Exercise 1: Suppose the (x_n) and (y_n) are sequences that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} x_n = \infty$. Show that $\lim_{n\to\infty} x_n/y_n = 0$.

Exercise 2: A number is algebraic if it is a solution of a polynomial equation,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Where each a_k is an integer, $n \geq 1$ and $a_n \neq 0$. Show that the collection of all algebraic number is countable.

Exercise 3: Let p be a fifth order polynomial, so $p(x) = \sum_{k=1}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Prove that there exists a solution of $p(x) = 0$.

Exercise 4: Let $\sum_{k=1}^{\infty} a_k$ be a series. Suppose moreover that $\lim_{k\to\infty} |a_k|^{\frac{1}{k}}$ exists and equals L. Show that the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Exercise 5: We say that a function $f : \mathbb{R} \to \mathbb{R}$ is periodic if there is a number L such that $f(x) = f(x + L)$ for all $x \in \mathbb{R}$. Show that a continuous, periodic function is uniformly continuous.

Exercise 6: Use the Nested interval Property to deduce the Axiom of Completeness without using any other form of the Axiom of Completeness.HINT: Look at the proof of the Bolzano-Weierstrass Theorem.

Exercise 7: Let (r_n) be an enumeration of the rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} & x = r_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Determine, with proof, where f is continuous.

Exercise 8: Let g be defined on an interval A, and let $c \in A$.

1. Explain why $g'(c)$ in Definition 5.2.1 could have been given by,

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

2. Assume A is open. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}$$

Exercise 9: Consider the function,

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} sin(kx).$$

Show that f is differentiable.

Exercise 10: Suppose that $f : [0,1] \to \mathbb{R}$ is twice differentiable, $f(0) > 0$, $f(1) = 1$, and $f'(1) < 1$. Suppose also that $f'' > 0$ on $[0,1]$. Show that there does not exist a solution of the equation $f(x) = x$ in $[0,1)$.

Exercise 11: Assume that, for each n, f_n is an integrable function on $[a, b]$. If $(f_n) \to f$ uniformly on $[a, b]$ prove that f is also integrable on this set.

Exercise 12: Let,

$$L(x) = \int_1^x \frac{1}{t} dt,$$

where we consider only $x > 0$.

1. What is $L(1)$? Explain why L is differentiable and find $L'(x)$.
2. Show that $L(xy) = L(x) + L(y)$
3. Show that $L(x/y) = L(x) - L(y)$
4. Let,

$$\gamma_n = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.