Exercise 1.3.9: (a) If $\sup A < \sup B$ then show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not necessarily the case if we we only assume $\sup A \leq \sup B$.

Proof (a). □

Example for (b):

Exercise 1.3.11 : Decide if the following statements are true. Give a short proof for the true statements and a counterexample for the false statements.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$ then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$ then $\sup A < \inf B$.

Proof. □

Exercise First Edition 1.4.1: Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$ then ab and $a + b \in \mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.

(c) Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about st and $s + t$ when $s, t \in \mathbb{I}$?

Proof. □

Exercise First Edition 1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A. Show that $s = \sup A$.

Proof. □

Exercise First Edition 1.4.3: Show that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Proof. □

Exercise First Edition 1.4.4: Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = b$.

Proof. □

Exercise First Edition 1.4.5: Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 (Density of Rational Numbers) by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof. $\square$