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Research Article

A Lyapunov Function and Global Stability for a Class of Predator-Prey Models

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We construct a new Lyapunov function for a class of predation models. Global stability of the positive equilibrium states of these systems can be established when the Lyapunov function is used.

1. Introduction

The dynamics of predator-prey systems are often described by differential equations, which represent time continuously. A common framework for such a model is [1–3]

$$\frac{dN}{dt} = Nf(N) - Pg(N,P),$$
$$\frac{dP}{dt} = h[g(N,P),P]P, \tag{1.1}$$

where N and P are prey and predator densities, respectively, $f(N)$ is the prey growth rate, $g(N)$ is the functional response, for example, the prey consumption rate by an average single predator, and $h[g(N),P]$ the per capita growth rate of predators (also known as the "predator numerical response"), which obviously increases with the prey consumption rate. The most widely accepted assumption for the numerical response with predator density restricting is as follows:

$$h\left[g(N,P),P\right]=\varepsilon g(N,P)-\delta P-\beta, \tag{1.2}$$

where β is a per capita predator death rate, ε the conversion efficiency of food into offspring, δ the density dependent rate [2]. And prototype of the prey growth rate $f(N)$ is the logistic growth

$$f(N)=r\left(1-\frac{N}{K}\right), \tag{1.3}$$

where $K>0$ is the carrying capacity of the prey.

When

$$g(N,P)=1-e^{-aN}, \tag{1.4}$$

where a is the efficiency of predator capture of prey, model (1.1) is called Ivlev-type predation model, due originally to Ivlev [4]. And Ivlev-type functional response is classified to the prey-dependent; that is, g is independent of predator P [2].

Both ecologists and mathematicians are interested in the Ivlev-type predator-prey model and much progress has been seen in the study of the model [5–13]. Of them, Xiao [8] gave global analysis of the following model:

$$\begin{aligned}\frac{dN}{dt}&=rN(1-N)-\left(1-e^{-aN}\right)P,\\ \frac{dP}{dt}&=P\left(1-e^{-aN}-d-\delta P\right).\end{aligned} \tag{1.5}$$

But, in paper [8], the author gave complex process to prove the global asymptotical stability of the positive equilibrium.

In this paper, we will establish a new Lyapunov function to prove the global stability of the positive equilibrium of model (1.5).

Our paper is organized as follows. In the next section, we discuss the existence, uniqueness of the positive equilibrium, and establish a new Lyapunov function to model (1.5). In Section 3, we will give some examples to show the robustness of our Lyapunov function.

2. Main Results

First of all, it is easy to verify that model (1.5) has two trivial equilibria (belonging to the boundary of $\mathbf{R}_+^2$, that is, at which one or more of populations has zero density or is extinct), namely, $E_0=(0,0)$ and $E_1=(1,0)$. For the positive equilibrium, set

$$rN(1-N)-\left(1-e^{-aN}\right)P=0, \qquad P\left(1-e^{-aN}-\delta P-d\right)=0, \tag{2.1}$$

which yields

$$N(1-N)=\frac{1}{r\delta}\left(\left(1-e^{-aN}\right)\left(1-e^{-aN}-d\right)\right). \tag{2.2}$$

We have the following Lemma regarding the existence of the positive equilibrium.

Lemma 2.1 (see [8]). *Suppose $1-e^{-a}>d$. Model* (1.5) *has a unique positive equilibrium $E^*=(N^*,P^*)$ if either of the following inequalities holds:*

(i) $d\geq 2(1-2e^{-a/2})$;

(ii) $d<2(1-2e^{-a/2})$, $(a/r\delta)(2(1-2e^{-a/2})-d))e^{-a/2}\geq 1+(2/a)\ln(1/2-d/4)$ *and* $(1/r\delta)((1/2+d/4)^2-d(1/2+d/4))<-(1/a)\ln(1/2+d/4)-1/a^2$.

Lemma 2.2. *Let $1-e^{-a}>d$, then $\Omega=\{(N,P)\mid 0\leq N\leq 1,\ 0\leq P\leq(1-e^{-a}-d)/\delta\}$ is a region of attraction for all solutions of model* (1.5) *initiating in the interior of the positive quadrant $\mathbf{R}_+^2$.*

Proof. Let $(N(t),P(t))$ be any solution of model (1.5) with positive initial conditions. Note that $dN/dt\leq N(1-N)$, by a standard comparison argument, we have

$$\limsup_{t\to\infty} N(t)\leq 1. \tag{2.3}$$

Then,

$$\frac{dP}{dt}=P\left(1-e^{-aN}-d-\delta P\right)\leq P\left(1-e^{-a}-d-\delta P\right). \tag{2.4}$$

Similarly, since $1-e^{-a}>d$, we have

$$\limsup_{t\to\infty} P(t)\leq\frac{1-e^{-a}-d}{\delta}. \tag{2.5}$$

On the other hand, for all $(N,P)\in\Omega$, we have $dN/dt|_{N=0}=0$ and $dP/dt|_{P=0}=0$. Hence, Ω is a region of attraction. As a consequence, we will focus on the stability of the positive equilibrium E^* only in the region Ω. □

In the following, we devote to the global stability of the positive equilibrium $E^*=(N^*,P^*)$ for model (1.5) by constructing a new Lyapunov function which is motivated by the work of Hsu [3].

Theorem 2.3. *If $a\leq 2$, the positive equilibrium $E^*=(N^*,P^*)$ of model* (1.5) *is globally asymptotically stable in the region Ω.*

Proof. For model (1.5), we construct a Lyapunov function of the form

$$V(N,P)=\int_{N^*}^{N}\frac{1-e^{-a\xi}-d-\delta P^*}{1-e^{-a\xi}}d\xi+\int_{P^*}^{P}\frac{\eta-P^*}{\eta}d\eta. \tag{2.6}$$

Note that $V(N,P)$ is non-negative, $V(N,P)=0$ if and only if $(N,P)=(N^*,P^*)$. Furthermore, the time derivative of V along the solutions of (1.5) is

$$\frac{dV}{dt}=\frac{1-e^{-aN}-d-\delta P^*}{1-e^{-aN}}\frac{dN}{dt}+\frac{P-P^*}{P}\frac{dP}{dt}. \tag{2.7}$$

Substituting the expressions of dN/dt and dP/dt defined in (1.5) into (2.7), we can obtain

$$\begin{aligned}\frac{dV}{dt}&=\left(1-e^{-aN}-d-\delta P^*\right)\left(\frac{rN(1-N)}{1-e^{-aN}}-P^*\right)-\delta(P-P^*)^2\\&=\left(1-e^{-aN}-d-\delta P^*\right)\left(\frac{rN(1-N)}{1-e^{-aN}}-\frac{rN^*(1-N^*)}{1-e^{-aN^*}}\right)-\delta(P-P^*)^2.\end{aligned} \tag{2.8}$$

Define

$$\phi(N)=1-e^{-aN}-d-\delta P^*, \tag{2.9}$$

then

$$\begin{aligned}\phi(N^*)&=0,\\ \phi'(N)&=1+ae^{-aN}>0.\end{aligned} \tag{2.10}$$

So,

$$(N-N^*)\left(\phi(N)-\phi(N^*)\right)>0. \tag{2.11}$$

Then we can get

$$\frac{dV}{dt}\leq 0. \tag{2.12}$$

If

$$\left(1-e^{-aN}-d-\delta P^*\right)\left(\frac{rN(1-N)}{1-e^{-aN}}-\frac{rN^*(1-N^*)}{1-e^{-aN^*}}\right)\leq 0 \tag{2.13}$$

holds, which is equivalent to

$$(N-N^*)\left(\frac{rN(1-N)}{1-e^{-aN}}-\frac{rN^*(1-N^*)}{1-e^{-aN^*}}\right)\leq 0. \tag{2.14}$$

Set $\varphi(N)=rN(1-N)/(1-e^{-aN})$, we obtain $\varphi(0)=0$ and

$$\varphi'(N)=\frac{r\left((1-2N)(1-e^{-aN})-aN(1-N)e^{-aN}\right)}{\left(1-e^{-aN}\right)^2}. \tag{2.15}$$

And set $\psi(N) = (1-2N)(1-e^{-aN}) - aN(1-N)e^{-aN}$, we can get $\psi(0) = 0$ and

$$\begin{aligned}\psi'(N) &= e^{-aN}\left(a^2N(1-N)+2\right)-2,\\ \psi''(N) &= ae^{-aN}\left(a^2N^2 - a(a+2)N + a - 2\right).\end{aligned} \tag{2.16}$$

In view of $a \le 2$, it follows that $\psi''(N) \le 0$ and $\psi'(N) \le \psi'(0) = 0$ in the region Ω. Then $\psi(N) \le 0$ is always true. It follows that $\varphi'(N) \le 0$, that is, $V' \le 0$. Consequently, the function $V(N,P)$ satisfies the asymptotic stability theorem [14]. Hence, $E^* = (N^*, P^*)$ is globally asymptotically stable. This completes the proof. □

3. Applications

In this paper, we construct a new Lyapunov function for proving the global asymptotical stability of model (1.5). The new Lyapunov function is useful not only to model (1.5), but also to other models.

In this section, we will give some examples to show the robustness of the Lyapunov function (2.6). The parameters of the following models are positive and have the same ecological meanings with those of in model (1.5).

Example 3.1. Considering the following Ivlev predator-prey model incorporating prey refuges (see [9]):

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right) - \left(1-e^{-a(1-m)N}\right)P,\\ \frac{dP}{dt} &= \left(1-e^{-a(1-m)N} - \delta P - d\right)P,\end{aligned} \tag{3.1}$$

where $m \in [0,1)$ is a refuge protecting of the prey. We can choose a Lyapunov functional as follows:

$$V(N,P) = \int_{N^*}^{N} \frac{1-e^{-a(1-m)\xi} - \delta P^* - d}{1-e^{-a(1-m)\xi}}\,d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta}\,d\eta. \tag{3.2}$$

The proof is similar to that of the Section 2.

Example 3.2. Considering the following predator-prey model with Rosenzweig functional response (see [10]):

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right) - bN^{\mu}P,\\ \frac{dP}{dt} &= (cN^{\mu} - \delta N - d)P,\end{aligned} \tag{3.3}$$

where $\mu \in (0,1]$ is the victim's competition constant. We can choose a Lyapunov functional as follows:

$$V(N,P) = \int_{N^*}^{N} \frac{cN^{\xi} - \delta P^* - d}{bN^{\xi}} d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta. \tag{3.4}$$

We omit the proof here.

Example 3.3. Considering the following model (1.1) with Holling-type functional response (see [11]):

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - bg(N)P, \\ \frac{dP}{dt} &= \left(g(N) - \delta P - d\right)P, \end{aligned} \tag{3.5}$$

where $g(N) = \alpha N/(\beta + N)$ is known as a Holling type-II function, $g(N) = \alpha N^2/(\beta + N^2)$ as a Holling type-III function and $g(N) = \alpha N^2/(\beta + \omega N + N^2)$ as a Holling type-IV function. We choose a Lyapunov function:

$$V(N,P) = \int_{N^*}^{N} \frac{g(\xi) - \delta P^* - d}{bg(\xi)} d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta. \tag{3.6}$$

For more details, we refer to [12].

Example 3.4. Considering the following diffusive Ivlev-type predator-prey model (see [13]):

$$\begin{aligned} \frac{\partial N}{dt} &= rN(1 - N) - \left(1 - e^{-aN}\right)P + d_1 \nabla^2 N, \\ \frac{\partial P}{dt} &= \left(\varepsilon\left(1 - e^{-aN}\right) - d\right)P + d_2 \nabla^2 P, \end{aligned} \tag{3.7}$$

where the nonnegative constants d_1 and d_2 are the diffusion coefficients of N and P, respectively. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the usual Laplacian operator in two-dimensional space, is used to describe the Brownian random motion.

Model (3.7) is to be analyzed under the following non-zero initial conditions

$$N(x,y,0) \geq 0, \quad P(x,y,0) \geq 0, \quad (x,y) \in \Omega = [0, Lx] \times [0, Ly], \tag{3.8}$$

and zero-flux boundary conditions:

$$\frac{\partial N}{\partial n} = \frac{\partial P}{\partial n} = 0. \tag{3.9}$$

In the above, n is the outward unit normal vector of the boundary $\partial\Omega$.

In order to give the proof of the global stability, we construct a Lyapunov function:

$$E(t) = \iint_{\Omega} V(N(t), P(t))\, dx\, dy, \tag{3.10}$$

where

$$V(N, P) = \int_{N^*}^{N} \frac{\varepsilon(1 - e^{-a\xi}) - d}{1 - e^{-a\xi}} d\xi + \int_{P^*}^{P} \frac{\eta - P^*}{\eta} d\eta. \tag{3.11}$$

Then, differentiating $E(t)$ with respect to time t along the solutions of model (3.7), we can obtain

$$\frac{dE(t)}{dt} = \iint_{\Omega} \frac{dV}{dt}\, dx\, dy + \iint_{\Omega} \left(\frac{\partial V}{\partial N} d_1 \nabla^2 N + \frac{\partial V}{\partial P} d_2 \nabla^2 P \right) dx\, dy. \tag{3.12}$$

Using Green's first identity in the plane, and considering the zero-flux boundary conditions (3.9), one can show that

$$\begin{aligned} \frac{dE(t)}{dt} &= \iint_{\Omega} \frac{dV}{dt}\, dx\, dy - \frac{d_1 \partial^2 V}{\partial N^2} \iint_{\Omega} \left[\left(\frac{\partial N}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 \right] dx\, dy \\ &\quad - \frac{d_2 \partial^2 V}{\partial P^2} \iint_{\Omega} \left[\left(\frac{\partial P}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 \right] dx\, dy \\ &\leq \iint_{\Omega} \frac{dV}{dt}\, dx\, dy. \end{aligned} \tag{3.13}$$

The remaining arguments are rather similar as Theorem 2.3.

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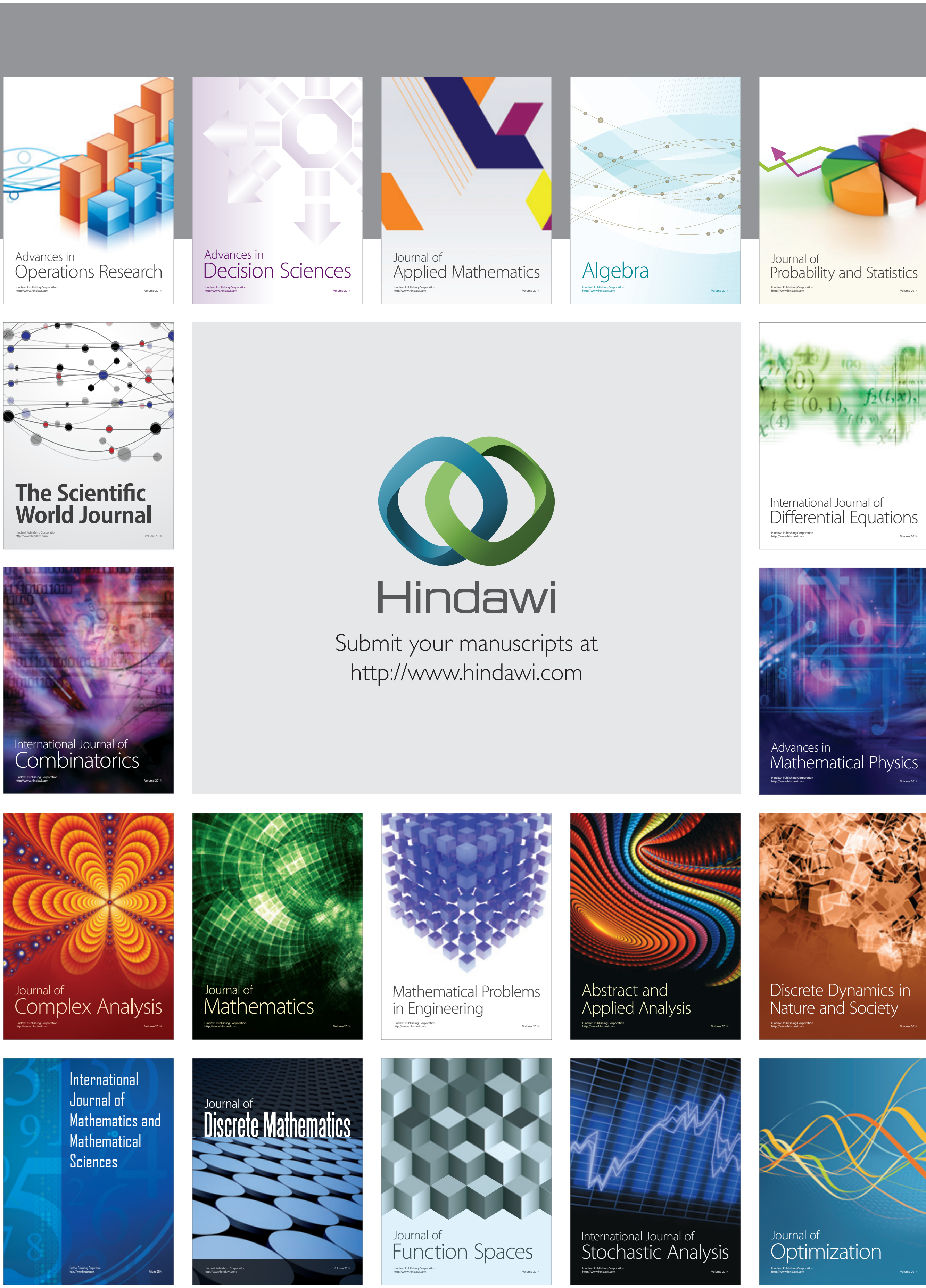

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