Exercise 1: Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in X$ and that $\lim_{x\to c} f(x)$ exist. Show that the limit is non-negative. Provide two proofs, one $\epsilon - \delta$ style, and the other using the sequential characterization of limits

Proof. Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in X$ and that $\lim_{x\to c} f(x) = L$. Also suppose for the sake of contradiction that $L < 0$. By the definition of a Functional Limit we know that for all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x-c| < \delta$ it follows that $|f(x)-L| < \epsilon$. Now consider $\epsilon = -L$. By substitution we get the following inequality,

$$|f(x) - L| < -L.$$

Expanding the inequality and solving for $f(x)$ we find,

$$\begin{aligned} L &< f(x) - L < -L, \\ 2L &< f(x) < 0. \end{aligned}$$

Since $2L < 0$ we find that our final inequality implies that $f(x) < 0$ and thus a contradiction. □

Proof. Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in X$ and that $\lim_{x\to c} f(x) = L$. Sequential Characterization of Limits we know that if $\lim_{x\to c} f(x) = L$ then for all sequences $(x_n) \subseteq X$ satisfying $(x_n) \to c$ it we know that $f(x_n) \to L$. By the Order Limit Theorem we know that if $f(x_n) \geq 0$ then $L \geq 0$. □

Exercise 2: Let a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges. Suppose that $|x| < M$. Show that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. Give an example to show that divergence is possible if $|x| = |M|$. Hint: $(a_n M^n)$ converges to zero and hence bounded.

Proof. Suppose that a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges and that for some $x \in \mathbb{R}$ $|x| < M$. Consider the series $\sum_{n=1}^{\infty} |a_n x^n|$, and through algebra we see that,

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n x^n| &= \sum_{n=1}^{\infty} |a_n||x|^n, \\ &= \sum_{n=1}^{\infty} |a_n| M^n M^{-n} |x|^n, \\ &= \sum_{n=1}^{\infty} |a_n| M^n \left(\frac{|x|}{M}\right)^n. \end{aligned}$$

Since the sequence $(a_n M^n)$ converges to zero its bounded, therefore there exists some $||a_n|M^n| < A$. Therefore we get the following inequality,

$$\sum_{n=1}^{\infty} |a_n x^n| \leq \sum_{n=1}^{\infty} A \left(\frac{|x|}{M}\right)^n.$$

Recall that $|x| < M$ and therefore we can surmise that $|\frac{|x|}{M}| < 1$. Thus it follows that,

$$\sum_{n=1}^{\infty} A \left(\frac{|x|}{M}\right)^n = \frac{A}{1 - \dfrac{|x|}{M}},$$

is a convergent geometric series. Thus by the Comparison Test we get that, $\sum_{n=1}^{\infty} |a_n x^n|$ is convergent and therefore $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. □

Solution:
To show that divergence is possible if $|x| = |M|$, let $M = -1$, $x = 1$ and $a_n = \dfrac{1}{n}$. By substitution this gives us the following,

$$\sum_{n=1}^{\infty} a_n M^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

A convergent alternating series. Substituting x we get,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

The famously divergent harmonic series.

Exercise 3: Suppose that $f : (0,1] \to \mathbb{R}$ is uniformly continuous. Show that $\lim_{x\to 0} f(x)$ exists.

Proof. Suppose that $f : (0,1] \to \mathbb{R}$ is uniformly continuous. Consider some $x_n \subseteq (0,1]$ such that $x_n \to c$. Since x_n is convergent, by Theorem 2.6.4 it is also a cauchy sequence. Recall that in Exercise 4.4.6(b) we proved that on a uniformly continuous function, if x_n is cauchy then $f(x_n)$ is also cauchy. By Theorem 2.6.4 we know that since $f(x_n)$ is cauchy it also converges, and thus there exists some L such that $f(x_n) \to L$.

Demonstrating that $f(x_n) \to L$ is the same for all sequences x_n, we first suppose $x_n \to c$ and $z_n \to c$ such that $f(x_n) \to L_x$ and $f(z_n) \to L_z$ where $L_x \neq L_z$. By the Algebraic Limit

Theorem we know that $|x_n - z_n| \to 0$ and we also know that since $L_x \neq L_z$ there must exists some ϵ_0 that has the following property,

$$|f(x_n) - f(z_n)| \geq \epsilon_0.$$

Thus by Theorem 4.4.5 f is not uniformly continuous and thus a contradiction. Therefore by Theorem 4.2.3 (Sequential Criterion for Functional Limits) we know that $\lim_{x\to 0} f(x)$ exists.

□

Exercise 4: Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq c|x - y|$$

1. Show that f is continuous on $\mathbb{R}$.

   Proof. Let $\epsilon > 0$. Now consider $\delta = \frac{\epsilon}{c}$ then for all $|x - y| < \delta$ we see that from the inequality above we get that,

$$\begin{aligned}
|f(x) - f(y)| &\leq c|x - y|, \\
\frac{1}{c}|f(x) - f(y)| &\leq |x - y|, \\
\frac{1}{c}|f(x) - f(y)| &< \delta \\
\frac{1}{c}|f(x) - f(y)| &< \frac{\epsilon}{c}, \\
|f(x) - f(y)| &< \epsilon.
\end{aligned}$$

   Thus by definition f is continuous on $\mathbb{R}$. □

2. Pick some point $y_1 \in \mathbb{R}$ and construct the sequence,

$$(y_1, f(y_1), f(f(y_1)), \ldots f^n(y_1)).$$

   In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a cauchy sequence. Hence we may let $y = \lim y_n$.

Proof. Suppose the the sequence above and note that by the previous inequality for any two elements in the sequence $f^n(y_1), f^m(y_1)$ where $n > m$ we get,

$$|f^n(y_1) - f^m(y_1)| \leq c|f^{n-1}(y_1) - f^{m-1}(y_1)|.$$

Continually applying the previous inequality to the right hand side, we get an upper bound for $|f^n(y_1) - f^m(y_1)|$,

$$\begin{aligned} |f^n(y_1) - f^m(y_1)| &\leq c|f^{n-1}(y_1) - f^{m-1}(y_1)| \\ &\leq c^2|f^{n-2}(y_1) - f^{m-2}(y_1)| \\ &\leq c^m|f^{n-m}(y_1) - f^{m-m}(y_1)| \\ &= c^m|f^{n-m}(y_1) - y_1|. \end{aligned}$$

Let $M = |f^{n-m}(y_1) - y_1|$, and note that since $0 < c < 1$ by Example 2.5.3 we know that $c^m M \to 0$. By the Order Limit Theorem we know that $|f^n(y_1) - f^m(y_1)|$ is convergent and therefore (y_n) is a cauchy sequence. □

3. Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

Proof. By the preceding problem we have shown that $y = \lim y_n$ where $y_n = f^n(y_1)$. Now consider $f(y)$, and by substitution we get that,

$$\begin{aligned} f(y) &= f(\lim(y_n)), \\ &= f(\lim(y_n)), \\ &= f(\lim f^n(y_1)), \\ &= \lim f^{n+1}(y_1)), \\ &= y. \end{aligned}$$

Thus we have shown that y is fixed. Now suppose there exists some x with the property that $f(x) = x$. By substitution into our initial inequality we get,

$$\begin{aligned} |f(x) - f(y)| &\leq c|x - y|, \\ |x - y| &\leq c|x - y|, \\ 1 &\leq c. \end{aligned}$$

Thus a contradiction, therefore it must be the case that y is unique. □

4. Finally prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \dots)$ converges to the y defined in part b.

Proof. Suppose some $x \in \mathbb{R}$ and y with the property that $y = f(y)$. By substitution we get the following inequality,

$$|f(x) - y| \leq c|x - y|.$$

Similarly to part b applying this inequality to our sequence yields an upper bound,

$$\begin{aligned} |f^n(x) - y| &\leq c|f^{n_1}(x) - y| \\ &\leq c^n|x - y|. \end{aligned}$$

Since $0 < c < 1$ by Example 2.5.3 we know that $c^m|x - y| \to 0$. By the Order Limit Theorem and Absolute Convergence we know that $f^n(x) - y \to 0$. Finally by the Algebraic Limit Theorem we know that $\lim f^n(x) \to y$.

□

Exercise 5: Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous and that $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 1} f(x) = \infty$. Show that f obtains a minimum on $(0, 1)$.

Proof. Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous and that $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 1} f(x) = \infty$. By the definition of infinite limit, for all $M > 0$ we can find a $\delta_0, \delta_1 > 0$ such that whenever $0 < |x-0| < \delta_0$ and $0 < |x-1| < \delta_1$ it follows that $f(x) > M$. let $a = 0+\delta_0$ and $b = 1-\delta_1$ now consider the closed interval $[a, b]$. Note that $[a, b] \subseteq (0, 1)$. By Example 3.2.9(ii) we know that $[a, b]$ is closed and by definition its bounded above by 1 and below by 0, thus $[a, b]$ is a compact set. By Theorem 4.4.1 we can conclude that f is continuous on $[a, b]$. By the Extreme Value Theorem there exists some $x_0 \in [a, b]$ such that $f(x_0) \leq f(x)$. Note that $x_0 \in (0, 1)$ and thus f obtains a minimum on $(0, 1)$ □

Exercise 6: Show that if $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous, then it has a continuous inverse function $f^{-1} : [f(a), f(b)] \to [a, b]$. Use this result to show that $x^{1/n}$ is continuous for each $n \in \mathbb{N}$.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous. Recall that to prove a function f has an inverse we must demonstrate that f is a bijection.

Suppose $x, y \in [a, b]$ such that $x \neq y$. Without loss of generality lets suppose that $x > y$. Since f is a strictly increasing function if $x > y$ then it follows that $f(x) > f(y)$ and therefore $f(x) \neq f(y)$. Thus f is an injection

Suppose $y \in [f(a), f(b)]$. By definition we know that $f(a) \leq y \leq f(b)$. Since f is continuous we know that by the Intermediate Value Theorem there exists some $x \in [a, b]$ where $f(x) = y$. Thus f is surjective on $f : [a, b] \to [f(a), f(b)]$. Since f is a bijection we know there exists an inverse function $f^{-1} : [f(a), f(b)] \to [a, b]$.

Recall that f is continuous and by definition for any $c \in [a, b]$, for all $\epsilon_0 > 0$ there exists a $\delta_0 > 0$ such that whenever $|x - c| < \delta_0$ it follows that $|f(x) - f(c)| < \epsilon_0$. Now consider f^1 and let $z \in [f(a), f(b)]$ with the property that $f(c) = z$. Consider $\delta = \epsilon_0$ then for all $y \in [f(a), f(b)]$ with the property that $f(x) = y$, $|y - z| < \delta$ implies,

$$\begin{aligned} |f^{-1}(y) - f^{-1}(z)| &= |f^{-1}(f(x)) - f^{-1}(f(c))|, \\ &= |x - c|, \\ &< \delta = \epsilon_0. \end{aligned}$$

Thus f^{-1} is continuous.

Using this result to show that $f(x) = x^{1/n}$ is continuous for each $nin\mathbb{N}$. Consider the strictly increasing and continuous function $g(x) = x^n$, and note that $f \circ g = (x^n)^{1/n} = x$ thus by our previous result $f(x) = x^{1/n}$ is continuous.

□

Exercise 7: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and that $f([0, 1]) \subseteq (0, 1)$. Prove that there is a solution of the equation $f(x) = x$.

Proof. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and that $f([0, 1]) \subseteq (0, 1)$. Consider a continuous function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. Clearly when $g(x) = 0$ we have a solution for $f(x) = x$. Consider $g(0) = f(0) - 0$, and suppose $g(0) \neq 0$ (otherwise we would have a solution). Then since $f(0) \in f([0, 1])$ which is a subset of $(0, 1)$ it must be the case that $g(0) > 0$. Similarly consider $g(1) = f(1) - 1$, and suppose $g(1) \neq 1$. Since $f(0) \in f([0, 1])$ which is a subset of $(0, 1)$ it follows that $g(1) < 1$. By The Intermediate Value Theorem that since g is continuous and $g(1) < 0 < g(0)$ there must exist a point $c \in (a, b)$ such that $g(c) = 0$.

□

Exercise 8: If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8 we saw that if f is continuous on $[a, b]$ then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a, b]$ with $f' \neq 0$ for all $x \in [a, b]$. Show that f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f'(x)}, \text{ where } y = f(x)$$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is one-to-one, continuous, differentiable function with an inverse function f^{-1} defined on $f^{-1} : [f(a), f(b)] \to [a, b]$ and the property that $f' \neq 0$. Let $f(c) \in [f(a), f(b)]$ and consider $f(y_n) \subseteq [f(a), f(b)]$ such that $f(y_n) \to f(c)$. By the definition of the derivative using the sequential characterization of a limit,

$$f'^{-1}(f(c)) = \lim_{n\to\infty} \frac{f'^{-1}(f(y_n)) - f'^{-1}(f(c))}{f(y_n) - f(c)}.$$

Simplifying to get our limit in terms of f rather than f^{-1} and solving using the algebraic limit theorem,

$$\begin{aligned}
f'^{-1}(f(c)) &= \lim_{n\to\infty} \frac{f'^{-1}(f(y_n)) - f'^{-1}(f(c))}{f(y_n) - f(c)}, \\
&= \lim_{n\to\infty} \frac{y_n - c}{f(y_n) - f(c)}, \\
&= \lim_{n\to\infty} \left(\frac{f(y_n) - f(c)}{y_n - c} \right)^{-1}, \\
&= \frac{1}{f'(c)}.
\end{aligned}$$

$\square$