Exercise 2.4.5 (Modified, with hints!): Suppose $x_1 = 2$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

1. Show that $x_n \geq 0$ for all n.

   Solution:
   We will proceed by induction on n. Suppose that for some $n \in \mathbb{N}$,

   $$x_n \geq 0.$$

   Recall that by definition we know that,

   $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

   Through some algebra we get,

   $$\begin{aligned} x_n &\geq 0 \\ x_n + \frac{2}{x_n} &\geq 0 \\ \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) &\geq 0 \\ x_{n+1} &\geq 0. \end{aligned}$$

   Thus we have shown through induction that $x_n \geq 0$ for all values of n.

2. Show that if $a > 0$ then $a + \frac{1}{a} \geq 2$. Hint: $(a-1)^2 \geq 0$. [Your proof should highlight the part where you use the hypothesis $a > 0$.]

   Solution:
   Suppose some $a \in \mathbb{R}$ such that $a > 0$. Note that the square of any real number is zero or positive, thus we get that,

   $$(a-1)^2 \geq 0.$$

   Through some algebra, and the fact that, $a > 0$ we get,

   $$\begin{aligned} (a-1)^2 &\geq 0, \\ a^2 - 2a + 1 &\geq 0, \\ a^2 + 1 &\geq 2a, \\ a + \frac{1}{a} &\geq 2. \end{aligned}$$

   Note that in the last step we can divide by a since $a > 0$ and for the same reason the direction of the inequality stays the same.

3. Show that if $b \neq 0$ then $b^2 + 4/b^2 \geq 4$. Hint: Use the previous item!

Solution:
Suppose some $b \in \mathbb{R}$ such that $b \neq 0$. Again note that the square of any real number is either zero or positive, thus we get,

$$(b^2 - 2)^2 \geq 0.$$

Through some algebra and the fact that $b \neq 0$ we get,

$$\begin{aligned} (b^2 - 2)^2 &\geq 0, \\ b^4 - 4b^2 + 4 &\geq 0, \\ b^4 + 4 &\geq 4b^2, \\ b^2 + 4/b^2 &\geq 4. \end{aligned}$$

Note that in the last step we can divide by b^2 since $b \neq 0$ and since $b^2 > 0$ the direction of the inequality is unchanged.

4. Show that $x_n^2 \geq 2$ for all n. Hint: Use the previous item!

Solution:
Note that $2^2 = 4 \geq 2$. We will proceed by induction on n. Suppose that for some $n \in \mathbb{N}$,

$$x_n^2 \geq 2.$$

Now recall the definition of x_{n+1},

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Squaring it, using the previous result, and the induction hypothesis we get,

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)^2, \\ &= \frac{1}{4}\left(x_n^2 + \frac{4}{x_n^2} + 4\right), \\ &\geq \frac{1}{4}(4 + 4), \\ &= 2. \end{aligned}$$

Note that since $x_n^2 \geq 2$ we know that $x_n \neq 0$ and therefore by the previous problem we know that,

$$(x_n^2 + \frac{4}{x_n^2}) \geq 4.$$

Thus by induction we have show that $x_n^2 \geq 2$ for all $n \in \mathbb{N}$.

5. Show that $x_n \geq x_{n+1}$ for all n. Hint: Use the previous item!

Solution:
Suppose that, $x_n^2 \geq 2$ and $x_n > 0$ for all $n \in \mathbb{N}$. Through some algebra we get,

$$\begin{aligned}
x_n^2 &\geq 2, \\
0 &\geq \frac{2 - x_n^2}{2x_n}, \\
0 &\geq \frac{1}{x_n} - \frac{x_n}{2}, \\
0 &\geq \frac{1}{x_n} + \frac{x_n}{2} - x_n, \\
0 &\geq \frac{1}{2}\left(\frac{2}{x_n} + x_n\right) - x_n, \\
x_n &\geq \frac{1}{2}\left(\frac{2}{x_n} + x_n\right), \\
x_n &\geq x_{n+1}.
\end{aligned}$$

Note that step 2 of the algebra relies on the fact that $x_n > 0$, and the last step is a substitution by definition. Thus we have shown that $x_n \geq x_{n+1}$.

6. Show that the sequence converges to a limit L.

Solution:
In step one we showed that the sequence a_n is bounded below by 0 and in the previous step we demonstrated that the sequence is monotone decreasing. Thus by the Monotone Convergence Theorem the sequence (a_n) must converge to some limit L.

7. Show that $L \neq 0$. Hint: If $x_n \to 0$ then $x_n^2 \to 0$.

Solution:
Suppose to the contrary that $L = 0$. Hence $x_n \to 0$. By the Algebraic Limit Theorem, consider computing $\lim(x_n^2)$,

$$\lim(x_n^2) = \lim(x_n)\lim(x_n) = L^2 = 0$$

Therefore by the definition of convergence we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n^2 - 0| = |a_n^2| = a_n^2 < \epsilon.$$

However we have shown previously that $a_n^2 \geq 2$ contradicting the convergence. Thus it must be the case that $L \neq 0$.

8. Show that $L^2 = 2$. Hint: $\lim x_{n+1} = \lim x_n$.

Solution:
Suppose that $\lim x_n = L$. Consider taking the limit of our definition of x_{n+1},

$$\lim x_{n+1} = \lim \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Note that by the fact that $\lim x_{n+1} = \lim x_n$, and the Algebreic Limit Theorem we can simplify the equation above,

$$\begin{aligned}
\lim x_{n+1} &= \lim \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),\\
L &= \frac{1}{2}\lim\left(x_n + \frac{2}{x_n}\right),\\
L &= \frac{1}{2}\left(\lim x_n + 2\lim \frac{1}{x_n}\right),\\
L &= \frac{1}{2}\left(L + \frac{2}{L}\right),\\
L &= \frac{L}{2} + \frac{1}{L},\\
2L &= L + \frac{2}{L},\\
L &= \frac{2}{L},\\
L^2 &= 2.
\end{aligned}$$

Exercise 2.5.5: Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Let (a_n) be a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$ and suppose to the contrary that (a_n) does not converge to a. Hence there exists an $\epsilon > 0$ so than for all $N \in \mathbb{N}$ there exists an $n \geq N$ where,

$$|a_n - a| \geq \epsilon.$$

Consider a subsequence (a_{i_j}) which satisfies the previous inequality. Note that by Theorem 2.5.2 it must converge to the same limit as (a_n) and by definition that limit is not a, Note that a_{i_j} converges to a and also does not converge to a. □

Exercise 2.5.6: Use a similar strategy to the on in Example 2.5.3 to show that $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit.(The results of 2.3.1 may be assumed.)

Proof. Suppose the sequence $b_n = b^{1/n}$ and let $b \geq 0$. First consider where $b < 1$, and note that in this case the sequence is bounded below by 1 since for all $b < 1$,

$$\begin{aligned} b &< 1, \\ b^{1/n} &< 1^{1/n} = 1. \end{aligned}$$

Now we will show that when $b < 1$, the sequence b_n is monotone increasing for all $n \in \mathbb{N}$, through induction on n. Suppose that for some $n \in \mathbb{N}$,

$$b^{1/n} \geq b^{1/(n-1)}.$$

Using some algebra on our induction hypothesis we get,

$$\begin{aligned} b^{1/n} &\geq b^{1/(n-1)}, \\ b^{1/(n+1)-1/(n)} b^{1/n} &\geq b^{1/(n+1)-1/(n)} b^{1/(n-1)}, \\ b^{1/(n+1)} &\geq b^{(n^2-1)/(n-1)(n+1)(n)}, \\ b^{1/(n+1)} &\geq b^{1/n}. \end{aligned}$$

Thus the sequence is monotone increasing when $b < 1$. By the Monotone Convergence Theorem, we know that when $b < 1$ the sequence b_n must converge to some limit L.

Now consider the case where $b \geq 1$, clearly the sequence wound then be bounded below by 1 by a similar argument. Now through a similar induction argument we know that when $b \geq 1$, the sequence b_n is monotone decreasing for all $n \in \mathbb{N}$ (replace the $\geq$ above with a $\leq$). Therefore by the Monotone Convergence Theorem we know that when $b \geq 1$ the sequence b_n must converge to some limit L. Thus $\lim b^{1/n}$ exists for all $b \geq 0$.

Now consider the subsequence, b_{2n} which has the same limit,

$$L = \lim b_{2n} = \lim b^{1/(n)} \lim b^{1/(n)} = L^2.$$

Therefore it follows that in order to satisfy the equation, $L = 1, 0$ thus when $b_n \neq 0$ we know that $L = 1$. □

Exercise 2.5.7: Extend the result proved in Example 2.5.3 to the case where $|b| < 1$; that is, show that $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Proof. Suppose that for $0 < b < 1$, the sequence $b_n = b^n$ converges to $\lim(b^n) = 0$. By the definition of convergence, we know that for $0 < b < 1$, and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|b_n - 0| = |b^n - 0| = |b^n| = b^n < \epsilon.$$

Now consider the intermediate step,

$$|b^n| = |b|^n < \epsilon.$$

Note that values of $-1 < b < 1$ arrive at the same convergence.

Suppose the sequence $b_n = b^n$ when $-1 < b < 1$. Note that for values of $b \in (-1, 1)$ we know that the sequence returns the following,

$$-|b|^n \leq b^n \leq |b|^n.$$

Taking the limit of the inequality, simplifying with the Algebraic Limit Theorem, and substituting the result from Example 2.5.3 we get,

$$\begin{aligned} \lim(-|b|^n) \leq & \lim(b^n) \leq \lim(|b|^n), \\ -\lim(b^n) \leq & b^n \leq \lim(b^n), \\ 0 \leq & b^n \leq 0. \end{aligned}$$

Thus by Squeeze Theorem it follows that $\lim b_n = 0$. □

Exercise 2.6.2: Give an example of each of the following, or argue that such a request is impossible.

1. A Cauchy sequence that is not monotone.

   Solution:
   Consider the alternating sequence,

   $$x_n = \frac{(-1)^n}{n^2}.$$

   The sequence converges and therefore it must be Cauchy, however it is clearly not monotone.

2. A Cauchy sequence with an unbounded subsequence.

Solution:
From the Cauchy Criterion we know that all Cauchy sequences are convergent, and any subsequence of a convergent sequence is also convergent.

3. A divergent monotone sequence with a Cauchy subsequence.

Solution:
Suppose divergent monotone sequence a_n with a Cauchy subsequence a_{n_i} where $\lim a_{n_i} = L$. let $a_j \in a_n$. Now consider the element a_{n_j}; the jth term of the Cauchy subsequence, since a_n is monotone(WLOG increasing) it must be the case that $a_j \leq a_{n_j}$. Recall that $a_j \leq a_{n_j} \leq L$ thus a_n is bounded above by L and by MCT is convergent.

4. An unbounded sequence containing a subsequence that is Cauchy.

Solution:
Consider the following sequence,

$$a_n = \begin{cases} \dfrac{1}{n^2} & \text{n is even} \\ n & \text{n is odd} \end{cases}$$

The sequence is unbounded however the subsequence of even index is convergent and is therefore Cauchy.

Exercise 2.6.7 (b): Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and demonstrate where the archimedean property is implicitly required.

Proof. Suppose that a sequence a_n is bounded. Since a_n is bounded there must exist an $M > 0$ such that, $|a_n| \leq M$ for all $n \in \mathbb{N}$. Bisecting the interval $[-M, M]$ into two closed intervals $[-M, 0]$ and $[0, M]$. It must be the case that at least one of those intervals contains an infinite number of terms of the sequence a_n, we will name that interval l_1. Performing the same operation again of splitting the interval in 2 on l_1 to define l_2 and so forth until l_n. We define a subsequence a_{n_i} where, $a_{n_i} \in l_i$. By construction we know that, for all $j > m \in \mathbb{N}$

$$|a_{n_j} - a_{n_m}| < \frac{2M}{2^m}.$$

Now we will show that,

$$\frac{2M}{2^n}$$

is convergent. Recall that in Supplemental Exercise 2 from HW4, we used the Archimedean Property to prove that,

$$\lim \frac{1}{2^n} = 0.$$

Therefore by the Algebraic Limit Theorem we know that,

$$\lim \frac{2M}{2^n} = 2M(0) = 0.$$

Therefore by the definition of convergence, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ where for all $m \geq N$

$$\frac{2M}{2^m} < \epsilon.$$

Thus it follows that,

$$\begin{aligned} |a_{n_j} - a_{n_m}| &< \frac{2M}{2^m}, \\ &\leq \frac{2M}{2^N}, \\ &< \epsilon. \end{aligned}$$

Therefore the subsequence a_{n_i} is Cauchy and by the Cauchy Criterion must converge. □