**Exercise 1:** Let $A$ and $B$ be non empty sets that are bounded above.Suppose $\sup A < \sup B$. Prove that there is an element in $B$ that is an upper bound for $A$.

*Proof.* Suppose that $A$ and $B$ be non empty sets that are bounded above and that $\sup A < \sup B$. Let $x = \sup A$ and $y = \sup B$. Now consider some $z$ such that, $0 < z < y - x$. Through some algebra we can see that, $x < y - z$ and therefore the term $y - z$ must be an upper bound for $A$ since it is larger than its least upper bound. Also note that $y - z < y$ and therefore $y - z$ must be contained in $B$. □

**Exercise 2:** In class we proved that $\mathbb{N}^2$ is countably infinite. Use this fact and a proof by induction to show that $\mathbb{N}^n$ is countably infinite for every $n \in \mathbb{N}$.

*Proof.* Consider the base case where $n = 1$, clearly $\mathbb{N}^1$ is countably infinite and we have proven that $\mathbb{N}^2$ is countably infinite. We will proceed by induction on $n$. Suppose there exists some $n \in \mathbb{N}$ such that $\mathbb{N}^n$ is countably infinite. By the induction hypothesis there exists some bijection $g : \mathbb{N}^n \to \mathbb{N}$. Now consider the bijection we proved in class $f : \mathbb{N}^2 \to \mathbb{N}$. Note that the composition of these two functions gives us, $f \circ g : \mathbb{N}^{n+1} \to \mathbb{N}$ and since $f \circ g$ is a composition of bijections it must also be a bijection. Thus by induction we have shown that for all $n \in \mathbb{N}$ $\mathbb{N}^n$ is countably infinite. □

**Exercise 3:** Compute,

$$\lim_{n\to\infty} \frac{3^n}{n!}.$$

A fully rigorous proof will involve a proof by induction.

*Proof.* □

**Exercise 4:** Let $(x_n)$ and $(y_n)$ be given, and define $(z_n)$ to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$. Prove that $(z_n)$ is convergent if and only if $(x_n)$ and $(y_n)$ are both convergent with $\lim x_n = \lim y_n$.

*Proof.* Suppose convergent sequences $(x_n)$ and $(y_n)$ such that $\lim x_n = \lim y_n = l$. consider a sequence $(z_n)$ such that $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$. Note that $(z_{2n}) = (y_n)$ and $(z_{2n-1}) =$

$(x_n)$. Let $\epsilon > 0$. Since $(x_n)$ and $(y_n)$ converge we know that there exists $N_x, N_y \in \mathbb{N}$ such that,

$$|x_n - l| < \epsilon,$$

$$|y_n - l| < \epsilon.$$

Consider an $N \in \mathbb{N}$ such that $N = \max(N_x, N_m)$. By substitution we get that for all odd and even values of $(z_n)$ we get,

$$|z_n - l| < \epsilon.$$

Thus $(z_n)$ is convergent. □

*Proof.* Suppose a sequence $(z_n) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$ is convergent to some limit $(z_n) \to l$. By Theorem 2.5.2 we know that all subsequences of $(z_n)$ must converge to the same limit. Consider some $a \in x_n$ such that $x_i = a$. Note that $x_i = z_{2i-1}$ and therefore $a \in z_n$. Similarly for some $b \in y_n$ such that $y_i = b$ we know that $y_i = z_{2i}$ and therefore $b \in z_n$. Thus both $x_n$ and $y_n$ are subsequences of $z_n$ and therefore $\lim z_n = \lim x_n = \lim y_n$. □

**Exercise 5:** Suppose $F$ is a collection of open intervals such that if $I, J \in F$ and $I \neq J$, then $I \cap J = \emptyset$. Prove that $F$ is countable.

*Proof.* Suppose $F$ is a collection of open intervals such that each interval is disjoint. Note since all intervals are disjoint it must follow that the pair of $\sup J$ and $\inf J$ is unique for all $J \in F$. Consider the sequence of ordered pairs,

$$a_n = (\inf(J_1), \sup(J_1)), \ldots, (\inf(J_n), \sup(J_n)).$$

Where for all $n \in \mathbb{N}$.

$$\sup J_n \leq \sup J_{n+1}$$

$$\inf J_n \leq \inf J_{n+1}$$

Now consider the function $f : F \to \mathbb{N}$ defined by $f(a_n) = n$. Clearly the function defined by this sequence is injective and therefore $F$ is countable.

□

**Exercise 6:** Let $(x_n)$ be a sequence converging to $L$. Define,

$$y_n = \frac{x_1 + \cdots + x_n}{n}.$$

That is $y_n$ is the average of the first $n$ terms of the $x_n$ sequence. Show that $y_n = L$ as well.

*Proof.* Suppose that the sequence $(x_n)$ is convergent to $L$. Therefore by definition for all $\epsilon > 0$,

$$|x_n - L| < \epsilon.$$

Now consider the expression,

$$|y_n - L| = |\frac{x_1 + \cdots + x_n}{n} - L|.$$

Through some algebra we get,

$$\begin{aligned} |y_n - L| &= \frac{1}{n}|(x_1 + \cdots + x_n) - nL|, \\ &= \frac{1}{n}|(x_1 - L) + \cdots + (x_n - L)|. \end{aligned}$$

By triangle inequality,

$$|y_n - L| \leq \frac{1}{n}|(x_1 - L)| + \cdots + |(x_n - L)|.$$

Since $(x_n)$ is convergent to $L$ we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - L| < \epsilon.$$

By substitution we know that,

$$\begin{aligned} |y_n - L| &< \frac{n\epsilon}{n}, \\ &< \epsilon. \end{aligned}$$

Thus $y_n$ is convergent with $y_n \to L$. □

**Exercise 7:** Use the Bolzano Weierstrass Theorem to prove the Monotone Convergence Theorem without assuming any other form of the Axiom of Completeness.

*Proof.* Consider $(a_n)$, a monotone and bounded sequence. Without loss of generality let's assume the $(a_n)$ is monotone increasing. By Bolzano Weierstrass we know that there exists a convergent subsequence of $(a_n)$, $(a_{n_k}) \to L$. Therefore for all $\epsilon > 0$ there exists an $K \in \mathbb{N}$ such that for all $k \geq K$,

$$|(a_{n_k})| < \epsilon.$$

$$\begin{aligned} |(a_{n_k}) - L| &< \epsilon \\ -\epsilon <&(a_{n_k}) - L < \epsilon \\ L - \epsilon <&(a_{n_k}) < \epsilon + L \end{aligned}$$

Let $N = n_K$, note that for all $n \geq N$ there exists a $k \geq K$, such that

$$a_{n_k} \leq a_n \leq a_{n_{k+1}}.$$

Thus we get the following inequality

$$L - \epsilon < a_{n_k} \leq a_n \leq a_{n_{k+1}} < \epsilon + L.$$

Therefore $a_n$ converges to $L$. □

**Exercise 8:** Suppose $(x_n)$ is a sequence and that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|.$$

Show that the sequence $(x_n)$ converges.

*Proof.* Suppose $(x_n)$ is a sequence and that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|.$$

□