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Permanence and boundedness of solutions of quasi-polynomial systems

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ABSTRACT

In this paper we consider analytical properties of a class of general nonlinear systems known as Quasi-Polynomial systems. We analyze sufficient conditions for permanence and boundedness of solutions, and illustrate our approach with an application to Lamb equations describing the evolution of electromagnetic modes in an optical maser.

1. Introduction

In recent years the authors analyzed several analytical properties of Quasi-Polynomial (QP) dynamical systems. For example, the problem of stability in the sense of Lyapunov [1], was the subject of references [2–5]. In [6] integrability properties of QP systems were analyzed through associated bi-linear non-associative algebras, and in [7] it was shown that any quasi-polynomial invariant of a QP system is related to a similar invariant of a Lotka–Volterra (LV) dynamical system. Applications to systems of biological interest was presented in [8]. More recently, in [9] we presented a connection between asymptotic stable interior fixed points of square ($m = n$), or isomonomial, QP systems and evolutionary stable states, a concept of evolutionary games.

The QP systems are defined as follows:

$$\dot{x}_i = l_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}; \quad i = 1, \ldots, n. \tag{1}$$

Here $x_i \in \Re^n$, with A and B real, constant rectangular matrices and $m \geq n$ is assumed [10]. This class of systems encompass many systems of interest [10–12] and was extensively studied in literature [2–13].

One interesting property is the possibility to map Eq. (1) into a quadratic Lotka–Volterra (LV) system [10]. Define new variables U_α, $\alpha = 1, \ldots, m$, satisfying:

$$x_i = \prod_{\beta=1}^{m} U_\beta^{D_{i\beta}}, \tag{2}$$

with D an invertible matrix with components $D_{i\beta}$. Now choose $D = \tilde{B}^{-1}$, where:

$$\tilde{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} & b_{1,n+1} & \cdots & b_{1,m} \\ B_{21} & B_{22} & \cdots & B_{2n} & b_{2,n+1} & \cdots & b_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} & b_{m,n+1} & \cdots & b_{m,m} \end{bmatrix}. \tag{3}$$

Parameters b_{jk} are arbitrary provided $\tilde{B}$ is invertible and reasons for its introduction can be found in [10–12]. By defining the auxiliary variables $x_k(t = 0) = 1$ for $k = n + 1, \ldots, m$, we have that $U_\alpha = \prod_{i=1}^{m} x_i^{D^{-1}_{\alpha i}}$ with $D = \tilde{B}^{-1}$. The variables U_α then satisfy the system of equations:

$$\dot{U}_\alpha = (\tilde{B}\tilde{l})_\alpha U_\alpha + U_\alpha \sum_{\beta=1}^{m} (\tilde{B}\tilde{A})_{\alpha\beta} U_\beta; \quad \alpha = 1, \ldots, m, \tag{4}$$

which is a quadratic LV type system in the variables U_α. Here

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$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{5}$$

and

$$\tilde{l} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

It is usual to define the $m \times m$ matrix $M \equiv BA$ and $\lambda \equiv Bl$. Note that:

$$BA = \tilde{B}\tilde{A}, \;\; Bl = \tilde{B}\tilde{l}.$$

Considering that $\prod_{\beta=1}^{m} U_\beta^{\tilde{B}^{-1}_{\alpha\beta}} = 1; \alpha = n+1, \ldots, m$ the dynamics of the n dimensional system (1) takes place in a manifold of the $m \geq n$ dimensional LV space. Further details on the properties of this mapping can be found in [10–12]. In [12] a generalization is presented that encompasses systems not originally in the QP format.

As shown in the above cited references, the LV canonical format is useful to analyze analytical properties of QP systems. In this paper we extend the results presented in [2–9] by considering sufficient conditions for uniformly bounded solutions and permanence of trajectories, which we present in the following section.

2. Permanence and uniformly bounded solutions

Let us consider the criteria for permanence in dynamical systems. The differential equations $\dot{x}_i = f_i(x_1, \ldots, x_n)$ is said to be permanent if there exists a constant $\delta > 0$ such that, if $x_i(0) > 0$, $\forall i = 1, \ldots, n$, then $\liminf_{t\to\infty} [x_i(t)] > \delta$, $i = 1, \ldots, n$, where δ does not depend on the initial condition $x_i(0)$ [14], see also section 12.2, [15]. For a permanent system, the boundaries of the phase space are repellents. On the other side, systems with bounded solutions obey $\limsup_{t\to\infty}[x_i(t)] \leq d_i$ for some constants d_i. A permanent system is linked to the existence of an Average Lyapunov Function [15]. We address the question of how the solutions of a QP system relates to permanent or bounded solutions in its associated LV system. We state the following theorem:

Theorem 1. *Considering a LV system with bounded solutions, then the following properties hold for the associated QP system:*

1. *If $\tilde{B}^{-1}_{ij} \geq 0$ for all $i = 1, \ldots, n$, the solutions of the corresponding QP system are bounded.*
2. *If $\tilde{B}^{-1}_{ij} \leq 0$ for $i = 1, \ldots, n$, the solutions of the corresponding QP system are permanent.*

Proof. To prove Theorem 1, note that, if the orbits in the LV system are bounded then there exists some $R_i > 0$ such that for all $t > 0$ and all i we have:

$$U_j(t) \leq R_j, \;\; R_j > 0 \, \forall j = 1, \ldots, m.$$

If $\tilde{B}^{-1}_{ij} \geq 0$ then

$$U_j(t) \leq R_j \Rightarrow U_j^{\tilde{B}^{-1}_{ij}}(t) \leq R_j^{\tilde{B}^{-1}_{ij}}.$$

We then have that

$$x_i(t) = \prod_{j=1}^{m} U_j^{B^{-1}_{ij}}(t) \leq \prod_{j=1}^{m} R_j^{B^{-1}_{ij}} \equiv \Delta_i,$$

for $\Delta_i > 0$. Thus $x_i(t) \leq \Delta_i$ for all $t > 0$.

On the other hand if $\tilde{B}^{-1}_{ij} \leq 0$ we have

$$U_j(t) \leq R_j \Rightarrow U_j^{\tilde{B}^{-1}_{ij}}(t) \geq R_j^{\tilde{B}^{-1}_{ij}},$$

which implies in

$$x_i(t) = \prod_{j=1}^{m} U_j^{B^{-1}_{ij}}(t) \geq \prod_{j=1}^{m} R_j^{B^{-1}_{ij}} = \Delta_i.$$

Thus $x_i(t) \geq \Delta_i$ for all $t > 0$.

It is important to note that, since $\tilde{B}$ is invertible, then in any row of $\tilde{B}^{-1}$ there is at least one non-null element, and the inequalities used above are always satisfied. This finishes the proof of the theorem. □

Permanent LV systems possess a unique interior fixed point and $(-1)^n \det M > 0$. QP systems with $m > n$ usually are mapped into a LV system with $\det M = \det BA = 0$ [4,5]. In order to have $\det M \neq 0$ we now restrict ourselves here to the case of square QP systems $(m = n)$. In this case we have $\tilde{B} = B$ and $\tilde{A} = A$:

Theorem 2. *Given a square QP system, if its associated LV system is permanent, then:*

1. *If $B^{-1}_{ij} \geq 0$ for all $i = 1, \ldots, n$, the solutions of the corresponding square QP system are permanent.*
2. *If $B^{-1}_{ij} \leq 0$ for all $i = 1, \ldots, n$, the solutions of the corresponding square QP system are bounded.*

Proof. Let the solutions of the LV system to be permanent, then for every $t > 0$:

$$U_j(t) \geq d_j, \;\; d_j > 0 \, \forall j = 1, \ldots, n.$$

If $B^{-1}_{ij} \geq 0$ then

$$U_j(t) \geq d_j \Rightarrow U_j^{B^{-1}_{ij}}(t) \geq d_j^{B^{-1}_{ij}}.$$

This in turn implies in

$$x_i(t) = \prod_{j=1}^{n} U_j^{B^{-1}_{ij}}(t) \geq \prod_{j=1}^{n} d_j^{B^{-1}_{ij}} = \delta_i.$$

Therefore exists $\delta_i > 0$ such that $x_i(t) \geq \delta_i$ for all $t > 0$. When $B^{-1}_{ij} \leq 0$ we have

$$U_j(t) \geq d_j \Rightarrow U_j^{B^{-1}_{ij}}(t) \leq d_j^{B^{-1}_{ij}}.$$

Thus

$$x_i(t) = \prod_{j=1}^{n} U_j^{B^{-1}_{ij}}(t) \leq \prod_{j=1}^{n} d_j^{B^{-1}_{ij}} = \delta_i.$$

Therefore $x_i(t) \leq \delta_i$ for all $t > 0$ and this finishes the proof. □

According to [15], for a permanent LV system there exists $\delta > 0$ such that $\delta < \liminf_{t\to\infty}[x_i(t)]$, $\forall i$. In this case there is also a constant R such that $\limsup_{t\to\infty}[x_i(t)] \leq R$, $\forall i$ provided $(x_1, \ldots, x_n) \in$

$\Re_+^n$. If this condition holds then the orbits of the LV system are uniformly bounded. The threshold δ is uniform and independent of the initial condition. Note that, if a given LV system is permanent and its trajectories are bounded, then a corresponding square QP system is also permanent with bounded solutions. We thus have the following corollary, which follows directly from Theorems 1 and 2:

Corollary 1. *Let a square QP system and its corresponding LV system. If the LV system is permanent with bounded solutions, then the corresponding square QP system is also permanent and its trajectories are bounded.*

Consider now the following result presented in [15], problem 13.6.3: let a n dimensional LV system of type $\dot{U}_i = \lambda_i U_i + U_i \sum_{j=1}^n M_{ij} U_j$ be uniformly bounded. Then it is permanent if there exists a vector $\vec{P} = (P_1, \ldots, P_n)$, $P_i > 0$, satisfying:

$$\sum_{i|U_i^*=0}^{n} P_i(\lambda_i + \sum_{j=1}^{n} M_{ij} U_j^*) > 0 \tag{7}$$

where $i|U_i^*$ is the sum over the *facial* fixed points U^* (fixed points obtained when we impose successively $U_i = 0, \forall i$).

From the above result we may state the following corollary, whose proof follows directly from Corollary 1:

Corollary 2. *Consider a square QP system with matrices A and B. If the associated LV system is uniformly bounded and $M = BA$, $\lambda = Bl$ satisfies the conditions $\sum_{i|U_i^*=0}^n P_i((Bl)_i + \sum_{j=1}^n (BA)_{ij} U_j^*) > 0$ for a given $\vec{P} = (P_1, \ldots, P_n)$, $P_i > 0$, then the solutions of the QP system are uniformly bounded and the system is permanent.*

Indeed, if the LV system has bounded solutions, according to Theorems 1 and 2 the QP system will be either permanent or possess bounded solutions. Given that the LV is also permanent, Theorem 1 guarantees that the associated square QP system is both permanent with bounded solutions. Note that the fixed points of the LV and the QP system are related by $U_\alpha = \prod_{i=1}^m x_i^{D_{\alpha i}^{-1}}$ with $D = \tilde{B}^{-1}$.

3. Lamb system

As a specific application, we consider the Lamb system [16]. The Lamb model is a theoretical description of an optical maser in which the electromagnetic fields are classically treated and the active medium made of thermally moving atoms acquiring nonlinear dipole moments. The variables x_i are the amplitudes of the transverse electric field. Competition between modes depend on parameters α_{ij} and may lead to regimes with one, two or three simultaneous modes:

$$\begin{aligned}
\dot{x}_1 &= x_1(\lambda_1 - \alpha_{11}x_1^2 - \alpha_{12}x_2^2 - \alpha_{13}x_3^2), \\
\dot{x}_2 &= x_2(\lambda_2 - \alpha_{21}x_1^2 - \alpha_{22}x_2^2 - \alpha_{23}x_3^2), \\
\dot{x}_3 &= x_3(\lambda_3 - \alpha_{31}x_1^2 - \alpha_{32}x_2^2 - \alpha_{33}x_3^2).
\end{aligned} \tag{8}$$

This system can be cast in the LV format defining $U_1 = x_1^2, U_2 = x_2^2, U_3 = x_3^2$ (the square amplitudes), which gives the following matrix M:

$$M = \begin{bmatrix} -2\alpha_{11} & -2\alpha_{12} & -2\alpha_{13} \\ -2\alpha_{21} & -2\alpha_{22} & -2\alpha_{23} \\ -2\alpha_{31} & -2\alpha_{32} & -2\alpha_{33} \end{bmatrix}. \tag{9}$$

The matrix B being:

$$B = \tilde{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{10}$$

The LV system associated to (9) are guaranteed to have bounded solutions if M is a **B** matrix ([15] section 15.2) and its initial condition is in the interior of the positive orthant. An M matrix is a **B** matrix iff, $\forall\, x \in \Re^n$ satisfying:

$$x \in \Re^n \,\|\, x_j \geq 0, \ \forall j = 1, \ldots, n,$$

but such that $x_k \neq 0$ at least for one $k \in (1, \ldots, n)$, there exists one index i such that $x_i > 0$ and $\sum_{j=1}^n M_{ij} x_j < 0$.

Let $x = (1, 0, 0)$. This implies $\sum_j M_{1j} x_j < 0 \Rightarrow -2\alpha_{11} < 0$ which is immediately satisfied. In the same way, if we choose $x = (0, 1, 0)$ we obtain $-2\alpha_{22} > 0$, which always holds by definition, and the same is true for $x = (0, 0, 1)$. Choosing $x = (0, 1, 1)$ we obtain that either $-2\alpha_{11} - 2\alpha_{12} < 0$ or $-2\alpha_{21} - 2\alpha_{22} < 0$ must hold. It is thus straightforward to verify M is a **B** matrix. In this way the LV system has uniformly bounded solutions. And since $\tilde{B}_{ij}^{-1} \geq 0$ the system (8) is uniformly bounded.

Given that the LV system has bounded solutions, permanence of the system may be analyzed with the help of 7. The facial fixed points of the system are:

$$\begin{aligned}
U_1 \equiv 0 \rightarrow \vec{q}_1 &= \left(0, \frac{\alpha_{23}\lambda_3 - \alpha_{33}\lambda_2}{\alpha_{32}\alpha_{23} - \alpha_{22}\alpha_{33}}, \frac{\alpha_{32}\lambda_2 - \alpha_{22}\lambda_3}{\alpha_{32}\alpha_{23} - \alpha_{22}\alpha_{33}}\right), \\
U_2 \equiv 0 \rightarrow \vec{q}_2 &= \left(\frac{\alpha_{13}\lambda_3 - \alpha_{33}\lambda_1}{\alpha_{31}\alpha_{13} - \alpha_{11}\alpha_{33}}, 0, \frac{\alpha_{31}\lambda_1 - \alpha_{11}\lambda_3}{\alpha_{31}\alpha_{13} - \alpha_{11}\alpha_{33}}\right), \\
U_3 \equiv 0 \rightarrow \vec{q}_3 &= \left(\frac{\alpha_{22}\lambda_1 - \alpha_{12}\lambda_2}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}}, \frac{\alpha_{11}\lambda_2 - \alpha_{21}\lambda_1}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}}\right).
\end{aligned} \tag{11}$$

This system is permanent if, for given λ_i and α_{ij} there exists a vector $\vec{P}$ that satisfies the set of conditions below. For the facial rest point (11) we have:

$$\begin{aligned}
& P_1\,(2\lambda_1 - 2\alpha_{12}q_1(2) - 2\alpha_{13}q_1(3)) \\
& + P_2\,(2\lambda_2 - 2\alpha_{22}q_1(2) - 2\alpha_{23}q_1(3)) \\
& + P_3\,(2\lambda_3 - 2\alpha_{32}q_1(2) - 2\alpha_{33}q_1(3)) > 0,
\end{aligned} \tag{12}$$

and substituting $q_1(2)$ and $q_1(3)$ from (11) leads to $P_1(2\lambda_1 - 2\alpha_{12}q_1(2) - 2\alpha_{13}q_1(3)) > 0$. Since $P_1 > 0$ this inequality can be written as:

$$\frac{2}{\Delta_{11}} \begin{vmatrix} \lambda_1 & \alpha_{12} & \alpha 13 \\ \lambda_2 & \alpha_{22} & \alpha_{23} \\ \lambda_3 & \alpha_{32} & \alpha_{33} \end{vmatrix} > 0, \tag{13}$$

where

$$\Delta_{11} = \begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix}. \tag{14}$$

For the other two facial rest points, using $P_2, P_3 > 0$, we obtain similar results:

$$\frac{2}{\Delta_{22}} \begin{vmatrix} \alpha_{11} & \lambda_1 & \alpha_{13} \\ \alpha_{21} & \lambda_2 & \alpha_{23} \\ \alpha_{31} & \lambda_3 & \alpha_{33} \end{vmatrix} > 0, \quad \frac{2}{\Delta_{22}} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \lambda_1 \\ \alpha_{21} & \alpha_{22} & \lambda_2 \\ \alpha_{31} & \alpha_{32} & \lambda_3 \end{vmatrix} > 0, \tag{15}$$

where

$$\Delta_{22} = \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{31} & \alpha_{33} \end{vmatrix}, \quad \Delta_{33} = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}. \tag{16}$$

We consider $\Delta_{ii} \neq 0$, $\forall i$ in order to have unique facial points.

Permanence is observed in several cases. For example, if vector $\Delta_{ii} > 0$, $\forall i$ we see that $(\lambda_1, \lambda_2, \lambda_3)$ lies in the convex hull of the columns of the matrix of the coefficients α_{ij}. Another general result is that, when the diagonal terms $\alpha_{ii} = 0$, $\forall i$, it is possible to show that (13) and (15) guarantee permanence if:

$$\lambda_i > \lambda_j \frac{a_{ik}}{a_{jk}} + \lambda_k \frac{a_{ij}}{a_{kj}},$$

with (i, j, k) in cyclic order. As a final result, if the off-diagonal terms $\alpha_{ij} = \alpha_{ik} = 0$ for $i \neq j \neq k$, it is possible to show that there is permanence if $\lambda_i > 0$. In fact, using Laplace (or co-factor) expansion in (13) we obtain:

$$\frac{2}{\Delta_{11}}\left(\lambda_1(-1)^{1+1}\begin{vmatrix}\alpha_{22} & \alpha_{23}\\ \alpha_{32} & \alpha_{33}\end{vmatrix} + \alpha_{12}(-1)^{1+2}\begin{vmatrix}\lambda_2 & \alpha_{23}\\ \lambda_3 & \alpha_{33}\end{vmatrix}\right) + \frac{2}{\Delta_{11}}\left(\alpha_{13}(-1)^{1+3}\begin{vmatrix}\lambda_2 & \alpha_{22}\\ \lambda_3 & \alpha_{32}\end{vmatrix}\right) > 0, \tag{17}$$

which reduces to $\lambda_1 > 0$ when $\alpha_{12} = \alpha_{13} = 0$. Similar results follow from (15) and thus we write the general result that if $\alpha_{ij} = \alpha_{ik} = 0$ for $i \neq j \neq k$ permanence is observed provided $\lambda_i > 0$.

4. Concluding remarks

In this paper we present results concerning analytical properties in a class of general nonlinear systems known as Quasi-Polynomial systems. The possibility of relating these systems to Lotka–Volterra equations allows the development of methods to analyze several analytical properties by extending results previously obtained in LV systems to encompass QP systems, an approach previously used by the authors [2–9]. Here we extended the range of such approach by analyzing sufficient conditions for uniformly bounded solutions and permanent orbits.

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