Light homework. You've been working hard!

Exercise Supplemental 1: Write up a nice proof of the Alternating Series Test.

Proof. Suppose a_n is monotone decreasing and converges, $a_n \to 0$. We will demonstrate that the sum,

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

Converges by showing that the partial sums with an even number of terms, s_{2j} and the partial sums with an odd number of terms, s_{2j+1} converge using the Monotone Convergence Theorem. First consider the sequence of partial sums with an odd amount of terms, s_{2j+1}. Consider the $j+1^{th}$ term,

$$s_{2j+1+1} = s_{2j+1} - a_{2j+2} + a_{2j+1+1}.$$

Since we know that the sequence a_n is monotone decreasing we know that $a_{2j+2} \geq a_{2j+1+1}$ and thus it follows that, s_{2j+1} is monotone decreasing. By construction we also see that $a_1 - a_2$ is a lower bound for the series s_{2j+1} thus by MCT it converges to L'.

Similarly consider the sequence of partial sums with an even number of terms ,s_{2j}. First note that the $s_{2(j+1}$ term,

$$s_{2(j+1)} = s_{2j} + a_{2j+1} - a_{2(j+1)}.$$

Again since the sequence a_n is monotone decreasing we know that $a_{2j+1} \geq a_{2(j+1)}$ and thus it follows that, s_{2j} is monotone increasing. By construction we also see that a_1 is an upper bound for the series s_{2j} thus by MCT it converges to L.

Now we will show that both even and odd subsequences converge to the same limit. Consider the following true expression,

$$s_{2j} + a_{2j+1} = s_{2j+1}$$

Now consider taking the limit of both sequences,

$$\lim(s_{2j} + a_{2j+1}) = \lim(s_{2j+1})$$

By the ALT and the fact that $a_n \to 0$ we know that,

$$\lim(s_{2j}) + \lim(a_{2j+1}) = \lim(s_{2j+1})$$

$$L + 0 = L'$$

Therefore since both the even and odd sub sequences converge to the same limit the sequence of all partial sums must converge.

□

Exercise Exercise 2.7.9: Given the series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if a_n satisfies,

$$\lim \frac{a_{n+1}}{a_n} = r < 1,$$

Then the series converges absolutely.

1. Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \leq N$ implies $|a_{n+1}| \leq |a_n|r'$.

   Solution:
   By the Ratio Test and the definition of convergence we know that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

   $$||\frac{a_{n+1}}{a_n}| - r| < \epsilon$$

   Note that since $r' > r$ we know that $r' - r > 0$. Now we let $\epsilon = r' - r$, and then through some algebra we get,

   $$\begin{aligned} ||\frac{a_{n+1}}{a_n}| - r| &< r' - r, \\ |\frac{a_{n+1}}{a_n}| - r &< r' - r, \\ |\frac{a_{n+1}}{a_n}| &< r', \\ \frac{|a_{n+1}|}{|a_n|} &< r', \\ |a_{n+1}| &< r'|a|. \end{aligned}$$

2. Why does $|a_N| \sum (r')^n$ converge

   Solution:
   Note that by Example 2.7.5 we know that $\sum |a_N|(r')^n$ is a convergent geometric series, since $r < r' < 1$.

3. Now show that if $\sum |a_n|$ converges and conclude that $\sum a_n$ converges.

   Proof. By the supposition of the ratio test we found in part 1 that for some $N \in Nats$ then for all $r < r' < 1$ and $n \geq N$,

   $$|a_{n+1}| \leq |a_n|r'.$$

Therefore we can surmise that for all $i \geq 0$ and $n \geq N$,

$$|a_{n+i}| \leq |a_n| r'^i.$$

Now consider the following sum,

$$\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} |a_{N+1}| \leq \sum_{i=1}^{\infty} |a_N| r'^i$$

Recall that in part 2 we showed that the right had side of the inequality converges, therefore by the Comparison Test we know that, $\sum |a_n|$ and by the Absolute convergence test we know that $\sum a_n$ must also converge. □

Exercise 3.2.2: Let,

$$A = \{(-1)^n + \frac{2}{n} | n \in Nats\},$$
$$B = \{x \in \mathbb{Q} | 0 < x < 1\}.$$

1. What are the limit points.

   Solution:
   Consider the following two subsequences of A,

   $$A_{odd} = (-1)^{2n+1} + \frac{2}{2n+1}$$
   $$A_{even} = (-1)^{2n} + \frac{2}{2n}$$

   Note that $A_{odd} \to -1$ and $A_{even} \to 1$ therefore by Theorem 3.2.5, 1 and -1 are limit points.

   For B consider the Density of $\mathbb{Q}$ in $\mathbb{R}$ and we see that for all $x \in (0, 1)$ there exists a sequence $a_n \in \mathbb{Q}$ and therefore by Theorem 3.2.5 we know that all $x \in (0, 1)$ are limit points.

2. Is the set open? or closed?

   For set A consider the $\sup A = 2$ and note that for all $V_\epsilon(2)$ contain upper bounds of A thus $V_\epsilon(2) \nsubseteq A$.

   For set B consider the Density of $\mathbb{R} - \mathbb{Q}$ in $\mathbb{R}$, and note that for all $x \in B$ there must exist an irrational number $i \in V_\epsilon(2)$.

3. Does the set contain any isolated points?

Solution:
A is composed of all isolated points except for $\{1\}$ and B is composed of all it's rational limit points therefore non are isolated.

4. Find the closure of the set

Solution:
$\overline{A} = A \cup \{-1\}$
$\overline{B} = 0 < x < 1$

Exercise 3.2.4: Let A be nonempty and bounded above so that $s = \sup A$ exists.

1. Show that $s \in \overline{A}$.

Proof. Suppose A is nonempty and bounded above so that $s = \sup A$ exists. Consider that sequence

$$a_n = s - \frac{1}{n}.$$

Clearly the sequence a_n converges to s, (a quick proof by def will show this) and since s is a supremum there must some sub sequence of a_n that is contained in A which also converges to s. Thus by Theorem 3.2.5 we know that s is a limit point and by definition $s \in \overline{A}$.

□

2. Can an open set contain its supremum.

Proof. Suppose for the sake of contradiction that there exists an open set A where $\sup A = s$ and $s \in A$. By the definition of open we know that for all $a \in A$ there exists an $\epsilon - neighborhood$ such that,

$$V_\epsilon(a) \subseteq A.$$

Now consider the $V_\epsilon(s)$ and recall that any $x \in \mathbb{R}$ where $x > s$ we know that $x \notin A$ and therefore it must be the case that

$$V_\epsilon(s) \nsubseteq A.$$

Thus an open set cannot contain its supremum.

□