

AMR for Fluids and Other Applications

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Overview

- 1 Finite Element Method: Convergence Theorem
- 2 Adaptivity Schemes and Firedrake/PETSc Compatibility

1 Finite Element Method: Convergence Theorem

2 Adaptivity Schemes and Firedrake/PETSc Compatibility

Definition (Linear Variational Problem)

Find $u \in H_{g_D}^1$ such that,

$$a(u, v) = F(v) \text{ for all } v \in H_0^1.$$

Where $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ is a bilinear form, and $F(\cdot) : H^1(\Omega) \rightarrow \mathbb{R}$ is a bounded linear form.

Theorem (Cea's Lemma; Ex: Elman et al. 2005)

Let u be the solution to a linear variational problem on H^1 and u_h be the finite element solution on S^h . If a is continuous and coercive then there exists constants $\gamma, \alpha > 0$ such that,

$$\|u - u_h\|_{H^1} \leq \frac{\gamma}{\alpha} \min_{v \in S^h} \|u - v\|_{H^1}.$$

Let $\pi_h(u)$ be the interpolant of u in S^h then,

$$\|u - u_h\|_{H^1} \leq \frac{\gamma}{\alpha} \|u - \pi_h(u)\|_{H^1}.$$

Theorem (1; Ex: Elman et al. 2005)

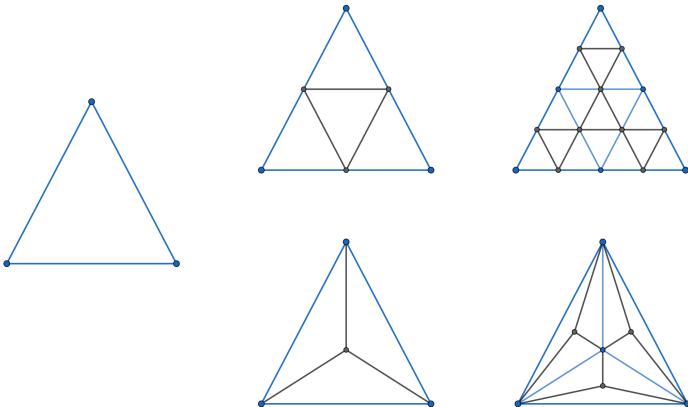
Let T_h be a triangulation and h_k be the largest length and θ_k be the minimum angle of $\Delta_k \in T_h$, then there exists some constant C_2

$$\|\nabla(u - \pi_h(u))\|_{L_2}^2 \leq C_2 \sum_{\Delta_k \in T_h} \frac{1}{\sin^2 \theta_k} h_k^2 \|D^2 u\|_{\Delta_k}^2.$$

- By estimation of interpolation error and the Bramble-Hilbert Lemma.

Definition (shape regularity; Ex: Elman et al. 2005)

A sequence of triangulations $\{T_h\}$ is shape regular if there exists a minimum angle $\theta_* \neq 0$ such that every element in T_h satisfies $\theta_T \geq \theta_*$.



Shape Regularity

- All together ...

$$\begin{aligned}
\|u - u_h\|_{H^1} &\leq \frac{\gamma}{\alpha} \|u - \pi_h(u)\|_{H^1} && \text{(Céa's Lemma),} \\
&\leq \frac{\gamma}{\alpha} \sqrt{1 + C_1} \|\nabla(u - \pi_h(u))\|_{L_2} && \text{(Poincaré-Friedrichs),} \\
&\leq \frac{\gamma}{\alpha} \sqrt{1 + C_1} \left(C_2 \sum_{\Delta_k \in \mathcal{T}_h} \frac{1}{\sin^2 \theta_k} h_k^2 \|D^2 u\|_{\Delta_k}^2 \right)^{\frac{1}{2}} && \text{(Th.1),} \\
&\leq \frac{\gamma}{\alpha} \sqrt{(1 + C_1) C_2} \frac{1}{\sin \theta_*} h \|D^2 u\|_{\Omega} && \text{(shape regularity).} \\
&= O(h).
\end{aligned}$$

- A different proof shows $O(h^2)$ for L_2 .

- A good S^h minimizes the interpolation error.

$$f(x, y) = \sqrt{1 - x^2} \quad \text{on} \quad \Omega = [-1, 1]^2$$

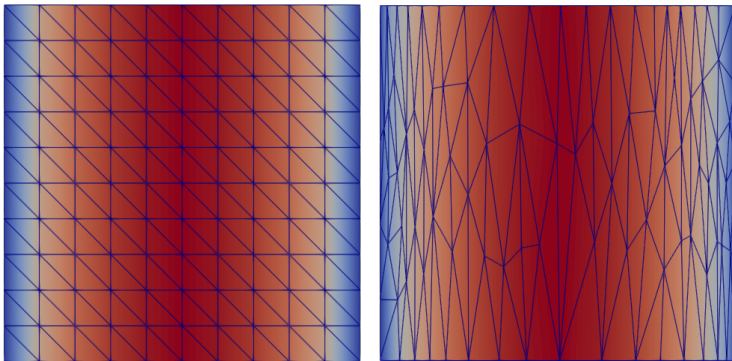


Figure: Interpolation of anisotropic f with roughly the same elements.

- A good refinement scheme should also focus on interpolation error.

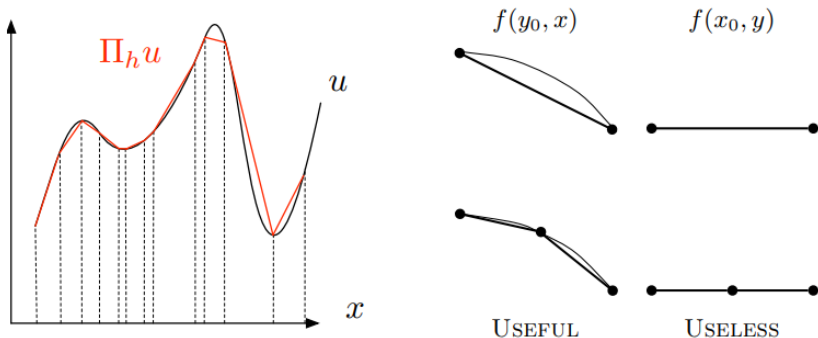


Figure: (Alauzet, 2010)

1 Finite Element Method: Convergence Theorem

2 Adaptivity Schemes and Firedrake/PETSc Compatibility

Tagging Schemes

- 1 **Solve:** Compute the solution on the current mesh.
- 2 **Estimate:** Estimate error for each element.
- 3 **Tag:** Tag elements for refinement/coarsening based on estimate.
- 4 **Refine:** Refine/coarsen mesh.

- Babuška-Rheinboldt error estimator (for Poisson),

$$\eta_K^2 = h_K^2 \int_K |f + \nabla^2 u_h|^2 dx + \frac{h_K}{2} \int_{\partial K \setminus \partial \Omega} \llbracket \nabla u_h \cdot \mathbf{n} \rrbracket^2 ds$$

- Refine and coarsen in a way where error is equidistributed (Bangerth & Rannacher, 2003)

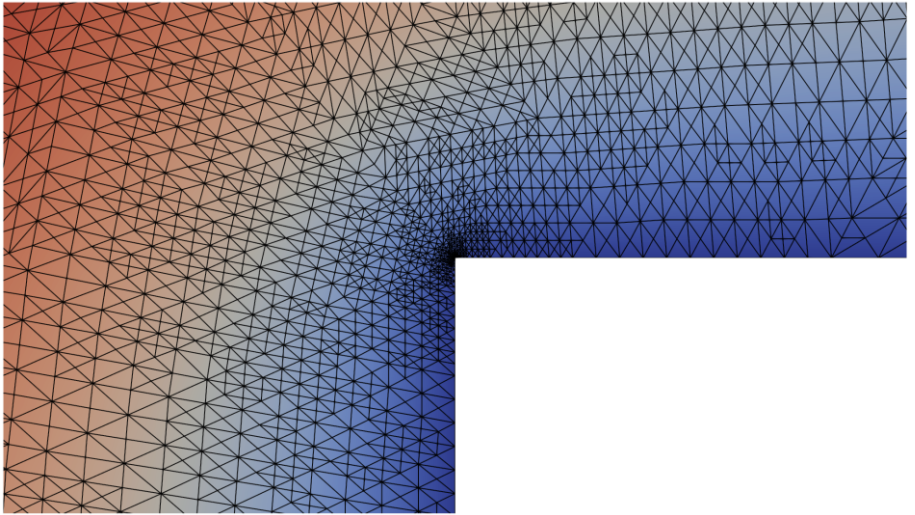


Figure: L-Shaped Homogeneous Dirichlet Poisson Problem.
(Farrell, 2024)

- Mark and refine functionality is implemented in Firedrake via Netgen through ngsPETSc (Zerbinati et al. 2024). 2D and 3D

```
1      import netgen
2      mesh = Mesh(ngmesh)
3      ...
4      AdaptedMesh = mesh.refine_marked_elements(indicator)
```

- The SBR algorithm (Plaza & Carey. 1998) is available in PETSc with bindings in petsc4py (or VIAMR). 2D only

```
1      import VIAMR
2      ...
3      AdaptedMesh = VIAMR.refinemarkedelements(mesh, indicator)
```

- Neither implementation can coarsen, so not ideal for time dependent problems.

Metric Based Adaptation

- Let $\Omega \subset \mathbb{R}^n$, and let $\mathbf{M} = \{\mathcal{M}(x)\}_{x \in \Omega}$ be a Riemannian Metric Space, where $\mathcal{M}(x) : \Omega \rightarrow \mathbb{R}^{n \times n}$ is an SPD matrix.
- Notions of distance, volume, and angle are derived from \mathbf{M} and used during mesh generation to drive adaptivity.

$$d_{\mathbf{M}}(a, b) = \int_0^1 \|\gamma'(t)\|_{\mathcal{M}} dt = \int_0^1 \sqrt{ab^T \mathcal{M}(a + tab) ab} dt.$$

$$|K|_{\mathbf{M}} = \int_K \sqrt{\det \mathcal{M}(x)} dx.$$

Geometric Interpretation

$$\mathcal{M}(x)^{-1/2} = U\Lambda^{-1/2}U^{-1}(x) \quad \text{where} \quad \Lambda = I(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \dots)$$

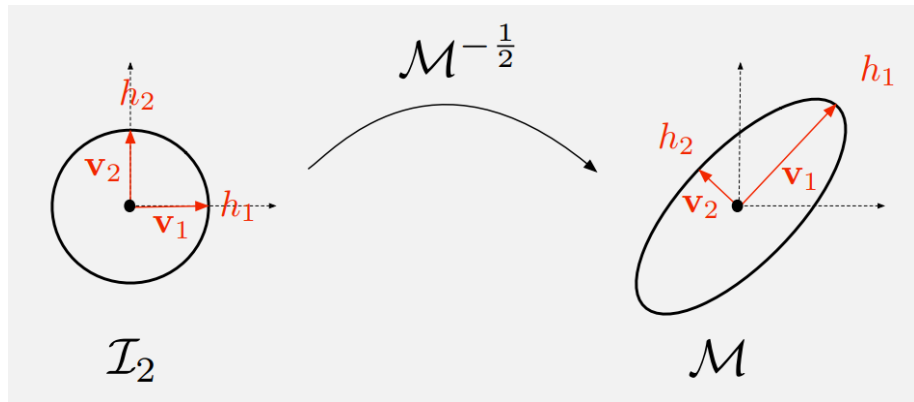


Figure: (Alauzet, 2010)

Geometric Interpretation

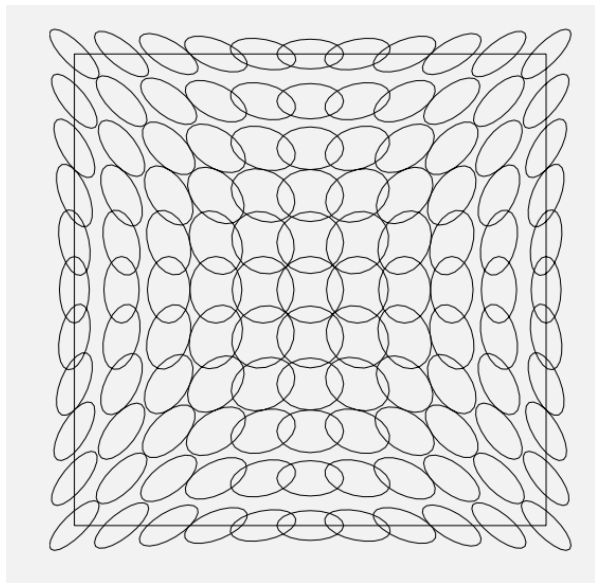


Figure: (Alauzet, 2010)

Geometric Interpretation

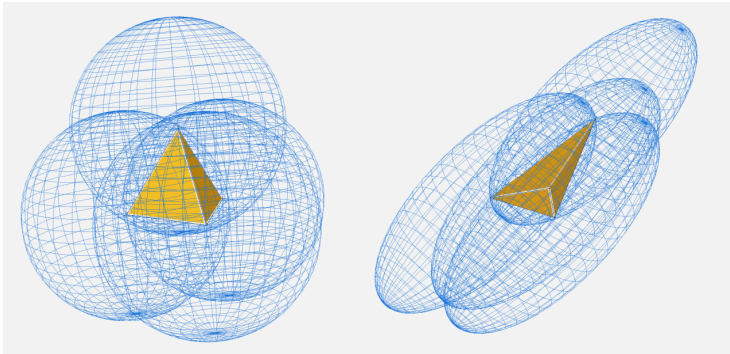


Figure: (Alauzet, 2010)

Isotropic Metrics and Operations

- Isotropic metrics should treat each dimension the same,

$$\mathcal{M}(x) = U(I\lambda(x))U^{-1}.$$

- We can intersect and average metrics to create new metrics.

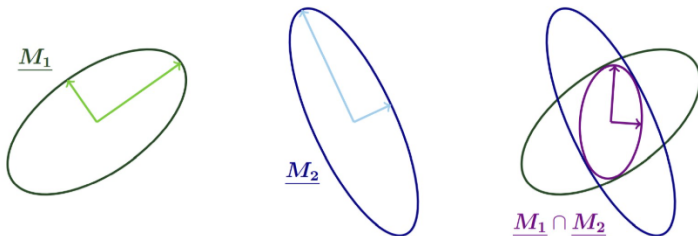


Figure: Metric Intersection (Wallwork, 2021)

Theorem (Alauzet, 2010)

Let u be a twice differentiable function on Ω with $H_u(x)$ its Hessian and let $|H_u(x)| = U |\Lambda| U^{-1}$. If \mathcal{H} is a unit mesh of Ω generated with respect to the metric,

$$\frac{c_n |H_u|}{\epsilon}(x)$$

where c_n depends on dimension of Ω then \mathcal{H} is optimal within ϵ w.r.t controlling linear interpolation error in L^∞ norm.

- Methods exists for interpolating discretely defined metrics and recovering Hessians from linear FE solutions.

- Parallel metric based adaptation is implemented in PETSc (Wallwork et al. 2022) and has been ported into Firedrake with the Animate library.

```
1      import animate
2      ...
3      P1_ten = TensorFunctionSpace(mesh, "CG", 1)
4      metric = RiemannianMetric(P1_ten)
5      metric.set_parameters(metric_params)
6      metric.compute_hessian(c)
7      metric.normalise()
8      adapted_mesh = adapt(mesh, metric)
```

Time Dependent Metric Adaptation

- 1 Perform hessian based adaptation on the initial time step.
- 2 Solve the problem on a specified sub-interval.
- 3 Compute the hessian based metric for each solution in the sub interval.
- 4 Intersect the metrics and adapt the mesh.
- 5 Transfer the solution to the new mesh (interp or project)
- 6 Re-solve the problem on the sub-interval.
- 7 Repeat 2-6 on the next sub-interval until we reach the end of the simulation.

Demo

Time Dependent Metric Adaptation (Metric Advection)

- ① Perform hessian based adaptation on the initial time step.
- ② Solve the fluid problem on a specified sub-interval.
- ③ Solve the advection equation on the initial metric with fluid velocities for the duration of the sub-interval.
- ④ Intersect the metrics and refine the mesh.
- ⑤ Re-solve the problem on the sub-interval.
- ⑥ Repeat 2-5 on the next sub-interval until we reach the end of the simulation.

Demo

SUPG Metric Advection

- Solve for metric M where,

$$\frac{\partial m_{ij}}{\partial t} + u \cdot \nabla m_{ij} = 0.$$

- Implicit-Euler weak form,

$$\int_{\Omega} \left(m^{n+1} \phi + \Delta t (u \cdot \nabla m^{n+1}) \phi \right) dx = \int_{\Omega} m^n \phi dx,$$

- SUPG stabilisation for CG finite elements modifies,

$$\phi \rightarrow \phi + \tau u \cdot \nabla \phi.$$

- The amount of added diffusion is controlled by τ .
- $\tau = \frac{h}{2|u|}$ for pure advection, $\tau = \frac{h|u|}{6K}$ for advection-diffusion.

References

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