Exercise Abbott 4.3.9: Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $k = \{x : h(x) = 0\}$. Show that k is a closed set.

Proof. Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $k = \{x : h(x) = 0\}$. Let x be a limit point of k, by Theorem 3.2.5 there exists a sequence $(a_n) \in k$ such that $lim a_n = x$ where $a_n \neq x$ for all $n \in \mathbb{N}$. By Theorem 4.3.2 (iii) since h is continuous for all $(a_n) \to x$ it follows that $h(a_n) \to h(x)$. Note that since $a_n \in k$ we know that $h(a_n) = 0$ for all n and therefore we know that $h(x) = 0$ an thus by definition we get that $x \in k$. Thus k contains all its limit points and is therefore closed. □

Exercise Supplemental 1: a) Show that a continuous function on all of $\mathbb{R}$ that equals zero on the rational numbers must be the zero function

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuos and that for all $q \in \mathbb{Q}$ we know that $f(q) = 0$. By Theorem 3.2.10 for every $x \in \mathbb{R}$ there exists a sequence $(q_n) \in \mathbb{Q}$ such that $(q_n) \to x$. By the continuity of f we know that $f(q_n) \to f(x)$, since all $q_n \in \mathbb{Q}$ by definition of f we know that $f(q_n) = 0$ and thus $f(x) = 0$ for all $x \in \mathbb{R}$. □

b) Suppose f and g are two continuous functions on the real numbers. Is it true that if $f(q) = g(q)$ for all $q \in \mathbb{Q}$, then f and g are the same function?

Proof. Suppose f and g are two continuous functions on the real numbers such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$. By Theorem 3.2.10 fir ever $x \in \mathbb{R}$ there exists a $(q_n) \in \mathbb{Q}$ such that $(q_n) \to x$. By the continuity of f and g we know that $f(q_n) \to f(x)$ and $g(q_n) \to g(x)$. Since $f(q) = g(q)$ we also get that $g(q_n) \to f(x)$ and $f(q_n) \to g(x)$. Finally by Theorem 2.2.7(Uniqueness of Limits) it must be the case that the limits are the same and we get $f(x) = g(x)$ for all $x \in \mathbb{R}$. □

Exercise Supplemental 2: Suppose $K \subseteq \mathbb{R}$ is compact. Show that there exists $x_M \in K$ such that $x_M \geq x$ for all $x \in K$. Then, with very little work, show that there exists $x_m \in K$ such that $x_m \leq x$ for all $x \in K$.

Proof. Suppose that $K \subseteq \mathbb{R}$ is compact. By the definition of compact we know that k is closed and bounded. Since K is bounded we know that there exists some $x_M = SupK$. Now consider every ϵ-neighborhood of x_M. By Lemma 1.3.8 we know that for every $\epsilon > 0$, there exists some $x \in K$ such that $x_M - \epsilon < x < x_M$. So we know that $x \in V_\epsilon(x_M) \cap K/\{x_M\}$

and thus by definition x_M is a limit point of K and since K is closed we know that $x_M \in K$

Since K is bounded there also exists an $x_m = Inf K$. By Lemma 1.3.8 we know that for every $\epsilon > 0$, there exists some $x \in K$ such that $x_m < x < x_m + \epsilon$ So we know that $x \in V_\epsilon(x_m) \cap K/\{x_m\}$ and thus by definition x_m is a limit point of K and since K is closed we know that $x_m \in K$ □

Exercise Abbott 4.3.7(a): Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on $\mathbb{R}$.

Proof. Consider the Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$ where,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Consider the $i \in \mathbb{I}$, by Theorem 3.2.10 (Density of $\mathbb{Q}$ in $\mathbb{R}$) we can construct a sequence $q_n \in \mathbb{Q}$ such that $q_n \to i$. Suppose for the sake of contradiction that f is continuous on all $i \in \mathbb{I}$ then by continuity it must be the case that since $q_n \to i$ then $f(q_n) \to f(i)$, however $f(q_n)$ is a constant sequence of 1 and $f(i) = 0$ therefore $f(q_n) \not\to f(i)$. Thus by contradiction f is not continuous on $\mathbb{I}$

Similarly consider $q \in \mathbb{Q}$, and constructing a sequence $i_n \in \mathbb{I}$ such that $i_n \to q$. Supposing that f is continuous on all $q \in \mathbb{Q}$ then by continuity we it must be the case that since $i_n \to q$ then $f(i_n) \to f(q)$ however $f(i_n)$ is a constant sequence of 0 and $f(q) = 1$ therefore by contradiction f is not continuous on $\mathbb{Q}$. □

Exercise Abbott 4.4.6: Give an example of each of the following, or state that such a request is impossible. for any that are impossible, supply a short explanation for why this is the case.

1. A continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a cauchy sequence.

   Solution:
   Let $f(x) = \frac{1}{x}$ and consider the cauchy sequence $x_n = \frac{1}{n!}$. Note that $f(x_n) = n!$ which is clearly not convergent and is therefore not a cauchy sequence.

2. A uniformly continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a cauchy sequence.

   Solution:
   Suppose $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) and for the sake of contradiction suppose $f(x_n)$ is not cauchy. By definition there exists some (bad) $epsilon$ such that for every $N \in \mathbb{N}$ for all $n, m \geq N$ we know that $|f(x_n) - f(x_m)| \geq \epsilon$. However since x_n is cauchy and f is continuous we know that $|x_n - x_m| \to 0$ then $|f(x_n) - f(x_m)| \to 0$ thus by contradiction f cannot be uniformly continuous.

3. A continuous function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a cauchy sequence.

   Solution:
   This request is impossible, note that the set $[0, \infty)$ is closed and therefore by Theorem 3.2.8 we know that $x_n \to L$ where $L \in [0, \infty)$. By continuity we get that $f(x_n) \to f(L)$ and since $f(x_n)$ is a convergent sequence it is also cauchy.

Exercise Abbott 4.4.9: A function $f : A \to \mathbb{R}$ is called *Lipschitz* is there exists a bound $M > 0$ such that,

$$|\frac{f(x) - f(y)}{x - y}| \leq M.$$

for all $x \neq y \in A$. Geometrically speaking, a function f is *Lipschitz* if there is a uniform bound on the magnitude of the sloped of lines drawn through any two points on the graph of f,

1. Show that if $f : A \to \mathbb{R}$ is *Lipschitz*, then it is uniformly continuous on A.

   Proof. Suppose that $f : A \to \mathbb{R}$ is *Lipschitz*. By the definition of a *Lipschitz* function we know that for all $x, y \in A$ there exists some M such that,

   $$|f(x) - f(y)| \leq M|x - y|.$$

   Let $\epsilon > 0$, now consider $\delta = \frac{\epsilon}{M}$ therefore whenever $|x - y| < \delta$ we get,

   $$\begin{aligned} |x - y| &< \delta, \\ M|x - y| &< M\delta, \\ |f(x) - f(y)| &< \epsilon. \end{aligned}$$

   Thus f is uniformly continuous on A. □

2. Is the converse statement true? Are all uniformly continuous functions necessarily *Lipschitz*?

Proof. Consider the function $f : [0, 1] \to [0, 1]$ such that $f(x) = \sqrt{x}$. We demonstrated in class that f is uniformly continuous. Suppose for the sake of contradiction that f is *Lipschitz*, we would get the following inequality for all $x, y \in [0, 1]$ and some $M > 0$,

$$|\sqrt{x} - \sqrt{y}| \leq M|x - y|$$

Let $x = 0$ and $y = \frac{1}{2M^2}$ we get the following by substitution,

$$\begin{gathered} |\sqrt{x} - \sqrt{y}| \leq M|x - y|, \\ \frac{1}{2M} \leq \frac{M}{4M^2}, \\ \frac{1}{2M} \leq \frac{1}{4M}. \end{gathered}$$

Thus by contradiction f is not *Lipschitz*. As demonstrated in class we can see that as x approaches 0 the slope increases to infinity and is therefore unbounded. $\square$