Exercise Abbott 5.2.5: Let,

$$f_a(x) = \begin{cases} x^a & x > 0 \\ 0 & x \leq 0 \end{cases}$$

1. For which values of a is f continuous at zero?

Proof. Recall that in order for f to be continuous at zero the right had limit of f as $x \to 0^+$ must be,

$$\lim_{x \to 0^+} x^a = 0.$$

Note that for any $a < 0$ the functional limit goes to infinity and for $a = 0$ we get $f(0) = 1$ for all $x > 0$ therefore it be that f is continuous for all $a < 0$. □

2. for which values of a is f differentiable at zero? In this case, is the derivative function continuous.

Proof. By definition, f is continuous at zero if the following limit exists,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

Clearly we know that the left hand limit is $\lim_{x \to 0^-} f'(x) = 0$, since the function is constant there. Now consider the right hand limit

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \frac{x^a}{x} = x^{a-1}.$$

Now let $b = a - 1$ and note like in the previous problem $0^b = 0$ for all $b < 0$, by substitution we get that $a < 1$. Thus we have shown that for all $a < 1$,

$$\lim_{x \to 0^+} f'(x) = 0 = \lim_{x \to 0^-} f'(x)$$

and thus f is differentiable at zero. □

3. For which values of a is f twice differentiable

Proof. Similarly to the previous problem we know that at $x = 0$ when the following limit exists,

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$$

Again since f is constant for all $x \leq 0$ we know that $\lim_{x \to 0^-} f''(x) = 0$. Now considering the right hand limit and substituting $\lim_{x \to 0^+} f'(x) = \frac{x^a}{x}$,

$$\lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \frac{f'(x)}{x} = \frac{x^a}{x}\frac{1}{x} = \frac{x^a}{x^2} = x^{a-2}.$$

By a similar algebraic argument as the previous problem we get that when $a > 2$,

$$\lim_{x \to 0^+} f''(x) = 0 = \lim_{x \to 0^-} f''(x)$$

and thus f is twice differentiable at zero.

□

Exercise Abbott 5.3.1(a): Recall that from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on a if there exists an $M > 0$ such that for all $x \neq y$ in A,

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

Show that if f is differentiable on closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable and $f' : [a, b] \to \mathbb{R}$ is continuous. By the Mean Value Theorem we know that since f is continuous and differentiable on $[a, b]$ there must exist some point $c \in (a, b)$, for all $x, y \in [a, b]$ that satisfies,

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Now note that f' is a continuous function defined on a compact set $[a, b]$ and therefore by the Extreme Value Theorem we know that there exists some $M \in |f'([a, b])|$ with the property that $|f'(x)| \leq M$. Thus we get the following,

$$\begin{aligned}\left|\frac{f(x) - f(y)}{x - y}\right| &= |f'(c)|,\\ &\leq M.\end{aligned}$$

□

Exercise Abbott 5.3.2: Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Proof. Suppose that f is differentiable on an interval A and that $f'(x) \neq 0$ on A. Since $f'(x) \neq 0$ we know that f must be either strictly increasing or strictly decreasing over A. Without loss of generality let's suppose f is strictly increasing over A. Suppose $x, y \in A$ such that $x \neq y$. Without loss of generality lets suppose that $x > y$. Since f is a strictly increasing function if $x > y$ then it follows that $f(x) > f(y)$ and therefore $f(x) \neq f(y)$. Thus f is one-to-one.

□

Solution:
For an example to show that the converse statement is not always true consider the piecewise function on the interval $[1, 3]$,

$$f(x) = \begin{cases} x & 1 \leq x \leq 2 \\ \frac{x}{2} - 1 & 2 < x \leq 3 \end{cases}$$

The function is trivially one-to-one, and since f is discontinuous at $x = 2$ we know by the contrapositive statement of Theorem 5.2.3 that f is not differentiable at $x = 2$.

Exercise Abbott 5.3.5(a): Supply the details for the proof of Cauchy's Generalized Mean Value Theorem

Proof. Suppose functions f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). We wish to demonstrate that there exists some $c \in (a, b)$ where,

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

Consider the function,

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Note that since h is composed of continuous and differentiable functions on the interval $[a, b]$ it is also continuous and differentiable on $[a, b]$. Applying the Algebraic Differentiability Theorem to h we get the following,

$$h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x).$$

Note that any solution c for the first equation must also give $h'(c) = 0$. Now consider applying the Mean Value Theorem on $h(x)$, we get that there exists some $c \in (a, b)$ such that,

$$\begin{aligned}
h'(c) &= \frac{([f(b) - f(a)]g(b) - [g(b) - g(a)]f(b)) - ([f(b) - f(a)]g(a) - [g(b) - g(a)]f(a))}{b - a} \\
&= \frac{f(b)g(b) - f(a)g(b) - g(b)f(b) - g(a)f(b) + f(b)g(a) + f(a)g(a) + g(b)f(a) - g(a)f(a)}{b - a} \\
&= \frac{0}{b - a} \\
&= 0
\end{aligned}$$

Thus since there exists some $c \in (a, b)$ with the property that $h'(c) = 0$ and that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

□

Exercise Abbott 5.3.11(a): Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hospital's rule.

Proof. Suppose the continuous functions f and g, defined over an interval A. Let $a \in A$ and suppose f and g are differentiable on $A\{a\}$. Also suppose that $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, and that the following limit exists,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

Since the limit exists by the definition of functional limit we know the following,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

By the definition of the Derivative,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}.$$

By the Algebraic Limit Theorem for Functional Limit and recalling that $f(a) = g(a) = 0$,

$$\begin{aligned}
\lim_{x\to a}\frac{f'(x)}{g'(x)} &= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}},\\
&= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)},\\
&= \lim_{x\to a}\frac{f(x)-0}{g(x)-0},\\
&= \lim_{x\to a}\frac{f(x)}{g(x)}.
\end{aligned}$$

□

Exercise Abbott 7.2.7: Let $f : [a, b] \to \mathbb{R}$ be monotone increasing on the set $[a, b]$ (i.e., $f(x) \le f(y)$ whenever $x < y$). Show that f is integrable on $[a, b]$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is monotone increasing on the set $[a, b]$. First note that by the Extreme Value Theorem and the fact that $[a, b]$ is a compact set we know that f is a bounded function, and further since it it monotone increasing we know that $f(a) \le f(x) \le f(b)$ for all $x \in [a, b]$.
Now suppose some partition P_ϵ with nodes $a = x_0 < x_1 < \cdots < x_n = b$, and note that since f is monotone increasing the upper and lower Riemann sums are bounded by the following (Left and right Riemann sums),

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x_i \le L(f, P_\epsilon) \le U(f, P_\epsilon) \le \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

Subtracting the bounded values we get that,

$$\sum_{i=1}^{n} f(x_i)\Delta x_i - \sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \sum_{i=1}^{n} f(x_i) - f(x_{i-1})\Delta x_i.$$

Let $M = Max\{\Delta x_i\}$, and through some algebra we can define an upperbound for the difference,

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x_i - \sum_{i=1}^{n} f(x_{i-1})\Delta x_i &= \sum_{i=1}^{n} f(x_i) - f(x_{i-1})\Delta x_i,\\
&\le \sum_{i=1}^{n} f(x_i) - f(x_{i-1})M,\\
&= M\sum_{i=1}^{n} f(x_i) - f(x_{i-1}),\\
&= M(f(b) - f(a)).
\end{aligned}$$

Let $\epsilon > 0$. Now consider a partition P_ϵ such that $M(f(b) - f(a)) < \epsilon$. Note that when we consider the difference between the upper and lower sums,

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &\leq \sum_{i=1}^{n} f(x_i)\Delta x_i - \sum_{i=1}^{n} f(x_{i-1})\Delta x_i, \\ &\leq M(f(b) - f(a),) \\ &< \epsilon. \end{aligned}$$

Thus by the Integrability Criterion f is integrable on $[a, b]$. $\square$