```
In[1]:= ClearAll["Global`*"]

In[2]:= k = 4

Out[2]= 4
```

This code attempts computing analytically the expectations for the various terms in Eq. 28 (Sec B1) of the Supplementary Information. The code computes the expectations for n = {2,3,...k} (with k small enough), and uses these values to verify the equations in Sec B1. To verify the equations for larger values of n, increase the value k above (the computing time grows exponentially; the values have been verified for n = 10)

Outer product

```
In[3]:= Op[a_, b_] := Outer[Times, a, b]
```

Build symmetric matrix with arbitrary coefficients

```
In[4]:= BuildB[n_] := Module[{A, B}, A = UpperTriangularize[Table[b[i, j], {i, 1, n}, {j, 1, n}], 1];
   B = A + Transpose[A];
   B]
```

Remove the mean from matrix B

```
In[5]:= BuildC[n_] := Module[{B, ones, m, K},
   B = BuildB[n];
   ones = Table[1, n];
   m = 1 / n Flatten[Transpose[B].ones]; K = B - Op[ones, m]; K]
```

This is the right hand side of the equations

```
In[6]:= Buildb[n_] := Table[1 / α, n]
```

Build the initial guess

```
In[7]:= Buildy0[n_] := Table[1 / α, n] - 1 / α ^ 2 BuildC[n].Table[1, n]
```

This is the matrix M (which depends on the correlation; this is the version for symmetric matrices, as specified in Sec B1)

```
In[8]:= BuildM[n_] := Module[{M},
   M = (IdentityMatrix[n] - 1 / α Transpose[BuildC[n]]).(IdentityMatrix[n] + 1 / α BuildC[n]); M]
```

These are the expectations for the powers of Bij

```
In[9]:= r6 := Flatten[Table[b[i, j] ^ 6 → μ6, {i, 1, n}, {j, 1, n}]]

In[10]:= r5 := Flatten[Table[b[i, j] ^ 5 → μ5, {i, 1, n}, {j, 1, n}]]

In[11]:= r4 := Flatten[Table[b[i, j] ^ 4 → μ4, {i, 1, n}, {j, 1, n}]]
```

```
In[12]:= r3 := Flatten[Table[b[i, j]^3 → μ3, {i, 1, n}, {j, 1, n}]]
```

```
In[13]:= r2 := Flatten[Table[b[i, j]^2 → μ2, {i, 1, n}, {j, 1, n}]]
```

```
In[14]:= rrem := Flatten[Table[b[i, j] → 0, {i, 1, n}, {j, 1, n}]]
```

Compute expectation given a matrix of expressions

```
In[15]:= ExpandMat[Z_] := Total[Total[ExpandAll[Z] /. r6 /. r5 /. r4 /. r3 /. r3 /. r2 /. r2 /. r2 /. rrem]]
```

```
In[16]:= ExpandExpression[x_] := ExpandAll[x] /. r6 /. r5 /. r4 /. r3 /. r3 /. r2 /. r2 /. r2 /. rrem
```

Having set up the problem, we compute all the terms in Eq 28

1. Term $b^T b$. First, we compute the expectation for this term for different values of n

```
In[17]:= btb = FullSimplify[Table[Buildb[n].Buildb[n], {n, 2, k}]]
```

Out[17]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2}, \frac{4}{\alpha^2}\right\}$$

The formula in Sec B1 reads

```
In[18]:= predictedbtb = Table[n / α^2, {n, 2, k}]
```

Out[18]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2}, \frac{4}{\alpha^2}\right\}$$

Make sure that the terms match

```
In[19]:= FullSimplify[btb - predictedbtb]
```

Out[19]=

```
{0, 0, 0}
```

2. Term $b^T y_0$. First, we compute the expectation for this term for different values of n

```
In[20]:= bty0 = FullSimplify[Table[Buildb[n].Buildy0[n], {n, 2, k}]]
```

Out[20]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2}, \frac{4}{\alpha^2}\right\}$$

The formula in Sec B1 reads

```
In[21]:= predictedbty0 = Table[n / α^2, {n, 2, k}]
```

Out[21]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2}, \frac{4}{\alpha^2}\right\}$$

Make sure that the terms match

In[22]:=
```
FullSimplify[bty0 - predictedbty0]
```
Out[22]=
```
{0, 0, 0}
```

3. Term $y_0^T\, y_0$. First, we compute the expectation for this term for different values of n

In[23]:=
```
y0ty0 = FullSimplify[Table[ExpandExpression[Buildy0[n].Buildy0[n]], {n, 2, k}]]
```
Out[23]=

$$\left\{\frac{2}{\alpha^2},\ \frac{3\,\alpha^2+2\,\mu 2}{\alpha^4},\ \frac{4\,\alpha^2+6\,\mu 2}{\alpha^4}\right\}$$

The formula in Sec B1 reads

In[24]:=
```
predictedy0ty0 = Table[n / α ^ 2 + 1 / α ^ 4 μ2 (n - 1) (n - 2), {n, 2, k}]
```
Out[24]=

$$\left\{\frac{2}{\alpha^2},\ \frac{3}{\alpha^2}+\frac{2\,\mu 2}{\alpha^4},\ \frac{4}{\alpha^2}+\frac{6\,\mu 2}{\alpha^4}\right\}$$

Make sure that the terms match

In[25]:=
```
FullSimplify[y0ty0 - predictedy0ty0]
```
Out[25]=
```
{0, 0, 0}
```

4. Term $b^T\, M\, y_0$. First, we compute the expectation for this term for different values of n

In[26]:=
```
btMy0 = Table[ExpandExpression[FullSimplify[Buildb[n].BuildM[n].Buildy0[n]]], {n, 2, k}]
```
Out[26]=

$$\left\{\frac{2}{\alpha^2},\ \frac{3}{\alpha^2}+\frac{2\,\mu 3}{3\,\alpha^5},\ \frac{4}{\alpha^2}+\frac{3\,\mu 3}{\alpha^5}\right\}$$

The formula in Sec B1 reads

In[27]:=
```
predictedbtMy0 = Table[n / α ^ 2 + μ3 (n - 1) (n - 2) ^ 2 / (n α ^ 5), {n, 2, k}]
```
Out[27]=

$$\left\{\frac{2}{\alpha^2},\ \frac{3}{\alpha^2}+\frac{2\,\mu 3}{3\,\alpha^5},\ \frac{4}{\alpha^2}+\frac{3\,\mu 3}{\alpha^5}\right\}$$

Make sure that the terms match

In[28]:=
```
FullSimplify[btMy0 - predictedbtMy0]
```
Out[28]=
```
{0, 0, 0}
```

5. Term $y_0^T\, M\, y_0$. First, we compute the expectation for this term for different values of n

In[29]:=
```
y0tMy0 = Table[ExpandExpression[FullSimplify[Buildy0[n].BuildM[n].Buildy0[n]]], {n, 2, k}]
```

Out[29]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2} - \frac{16\,\mu 2^2}{3\,\alpha^6} + \frac{4\,\mu 3}{3\,\alpha^5} - \frac{2\,\mu 4}{9\,\alpha^6}, \frac{4}{\alpha^2} - \frac{33\,\mu 2^2}{2\,\alpha^6} + \frac{6\,\mu 3}{\alpha^5} - \frac{3\,\mu 4}{2\,\alpha^6}\right\}$$

The formula in Sec B1 reads

In[30]:=
```
predictedy0tMy0 = Table[(n) / α ^ 2 - μ4 (n - 1) (n - 2) ^ 3 / (n ^ 2 α ^ 6) +
    μ3 2 (n - 1) (n - 2) ^ 2 / (n α ^ 5) - μ2 ^ 2 (n - 1) (n - 2) (n ^ 3 - 5 n ^ 2 + 18 n - 12) / (n ^ 2 α ^ 6), {n, 2, k}]
```

Out[30]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2} - \frac{16\,\mu 2^2}{3\,\alpha^6} + \frac{4\,\mu 3}{3\,\alpha^5} - \frac{2\,\mu 4}{9\,\alpha^6}, \frac{4}{\alpha^2} - \frac{33\,\mu 2^2}{2\,\alpha^6} + \frac{6\,\mu 3}{\alpha^5} - \frac{3\,\mu 4}{2\,\alpha^6}\right\}$$

Make sure that the terms match

In[31]:=
```
FullSimplify[y0tMy0 - predictedy0tMy0]
```

Out[31]=
```
{0, 0, 0}
```

6. Term $\boldsymbol{y_0^T M^T M y_0}$. First, we compute the expectation for this term for different values of n

In[32]:=
```
y0tMtMy0 = Table[ExpandExpression[
    FullSimplify[Buildy0[n].Transpose[BuildM[n]].BuildM[n].Buildy0[n]]], {n, 2, k}]
```

Out[32]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2} + \frac{16\,\mu 2^2}{3\,\alpha^6} + \frac{104\,\mu 2^3}{9\,\alpha^8} + \frac{4\,\mu 3}{3\,\alpha^5} + \frac{208\,\mu 2\,\mu 3}{27\,\alpha^7} - \frac{20\,\mu 3^2}{27\,\alpha^8} + \frac{2\,\mu 4}{9\,\alpha^6} + \frac{236\,\mu 2\,\mu 4}{27\,\alpha^8} - \frac{4\,\mu 5}{27\,\alpha^7} + \frac{2\,\mu 6}{27\,\alpha^8},\right.$$
$$\left.\frac{4}{\alpha^2} + \frac{33\,\mu 2^2}{2\,\alpha^6} + \frac{57\,\mu 2^3}{\alpha^8} + \frac{6\,\mu 3}{\alpha^5} + \frac{15\,\mu 2\,\mu 3}{\alpha^7} - \frac{9\,\mu 3^2}{\alpha^8} + \frac{3\,\mu 4}{2\,\alpha^6} + \frac{129\,\mu 2\,\mu 4}{4\,\alpha^8} - \frac{3\,\mu 5}{2\,\alpha^7} + \frac{3\,\mu 6}{4\,\alpha^8}\right\}$$

The formula in Sec B1 reads

In[33]:=
```
predictedy0tMtMy0 = Table[n / α ^ 2 + μ6 (n - 1) (n - 2) ^ 4 / (n ^ 3 α ^ 8) - μ5 2 (n - 1) (n - 2) ^ 4 / (n ^ 3 α ^ 7) -
    μ2 μ3 4 (n - 1) (n - 2) (n ^ 4 - 12 n ^ 3 + 49 n ^ 2 - 88 n + 40) / (n ^ 3 α ^ 7) +
    μ4 μ2 2 (-2 + n) (-1 + n) (92 - 140 n + 76 n^2 - 17 n^3 + 2 n^4) / (n ^ 3 α ^ 8) + μ4 (n - 1) (n - 2) ^ 3 / (n ^ 2 α ^ 6) +
    μ2 ^ 3 (n - 1) (n - 2) (-432 + 724 n - 392 n^2 + 105 n^3 - 17 n^4 + 2 n^5) / (n ^ 3 α ^ 8) +
    μ3 ^ 2 2 (n - 1) (n - 2) (64 - 92 n + 56 n^2 - 14 n^3 + n^4) / (n ^ 3 α ^ 8) + μ3 2 (n - 1) (n - 2) ^ 2 / (n α ^ 5) +
    μ2 ^ 2 (-2 + n) (-1 + n) (-12 + 18 n - 5 n^2 + n^3) / (n ^ 2 α ^ 6), {n, 2, k}]
```

Out[33]=

$$\left\{\frac{2}{\alpha^2}, \frac{3}{\alpha^2} + \frac{16\,\mu 2^2}{3\,\alpha^6} + \frac{104\,\mu 2^3}{9\,\alpha^8} + \frac{4\,\mu 3}{3\,\alpha^5} + \frac{208\,\mu 2\,\mu 3}{27\,\alpha^7} - \frac{20\,\mu 3^2}{27\,\alpha^8} + \frac{2\,\mu 4}{9\,\alpha^6} + \frac{236\,\mu 2\,\mu 4}{27\,\alpha^8} - \frac{4\,\mu 5}{27\,\alpha^7} + \frac{2\,\mu 6}{27\,\alpha^8},\right.$$
$$\left.\frac{4}{\alpha^2} + \frac{33\,\mu 2^2}{2\,\alpha^6} + \frac{57\,\mu 2^3}{\alpha^8} + \frac{6\,\mu 3}{\alpha^5} + \frac{15\,\mu 2\,\mu 3}{\alpha^7} - \frac{9\,\mu 3^2}{\alpha^8} + \frac{3\,\mu 4}{2\,\alpha^6} + \frac{129\,\mu 2\,\mu 4}{4\,\alpha^8} - \frac{3\,\mu 5}{2\,\alpha^7} + \frac{3\,\mu 6}{4\,\alpha^8}\right\}$$

Make sure that the terms match

In[34]:=
```
FullSimplify[y0tMtMy0 - predictedy0tMtMy0]
```

Out[34]=
```
{0, 0, 0}
```