Exercise Supplemental 1: Suppose $(a_n) \to a$ and $a \neq 0$. Show that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \neq 0$.

Proof. Suppose that the sequence $(a_n) \to a$ and $a \neq 0$. Since the sequence (a_n) converges we know that for all, $\epsilon \in \mathbb{R}$, where $\epsilon < 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a| < \epsilon.$$

Consider an $\epsilon < a$ then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned} |a_n - a| &< \epsilon, \\ a - \epsilon &< a_n < a + \epsilon, \\ 0 &< a_n < a + \epsilon. \end{aligned}$$

Thus we have shown that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \neq 0$. □

Exercise Supplemental 2:

1. Show that if $a, b \geq 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$.

   Proof. Let that $a, b \geq 0$, now suppose $\sqrt{a} \leq \sqrt{b}$. Through some algebra,

   $$\begin{aligned} a &= \sqrt{a}\sqrt{a} \\ &\leq \sqrt{a}\sqrt{b} \\ &\leq \sqrt{b}\sqrt{b} \\ &= b \end{aligned}$$

   Thus we have shown that $a \leq b$, and thus by contrapositive if $a, b \geq 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$. □

2. Exercise 2.3.1(a) If $(x_n) \to 0$, show that $\sqrt{(x_n)} \to 0$

   Proof. Suppose the convergent sequence (x_n) such that $(x_n) \to 0$. Recall by the definition of convergent for all $\epsilon > 0$ we know that there exists an $N \in \mathbb{N}$ such that when $n \geq N$,

   $$|x_n| < \epsilon.$$

   Note that since this inequality is true for all $\epsilon > 0$, its also true for ϵ^2 which leaves us with,

   $$\begin{aligned} x_n &< \epsilon^2 \\ \sqrt{x_n} &< \epsilon \end{aligned}$$

   Thus we have shown that $\sqrt{(x_n)} \to 0$. □

Exercise 2.3.3: Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$ then $\lim y_n = l$ as well.

Proof. Suppose that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and that $\lim x_n = \lim z_n = l$. Let $\epsilon > 0$, since both x_n and z_n converge we know that there exists $N_x, N_z \in \mathbb{N}$ such that for all $n_x \geq N_x$, $n_z \geq N_z$, the following are true,

$$|x_{n_x} - l| \leq \epsilon$$

$$|z_{n_z} - l| \leq \epsilon$$

Now let $N = max\{N_x, N_z\}$, to ensure that the above inequalities apply. Therefore for all $n \geq N$,

$$-\epsilon < x_n - l < z_n - l < \epsilon.$$

Recall, that through algebra we get,

$$x_n \leq y_n \leq z_n,$$
$$x_n - l \leq y_n - l \leq z_n - l.$$

Therefore the following is true,

$$-\epsilon < x_n - l \leq y_n - l \leq z_n - l < \epsilon,$$
$$-\epsilon < y_n - l < \epsilon,$$
$$|y_n - l| < \epsilon.$$

Thus we have shown that $\lim y_n = l$. □

Exercise 2.3.10: Consider the following list of conjectures. Provide a short proof for those that sre true and a counterexample for any that are false.

1. If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$

   Proof. Consider $a_n = (-1)^{n+1}$ and $b_n = (-1)^n$. Clearly the following equation is true over all values of n,

   $$a_n - b_n = 0.$$

   Therefore $\lim(a_n - b_n) = 0$, yet $\lim a_n \neq \lim b_n$. □

2. If $(b_n) \to b$, then $|b_n| \to |b|$

Proof. Suppose $(b_n) \to b$. Consider that through the triangle inequality we know that (Exercise 1.2.6d),

$$|b_n - b| \geq ||b_n| - |b||.$$

Since $(b_n) \to b$ we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|b_n - b| < \epsilon.$$

Thus it follows simply that,

$$||b_n| - |b|| < \epsilon.$$

Thus we have shown that $|b_n| \to |b|$. □

3. If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

Proof. Suppose $(a_n) \to a$ and $(b_n - a_n) \to 0$. Rewriting the expression $|b_n - a|$,

$$|b_n - a| = |b_n - a_n + a_n - a|.$$

By the triangle inequality,

$$|b_n - a_n + a_n - a| \leq |b_n - a_n| + |a_n - a|.$$

Since $(a_n) \to a$ and $(b_n - a_n) \to 0$ we know that for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a| < \frac{\epsilon}{2},$$
$$|a_n - b_n| < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} |b_n - a| &\leq |b_n - a_n| + |a_n - a|, \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}, \\ &< \epsilon. \end{aligned}$$

Thus we have shown that, $(b_n) \to a$. □

4. If $a_n \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$ then $(b_n) \to b$.

Proof. Suppose $a_n \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$. Since $a_n \to 0$ we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - 0| = |a_n| < \epsilon.$$

Therefore we chain these inequalities and get,

$$|b_n - b| \leq |a_n| < \epsilon$$

Thus we have shown that, $(b_n) \to b$. □

Exercise Supplemental 3: Show that if $|b_n| \to 0$, then $b_n \to 0$. Then show that this statement is false if we replace 0 with any other real number.

Proof. Suppose the sequence $|b_n| \to 0$. Since $|b_n|$ converges we know that for all $\epsilon > 0$ there exists an $n \in \mathbb{N}$ where for all $n \geq N$,

$$||b_n| - 0| \leq \epsilon.$$

Rewriting the expression,

$$||b_n| - 0| = ||b_n|| = |b_n| = |b_n - 0|.$$

Therefore the following inequality still holds,

$$|b_n - 0| \leq \epsilon.$$

Thus we have shown that, $b_n \to 0$.

Suppose we where to replace 0 with a 1 and consider the sequence,

$$b_n = (-1)^n.$$

Clearly $|b_n| \to 1$ however $b_n \to -1$ thus the statement does not hold. □

Exercise Supplemental 4: Consider the series $\sum_{n=1}^{\infty} 1/n^2$. Give a careful proof by induction that the partial sums

$$s_k = \sum_{n=1}^{k} 1/n^2$$

satisfy $s_k \leq 2 - 1/k$.

Proof. Consider the case where $k = 1$,

$$s_1 = \frac{1}{1^2} = 1.$$

Clearly,

$$s_1 = 1 \leq 2 - \frac{1}{1} = 1.$$

We will now proceed by induction on k. Suppose there exists some $k \in \mathbb{N}$ such that,

$$s_k \leq 2 - 1/k$$

Note that by the definition of s_k we know that,

$$s_{k+1} = s_k + \frac{1}{(k+1)^2}.$$

From our induction hypothesis and using the same algebraic argument as example 2.4.4. we get that,

$$\begin{aligned} s_{k+1} &= s_k + \frac{1}{(k+1)^2} \\ &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}, \\ &< 2 - \frac{1}{k} + \frac{1}{(k)(k+1)}, \\ &= 2 - \frac{1}{k} + (\frac{1}{(k)} - \frac{1}{(k+1)}), \\ &= 2 - \frac{1}{(k+1)}. \end{aligned}$$

Thus we have proven through induction that the partial sums s_k satisfy $s_k \leq 2 - 1/k$. □

Exercise 2.4.3(a): Show that the following sequence converges and find the limit,

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

Proof. First we will prove that the sequence is bounded above by 2 using induction. Note that the sequence a_n written in the form of a recurrence relation,

$$a_{n+1} = \sqrt{2 + a_n} \tag{1}$$

Note that when $n = 1$ we see that,

$$a_1 = \sqrt{2} < 2. \tag{2}$$

now suppose that for some $n \in \mathbb{N}$ the following is true,

$$a_n \leq 2.$$

Consider the term a_{n+1} by the definition,

$$\begin{aligned} a_{n+1} &= \sqrt{2 + a_n}, \\ &\leq \sqrt{2+2}, \\ &\leq 2. \end{aligned}$$

Therefore by induction for all $n \in \mathbb{N}$ we have shown that $a_n \leq 2$ and thus the sequence a_n is bounded above by 2.

Now we will prove that the sequence is monotone increasing through induction. First note that,

$$a_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = a_1$$

Now suppose that for some $n \in \mathbb{N}$,

$$a_n \geq a_{n-1}.$$

Consider the term a_{n+1} by the definition,

$$a_{n+1} = \sqrt{2 + a_n} \geq \sqrt{2 + a_{n-1}} = a_n.$$

Thus we have shown that for all $n \in \mathbb{N}$ that $a_n \geq a_{n-1}$.

By the Monotone convergence theorem we can be certain that the series converges. To find where it converges consider the fizxed point equation,

$$\phi(x) = \sqrt{2 + x}.$$

Finding the fixed points for ϕ,

$$\begin{aligned} x &= \sqrt{2 + x}, \\ x^2 &= 2 + x, \\ x^2 - x - 2 &= 0, \\ (x - 2)(x + 1) &= 0. \end{aligned}$$

Since the sequence only produces positive real numbers we know that the series must converge to a value of 2. $\square$