Exercise 1: Suppose the (x_n) and (y_n) are sequences that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = \infty$. Show that $\lim_{n\to\infty} x_n/y_n = 0$.

Proof. Suppose that (x_n) and (y_n) are sequences that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} x_n = \infty$. First we will show that the series $\frac{1}{y_n}$ converges to 0. Let $\epsilon > 0$. Let $\epsilon > \frac{1}{y_N}$. Since y_n diverges there must exists some $N \geq n$ such that $\frac{1}{y_n} \geq \frac{1}{y_N}$, therefore,

$$\begin{aligned} |\frac{1}{y_n} - 0| &= \frac{1}{y_n}, \\ &\leq \frac{1}{y_N}, \\ &< \epsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} \frac{1}{y_n} = 0$. By ALT we know that $\lim_{n\to\infty} x_n/y_n = L(0) = 0$. □

Exercise 2: A number is algebraic if it is a solution of a polynomial equation,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Where each a_k is an integer, $n \geq 1$ and $a_n \neq 0$. Show that the collection of all algebraic number is countable.

Proof. Suppose an n degree polynomial with a non-zero leading term. Note that the set of roots to this polynomial is finite and by the Fundamental Theorem of Algebra there are at most n real roots. Let A_n be the set of all solutions to an n degree polynomial with a non-zero leading term. Note that since $a_n \in \mathbb{Z}$, by the multiplication property an upper bound on $|A_n|$ is given by,

$$|A_n| \leq n|\mathbb{Z}|^n.$$

Since A_n is bounded above by a countable product, A_n must also be countable. By Definition the set of all algebraic numbers is the set of all roots of all polynomial equations that have order $n \geq 1$ and non- zero leading terms. Thus the set of algebraic numbers is given by,

$$\bigcup_{i=1}^{\infty} A_i.$$

Therefore by Theorem 1.5.8 (ii) the set of all algebraic numbers is countable.

□

Exercise 3: Let p be a fifth order polynomial, so $p(x) = \sum_{k=0}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Prove that there exists a solution of $p(x) = 0$.

Proof. Suppose that p be a fifth order polynomial, so $p(x) = \sum_{k=1}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Consider the following factorization of p,

$$p(x) = x^5(a_5 + \frac{a_4}{x} + \frac{a_3}{x^2} + \frac{a_2}{x^3} + \frac{a_1}{x^4} + \frac{a_0}{x^5}).$$

By the ALT we can see that the limit as $x \to \pm\infty$ of the summand term, we get,

$$\lim_{x\to\infty}(a_5 + \frac{a_4}{x} + \frac{a_3}{x^2} + \frac{a_2}{x^3} + \frac{a_1}{x^4} + \frac{a_0}{x^5}) = a_5,$$

$$\lim_{x\to-\infty}(a_5 + \frac{a_4}{x} + \frac{a_3}{x^2} + \frac{a_2}{x^3} + \frac{a_1}{x^4} + \frac{a_0}{x^5}) = a_5.$$

Looking at the limit of $p(x)$ as $x \to \pm\infty$ we see that,

$$\lim_{x\to\infty} p(x) = x^5 a_5 = \infty,$$

$$\lim_{x\to-\infty} p(x) = x^5 a_5 = -\infty.$$

Thus there exists some $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$. Note that since $p(x)$ is a polynomial it is continuous on the domain $[a, b]$. Since $f(a) < 0 < f(b)$, then by the Intermediate Value Theorem there must exists a $c \in (a, b)$ such that $p(c) = 0$. □

Exercise 4: Let $\sum_{k=1}^{\infty} a_k$ be a series. Suppose moreover that $\lim_{k\to\infty} |a_k|^{\frac{1}{k}}$ exists and equals L. Show that the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Proof. Suppose the series $\sum_{k=1}^{\infty} a_k$ and that the $\lim_{k\to\infty} |a_k|^{\frac{1}{k}}$ exists and equals L. Note that by the ALT we know that since $\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = L$

$$\lim_{k\to\infty} |a_k|^{\frac{1}{k}^{k}} = \lim_{k\to\infty} |a_k| = L^k$$

Let $L < 1$. Note that if $L < 1$ there exist some r such that $L < r < 1$ and furthermore we know that, $L^k < r^k < 1$. □

Exercise 5: We say that a function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there is a number L such that $f(x) = f(x + L)$ for all $x \in \mathbb{R}$. Show that a continuous, periodic function is uniformly continuous.

Exercise 6: Use the Nested interval Property to deduce the Axiom of Completeness without using any other form of the Axiom of Completeness.HINT: Look at the proof of the Bolzano-Weierstrass Theorem.

Exercise 7: Let (r_n) be an enumeration of the rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{n} & x = r_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Determine, with proof, where f is continuous.

Proof. Suppose (r_n) is an enumeration of the rational numbers, and we define a function $f : \mathbb{R} \to \mathbb{R}$. Let $c \in \mathbb{Q}$. Now consider the construction of a sequence $x_n \to c$ where $x_n \notin \mathbb{R}$. Note that the function limit of the sequence $f(x) \to 0$ while $f(c) = \frac{1}{n}$ for some n. Since $x_n \to c$ and $f(x_n) \not\to f(c)$ we get that by Corollary 4.3.3 f is not continuous at $c \in \mathbb{Q}$.

Now let $c \notin \mathbb{Q}$. Suppose some sequence $x_n \to c$. Note that x_n is either comprised entirely of rational numbers, or irrational numbers or a mic of both. In all cases $f(x_n) \to 0$ and since $f(c) = 0$ by the sequential definition of continuity $f(x)$ is continuous at $c \notin \mathbb{Q}$. □

Exercise 8: Let g be defined on an interval A, and let $c \in A$.

1. Explain why $g'(c)$ in Definition 5.2.1 could have been given by,

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

2. Assume A is open. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}$$

Exercise 9: Consider the function,

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} sin(kx).$$

Show that f is differentiable.

Exercise 10: Suppose that $f : [0,1] \to \mathbb{R}$ is twice differentiable, $f(0) > 0$, $f(1) = 1$, and $f'(1) < 1$. Suppose also that $f'' > 0$ on $[0,1]$. Show that there does not exist a solution of the equation $f(x) = x$ in $[0,1)$.

Exercise 11: Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n) \to f$ uniformly on $[a,b]$ prove that f is also integrable on this set.

Exercise 12: Let,

$$L(x) = \int_1^x \frac{1}{t} dt,$$

where we consider only $x > 0$.

1. What is $L(1)$? Explain why L is differentiable and find $L'(x)$.
2. Show that $L(xy) = L(x) + L(y)$
3. Show that $L(x/y) = L(x) - L(y)$
4. Let,

$$\gamma_n = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.