Exercise 1.2.6: Use the *triangle inequality* to establish the following inequalities:

(a) $|a - b| \leq |a| + |b|$

Proof: (Direct) Suppose $a, b \in \mathbb{R}$. Note that,

$$|a - b| = |a + (-b)|.$$

By the *triangle inequality* we know that,

$$|a + (-b)| \leq |a| + |(-b)|.$$

Note,

$$|a| + |(-b)| = |a| + |b|.$$

Therefore by substitution we arrive at,

$$|a - b| \leq |a| + |b|$$

□

(b) $||a| - |b|| \leq |a - b|$.

Proof: (Direct) Suppose $a, b \in \mathbb{R}$. Note that,

$$a = (a - b) + b.$$

Therefore,

$$|a| = |(a - b) + b|.$$

Thus by *triangle inequality* we know that,

$$\begin{aligned} |a - b + b| &\leq |(a - b)| + |b|, \\ |a| &\leq |(a - b)| + |b|, \\ |a| - |b| &\leq |a - b|. \end{aligned}$$

Now consider,

$$b = (b - a) + a.$$

Therefore we can surmise,

$$|b| = |(b - a) + a|.$$

Thus by *triangle inequality* we know that,

$$\begin{aligned} |b - a + a| &\leq |(b - a)| + |a|, \\ |b| &\leq |(b - a)| + |a|, \\ |b| - |a| &\leq |b - a|, \end{aligned}$$

Therefore it follows that,

$$||a| - |b|| \leq |a - b|.$$

□

Exercise 1.2.7(b), (d): Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x) : x \in A\}$.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
Proof: (Direct) Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Let $A = \mathbb{R}_{\leq 0}$ and $B = \mathbb{R}_{\geq 0}$. Note.

$$f(A \cap B) = \{0\}$$

and,

$$f(A) \cap f(B) = \mathbb{R}_{\geq 0}$$

Thus $f(A \cap B) \neq f(A) \cap f(B)$.

(d) Form and prove a conjecture concerning $f(A \cup B)$ and $f(A) \cup f(B)$.

Conjecture: Let $f : \mathbb{R} \to \mathbb{R}$, if $A, B \subset \mathbb{R}$ then $f(A \cup B) \subset f(A) \cup f(B)$

Proof: (Direct) Suppose $f : \mathbb{R} \to \mathbb{R}$, $A, B \subset \mathbb{R}$, and $y \in f(A \cup B)$. By the definition of the set $f(A \cup B)$ we know that there exists some $x \in A \cup B$ such that $y = f(x)$. Note that $x \in A, B$ and it therefore must follow that $y \in f(A), f(B)$. Thus $y \in f(A) \cup f(B)$ and $f(A \cup B) \subset f(A) \cup f(B)$.

□

Exercise 1.2.11: Form the logical negation of each claim. Do not use the easy way out: "It is not the case that..." is not permitted

(a) For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + (1/n) < b$.

(b) There exist a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution:

(a) There exists $a, b \in \mathbb{R}$ where $a < b$ and for all $n \in \mathbb{N}$, $a + (1/n) < b$.

(b) For all real numbers $x > 0$, there exists $n \in \mathbb{N}$ such that $x < \frac{1}{n}$

(c) If $x \in \mathbb{R}$ then there exists $a, b \in \mathbb{R}$ such that $a < b$ and $x < a$ and $x > b$

Exercise [1.2 Supplement]: Show that the sequence $(x_1, x_2, x_3, \ldots)$ defined in Example 1.2.7 is bounded above by 2. That is, show that for every $i \in \mathbb{N}$, $x_i \leq 2$.

Proof. (Induction):

Base Case: Let $n = 1$,

$$x_n = 1.$$

By definition, and obviously $1 \leq 2$.

Induction Hypothesis: Suppose that for some $n \in \mathbb{N}$,

$$x_n \leq 2$$

By definition we know that,

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + 1, \\ 2(x_{n+1} - 1) &= x_n. \end{aligned}$$

By our Induction hypothesis we know that,

$$\begin{aligned} 2(x_{n+1} - 1) &\leq 2, \\ (x_{n+1} - 1) &\leq 1, \\ x_{n+1} &\leq 2. \end{aligned}$$

Thus by Induction we have shown that for every $i \in \mathbb{N}$, $x_i \leq 2$.

$\square$

Exercise 1.3.5: As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in \mathbb{R}\}$

(a) If $c \geq 0$, show that $\sup(cA) = c\sup(A)$.

(b) Postulate a similar statement for $\sup(cA)$ when $c < 0$.

Proof (a). Suppose some $s \in \mathbb{R}$ such that $s = sup(A)$. By the definition of supremum we know that for all $a \in A$, $a \leq s$. Multiplying by $c \in \mathbb{R}$ on both sides we get, $ca \leq cs$, by the definition of upper bound and the set cA whe know that $csup(A)$ is an upper bound for cA.

Case 1: $c = 0$ Note if c is $c = 0$ then $cA = \{0\}$ and subsequently, $\sup(cA) = c\sup(A)$.

Case 2: $c > 0$
Let b be an arbitrary upper bound for the set cA. Note by definition,

$$ca \leq b$$
$$a \leq \frac{b}{c}$$

Since s is the least upper bound for the set A we can surmise that, therefore we know that $\frac{b}{c}$ is an upper bound for the set A and,

$$s \leq \frac{b}{c}$$

and therefore,

$$sc \leq b.$$

Thus sc is the least upper bound for the set cA, and

$$\sup(cA) = c\sup(A).$$

□

[Postulate] If $c < A$ and A is a bounded set, then

$$sup(cA) = cinf(A)$$

Exercise 1.3.7: Prove that if a is an upper bound for A and if a is also an element of A, then $a = \sup A$.

Proof. (Contradiction): Suppose that a is an upper bound for A and a is also an element of A, and $a \neq sup(A)$. Let $b = sup(A)$, note that by definition $b < a$ and $b \geq c$ for all $c \in A$. Also note that $a \in A$ and recall $b < a$. Thus $b = sub(A)$ and $b \neq sub(A)$. □

Exercise 1.3.8: Compute, without proof, the suprema and infima of the following sets.

(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : n, m \in \mathbb{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbb{N}\}$.

(d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.

Solution:

(a) Infimum: 0

Supremum: 1

(b) Infimum: -1

Supremum: 1

(c) Infimum: $\frac{1}{4}$

Supremum: $\frac{1}{3}$

(d) Infimum: 0

Supremum: 1