Computational Geometry

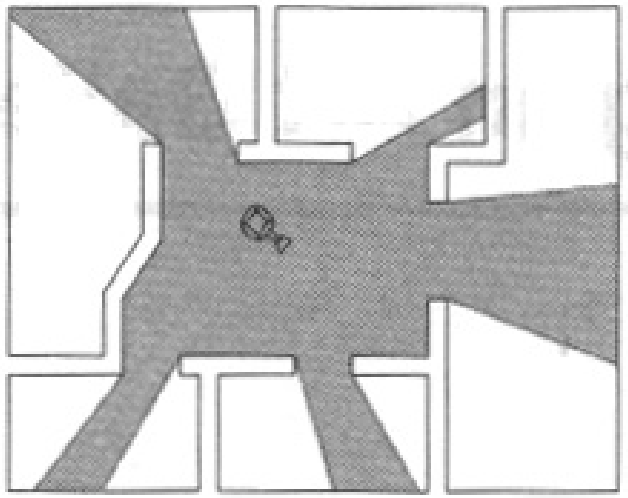
6. Polygon Triangulation

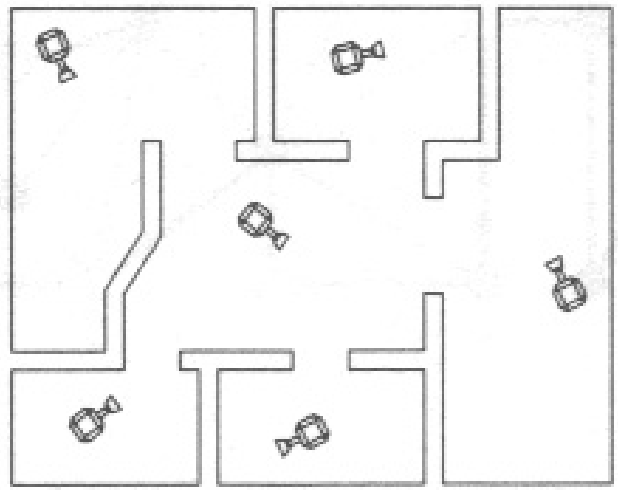
6.1 The Art-Gallery Problem

A museum shall be equipped with cameras, each can observe an angle of 360°, covering the whole area of an art gallery.

The *Art-Gallery-Problem* is, to determine

- the smallest number of cameras, and
- where these cameras should be placed.





6.1 The Art-Gallery Problem

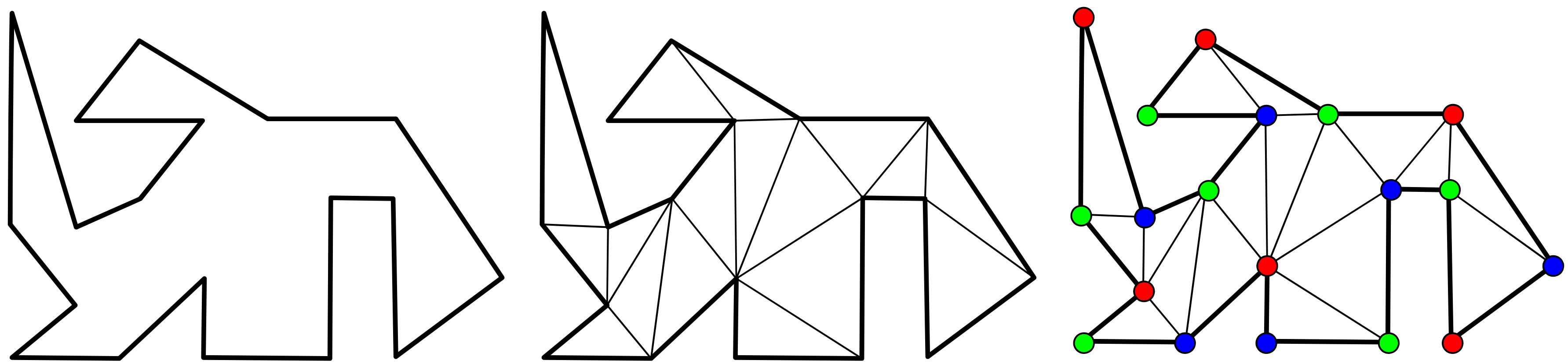
- The area of the gallery is supposed to be a simple polygon P (i.e. a single component without holes) with n vertices, where adjacent vertices are not collinear.
- Determining the minimal number of cameras and their positions is NP-hard for arbitrary polygons.
- **But:** For P with n vertices there is an upper bound for the number of cameras: $\lfloor n/3 \rfloor$.

Partitioning the polygon into triangles (*triangulation*) can solve this problem efficiently.

6.1 The Art-Gallery Problem

Idea

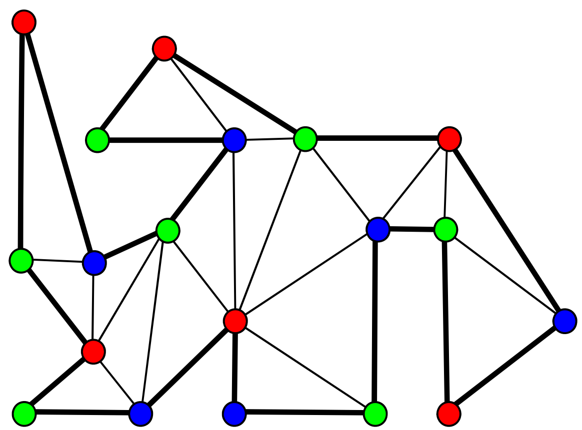
- Triangulate the polygon.
- Construct a 3-coloring (red, green, blue) of the vertices, such that the vertices of a triangle have different colors.
- Place cameras at the vertices with the least frequent color. This color occurs in every triangle exactly once.



6.1 The Art-Gallery Problem

Questions

- Is it possible to partition every simple polygon without self-intersections into triangles?
- Is there always a 3-coloring of the vertices?
- Is both efficiently computable?



6.1 The Art-Gallery Problem

Proposition 1

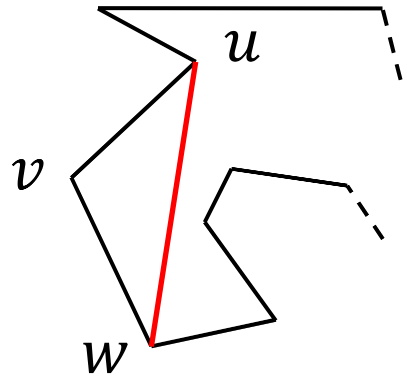
Every simple polygon with n vertices has a triangulation and every triangulation has exactly $n - 2$ triangles.

Proof (Induction on n)

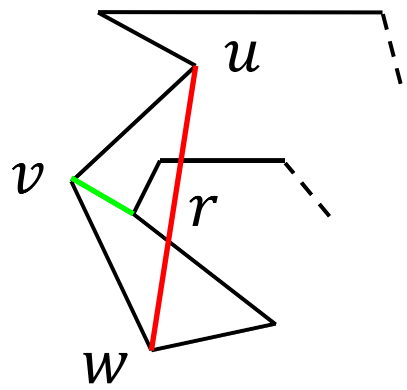
- **Start:** $n = 3$.
- **Assumption:** Proposition 1 holds for all $m < n, n > 3$.
- **Step:** Let P be a polygon with n vertices, and u, v, w a sequence of adjacent vertices, where v has the smallest x-coordinate (and among those the smallest y-coordinate).

6.1 The Art-Gallery Problem

1. If the line segment $\overline{uw}$ lies in the interior of P, the polygon can be subdivided along $\overline{uw}$.



2. Otherwise P has at least on vertex in interior of the triangle uvw.
   - Chose among those the vertex r with maximal distance to the edge $\overline{uw}$.
   - Then $\overline{vr}$ is in the interior of P.
     - If $\overline{vr}$ were intersected, there would be a vertex r' with larger distance to $\overline{uw}$.



6.1 The Art-Gallery Problem

- Subdivide the polygon along $\overline{uw}$ or $\overline{vr}$ into two polygons with m_1 und m_1 vertices $(m_1 + m_2 = n + 2)$
- From the induction assumption both can be triangulated.
- The total number of triangles is

$$(m_1 - 2) + (m_2 - 2) = n - 2. \qquad \square$$

Remark

- Placing the cameras at inner edges, every camera can observe two triangles.
- Placing the cameras at the vertices, every camera can observe at least two triangles.

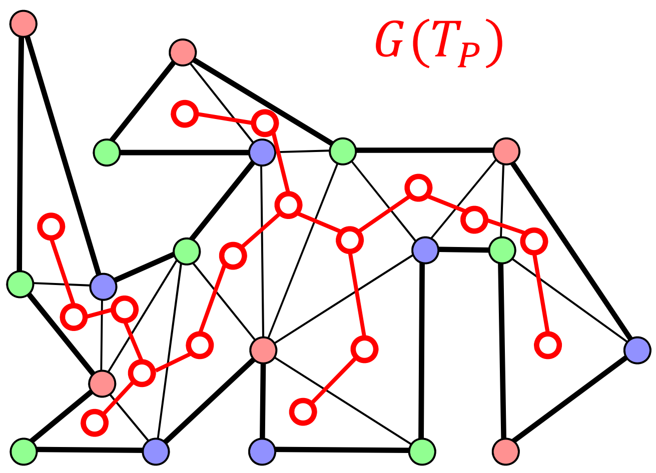
6.1 The Art-Gallery Problem

Proposition 2 (Art-Gallery-Theorem)

For a simple polygon P with n vertices $\lfloor n/3 \rfloor$ cameras are sufficient and in some cases also necessary, such that every inner point is visible from at least one camera

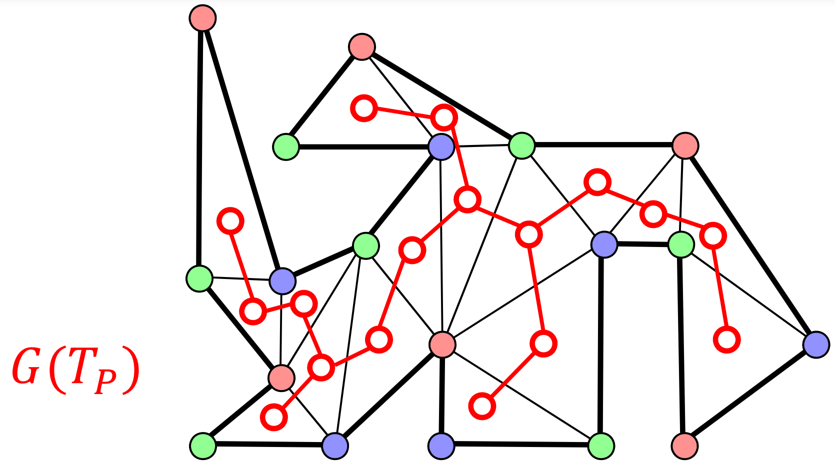
Proof

1. First prove that a 3-coloring of the triangulation T_P of P always exists.
   - The dual graph $G(T_P)$ of T_P has one node for every triangle in T_P.
   - The nodes of edge-adjacent triangles are connected by an edge in $G(T_P)$.



6.1 The Art-Gallery Problem

- Each edge of $G(T_P)$ intersects one inner edge of T_P, which partitions the polygon P in two separate sub-polygons.



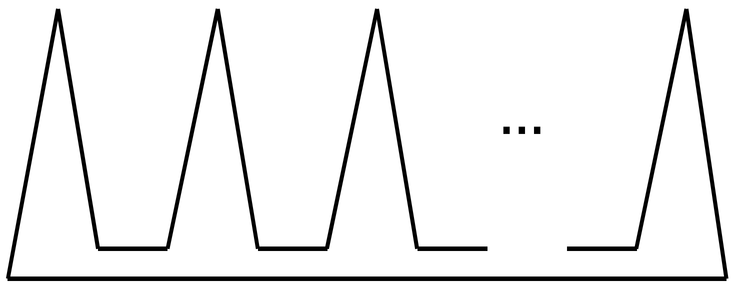
- Thus, $G(T_P)$ does not have cycles, i.e. it is a tree.
- For the 3-coloring start with an arbitrary triangle and follow recursively the edges of $G(T_P)$.
- For every triangle there remains exactly one vertex to color, so that its color is determined uniquely.

6.1 The Art-Gallery Problem

2. The number of vertices with the least frequent color is at most $\lfloor n/3 \rfloor$.

   Because every triangle has one vertex of this color, $\lfloor n/3 \rfloor$ cameras are sufficient.

3. $n/3$ cameras are also necessary, because a polygon with k peaks ($n = 3k$), each of which is completely visible from only one camera, can be constructed:



$\square$

6.2 Monotone Partitioning

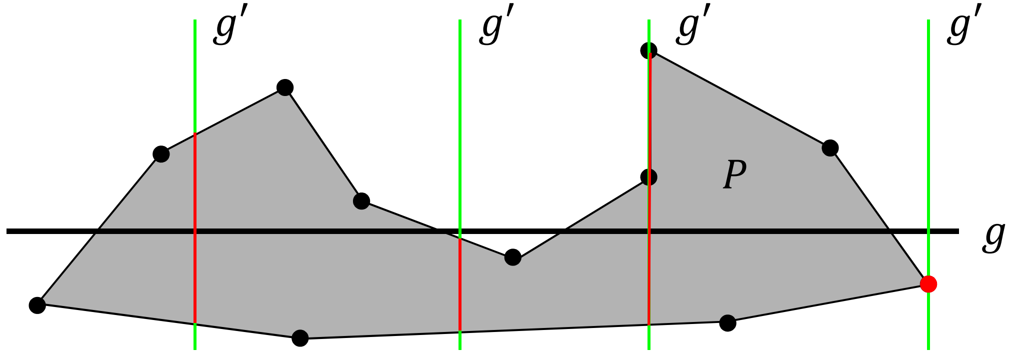
- The proof of Proposition 1 gives a recursive algorithm to construct a triangulation.
  - Searching for a diagonal to subdivide the polygon takes $O(n)$, yielding a total run time of $O(n^2)$.
- Convex polygons can be triangulated in $O(n)$.
  - **But:** Partitioning a polygon into convex regions is as complex as computing a triangulation.
- An efficient approach is the partitioning into *monotone polygons*.

6.2 Monotone Partitioning

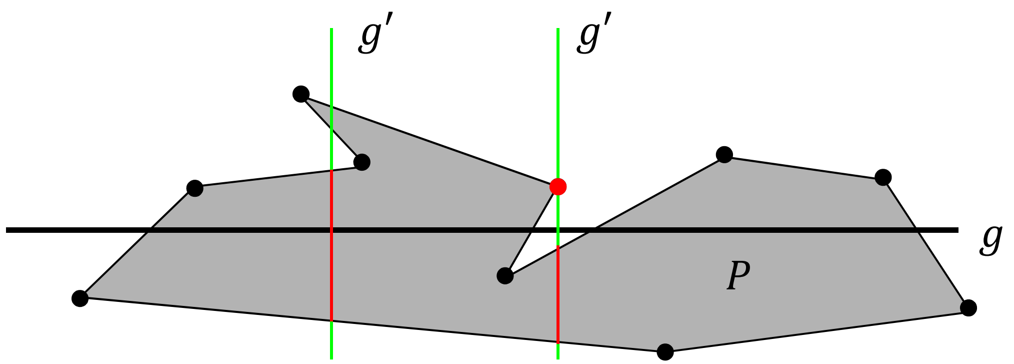
Definition 1

A polygon P is called *monotone* with respect to a line g, if for **every** perpendicular line $g' \perp g$ the intersection is $P \cap g'$ connected.

A polygon monotone with respect to the y-axis is called *y-monotone*.



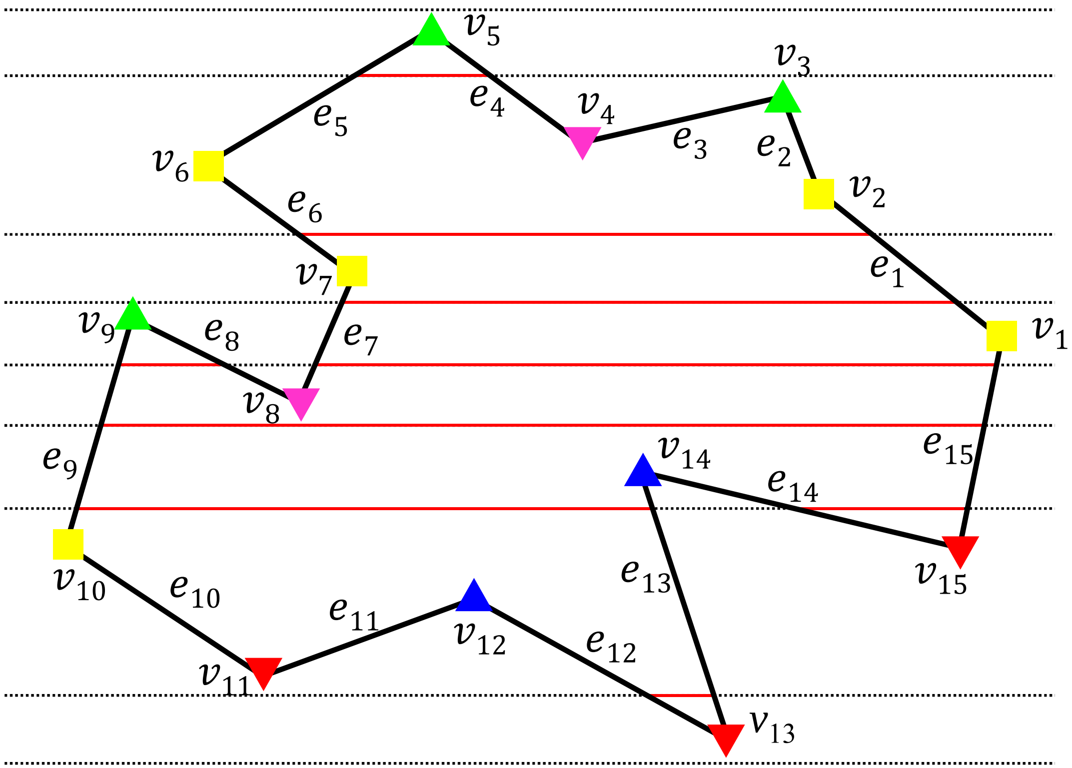
- In other words, $P \cap g'$ is either a line segment, a point, or empty.



6.2 Monotone Partitioning

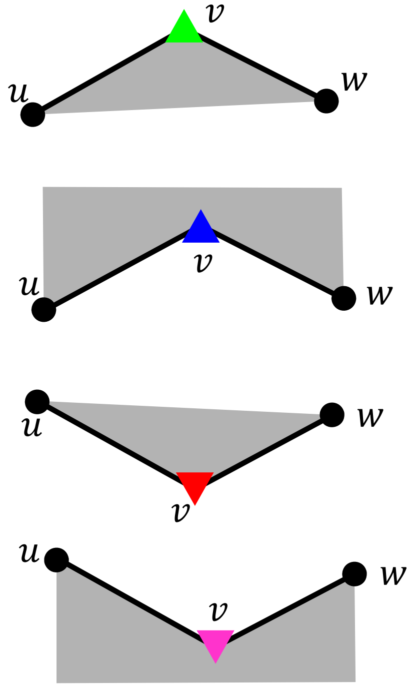
Approach: 1. Partition P in y-monotone pieces, 2. which are triangulated top to bottom.

- For the monotone partitioning the corners are classified into five categories:
  - ▲ Start vertex
  - ▼ End vertex
  - ■ Regular vertex
  - ▲ Split vertex
  - ▼ Merge vertex



6.2 Monotone Partitioning

- **Definition 2**
  A point u is *below* of v ($u < v$), if $u_y < v_y$ or $u_y = v_y$ and $u_x > v_x$. A point u is *above* of v, if $v < u$.
- **Definition 3**
  A vertex v of a polygon with inner angle $a(v)$ and neighbors u, w is called



  - ▲ ***start vertex***, if $u < v, w < v$ and $a(v) < \pi$
  - ▲ ***split vertex***, if $u < v, w < v$ and $a(v) > \pi$
  - ▼ ***end vertex***, if $u > v, w > v$ and $a(v) < \pi$
  - ▼ ***merge vertex***, if $u > v, w > v$ and $a(v) > \pi$
  - ■ ***regular vertex***, otherwise.

6.2 Monotone Partitioning

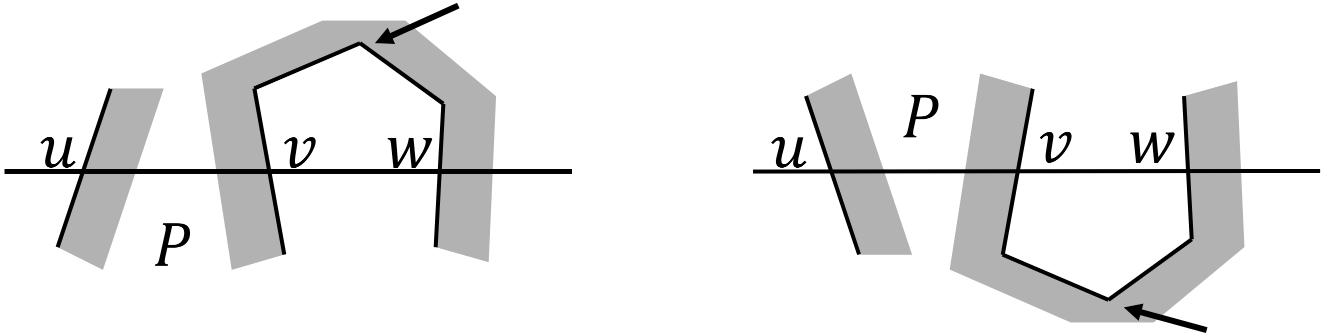
Lemma 3

A polygon P is y-monotone, if it has no split and merge vertices.

Proof

- Assume P is not y-monotone. Then P should have a split or merge vertex.
- If P is not y-monotone, there is a horizontal line g, that intersects or touches P in at least three points u, v, w (from left to right): w.l.o.g. a line segment $\overline{uv}$ and a point w.

6.2 Monotone Partitioning



- Traversing the polygon chain starting in v in both directions, the next intersections with g are u and w.
- If the polygon chain $u..v$ has an end (start) vertex, the polygon chain $v..w$ must have a split (merge) vertex.
  - Analog if $v..w$ has an end or start vertex.
- Thus, P has at least one split or merge vertex.

□

6.2 Monotone Partitioning

- The partitioning into y-monotone polygons can be computed by inserting suitable diagonals to get rid of the split and merge vertices.
  - These diagonals should go upward from a split vertex and downward from a merge vertex (without intersection).
  - This way they become regular vertices of the resulting sub-polygons,
    - because in both sub-polygons the predecessor and successor vertices of the former split/merge vertex lie on different sides (above and below) of the former split/merge vertex.

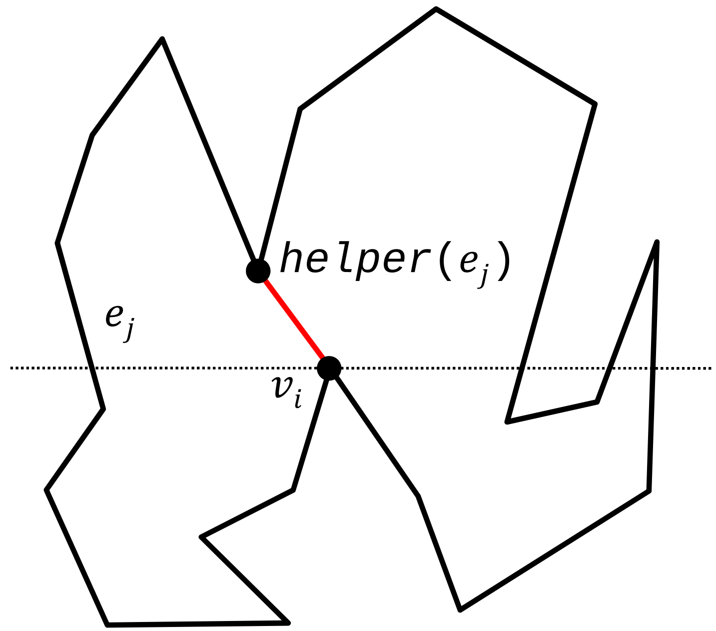
6.2 Monotone Partitioning

- Use a horizontal sweep-line moving from top to bottom.
  - At every vertex an event is triggered and processed.
  - The vertices v_i of P are stored in a priority queue sorted from top to bottom, i.e. the priority is the y-coordinate.
  - Edges $e_i = \overline{v_i v_{i+1}}$ are stored in a doubly linked list, to compute neighbors and partitionings in constant time.

6.2 Monotone Partitioning

Goal: Add diagonal from a split vertex v_i to a vertex above.

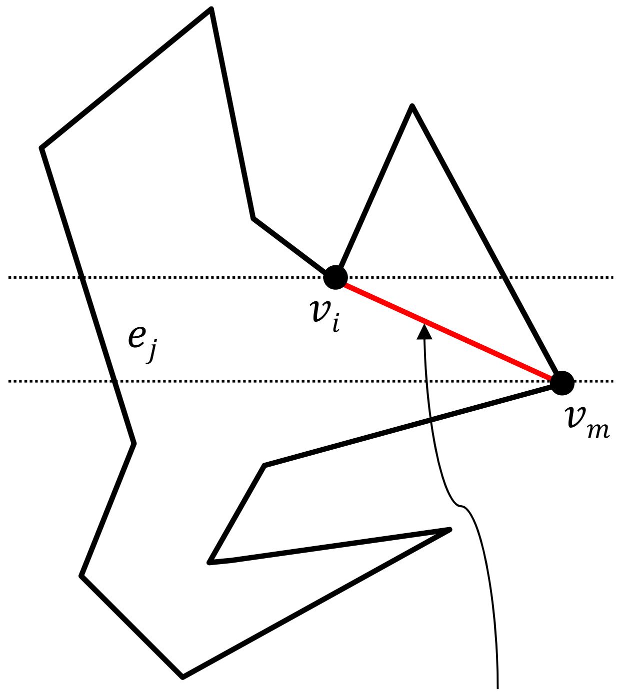
- Let v_i be a split vertex on the sweep-line and e_j the edge left of it on the sweep-line.
- The vertex `helper`(e_j) is the lowest vertex above the sweep-line, such that the horizontal line segment from e_j to `helper`(e_j) lies completely in P.
- Note, that the upper end point of e_j is always a candidate for `helper`(e_j).
- Insert diagonal from v_i to `helper`(e_j).



6.2 Monotone Partitioning

Goal: Add diagonal from a merge vertex v_i to a vertex below.

- Difficult, because the area below a merge vertex has not been explored by the sweep line as it reaches the merge vertex.
- Merge vertices are stored initially as `helper` of the next left edge (here e_j).
- As soon as e_j gets a new `helper`, this is connected with the merge vertex v_i.



Diagonal is added when the sweep line reaches v_m.

6.2 Monotone Partitioning

- The edges and their `helpers` are stored efficiently in a search tree.
- The sweep-line algorithm operates directly on the doubly linked list of the edges $e_i = \overline{v_i v_{i+1}}$.

Algorithm 1: Partition in y-monotone polygons
Input: Simple polygon P in doubly linked edge list D. **Output:** Partitioning of P in y-monotone pieces, stored in D.

```
Fill priority queue Q from top to bottom with vertices;
Initialize empty search tree T;
while (Q is not emmty) {
   Take topmost vertex v_i from Q;
   Process the event corresponding to the type of v_i;
}
```

6.2 Monotone Partitioning

- Events are processed depending on the type of the vertex.

StartVertexEvent(v_i)
1: Add e_i with helper(e_i):=v_i to T;

EndVertexEvent(v_i)
1: **if** (helper(e_{i-1}) is a merge vertex) **then** { 2: Add diagonal from v_i to helper(e_{i-1}) to D; 3: } 4: Remove e_{i-1} from T;

SplitVertexEvent(v_i)
1: Search in T for the edge e_j left of v_i; 2: Add diagonal from v_i to helper(e_j) to D; 3: helper(e_j):=v_i; 4: Add e_i with helper(e_i):=v_i to T;

6.2 Monotone Partitioning

MergeVertexEvent(v_i)

if (helper(e_{i-1}) is a merge vertex) **then** {
    Add diagonal from v_i to helper(e_{i-1}) to D;
}
Remove e_{i-1} from T;
Search in T for the edge e_j left of v_i;
if (helper(e_j) is a merge vertex) **then** {
    Add diagonal from v_i to helper(e_j) to D;
}
helper(e_j):=v_i;

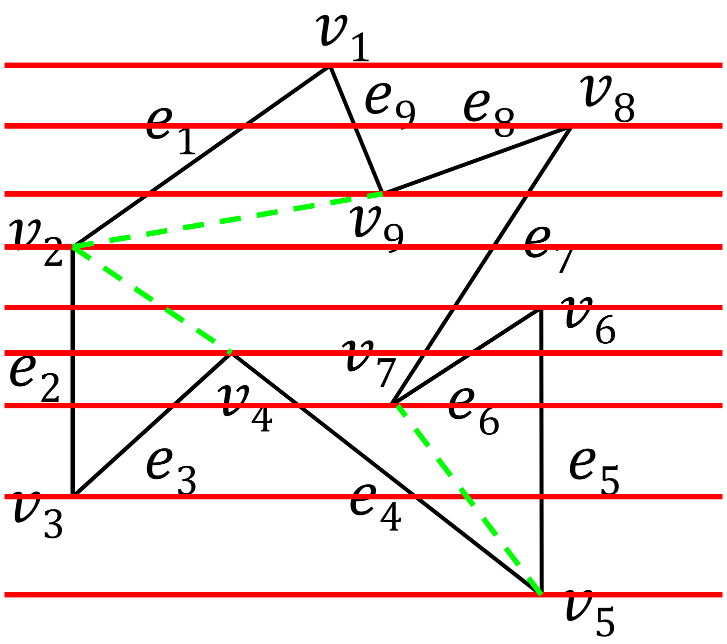
6.2 Monotone Partitioning

RegularVertexEvent(v_i)

```
if (the interior of P is right of v_i) then {
   if (helper(e_{i-1}) is a merge vertex) then {
      Add diagonal from v_i to helper(e_{i-1}) to D;
   }
   Remove e_{i-1} from T;
   Add e_i with helper(e_i):=v_i to T;
} else {
   Search in T for the edge e_j left of v_i;
   if (helper(e_j) is a merge vertex) then {
      Add diagonal from v_i to helper(e_j) to D;
   }
   helper(e_j):=v_i;
}
```

6.2 Monotone Partitioning

Example



event	content of T
v_1	e_1v_1
v_8	e_1v_1 e_8v_8
v_9	e_1v_9
v_2	e_2v_2
v_6	e_2v_2 e_6v_6
v_4	e_2v_4 e_4v_4 e_6v_6
v_7	e_2v_4 e_4v_7
v_3	e_4v_7
v_5	

- What is the content of the search tree T after each event?
- Which diagonals are added?

6.2 Monotone Partitioning

Lemma 4

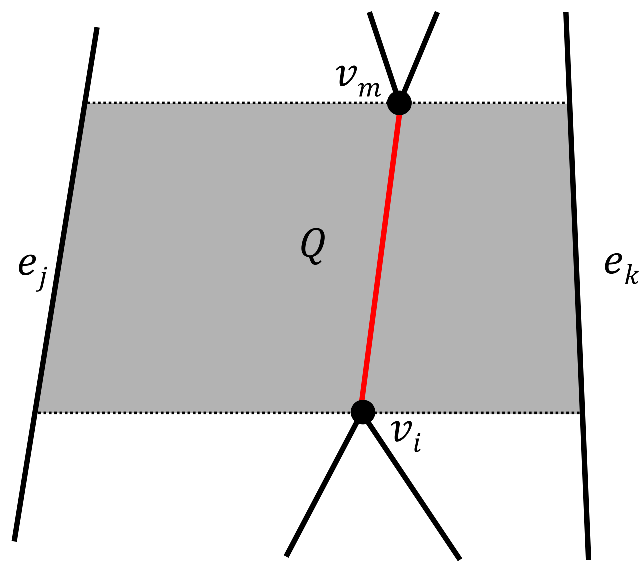
Algorithm 1 partitions P into y-monotone sub-polygons by adding non-intersecting diagonals.

Proof

- All split vertices are removed by upward and all merge vertices by downward diagonals. Thus, by Lemma 3 all remaining sub-polygons are y-monotone.
- That there are no intersections is proved for the example of diagonals generated by `SplitVertexEvent`.

6.2 Monotone Partitioning

- Let $\overline{v_m v_i}$ be the new line segment in `SplitVertexEvent`, e_j and e_k are the edges left and right of v_i, and `helper(`e_j`)` $= v_m$.
- Consider the area Q between e_j and e_k, bounded from below and above by lines parallel to the x-axis though v_i und v_m.
- Because v_m is the last event before v_i relative to e_j, Q does not contain any further edges or vertices.
- Thus, the diagonal has no intersections.



- The other events are treated analogously. □

6.2 Monotone Partitioning

Proposition 5

A simple polygon with n vertices can be partitioned into y-monotone pieces in $O(n \log n)$.

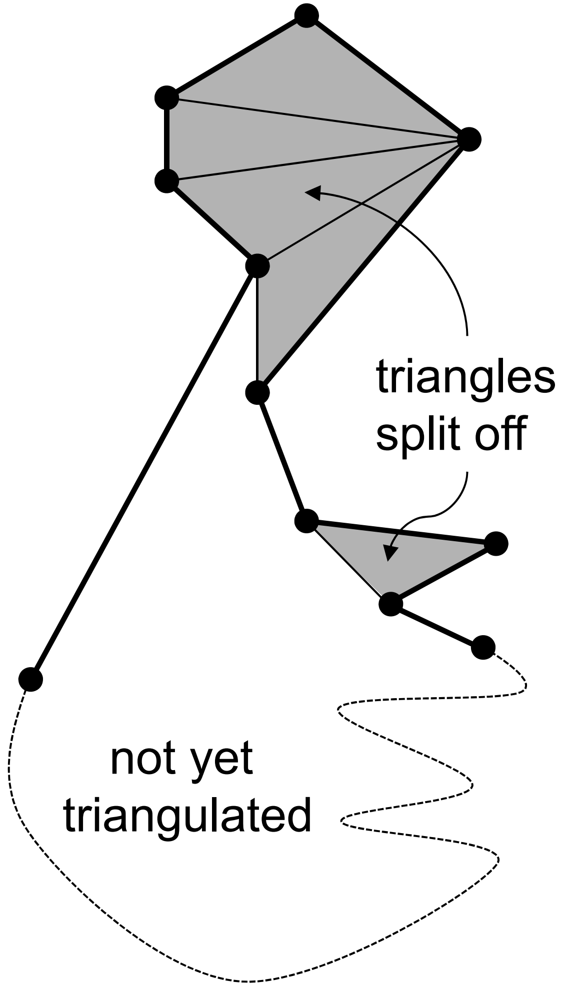
Proof

- Constructing Q takes $O(n \log n)$, and T is initialized in $O(1)$.
- Every event takes at most one operation at queue Q, two insertions on list D and one search, one insertion and one removal on tree T, which takes in total $O(\log n)$.
- Because there are n events, the total run time is $O(n \log n)$.

□

6.3 Triangulation of monotone Polygons

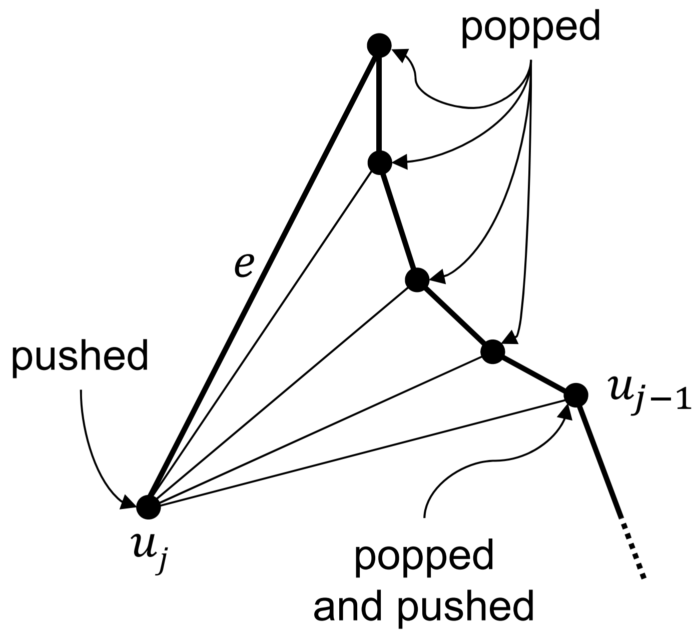
- To triangulate a y-monotone polygon the left and the right boundaries are processed top to bottom and the vertices a connected accordingly.
- Problems are caused by *reflex* vertices with an inner angle of $\alpha(v) > \pi$.
  - The vertices are stored in a stack, containing the not yet triangulated piece of the polygon above the sweep-line.
  - The lowest element of the stack belongs to the opposite side of the polygon.



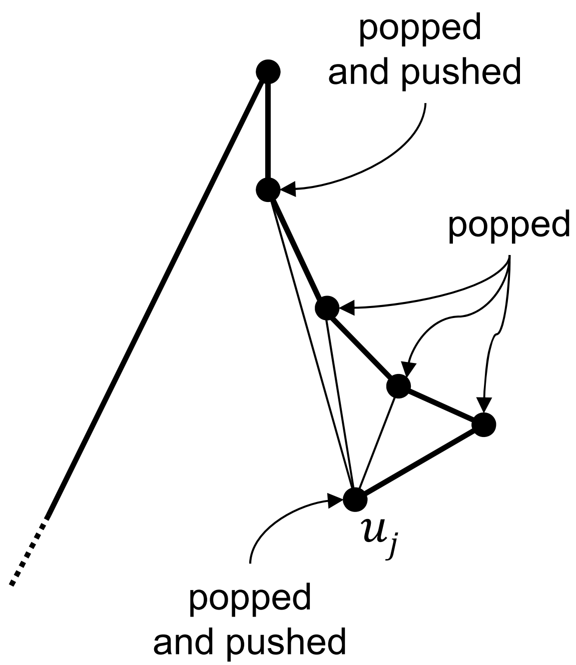
6.3 Triangulation of monotone Polygons

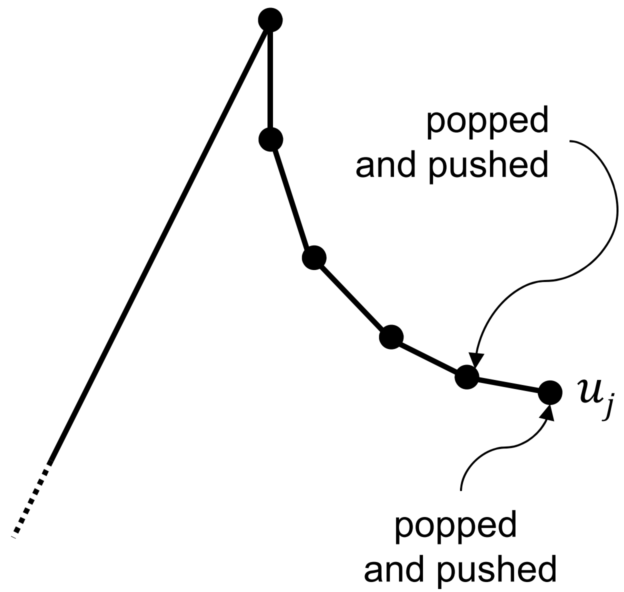
- Process the points $u_1, \dots, u_n$ of the y-monotone polygon from top to bottom.
- For a new point u_j, test if it lies on the opposite side of P as the top-most element on the stack.

1. In this case,
   a) all points from the stack can be connected to u_j except for the last one.
   b) Then push u_{j-1} and u_j back on the stack (u_j on top).



2. Otherwise,
   a) pop u_j from the stack and connect it to as many points on the stack as possible (i.e. the diagonal is inside P).
   b) The last of those and u_j are pushed back on the stack.





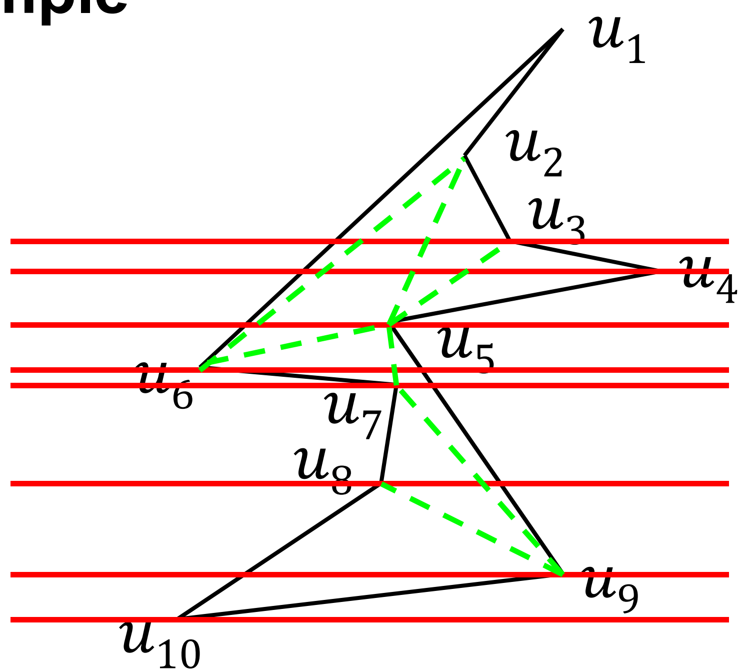
6.3 Triangulation of monotone Polygons

Algorithm 2: Triangulate y-monotone polygon

Input: A y-monotone polygon P in a doubly linked edge list D.
Output: Triangulation of P in D.

Sort vertices top to bottom $u_1, \dots, u_n$;
Initialize empty stack S; S.push(u_1); S.push(u_2);
for $(j = 3, \dots, n-1)$ {
    if (u_j and S.top() are on different sides) **then** {
        Pop all vertices from S and add their diagonals to u_j to D, except for the last one;
        S.push(u_j); S.push(u_{j-1});
} **else** {
        Pop one vertices from S;
        Pop the other vertices from S and add their diagonals to u_j to D, as long as this diagonal lies inside P;
Push the last poped vertex back; S.push(u_j);
} }
Pop all vertices from S and add their diagonals to u_n to D, except for the first and last one;

Example



j	stack S
	$u_2 u_1$
3	$u_3 u_2 u_1$
4	$u_4 u_3 u_2 u_1$
5	$u_5 u_2 u_1$
6	$u_6 u_5$
7	$u_7 u_5$
8	$u_8 u_7 u_5$
9	$u_9 u_8$
10	

- What is the content of the stack after each iteration?
- Which vertices are connected?

Lemma 6

A y-monotone polygon with n vertices can be triangulated in linear run time.

Proposition 7

A polygon with n vertices can be triangulated in $O(n \log n)$ using $O(n)$ of memory.

- This follows from the combination of Algorithms 1 and 2.

Remarks

- Algorithm 1 is also correct for polygons with holes.
- For triangulation of arbitrary polygons $\Omega(n \log n)$ is a lower bound [1].
- Simple polygons can be triangulated in $O(n)$.
  - A very complicated linear algorithm is due to Chazelle [2], 1991.
- The problem to triangulate (*tetrahedralization*) a 3d polytope is in general not solvable without auxiliary inner vertices.
  - There are efficient algorithms for such polytopes, but
  - to decide if auxiliary inner vertices are necessary is NP-complete.

6.4 Literature

[1] Marc de Berg et al., *Computational Geometry: Algorithms and Applications*, 2nd Edition, Springer, 2000, Chapter 3.

[2] B. Chazelle, *Triangulating a simple polygon in linear time*, Discrete Computational Geometry, 6:485-524, 1991.