Exercise 1.4.7: Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ contradicts the assumption that $\alpha = \sup A$.

Proof. Consider the set,

$$A = \{a \in \mathbb{R} : a^2 < 2\}.$$

Let $\alpha = \sup A$. Suppose to the contrary that $\alpha^2 > 2$. Consider an element of A that os larger smaller than α,

$$\begin{aligned}(\alpha - \frac{1}{n})^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}, \\ &> \alpha^2 - \frac{2\alpha}{n}, \\ &> \alpha^2 - (\alpha^2 - 2), \\ &= 2.\end{aligned}$$

Thus we have shown that $(\alpha - \frac{1}{n})$ is greater than a for all $a \in A$ and therefore $(\alpha - \frac{1}{n})^2$ is an upper bound. Since $(\alpha - \frac{1}{n}) < \alpha$ we have contradicted $\alpha = \sup A$.

□

Exercise Supplemental 1: Give a from-scratch proof of the following facts:

(a) If $f : A \to B$ has an inverse function g, then f is injective.

(b) If $f : A \to B$ has an inverse function g, then f is surjective.

Proof (a). □

Proof (b). □

Exercise Supplemental 2: Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.

Exercise 1.5.10 (a) (c): (Wait until after Wednesday to start this one)

(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(c) Determine, with proof, if the same statement remains true replacing uncountable with infinite.

Proof (a). □

Proof (b). □

Exercise Supplemental 3: (Wait until after Wednesday to start this one) Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Use the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite to show that $\cup_{k=1}^{\infty} A_k$ is at most countable. Hint: take advantage of surjections.

Proof. □