Exercise Abbott 7.4.5: Let f and g be integrable functions on $[a, b]$.

1. Show that if P is any partition of $[a, b]$, then

$$U(f + g, P) \leq U(f, P) + U(f, P).$$

Proof. First, note the following about the additions of functions,

$$(f + g)(x) = f(x) + g(x) \leq Maxf(x) + Maxg(x).$$

Recall the series definition of upper sums for where $M_{f+g,i} = \sup f(x) + f(g)$, $M_{f,i} = \sup f(x)$, and $M_{g,i} = \sup g(x)$, with $x \in \Delta x_i$.

$$U(f + g, P) = \sum_{i=1}^{n} M_{f+g,i} \Delta x_i.$$

Note that by the previous inequality we know that on every sub-interval I_i,

$$M_{f+g,i} \leq M_{f,i} + M_{g,i}.$$

Thus by substitution we get the following,

$$\begin{aligned} U(f + g, P) &= \sum_{i=1}^{n} M_{f+g,i} \Delta x_i \\ &\leq \sum_{i=1}^{n} (M_{f,i} + M_{g,i}) \Delta x_i \\ &= \sum_{i=1}^{n} M_{f,i} \Delta x_i + \sum_{i=1}^{n} M_{g,i} \Delta x_i \end{aligned}$$

$$U(f, P) + U(f, P).$$

□

As an example for the strict inequality, consider $f(x) = x$ and $g(x) = -x$ on $[-1, 1]$. Note that $f + g(x) = 0$, so $U(f + g, P) = 0$. Also note that on each sub-interval I_i,

$$M_{f,i} + M_{g,i} = f(x_i) + g(x_{1-i}).$$

Since f and g are strictly increasing and strictly decreasing respectively we know that,

$$f(x_i) + g(x_{1-i}) > 0$$

Thus we get that $0 < U(f, P) + U(f, P)$ and finally that $U(f + g, P) < U(f, P) + U(f, P)$.

2. Review the proof of Theorem 7.4.2(ii), and provide an argument for part (i) of this theorem.

Proof. Suppose f and g are integrable functions on the interval $[a, b]$. Recall that in the previous problem we demonstrated that,

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

By a similar argument we will now demonstrate that for all partitions P of $[a, b]$,

$$L(f, P) + L(g, P) \leq L(f + g, P).$$

By the definition of the addition of functions we know that,

$$(f + g)(x) = f(x) + g(x) \geq Minf(x) + Ming(x).$$

Applying this inequality to our series definition for lower sums we get the following,

$$m_{f,i} + m_{g,i} \leq m_{f+g,i}.$$

Thus,

$$L(f, P) + L(g, P) \leq L(f + g, P).$$

Finally we have the following chain inequality for all partitions P,

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P).$$

Since f and g are both integrable we have that for some partition P of $[a, b]$,

$$L(f, P) = U(f, P),$$

$$L(g, P) = U(g, P),$$

Thus it must be the case that,

$$L(f + g, P) = U(f + g, P)$$

Therefore $f + g$ is integrable with,

$$\int_a^b f + g = U(f + g, P) = U(f, P) + U(g, P) = \int_a^b f + \int_a^b g.$$

□

Exercise Abbott 7.5.1: 1. Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is f differentiable? Where does $F'(x) = f(x)$?

Solution:
Consider $f(x) = |x|$ as a piecewise function we get,

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Applying our definition of F we get the following piecewise function

$$F(x) = \begin{cases} \int_{-1}^{x} -x & x < 0 \\ \frac{1}{2} + \int_{0}^{x} x & x \geq 0 \end{cases}.$$

Using FTC to evaluate the inside integrals we get,

$$F(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}x^2 & x < 0 \\ \frac{1}{2} + \frac{1}{2}x^2 & x \geq 0 \end{cases}.$$

By our definition of F through FTC(ii) we get that F is differentiable and continuous everywhere. With piecewise differentiation we get that for all x,

$$F'(x) = f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

2. Repeat part 1 with the following function,

$$f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$$

Solution:
Using our definition of F to get the following piecewise function,

$$F(x) = \begin{cases} \int_{-1}^{x} 1 & x < 0 \\ 1 + \int_{0}^{x} 2 & x \geq 0 \end{cases}.$$

Using FTC(i) to evaluate the inside integrals,

$$F(x) = \begin{cases} x + 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}.$$

With our definition of F by FTC(ii) we get that F is is continuous every and differentiable everywhere f is continuous. Thus F is continuous on all $x \neq 0$.

Exercise Abbott 7.5.4: Show that if $f : [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a,b]$, then $f(x) = 0$ everywhere on $[a,b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all $x \in [a,b]$. Let,

$$F(x) = \int_a^x f.$$

Since f is continuous everywhere on $[a,b]$ by FTC(ii) we get that $F'(x) = f(x)$ for all $x \in [a,b]$. Note that $F(x)$ is a constant function therefore it follows that $F'(x) = 0 = f(x)$. □

For an example where f is not continuous consider the following function on the interval $[0,1]$,

$$f(x) = \begin{cases} 1 & x = \dfrac{1}{n} \\ 0 & otherwise \end{cases}.$$

Clearly the function is discountinuous. We will show that f is Riemann Integrable and that $\int_a^x f = 0$ in the next problem.

Exercise Supplemental 1:

1. Use Theorems 7.3.2 and 7.4.1 to show that if f is continuous on $[a,b]$ except at finitely many points, then f is Riemann integrable. The proof is by induction!

   Proof. Suppose f is continuous on $[a,b]$ except at finitely at many points $S \subseteq [a,b]$. Note that since S is finite its elements can be written in the form of $s_1 < s_2 < s_3 < \cdots < s_n \in S$. Consider the interval $[a, s_1]$. Note that f is continuous on $[a, s_1]$ since $[a, s_1] \cap S\{s_1\} = \emptyset$. By Theorem 7.2.9 we know that since f continuous on $[a, s_1]$ it is also integrable on $[a, s_1]$. Suppose there exists some valid $n \in \mathbb{N}$ where f is integrable on the interval $[a, s_n]$. Consider the interval $[s_n, s_{n+1}]$. Again note that f is continuous on $[s_n, s_{n+1}]$ since $[s_n, s_{n+1}] \cap S\{s_n, s_n + 1\} = \emptyset$. By Theorem 7.2.9 f is integrable on $[s_n, s_{n+1}]$. Since f is integrable on $[a, s_n]$ and $[s_n, s_{n+1}]$, by Theorem 7.4.1 f is integrable on $[a, s_{n+1}]$. Thus by induction we know that f is integrable on $[a, s_n]$ for all $s_n \in S$ and therefore by Theorem 7.3.2 we get that f is integrable on $[a,b]$. □

2. Define g on $[0, 1]$ by

$$g(x) = \begin{cases} 1 & x = 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Determine (with proof) if g is Riemann integrable or not.

Proof. Let $\epsilon > 0$ and consider the interval $[\epsilon, 1]$. Note that the set of discontinuities in $[\epsilon, 1]$ is finite. By the previous result we know that g is Riemann integrable on $[\epsilon, 1]$. Now let P be a partition on $[0, 1]$ and P_ϵ be a partition on $[\epsilon, 1]$. Note that by the density of the Irrational numbers there exists an $r \in \mathbb{I}$ inside every sub-interval I_i. Therefore it must follow that for all P, and P_ϵ,

$$L(f, P) = L(f, P_\epsilon) = 0.$$

Since $[0, 1]$ is a refinement of $[\epsilon, 1]$ by Lemma 7.2.3 we get that,

$$U(f, P_\epsilon) \geq U(f, P)$$

Recall that since g is Riemann integrable on $[\epsilon, 1]$ we know that for some P_i partition of $[\epsilon, 1]$,

$$L(f, P_i) = U(f, P_i) = 0.$$

From the previous inequality it must be the case that,

$$U(f, P_i) = U(f, P) = L(f, P) = 0$$

Thus g is Riemann integrable on $[0, 1]$ with a value of 0. □