Exercise Supplemental 1: Show that the sequence $(-1)^n$ does not converge.

Proof. Suppose for the sake of contradiction that the sequence $(-1)^n$ converges to l. By the definition of converges we know that for all tolerances $\epsilon \in \mathbb{R}$ there exists some $N \in \mathbb{N}$ such that for all $n \leq N$,

$$|L - (-1)^n| < \epsilon.$$

Consider $\epsilon = 2$ and suppose $(-1)^n = 1$, then,

$$\begin{aligned} |L - 1| &< \frac{1}{2}, \\ -\frac{1}{2} < &L - 1 < \frac{1}{2}, \\ \frac{1}{2} < &L < \frac{3}{2}. \end{aligned}$$

Now suppose that $(-1)^n = -1$

$$\begin{aligned} |L + 1| &< \frac{1}{2}, \\ -\frac{1}{2} < &L + 1 < \frac{1}{2}, \\ -1 < &L < -\frac{1}{2}. \end{aligned}$$

Clearly L cannot exist in both $(-1, -\frac{1}{2})$ and $(\frac{1}{2}, \frac{3}{2})$ thus a contradiction. □

Exercise Supplemental 2:

(a) Show that for all $n \in \mathbb{N}$, $2^n \geq n$.

(b) Show that $\lim_{n\to\infty} 1/2^n = 0$.

Part (a). Consider the case where $n = 1$. Clearly,

$$\begin{aligned} 2^{(1)} &= 2 \\ &\geq (1). \end{aligned}$$

Now we will proceed by induction on n. Suppose there exists some $n \in \mathbb{N}$ such that,

$$2^n \geq n.$$

Now note that,

$$\begin{aligned} 2^n &\geq n, \\ 2^n + 1 &\geq n + 1, \\ 2^n + 2^n &\geq n + 1, \\ 2^n 2 &\geq n + 1, \\ 2^{n+1} &\geq n + 1. \end{aligned}$$

Thus by induction we have shown that for all $n \in \mathbb{N}$ $2^n \geq n$. □

Part (b). Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. Then for all $n \geq N$,

$$\begin{aligned}|0 - \frac{1}{2^n}| &= \frac{1}{2^n},\\ &\leq \frac{1}{2^N},\\ &< \epsilon.\end{aligned}$$

Thus the sequence $\frac{1}{2^n}$ converges to 0. $\square$

Exercise 2.2.2: From the definition, compute the given limits.

a.

$$\lim \frac{2n+1}{5n+4} = \frac{2}{5}$$

b.

$$\lim \frac{2n^2}{n^3+3} = 0$$

c.

$$\lim \frac{sin(n^2)}{n^{\frac{1}{3}}} = 0$$

Part (a). let $\epsilon > 0$. Note that,

$$|\frac{2}{5} - \frac{2n+1}{5n+4}| = \frac{3}{5(5n+4)} < \frac{3}{5n}.$$

Through Theorem 1.4.2(ii) we can pick $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{5\epsilon}{3}$. Then for all $n \geq N$,

$$\begin{aligned}|\frac{2}{5} - \frac{2n+1}{5n+4}| &= \frac{3}{5(5n+4)},\\ &< \frac{3}{5n},\\ &< \frac{3}{5N},\\ &< \epsilon.\end{aligned}$$

Thus the sequence $\frac{2n+1}{5n+4}$ converges to $\frac{2}{5}$. $\square$

Part (b). let $\epsilon > 0$. Note that,

$$|0 - \frac{2n^2}{n^3+3}| = \frac{2n^2}{n^3+3} \leq \frac{2n^2}{n^3} = \frac{2}{n}.$$

Through Theorem 1.4.2(ii) we can pick $N \in \mathbb{N}$ such that $\dfrac{2}{N} < \epsilon$. Then for all $n \geq N$,

$$\begin{aligned} |0 - \frac{2n^2}{n^3+3}| &= \frac{2n^2}{n^3+3}, \\ &\leq \frac{2}{N}, \\ &< \epsilon. \end{aligned}$$

Thus the sequence $\dfrac{2n^2}{n^3+3}$ converges to 0. □

Part (c). let $\epsilon > 0$. Note that the inequality,

$$|0 - \frac{sin(n^2)}{n^{\frac{1}{3}}}| = \frac{sin(n^2)}{n^{\frac{1}{3}}} \leq \frac{1}{n^{\frac{1}{3}}}.$$

Through Theorem 1.4.2(ii) we can pick $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon^3$. Then for all $n \geq N$,

$$\begin{aligned} |0 - \frac{sin(n^2)}{n^{\frac{1}{3}}}| &= \frac{sin(n^2)}{n^{\frac{1}{3}}}, \\ &\leq \frac{1}{n^{\frac{1}{3}}}, \\ &\leq \frac{1}{N^{\frac{1}{3}}}, \\ &< \epsilon. \end{aligned}$$

Thus the sequence $\dfrac{sin(n^2)}{n^{\frac{1}{3}}}$ converges to 0. □

Exercise 2.2.3: Describe what needs to be shown to disprove the given statements.

Solution:

(a) Find a college in the United States where every student is less than 7 feet tall.

(b) Find a college in the United States where no professor gives their students an A or B.

(c) show that for all colleges in the United States, there exists some student who is less than 6 feet tall.

Exercise 2.2.6: Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$ and prove that $a = b$

Proof. Suppose (a_n) is a convergent series where $(a_n) \to a$ and also that $(a_n) \to b$. By the definition of convergence we know that there exist some $\epsilon > 0$ where for $N_a \in \mathbb{N}$, and that for all $n \geq N_a$ then,

$$|a - a_n| < \frac{\epsilon}{2}$$

Likewise there exists some $N_b \in \mathbb{N}$, where for all $n \geq N_b$ such that,

$$|b - a_n| < \frac{\epsilon}{2}$$

If $N = max\{N_a, N_b\}$ then for all $n \geq N$ we know that both inequalities hold. Now through some algebra and the triangle inequality we get,

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |a - a_n| + |a_n - b| \\ &< \epsilon. \end{aligned}$$

Note that we have shown that,

$$|a - b| < \epsilon$$

is true for all $\epsilon > 0$ and thus as a consequence it must be the case that,

$$\begin{aligned} |a - b| &= 0, \\ a &= b. \end{aligned}$$

□

Exercise 2.2.5(a): Determine, with a proof, $\lim_{n\to\infty}[[5/n]]$.

Solution:
Claim: From calculating the first few numbers in the sequence I get,

$$5, 2, 1, 1, 1, 0, 0$$

Therefore I claim that $\lim_{n\to\infty}[[5/n]] = 0$

Proof. Let $\epsilon > 0$. Note that as long as we go out more than 5 elements in the sequence then the convergence condition is satisfied. Let $N = 6$ and note that for all $n \geq N$,

$$\begin{aligned} |0 - [[5/n]]| &= [[5/n]], \\ &= [[5/N]], \\ &= 0, \\ &< \epsilon. \end{aligned}$$

Thus the sequence $[[5/n]]$ converges to 0. □

Exercise 2.3.9(a)(c):

(a) If (a_n) is a bounded sequence and $b_n \to 0$, show $a_n b_n \to 0$.

(c) Prove Theorem 2.3.3(iii) for the case $a = 0$.

Solution:

(a) *Proof.* Suppose that (a_n) is a bounded sequence and $b_n \to 0$. Since (a_n) is bounded, there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all n. Since $b_n \to 0$ we know that for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|0 - b_n| = |b_n| < \frac{\epsilon}{M}.$$

Through some algebra we can see that,

$$\begin{aligned}
|a_n b_n| &= |a_n||b_n|, \\
&\leq M|b_n|, \\
&< M\frac{\epsilon}{M}, \\
&= \epsilon.
\end{aligned}$$

Note that we have shown that, $|a_n b_n| < \epsilon$ thus $a_n b_n \to 0$.

□

(c) *Proof.* Suppose a sequence (a_n) and (b_n) such that $a_n \to 0$ and $b_n \to b$. Note that since b_n converges it must be bounded and therefore from the prevous proof we get that,

$$\lim a_n b_n = 0 = 0b.$$

□