1,\ \sigma=1$	$-\rho U_3U_5+U_2U_5+U_2U_4$	$-(U_1+U_5)$	Not found
7	$\beta=1,\ \sigma=1/3$	$4U_2^2+8U_1U_2-12\rho U_2U_3$	$-2/3(U_1+U_2)$	Not found
		$+12U_2U_4-9U_4U_5$		
8	$\sigma=1,\ \beta=4$	$4U_2(U_4U_2+4U_1^2+U_4U_5$	$-U_2+\rho U_3-U_4-U_5$	Not found
		$-2U_1U_4)-U_4^2U_5+4\rho U_2U_3(U_4-4U_2)$		

TABLE XII. Semi-invariants and invariants of the Rikitake system.

No.	Parameter conditions	Semi-invariant	Eigenvalue	QP-invariant
1	$\beta=0$	U_2-U_4	$1/49(U_2-U_4+\alpha U_6-U_7)$	Not found
2	$\beta=\mu=0$	U_2-U_4	$1/49(U_2-U_4+\alpha U_6-U_7)$	U_6
		$U_2-2\beta U_3+U_7$	$1/49(-U_2-\beta U_3+U_4-\beta U_5+U_7)$	$U_2^{1/49}U_3^{-2/49}$
3	$\alpha=\mu=0$	$-U_4+2\beta U_5-U_7$	$1/49(-U_2-\beta U_3+U_4-\beta U_5+U_7)$	$U_2^{1/49}$
		$U_2U_7-2\beta U_3U_4-U_4U_7$	0	SI

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