Exercise 1.2.6: Use the *triangle inequality* to establish the following inequalities:

(a) $|a - b| \leq |a| + |b|$

Proof: (Direct) Suppose $a, b \in \mathbb{R}$. Note that,

$$|a - b| = |a + (-b)|.$$

By the *triangle inequality* we know that,

$$|a + (-b)| \leq |a| + |(-b)|.$$

Note,

$$|a| + |(-b)| = |a| + |b|.$$

Therefore by substitution we arrive at,

$$|a - b| \leq |a| + |b|$$

□

(b) $||a| - |b|| \leq |a - b|$.

Proof: (Direct) Suppose $a, b \in \mathbb{R}$. Note that,

$$a = (a - b) + b.$$

Therefore,

$$|a| = |(a - b) + b|.$$

Thus by *triangle inequality* we know that,

$$\begin{aligned} |a - b + b| &\leq |(a - b)| + |b|, \\ |a| &\leq |(a - b)| + |b|, \\ |a| - |b| &\leq |a - b|. \end{aligned}$$

$$|b| = |(b - a) + a|.$$

Thus by *triangle inequality* we know that,

$$\begin{aligned} |b - a + a| &\leq |(b - a)| + |a|, \\ |b| &\leq |(b - a)| + |a|, \\ |b| - |a| &\leq |b - a|. \end{aligned}$$

Since

$$|a - b| = |b - a|,$$

We get,

$$||a| - |b|| \leq |a - b|$$

□

Exercise 1.2.7(b), (d): Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x) : x \in A\}$.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
Proof: (Direct) Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Let $A = \mathbb{R}_{\leq 0}$ and $B = \mathbb{R}_{\geq 0}$. Note.

$$f(A \cap B) = \{0\}$$

and,

$$f(A) \cap f(B) = \mathbb{R}_{\geq 0}$$

Thus $f(A \cap B) \neq f(A) \cap f(B)$.

(d) Form and prove a conjecture concerning $f(A \cup B)$ and $f(A) \cup f(B)$.

Conjecture: Let $f : \mathbb{R} \to \mathbb{R}$, if $A, B \subset \mathbb{R}$ then $f(A \cup B) \subset f(A) \cup f(B)$

Proof: (Direct) Suppose $f : \mathbb{R} \to \mathbb{R}$, $A, B \subset \mathbb{R}$, and $y \in f(A \cup B)$. By the definition of the set $f(A \cup B)$ we know that there exists some $x \in A \cup B$ such that $y = f(x)$. Note that $x \in A, B$ and it therefore must follow that $y \in f(A), f(B)$. Thus $y \in f(A) \cup f(B)$ and $f(A \cup B) \subset f(A) \cup f(B)$.

□

Exercise 1.2.8: Form the logical negation of each claim. Do not use the easy way out: "It is not the case that..." is not permitted

(a) For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + (1/n) < b$.

(b) Between every two distinct real numbers there is a rational number.

(c) For all natural numbers $n \in \mathbb{N}$, $\sqrt{n}$ is either a natural number or is an irrational number.

(d) Given any real number $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.

Solution:

(a) There exists $a, b \in \mathbb{R}$ where $a < b$ and for all $n \in \mathbb{N}$, $a + (1/n) < b$.

(b) If $x \in \mathbb{R}$ then there exists $a, b \in \mathbb{R}$ such that $a < b$ and $x < a$ and $x > b$

(c) There exists some $n \in \mathbb{N}$ such that $\sqrt{n}$ is both a natural number and irrational

(d) There exists some $x \in \mathbb{R}$ such that $n > x$ for every $n \in \mathbb{N}$

Exercise 1.2: Show that the sequence $(x_1, x_2, x_3, \ldots)$ defined in Example 1.2.7 is bounded above by 2. That is, show that for every $i \in \mathbb{N}$, $x_i \leq 2$.

Proof. □

Exercise 1.3.4: Assume that A and B are nonempty, bounded above, and satisfy $B \subseteq A$. Show that $\sup B \leq \sup A$.

Proof. □

Exercise 1.3.5: Let A be bounded above and let $c \in \mathbb{R}$. Define the sets $c + A = \{a + c : a \in A\}$ and $cA = \{ca : a \in A\}$.

(a) Show that $\sup(c + A) = c + \sup(A)$.

(b) If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.

(c) Postulate a similar statment for $\sup(cA)$ when $c < 0$.

Proof (a). □

Proof (b). □

Statement for part (c):

Exercise 1.3.6: Compute, without proof, the suprema and infima of the following sets.

(a) $\{n \in \mathbb{N} : n^2 < 10\}$.

(b) $\{n/(n+m) : n, m \in \mathbb{N}\}$.

(c) $\{n/(2n+1) : n \in \mathbb{N}\}$.

(d) $\{n/m : m, n \in \mathbb{N} \text{ with } m + n \leq 10\}$.

Solution:

(a)

(b)

(c)

(d)

Exercise 1.3.7: Prove that if a is an upper bound for A an dif a is also an element of A, then $a = \sup A$.

Proof. □

Exercise 1.3.8: If $\sup A < \sup B$ then show that there exists an element $b \in B$ that is an upper bound for A.

Proof. □