Exercise 1: Let A and B be non empty sets that are bounded above.Suppose $\sup A < \sup B$. Prove that there is an element in B that is an upper bound for A.

Proof. Suppose that A and B be non empty sets that are bounded above and that $\sup A < \sup B$. Let $x = \sup A$ and $y = \sup B$. Now consider some z such that, $0 < z < y - x$. Through some algebra we can see that, $x < y - z$ and therefore the term $y - z$ must be an upper bound for A since it is larger than its least upper bound. Also note that $y - z < y$ and therefore $y - z$ must be contained in B. □

Exercise 2: In class we proved that $\mathbb{N}^2$ is countably infinite. Use this fact and a proof by induction to show that $\mathbb{N}^n$ is countably infinite for every $n \in \mathbb{N}$.

Proof. Consider the base case where $n = 1$, clearly $\mathbb{N}^1$ is countably infinite and we have proven that $\mathbb{N}^2$ is countably infinite. We will proceed by induction on n. Suppose there exists some $n \in \mathbb{N}$ such that $\mathbb{N}^n$ is countably infinite. By the induction hypothesis there exists some bijection $g : \mathbb{N}^n \to \mathbb{N}$. Now consider the bijection we proved in class $f : \mathbb{N}^2 \to \mathbb{N}$. Note that the composition of these two functions gives us, $f \circ g : \mathbb{N}^{n+1} \to \mathbb{N}$ and since $f \circ g$ is a composition of bijections it must also be a bijection. Thus by induction we have shown that for all $n \in \mathbb{N}$ $\mathbb{N}^n$ is countably infinite. □

Exercise 3: Compute,

$$\lim_{n\to\infty} \frac{3^n}{n!}.$$

A fully rigorous proof will involve a proof by induction.

Proof. We will proceed by induction to prove the following inequality for all $n \in \mathbb{N}$ when $n \geq 9$,

$$\frac{1}{n!} \leq \frac{1}{4^n}.$$

Consider the base case where $n = 9$,

$$\frac{1}{4^n} = \frac{1}{4^9} \leq \frac{1}{9!} = \frac{1}{n!}.$$

Now suppose there exists some $n \in \mathbb{N}$ where $n \geq 9$ such that,

$$\frac{1}{n!} \leq \frac{1}{4^n}.$$

Now consider $\frac{1}{n!}$, and by substituting our induction hypothesis,

$$\begin{aligned}\frac{1}{n+1!} &= \frac{1}{n+1}\frac{1}{n!}\\ &\leq \frac{1}{n+1}\frac{1}{4^n}\\ &\leq \frac{1}{4}\frac{1}{4^n}\\ &\leq \frac{1}{4^{n+1}}.\end{aligned}$$

Therefore for large n we know that,

$$\frac{3^n}{n!} \leq \frac{3^n}{4^n} = \frac{3}{4}^n.$$

Furthermore it must also be the case that,

$$\lim_{n\to\infty}\frac{3^n}{n!} \leq \lim_{n\to\infty}\frac{3}{4}^n.$$

Clearly, $\frac{3^n}{n!}$ is bounded below by 0 and since,

$$\lim_{n\to\infty}\frac{3}{4}^n \to 0.$$

It must be the case that,

$$\lim_{n\to\infty}\frac{3^n}{n!} = 0.$$

□

Exercise 4: Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Proof. Suppose convergent sequences (x_n) and (y_n) such that $\lim x_n = \lim y_n = l$. consider a sequence (z_n) such that $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$. Note that $(z_{2n}) = (y_n)$ and $(z_{2n-1}) = (x_n)$. Let $\epsilon > 0$. Since (x_n) and (y_n) converge we know that there exists $N_x, N_y \in \mathbb{N}$ such that,

$$|x_n - l| < \epsilon,$$

$$|y_n - l| < \epsilon.$$

Consider an $N \in \mathbb{N}$ such that $N = \max(N_x, N_m)$. By substitution we get that for all odd and even values of (z_n) we get,

$$|z_n - l| < \epsilon.$$

Thus (z_n) is convergent. □

Proof. Suppose a sequence $(z_n) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$ is convergent to some limit $(z_n) \to l$. By Theorem 2.5.2 we know that all subsequences of (z_n) must converge to the same limit. Consider some $a \in x_n$ such that $x_i = a$. Note that $x_i = z_{2i-1}$ and therefore $a \in z_n$. Similarly for some $b \in y_n$ such that $y_i = b$ we know that $y_i = z_{2i}$ and therefore $b \in z_n$. Thus both x_n and y_n are subsequences of z_n and therefore $\lim z_n = \lim x_n = \lim y_n$. □

Exercise 5: Suppose F is a collection of open intervals such that if $I, J \in F$ and $I \neq J$, then $I \cap J = \emptyset$. Prove that F is countable.

Proof. Suppose F is a collection of open intervals such that each interval is disjoint. By Theorem 1.4.3 (The Density of $\mathbb{Q}$ in $\mathbb{R}$) we know that there must exist at least one rational number r in side of each open interval. Define A as a set containing the compliment of the union of all the open sets in F,

$$(\bigcup_F J)^c \in A.$$

Note that $|A| = 1$, and that,

$$(\bigcup_F J) \cup (\bigcup_A I) = \mathbb{R}.$$

Consider the function, $f : \mathbb{Q} \to F \cup A$ such that $f(r) = J$ when $r \in J$. Now we will show that f is a surjective function. Consider some $J \in F \cup A$, and note that by Theorem 1.4.3 there must exist some rational number $r \in J$ and thus f is surjective. It then follows that $F \cup A$ is at most countable and since F and A are also disjoint, $|F| = |F \cup A| - 1$. Thus F is also at most countable. □

Exercise 6: Let (x_n) be a sequence converging to L. Define,

$$y_n = \frac{x_1 + \cdots + x_n}{n}.$$

That is y_n is the average of the first n terms of the x_n sequence. Show that $y_n = L$ as well.

Proof. Suppose that the sequence (x_n) is convergent to L. Therefore by definition for all $\epsilon > 0$,

$$|x_n - L| < \epsilon.$$

Now consider the expression,

$$|y_n - L| = |\frac{x_1 + \cdots + x_n}{n} - L|.$$

Through some algebra we get,

$$\begin{aligned} |y_n - L| &= \frac{1}{n}|(x_1 + \cdots + x_n) - nL|, \\ &= \frac{1}{n}|(x_1 - L) + \cdots + (x_n - L)|. \end{aligned}$$

By triangle inequality,

$$|y_n - L| \leq \frac{1}{n}|(x_1 - L)| + \cdots + |(x_n - L)|.$$

Since (x_n) is convergent to L we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - L| < \epsilon.$$

By substitution we know that,

$$\begin{aligned} |y_n - L| &< \frac{n\epsilon}{n}, \\ &< \epsilon. \end{aligned}$$

Thus y_n is convergent with $y_n \to L$. □

Exercise 7: Use the Bolzano Weierstrass Theorem to prove the Monotone Convergence Theorem without assuming any other form of the Axiom of Completeness.

Proof. Consider (a_n), a monotone and bounded sequence. Without loss of generality let's assume the (a_n) is monotone increasing. By Bolzano Weierstrass we know that there exists a convergent subsequence of (a_n), $(a_{n_k}) \to L$. Therefore for all $\epsilon > 0$ there exists an $K \in \mathbb{N}$ such that for all $k \geq K$,

$$|(a_{n_k})| < \epsilon.$$

$$\begin{aligned} |(a_{n_k}) - L| &< \epsilon \\ -\epsilon <&(a_{n_k}) - L < \epsilon \\ L - \epsilon <&(a_{n_k}) < \epsilon + L \end{aligned}$$

Let $N = n_K$, note that for all $n \geq N$ there exists a $k \geq K$, such that

$$a_{n_k} \leq a_n \leq a_{n_{k+1}}.$$

Thus we get the following inequality

$$L - \epsilon < a_{n_k} \leq a_n \leq a_{n_{k+1}} < \epsilon + L.$$

Therefore a_n converges to L. □

Exercise 8: Suppose (x_n) is a sequence and that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|.$$

Show that the sequence (x_n) converges.

Proof. Suppose (x_n) is a sequence and that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|.$$

Note that by expansion and iterative substitution, we get the following expression,

$$|x_n - x_{n_1}| = \frac{1}{2^{n-1}}|x_2 - x_1|.$$

Let $M = |x_2 - x_1|$ and recall that in Homework 4, supplemental exercise 2 we proved that,

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Thus it follows by Algebraic Limit Theorem that,

$$\lim |x_n - x_{n_1}| = \lim \frac{M}{2^n} = M0 = 0.$$

Now consider the following, where $m, n \in \mathbb{N}$ and $m \geq n$,

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n|.$$

By Triangle Inequality we get that,

$$|x_m - x_n| \leq \sum_{i=1}^{m-n} |x_{1+n} - x_{1+n-1}|.$$

Using the given property we can get each term in the sum as a factor of $|x_{n+1} - x_n|$,

$$|x_m - x_n| \leq \sum_{i=1}^{m-n} \frac{1}{2^{i-1}}|x_{n+1} - x_n|.$$

Since the sum is a constant term let,

$$C = \sum_{i=1}^{m-n} \frac{1}{2^{i-1}}.$$

Let $\epsilon > 0$. Consider an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_{n+1} - x_n| < \frac{\epsilon}{C}.$$

Therefore we get the following,

$$|x_m - x_n| \leq C|x_{n+1} - x_n|,$$
$$< C\frac{\epsilon}{C},$$
$$< \epsilon.$$

Therefore the sequence x_n is Cauchy and converges. □

Exercise 9: Let (a_n) and (b_n) be sequences with $b_n \geq 0$ for all n and $\lim b_n = 0$. We say that $a_n = O(b_n)$ If there is a constant C such that $|a_n| \leq Cb_n$ for all n. Roughly speaking, $a_n = O(b_n)$ if the sequence a_n converges to zero at least as fast as teh sequence b_n.
Suppose a_n and b_n are sequences with $b_n > 0$. Suppose also that $\lim fraca_nb_n = L$ for some number L. Prove that $a_n = O(b_n)$.

Proof. Suppose that there exists sequences a_n and b_n such that $b_n \geq 0$ for all n, $\lim b_n = 0$ and $\lim fraca_nb_n = L$. By the definition of convergence we know that for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ where for all $n \geq N$,

$$|\frac{a_n}{b_n} - L| < \epsilon.$$

Through some more algebra we get,

$$|\frac{a_n}{b_n} - L| < \epsilon,$$
$$-\epsilon < \frac{a_n}{b_n} - L < \epsilon,$$
$$L - \epsilon < \frac{a_n}{b_n} < \epsilon + L,$$
$$b_n(L - \epsilon) < a_n < b_n(\epsilon + L).$$

Now note that $(L - \epsilon)$ and $(\epsilon + L)$ are constant terms and thus we get that $a_n = O(b_n)$. □

Exercise 10: Suppose (a_n) and (b_n) are sequences with $b_n > 0$ and $a_n = O(b_n)$.

1. Suppose that $\sum b_n$ converges. Prove that $\sum a_n$ converges also.

   Proof. Suppose that (a_n) and (b_n) are sequences with $b_n > 0$ and $a_n = O(b_n)$ and that $\sum b_n$ converges. Note that since $a_n = O(b_n)$ we know that $|a_n| \leq Cb_n$ for some constant C. Summing over all n we get the following inequality,

   $$-C\sum b_n \leq \sum_{i=1}^{n} a_n \leq C\sum b_n.$$

Let $\lim \sum b_n \to L$. Therefore by the Algebraic Limit Theorem for Series we know that,

$$\lim(-C \sum b_n) = -C \lim(\sum b_n) = -CL,$$
$$\lim(C \sum b_n) = C \lim(\sum b_n) = CL.$$

It then follows that by Squeeze Theorem we have,

$$-CL \leq \lim \sum a_n \leq CL,$$
$$\lim |\sum_{i=1}^{n} a_n| = CL.$$

Thus $\sum a_n$ is absolutely convergent, and therefore must also be convergent. □

2. Suppose that $\sum a_n$ diverges, show that $\sum b_n$ also diverges.

Proof. Consider the contrapositive statement then refer to part 1. □

3. Determine if the following series converges,

$$\sum_{n=1}^{\infty} \sqrt{\frac{n^3 - 3n + 2}{8n^4 + n^2 + 22}}.$$

Proof. Note that through algebra the following is true,

$$a_n = \sqrt{\frac{n^3 - 3n + 2}{8n^4 + n^2 + 22}} \leq \sqrt{\frac{n^3}{8n^4}} = \sqrt{\frac{1}{8n}} = b_n.$$

Now recall that the the sum,

$$\sum_{n=1}^{\infty} \sqrt{\frac{1}{8n}} = \frac{1}{8} \sum_{n=1}^{\infty} \sqrt{\frac{1}{n}},$$

is a divergent, positive p series. Now consider the Limit Comparison Test,

$$\lim \frac{a_n}{b_n} = \lim \sqrt{\frac{n^3 - 3n + 2(8n)}{8n^4 + n^2 + 22}} = 1.$$

Thus both series must either diverge or converge and since we know $\sum b_n$ is a divergent p series it must be the case that $\sum a_n$ converges. □

Note that Limit Comparison Test is a corollary of the Comparison Test/Theorem 2.7.4 the following is a quick proof courtesy of *Calculus Early Trancendentals* by James Stewart, *Proof.* Suppose $\sum a_n$ and $\sum b_n$ are positive series and let,

$$\lim \frac{a_n}{b_n} = c,$$

where $0 < c < \infty$. Now let M, m be positive real numbers such that $m < c < M$. Since the aforementioned series converges to c there exists some $N \in \mathbb{N}$ where for all $n \geq N$,

$$m < \frac{a_n}{b_n} < M.$$

Simply multiplying both sides of the inequality by b_n yields,

$$mb_n < a_n < Mb_n.$$

Note that by ALT for Series we know that if $\sum b_n$ converges so does $\sum Mb_n$ and therefore by Theorem 2.7.4 (i) $\sum a_n$ converges. Similarly if $\sum b_n$ diverges so does $\sum mb_n$ and therefore by Theorem 2.7.4 (ii) $\sum a_n$ must also diverge.

□